Effective algebraic degeneracy

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I – Statement of the main result

- **Complex projective algebraic hypersurface :**

$$X^n \subset \mathbb{P}^{n+1}(\mathbb{C})\,.$$

- **Homogeneous coordinates :**

$$\begin{aligned} P_{n+1}(\mathbb{C}) &= \big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}\big] \\ &= \{(z_0, z_1, \ldots, z_n, z_{n+1})\}\big/ z \sim \lambda\, z, \end{aligned}$$

where at least one z_i is nonzero.

- **In other words :** For every $\lambda \in \mathbb{C}$, $\lambda \neq 0$:

$$\big[\lambda z_0 \colon \lambda z_1 \colon \cdots \colon \lambda z_n \colon \lambda z_{n+1}\big] = \big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}\big]\,.$$

- **Homogeneous degree d polynomial :**

$$P := \sum_{\beta_0+\beta_1+\cdots+\beta_n+\beta_{n+1}=d} \mathsf{coeff} \cdot z_0^{\beta_0} z_1^{\beta_1} \cdots z_n^{\beta_n} z_{n+1}^{\beta_{n+1}}\,.$$

- **Complex projective algebraic hypersurface :**

$$\begin{aligned} X = \Big\{ \big[z_0 \colon z_1 \colon \ldots \colon z_n \colon z_{n+1}\big] \in \mathbb{P}^{n+1} \colon \\ P(z_0, z_1, \ldots, z_n, z_{n+1}) = 0 \Big\}\,. \end{aligned}$$

- **Canonical line bundle :**

$$K_X := \Lambda^n T_X^*.$$

- **General type :** As $m \to \infty$:

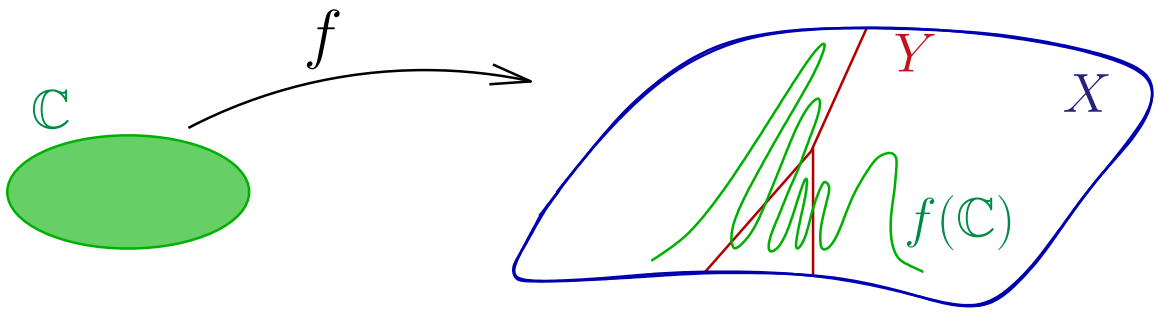
$$h^0\big(X,\, K_X^{\otimes m}\big) \sim c \cdot m^{\dim X}\,.$$

for a certain constant $c > 0$

• **Equivalent characterization :**

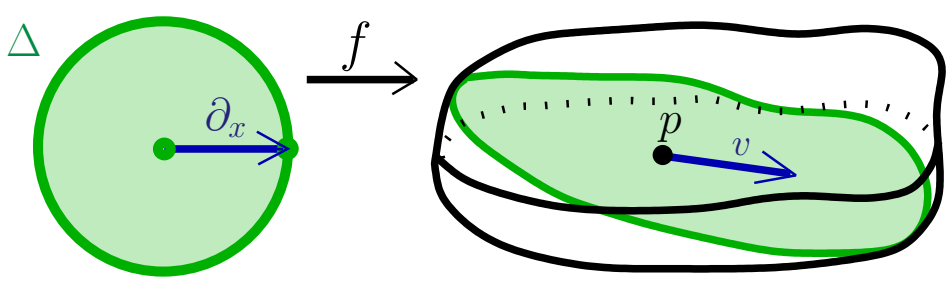
$$\boxed{\deg X \geqslant \dim X + 3}\,.$$

• **(Strong) Conjecture of Green-Griffiths (1979) :** If the projective algebraic hypersurface $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is generic of degree $d \geqslant n+3$, then there exists a proper algebraic subvariety $Y \subset X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ is necessarily completely contained inside Y, namely : $f(\mathbb{C}) \subset Y$.



• **Kobayashi-Royden infinitesimal metric :** Let M be a complex manifold, p a point of M. One defines the pseudo-norm $\mathsf{Kob}_p(v)$ of a vector $v \in T_pM$ as inversely proportional to the "infinitesimal size" of the "biggest" holomorphic discs passing through (p, v) :

$$\mathsf{Kob}_p(v) := \inf\left\{\alpha : \exists f : \Delta \to M,\ f(0) = p,\ f_*(\partial_x) = \frac{v}{\alpha}\right\}.$$



• The manifold M is said **hyperbolic** (in the sense of Kobayashi) if the integrated pseudodistance :

$$\inf_{\gamma:p\to q}\int_0^1 \mathsf{Kob}_{\gamma(t)}\big(\gamma'(t)\big)\,dt =: \mathsf{pseudodist}_{\mathsf{Kob}}(p,q)$$

(which satisfies automatically the triangle inequality) sets at finite positive distance every pair of distinct points :

$$\mathsf{pseudodist}_{\mathsf{Kob}}(p,q) > 0 \quad \text{as soon as} \quad p \neq q\,.$$

• **Brody theorem (1978) :** *A complex projective algebraic* $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *is Kobayashi-hyperbolic* if and only if *every entire holomorphic curve* $f:\mathbb{C}\to X$ *is* constant.

• **Kobayashi hyperbolicity conjecture (1970) :** A complex projective algebraic $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is **hyperbolic**, namely [Brody] : « every entire holomorphic curve $f:\mathbb{C}\to X$ is necessarily **constant** », whenever

$$\deg X \geqslant 2n+1\,,$$

provided X is generic.

• **Siu 2002, 2004 :** There exists $d_n \gg 1$ such that the generic hypersurfaces $X^n \subset \mathbb{P}^{n+1}$ of degree :

$$\deg X \geqslant d_n$$

are Kobayashi-hyperbolic.

• **Some last decade previous results :**

- **Dimension 2 :** $X^2 \subset \mathbb{P}^3(\mathbb{C})$: Green-Griffiths + Kobayashi :
  - ☐ SIU-YEUNG, 1996 : $d \geqslant 10^{13}$.
  - ☐ MCQUILLAN, 1999 : $d \geqslant \mathbf{36}$.
  - ☐ DEMAILLY-EL GOUL, 2000 : $d \geqslant \mathbf{21}$.
  - ☐ PAŬN, 2008 : $d \geqslant \mathbf{18}$.
- **Dimension 3 :** $X^3 \subset \mathbb{P}^4(\mathbb{C})$: Algebraic degeneracy :
  - ☐ ROUSSEAU 2007 : $d \geqslant \mathbf{593}$.

Theorem. [DMR, 5 feb. 2008] *If $X \subset \mathbb{P}^{n+1}$ is a generic complex projective algebraic hypersurface, there exists a proper algebraic subvariety $Y \subsetneqq X$ such that $f(\mathbb{C}) \subset Y$ for every nonconstant entire holomorphic curve :*

- *for $\dim X = \mathbf{4}$, whenever $\deg X \geqslant \mathbf{3203}$;*
- *for $\dim X = \mathbf{5}$, whenever $\deg X \geqslant \mathbf{35355}$;*
- *for $\dim X = \mathbf{6}$, whenever $\deg X \geqslant \mathbf{172925}$.*

Theorem. [DIVERIO-M.-ROUSSEAU, 17 nov. 2008] *In arbitrary dimension $n \geqslant 2$ with $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ generic, strong algebraic degeneracy of nonconstant entire holomorphic curves holds whenever :*

$$\deg X \geqslant n^{(n+1)^{(n+5)}}.$$

II – Bundles of invariant jets

- **General structure of the proof :**

Step 1 : Entire holomorphic curves $f : \mathbb{C} \to X$ must satisfy (algebraic) differential equations.

Step 2 : A plethora of such differential equations implies that entire curves degenerate inside some fixed $Y \subsetneqq X$.

- **Two main jet bundles :**
  - ☐ Green-Griffiths jets ;
  - ☐ Demailly jets.

- **Germ of holomorphic curve :**

$$f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0) \simeq (X, x).$$

- **Algebraic differential operator of order k :**

$$\mathsf{P}\big(f', f'', \ldots, f^{(k)}\big) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}^n} \mathsf{p}_{\alpha_1 \alpha_2 \ldots \alpha_k}(f) \cdot$$
$$\cdot\, (f')^{\alpha_1} (f'')^{\alpha_2} \ldots (f^{(k)})^{\alpha_k},$$

the sum being *finite*, where the $\mathsf{p}_{\alpha_1 \alpha_2 \ldots \alpha_k}(z)$ are holomorphic functions, and where :

$$(f^{(i)})^{\alpha_i} = (f_1^{(i)})^{\alpha_{i,1}} \ldots (f_n^{(i)})^{\alpha_{i,n}}.$$

- **Definition :** Denote $\mathscr{E}_{k,m}^{GG}$ the bundle (introduced by Green-Griffiths) whose sections are differential operators of order k that are homogeneous of weight :

$$m = |\alpha_1| + 2|\alpha_2| + \cdots + k|\alpha_k| \,.$$

• **Demailly-Semple invariant jets :** In local coordinates $z = (z_1, \dots, z_n)$ centered in $x \in X$, a differential operator :

$$\mathsf{P} = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} \mathsf{p}_\alpha(f)(f')^{\alpha_1} \dots (f^{(k)})^{\alpha_k}$$

is said to be **invariant under local reparametrizations** $\phi\colon (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$ if :

$$\mathsf{P}\big((f \circ \phi)', \dots, (f \circ \phi)^{(k)}\big) = \phi'(0)^m P\big(f', \dots, f^{(k)}\big) .$$

• **Structure :** Make a subbundle $\mathscr{E}^{DS}_{k,m}$ of $\mathscr{E}^{GG}_{k,m}$.

• **Examples :**

- ☐ Jets of order **1** :

$$\mathsf{E}^n_1 = \mathbb{C}\big[f'_1, f'_2, \dots, f'_n\big].$$

- ☐ Jets of order **2** :

f'_i and the : $$\begin{vmatrix} f'_{i_1} & f'_{i_2} \\ f''_{i_1} & f''_{i_2} \end{vmatrix} = \begin{vmatrix} f'_{i_1} & f'_2 \\ f''_1 + \phi'' f'_{i_1} & f''_2 + \phi'' f'_{i_2} \end{vmatrix}$$

• **Concretely :** Set $g_i := f_i \circ \phi$ and compute :

$$\begin{aligned}
g'_i &= \phi' f'_i, \\
g''_i &= \phi'' f'_i + \phi'^2 f''_i, \\
g'''_i &= \phi''' f'_i + 3\,\phi''\phi' f''_i + \phi'^3 f'''_i, \\
g''''_i &= \phi'''' f'_i + 4\,\phi'''\phi' f''_i + 3\,\phi''^2 f''_i + 6\,\phi''\phi'^2 f'''_i + \phi'^4 f''''_i, \\
g'''''_i &:= \phi''''' f'_i + 5\,\phi''''\phi' f''_i + 10\,\phi'''\phi'' f''_i + 15\,\phi''^2\phi' f'''_i + \\
&\qquad + 10\,\phi'''\phi'^2 f'''_i + 10\,\phi''\phi'^3 f''''_i + \phi'^5 f'''''_i.
\end{aligned}$$

$$\mathsf{P}\big(g', g'', g''', g'''', g'''''\big) = \big(\phi'\big)^m \mathsf{P}\big(f', f'', f''', f'''', f'''''\big).$$

- **Step 1 :** Construct **at least one** differential equation.

Theorem. [BLOCH, AHLFORS, GREEN-GRIFFITHS, DEMAILLY, SIU] *Let X be a complex projective algebraic hypersurface, let A be an ample line bundle on X — just take $A = \mathscr{O}_X(1)$ — and let :*

$$\mathsf{P} \in H^0\big(X, \mathscr{E}_{k,m}^{GG \text{ or } DS} \otimes A^{-1}\big)$$

be a global section. Then every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ satisfies the corresponding differential equation :

$$\mathsf{P}\big(f', \dots, f^{(k)}\big) \equiv 0.$$

- **Algebraic difficulties :**

☐ No such $\mathsf{P}(j^k f)$ exist when : jet order $<$ dimension.

☐ Explore the cohomology of jet bundles.

☐ Understand the condition of invariancy in the definition of Demailly invariant jets.

- **Known algebraic descriptions of $\mathscr{E}_{k,m}^{DS}$:**

☐ $n = 2$, $k = 3$: DEMAILLY (unpublished) ; ROUSSEAU. **5**.

☐ $n = 3$, $k = 3$: ROUSSEAU. **16**.

☐ $n = 2$, $k = 4$: DEMAILLY-EL GOUL (unpublished) ; M. **9**.

☐ $n = \mathbf{2}$, $k = \mathbf{5}$: M. **56**.

☐ $n = \mathbf{4}$, $k = \mathbf{4}$: M. **2835**.

III - Explicit algebras of invariants

• Joël M. *Jets de Demailly-Semple d'ordres 4 et 5 en dimension 2*, Int. J. Contemp. Math. Sciences, **3** (2008) no. 18, 861–933.

• Joël M. *An algorithm to generate all polynomials in the k-jet of a holomorphic disc $D \to \mathbb{C}^n$ that are invariant under source reparametrization*, arxiv.org/abs/0808.3547/, 103 pages.

• Joël M. *Low pole order frames on vertical jets of the universal hypersurface*, Ann. Inst. Fourier (Grenoble), to appear, 31 pages.

• Simone Diverio, Joël M. and Erwan Rousseau, *Effective algebraic degeneracy*, arxiv.org/abs/0811.2346/, 47 pages.

First main goal

Explain Green-Griffiths algebraic degeneracy for $\dim X = \mathbf{4}$ *whenever* $\deg X \geqslant \mathbf{3203}$.

Second main goal

Explain Green-Griffiths algebraic degeneracy for $\dim X = \mathbf{n}$ *whenever* $\deg X \geqslant n^{(n+1)^{n+5}}$.

III-2 – A general algorithm

Theorem. [M., 2008] *Construction of a complete algorithm which generates all jet polynomials invariant under reparametrization, in arbitrary dimension n and for jets of arbitrary order k.*

• **First illustration :** Dimension $n = 3$ and jet of order $k = 3$ [Rousseau 2006 ; M. 2007] : $\boxed{\textbf{4}\text{ bi-invariants}}$.

• **9 jet variables :**

$$\left(f_1', f_2', f_3', f_1'', f_2'', f_3'', f_1''', f_2''', f_3'''\right).$$

• **Double trick of reparametrizing by f_1^{-1} and of requiring unipotent invariance yields an initial form :**

$$\boxed{\mathsf{P}\big(j^3 f\big) = \sum\nolimits_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}_a\Big(\Lambda^3, \Lambda^5, D^6\Big).}$$

• **Explicit expressions :** Each polynomial P_a depends upon the following basic bi-invariants :

$$\Lambda^3 := \begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix} =: \Delta_{1,2}^{',''},$$

$$\Lambda^5 := \Delta_{1,2}^{','''} f_1' - 3\,\Delta_{1,2}^{',''} f_1'',$$

$$D^6 := \begin{vmatrix} f_1' & f_2' & f_3'' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix} =: \Delta^{','','''}.$$

• **Observation : these 3 bi-invariants are algebraically independent, even after setting $f_1' = 0$:**

$$\Lambda^3\big|_0 = -f_2' f_1'',$$
$$\Lambda^5\big|_0 = f_2'\, f_1''\, f_1'',$$
$$D^6\big|_0 = \begin{vmatrix} 0 & f_2' & f_3'' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}.$$

Theorem. *Then in the sum :*

$$\mathsf{P}\big(j^3 f\big) = \sum\nolimits_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(\Lambda^3,\, \Lambda^5,\, D^6\Big),$$

there are in fact no negative powers of f_1', so that :

$$\mathsf{UDS}_3^3 = \mathbb{C}\big[f_1',\, \Lambda^3,\, \Lambda^5,\, D^6\big]\,.$$

Proof. Ortherwise, get by chasing the denominators :

$$(f_1')^{-a_{\min}}\, \mathsf{P}\big(j^3 f\big) = \mathsf{P}_{a_{\min}}\big(\Lambda^3,\, \Lambda^5,\, D^6\big) + f_1' \times \text{remainder}\,.$$

Then set $f_1' = 0$ hence get :

$$0 \equiv \mathsf{P}_{a_{\min}}\big(\Lambda^3\big|_0,\, \Lambda^5\big|_0,\, D^6\big|_0\big),$$

whence $\mathsf{P}_{-a_{\min}} = 0$ by the algebraic independence of $\Lambda^3\big|_0$, $\Lambda^5\big|_0$, $D^6\big|_0$, which visibly contradicts the definition of $-a_{\min}$. $\square$

• **Anticipated reinterpretation :**

$$\text{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ D^6\big|_0\Big) = \{0\}\,.$$

Corollary. [Rousseau 2006 ; M. 2007] *By polarizing these* **4** *bi-invariants, one obtains that the full algebra* DS^3_3 *of invariants under reparametrization is generated by the* **16** *bi-invariants :*

$$f'_i,\ \Lambda^3_{i,j},\ \Lambda^5_{i,j;\,k},\ D^5.$$

• **Second illustration :** dimension $n = 4$ and jet order $k = 4$.

• **16 jet variables :**

$$\left(f'_1, f'_2, f'_3, f'_4, f''_1, f''_2, f''_3, f''_4, f'''_1, f'''_2, f'''_3, f'''_4, f''''_1, f''''_2, f''''_3, f''''_4\right).$$

• **Same double trick and some computations provide an initial representation :**

$$\mathsf{P}\big(j^4 f\big) = \sum\nolimits_{-\frac{3}{4}m \leqslant a \leqslant m} (f'_1)^a\, \mathsf{P}_a\Big(\Lambda^3,\ \Lambda^5,\ \Lambda^7,\ D^6,\ D^8,\ N^{10},\ W^{10}\Big).$$

• **Explicit expressions :** Each polynomial P_a depends upon the following seven basic bi-invariants :

$$\Lambda^7 := \big(\Delta^{',''''}_{1,2} + 4\,\Delta^{'','''}_{1,2}\big)\, f'_1 f'_1 - \\ - 10\,\Delta^{','''}_{1,2}\, f'_1 f''_1 + 15\,\Delta^{',''}_{1,2}\, f''_1 f''_1,$$

$$D^8 := f'_1 \begin{vmatrix} f'_1 & f'_2 & f''_3 \\ f''_1 & f''_2 & f''_3 \\ f''''_1 & f''''_2 & f''''_3 \end{vmatrix} - 6\, f''_1 \begin{vmatrix} f'_1 & f'_2 & f''_3 \\ f''_1 & f''_2 & f''_3 \\ f'''_1 & f'''_2 & f'''_3 \end{vmatrix},$$

$$N^{10} := \Delta^{\prime,\prime\prime\prime,\prime\prime\prime\prime}_{1,2,3}\, f_1' f_1' - 3\,\Delta^{\prime,\prime\prime,\prime\prime\prime\prime}_{1,2,3}\, f_1' f_1'' + \\ + 4\,\Delta^{\prime,\prime\prime,\prime\prime\prime}_{1,2,3}\, f_1' f_1''' + 3\,\Delta^{\prime,\prime\prime,\prime\prime\prime}_{1,2,3}\, f_1'' f_1'',$$

and finally, the **Wronskian** :

$$W^{10} := \begin{vmatrix} f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ f_1''' & f_2''' & f_3''' & f_4''' \\ f_1'''' & f_2'''' & f_3'''' & f_4'''' \end{vmatrix}.$$

- **Starting point for the algorithm :**

$$\boxed{\mathsf{P}\big(j^5 f\big) = \sum\nolimits_{-\frac{3}{4}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(\Lambda^3,\, \Lambda^5,\, \Lambda^7,\, D^6,\, D^8,\, N^{10},\, W^{10}\Big)}$$

- **Presence of negative powers of f_1' !**

□ **Compute the Ideal of Relations of these seven bi-invariants restricted to $\{f_1' = 0\}$:**

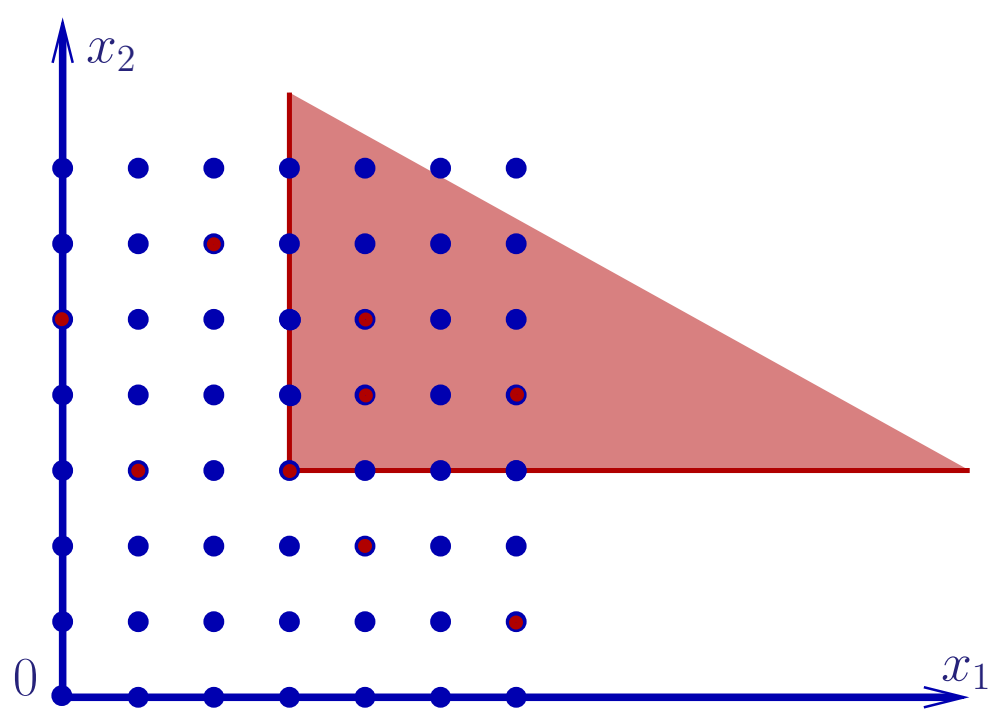
$$\text{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ D^6\big|_0,\ D^8\big|_0,\ N^{10}\big|_0,\ W^{10}\big|_0\Big),$$

namely a generating set of the ideal of all polynomials in seven variables that give zero, identically, after substituting these four restricted invariants.

Get the six relations valuable for $f_1' = 0$:

$$
\begin{aligned}
0 &\overset{a}{\equiv} 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7\Big|_0,\\
0 &\overset{b}{\equiv} 2\,\Lambda^5 D^6 - \Lambda^3 D^8\Big|_0,\\
0 &\overset{c}{\equiv} \Lambda^7 D^6 - 5\,\Lambda^3 N^{10}\Big|_0,\\
0 &\overset{d}{\equiv} \Lambda^5 D^8 - 6\,\Lambda^3 N^{10}\Big|_0,\\
0 &\overset{e}{\equiv} \Lambda^7 D^8 - 10\,\Lambda^5 N^{10}\Big|_0,\\
0 &\overset{f}{\equiv} D^8 D^8 - 12\,D^6 N^{10}\Big|_0.
\end{aligned}
$$

Gröbner bases ; Dickson lemma ; S-polynomials



□ **So without setting $f_1' = 0$, there should be six remainders that are a multiple of f_1' :**

$$0 \overset{a}{\equiv} 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7 + f_1' \times \text{something},$$
$$0 \overset{b}{\equiv} 2\,\Lambda^5 D^6 - \Lambda^3 D^8 + f_1' \times \text{something},$$
$$0 \overset{c}{\equiv} \Lambda^7 D^6 - 5\,\Lambda^3 N^{10} + f_1' \times \text{something},$$
$$0 \overset{d}{\equiv} \Lambda^5 D^8 - 6\,\Lambda^3 N^{10} + f_1' \times \text{something},$$
$$0 \overset{e}{\equiv} \Lambda^7 D^8 - 10\,\Lambda^5 N^{10} + f_1' \times \text{something},$$
$$0 \overset{f}{\equiv} D^8 D^8 - 12\,D^6 N^{10} + f_1' \times \text{something}.$$

□ **Each something necessarily also is a bi-invariant.**

□ **Find the maximal power of f_1' which factors each something.**

- **Get the six completed expressions :**

$$0 \overset{a}{\equiv} 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7 + f_1' f_1' M^8,$$
$$0 \overset{b}{\equiv} 2\,\Lambda^5 D^6 - \Lambda^3 D^8 + \tfrac{1}{3}\, f_1' E^{10},$$
$$0 \overset{c}{\equiv} \Lambda^7 D^6 - 5\,\Lambda^3 N^{10} + f_1' L^{12},$$
$$0 \overset{d}{\equiv} \Lambda^5 D^8 - 6\,\Lambda^3 N^{10} + f_1' L^{12},$$
$$0 \overset{e}{\equiv} \Lambda^7 D^8 - 10\,\Lambda^5 N^{10} - f_1' Q^{14},$$
$$0 \overset{f}{\equiv} D^8 D^8 - 12\,D^6 N^{10} - f_1' R^{15}.$$

- **Test whether or not the obtained bi-invariants :**

M^8 E^{10} L^{12} Q^{14} R^{15}

belong or do not belong to the algebra generated by the previously known bi-invariants :

f_1' Λ^3 Λ^5 Λ^7 D^6 D^8 N^{10} W^{10}

- **Here : none of the above 13 bi-invariants is equal to a polynomial with respect to the 12 remaining ones.**

- **Compute the explicit expressions :**

$$M^8 := \frac{-5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7}{f_1' f_1'}$$
$$= 3\,\Delta_{1,2}^{',''''}\,\Delta_{1,2}^{',''} + 12\,\Delta_{1,2}^{'','''}\,\Delta_{1,2}^{',''} - 5\,\Delta_{1,2}^{','''}\,\Delta_{1,2}^{','''},$$

$$E^{10} := \frac{-6\,\Lambda^5\,D^6 + 3\,\Lambda^3\,D^8}{f_1'}$$
$$= 3\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2} - 6\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\,,$$

$$L^{12} := \frac{-\Lambda^7 D^6 + 5\,\Lambda^3 N^{10}}{f_1'}$$
$$= -\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''''}_{1,2}\;f_1' - 4\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{'',\,'''}_{1,2}\;f_1' + 5\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1' + 10\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\,f_1'' -$$
$$- 15\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'' + 20\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'''\,,$$

$$Q^{14} := \frac{\Lambda^7 D^8 - 10\,\Lambda^5 N^{10}}{f_1'}$$
$$= -10\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1' f_1' + \Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''''}_{1,2}\;f_1' f_1' + 4\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{'',\,'''}_{1,2}\;f_1' f_1' +$$
$$+ 20\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1' f_1'' + 30\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1' f_1'' - 6\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,''''}_{1,2}\;f_1' f_1'' -$$
$$- 24\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{'',\,'''}_{1,2}\;f_1' f_1'' - 40\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1' f_1''' - 75\,\Delta^{',\,'',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'' f_1'' +$$
$$+ 30\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1'' f_1'' + 120\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'' f_1'''\,,$$

$$R^{15} := \Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;f_1' - 12\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;f_1' + 24\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;f_1'' -$$
$$- 48\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;f_1'''\,,$$

□ **Then restart the process with the 12 bi-invariants.**

$$\text{Ideal-Rel}\left(\Lambda^3\big|_0,\;\Lambda^5\big|_0,\;\Lambda^7\big|_0,\;D^6\big|_0,\;D^8\big|_0,\;N^{10}\big|_0,\;M^8\big|_0,\;E^{10}\big|_0,\;L^{12}\big|_0,\;Q^{14}\big|_0,\;R^{15}\big|_0,\;W^{10}\big|_0\right)$$

□ **Compute the Ideal of Relations of these bi-invariants restricted to $\{f_1' = 0\}$:**

$$0 \overset{g}{\equiv} 4\,D^8Q^{14} - 5\,\Lambda^7R^{15} - f_1'X^{21},$$

$$0 \overset{h}{\equiv} 24\,D^6Q^{14} - 25\,\Lambda^5R^{15} + f_1'V^{19},$$

$$0 \overset{i}{\equiv} L^{12}L^{12} + E^{10}Q^{14} - f_1'M^8R^{15},$$

$$0 \overset{j}{\equiv} 8\,N^{10}L^{12} + \Lambda^7R^{15} + f_1'X^{21},$$

$$0 \overset{k}{\equiv} 4\,D^8L^{12} + 5\,\Lambda^5R^{15} - f_1'V^{19},$$

$$0 \overset{l}{\equiv} 8\,D^6L^{12} + 5\,\Lambda^3R^{15} - \tfrac{1}{3}\,f_1'U^{17},$$

$$0 \overset{m}{\equiv} \Lambda^7L^{12} + \Lambda^5Q^{14} - 2\,f_1'M^8N^{10},$$

$$0 \overset{n}{\equiv} 5\,\Lambda^5L^{12} + 3\,\Lambda^3Q^{14} - f_1'D^8M^8,$$

$$0 \overset{o}{\equiv} 8\,N^{10}E^{10} + \Lambda^5\,R^{15} - f_1'V^{19},$$

$$0 \overset{p}{\equiv} 4\,D^8E^{10} + 3\,\Lambda^3R^{15} - f_1'U^{17},$$

$$0 \overset{q}{\equiv} 5\,\Lambda^7E^{10} + 3\,\Lambda^3Q^{14} - 6f_1'D^8M^8,$$

$$0 \overset{r}{\equiv} 5\,\Lambda^5E^{10} - 3\,\Lambda^3L^{12} - 6\,f_1'D^6M^8,$$

$$0 \overset{s}{\equiv} 8\,\Lambda^5N^{10}Q^{14} - \Lambda^7\Lambda^7R^{15} + f_1'Q^{14}Q^{14} + 4\,f_1'N^{10}N^{10}M^8,$$

$$0 \overset{t}{\equiv} 24\,\Lambda^3N^{10}Q^{14} - 5\,\Lambda^5\Lambda^7R^{15} - 5\,f_1'L^{12}Q^{14} + 2\,f_1'M^8D^8N^{10}.$$

□ **Get 3 new bi-invariants :**

$$U^{17}, \qquad V^{19}, \qquad X^{21},$$

that are defined explicitly by :

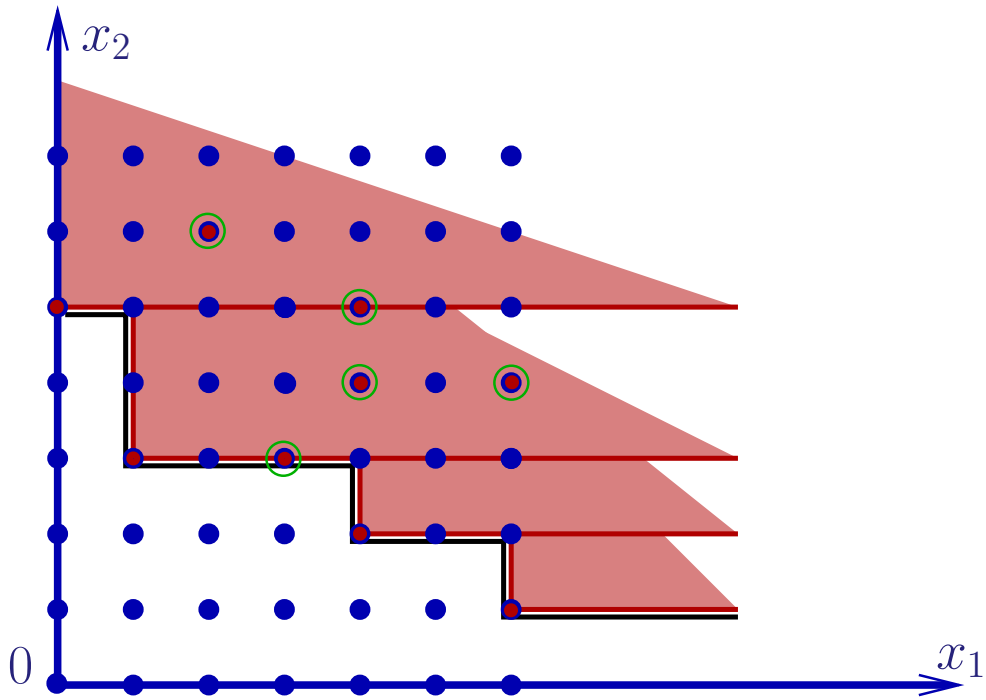
$$
\begin{aligned}
U^{17} &= \frac{4\,D^8E^{10}+3\,\Lambda^3R^{15}}{f_1'}\\
&= 15\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2} - 36\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2} -\\
&\quad - 24\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2} + 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,,
\end{aligned}
$$

$$
\begin{aligned}
V^{19} &= \frac{8\,N^{10}E^{10}+\Lambda^5R^{15}}{f_1'}\\
&= 24\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1' - 60\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1' +\\
&\quad + \Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1' - 75\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1'' +\\
&\quad + 36\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1'' + 168\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'' -\\
&\quad - 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'' + 96\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''' -\\
&\quad - 240\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1''',
\end{aligned}
$$

$$
\begin{aligned}
X^{21} &= \frac{4\,D^8Q^{14}-5\,\Lambda^7R^{15}}{f_1'}\\
&= -40\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1' - 4\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1' -\\
&\quad - 4\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1' + 60\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1' +\\
&\quad + 240\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1' + 130\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1'' +\\
&\quad + 120\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1'f_1'' - 168\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1'' -\\
&\quad - 668\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1'' - 360\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1'' -\\
&\quad - 160\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1''' + 240\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1''' +\\
&\quad + 960\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1''' - 375\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''f_1'' +\\
&\quad + 840\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1''f_1'' + 180\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''f_1'' +\\
&\quad + 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1''f_1'' + 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1''f_1'' -\\
&\quad - 1440\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1''f_1''' + 480\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''f_1'''.
\end{aligned}
$$

□ **Termination of the algorithm : Use plain lexicographic ordering of the 14 bi-invariants :**

$$\Lambda^3 > \Lambda^5 > \Lambda^7 > D^6 > D^8 > N^{10} > M^8 > E^{10} > L^{12} > \\ > Q^{14} > R^{15} > U^{17} > V^{19} > X^{21},$$



□ **Obtain 41 completed syzygies between these 14 bi-invariants :**

$$0 \overset{1}{\equiv} -5\,\Lambda^5\Lambda^5 + 3\,\underline{\Lambda^3\Lambda^7}_{\mathsf{LT}} - f_1' f_1' M^8,$$

$$0 \overset{2}{\equiv} -2\,\Lambda^5 D^6 + \underline{\Lambda^3 D^8}_{\mathsf{LT}} - \tfrac{1}{3} f_1'\, E^{10},$$

$$0 \overset{3}{\equiv} -\Lambda^7 D^6 + 5\,\underline{\Lambda^3 N^{10}}_{\mathsf{LT}} - f_1' L^{12},$$

$$0 \overset{4}{\equiv} -5\,\Lambda^5 E^{10} + 3\,\underline{\Lambda^3 L^{12}}_{\mathsf{LT}} + 6\, f_1' D^6 M^8,$$

$$0 \overset{5}{\equiv} 5\,\Lambda^7 E^{10} + 3\,\underline{\Lambda^3 Q^{14}}_{\mathsf{LT}} - 6\, f_1' D^8 M^8,$$

$$0 \overset{6}{\equiv} 4\, D^8 E^{10} + 3\,\underline{\Lambda^3 R^{15}}_{\mathsf{LT}} - f_1' U^{17},$$

$$0 \overset{7}{\equiv} -36\, D^6 D^6 M^8 - 5\, E^{10} E^{10} + 3\, \underline{\Lambda^3 U^{17}}_{\mathsf{LT}} + 0,$$
$$0 \overset{8}{\equiv} -5\, E^{10} L^{12} - 6\, D^6 D^8 M^8 + 3\, \underline{\Lambda^3 V^{19}}_{\mathsf{LT}} + 0,$$
$$0 \overset{9}{\equiv} 5\, L^{12} L^{12} + 3\, \underline{\Lambda^3 X^{21}}_{\mathsf{LT}} + M^8 D^8 D^8 + 0,$$
$$0 \overset{10}{\equiv} -6\, \Lambda^7 D^6 + 5\, \underline{\Lambda^5 D^8}_{\mathsf{LT}} - f_1' L^{12},$$
$$0 \overset{11}{\equiv} -\Lambda^7 D^8 + 10\, \underline{\Lambda^5 N^{10}}_{\mathsf{LT}} + f_1' Q^{14},$$
$$0 \overset{12}{\equiv} \underline{\Lambda^5 L^{12}}_{\mathsf{LT}} - \Lambda^7 E^{10} + f_1' D^8 M^8,$$

$$0 \overset{13}{\equiv} \Lambda^7 L^{12} + \underline{\Lambda^5 Q^{14}}_{\mathsf{LT}} - 2\, f_1' M^8 N^{10},$$
$$0 \overset{14}{\equiv} 8\, N^{10} E^{10} + \underline{\Lambda^5 R^{15}}_{\mathsf{LT}} - f_1' V^{19},$$
$$0 \overset{15}{\equiv} \underline{\Lambda^5 U^{17}}_{\mathsf{LT}} - E^{10} L^{12} - 6\, D^6 D^8 M^8 + 0,$$
$$0 \overset{16}{\equiv} \underline{\Lambda^5 V^{19}}_{\mathsf{LT}} - M^8 D^8 D^8 - L^{12} L^{12} + f_1' M^8 R^{15},$$
$$0 \overset{17}{\equiv} \underline{\Lambda^5 X^{21}}_{\mathsf{LT}} - L^{12} Q^{14} + 2\, D^8 N^{10} M^8 + 0,$$
$$0 \overset{18}{\equiv} 8\, N^{10} L^{12} + \underline{\Lambda^7 R^{15}}_{\mathsf{LT}} + f_1' X^{21},$$

$$0 \overset{19}{\equiv} -L^{12} L^{12} + \underline{\Lambda^7 U^{17}}_{\mathsf{LT}} - 5\, M^8 D^8 D^8 + 0,$$
$$0 \overset{20}{\equiv} L^{12} Q^{14} + \underline{\Lambda^7 V^{19}}_{\mathsf{LT}} - 10\, D^8 M^8 N^{10} + 0,$$

$$0 \overset{21}{\equiv} 20\, N^{10} N^{10} M^8 + Q^{14} Q^{14} + \underline{\Lambda^7 X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{22}{\equiv} 6\, \underline{D^6 M^8 R^{15}}_{\mathsf{LT}} + L^{12} U^{17} - E^{10} V^{19} + 0,$$
$$0 \overset{23}{\equiv} 5\, \underline{D^8 M^8 R^{15}}_{\mathsf{LT}} - Q^{14} U^{17} - L^{12} V^{19} + 0,$$
$$0 \overset{24}{\equiv} 10\, \underline{N^{10} M^8 R^{15}}_{\mathsf{LT}} - Q^{14} V^{19} + L^{12} X^{21} + 0,$$

$$0 \overset{25}{\equiv} 5\,\underline{M^8R^{15}R^{15}}_{\mathsf{LT}} + V^{19}V^{19} + U^{17}X^{21} + 0,$$

$$0 \overset{26}{\equiv} -D^8D^8 + 12\,\underline{D^6N^{10}}_{\mathsf{LT}} + f_1'R^{15},$$

$$0 \overset{27}{\equiv} -5\,D^8E^{10} + 6\,\underline{D^6L^{12}}_{\mathsf{LT}} + f_1'U^{17},$$

$$0 \overset{28}{\equiv} 3\,\underline{D^6Q^{14}}_{\mathsf{LT}} + 25\,N^{10}E^{10} - 3\,f_1'V^{19},$$

$$0 \overset{29}{\equiv} 5\,E^{10}R^{15} - D^8U^{17} + 6\,\underline{D^6V^{19}}_{\mathsf{LT}} + 0,$$

$$0 \overset{30}{\equiv} -3\,L^{12}R^{15} + N^{10}U^{17} + 3\,\underline{D^6X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{31}{\equiv} -10\,N^{10}E^{10} + \underline{D^8L^{12}}_{\mathsf{LT}} + f_1'V^{19},$$

$$0 \overset{32}{\equiv} \underline{D^8Q^{14}}_{\mathsf{LT}} + 10\,N^{10}L^{12} + f_1'X^{21},$$

$$0 \overset{33}{\equiv} -2\,N^{10}U^{17} + \underline{D^8V^{19}}_{\mathsf{LT}} + L^{12}R^{15} + 0,$$

$$0 \overset{34}{\equiv} Q^{14}R^{15} + 2\,N^{10}V^{19} + \underline{D^8X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{35}{\equiv} -2\,L^{12}N^{10}U^{17} + R^{15}L^{12}L^{12} + 10\,\underline{V^{19}N^{10}E^{10}}_{\mathsf{LT}} - f_1'V^{19}V^{19},$$

$$0 \overset{36}{\equiv} 2\,N^{10}U^{17}Q^{14} - R^{15}L^{12}Q^{14} + 10\,\underline{V^{19}N^{10}L^{12}}_{\mathsf{LT}} + f_1'V^{19}X^{21},$$

$$0 \overset{37}{\equiv} 10\,\underline{N^{10}L^{12}X^{21}}_{\mathsf{LT}} - R^{15}Q^{14}Q^{14} - 2\,Q^{14}N^{10}V^{19} + f_1'X^{21}X^{21},$$

$$0 \overset{38}{\equiv} 2\,\underline{N^{10}U^{17}X^{21}}_{\mathsf{LT}} - X^{21}L^{12}R^{15} + V^{19}Q^{14}R^{15} + 2\,N^{10}V^{19}V^{19} + 0,$$

$$0 \overset{39}{\equiv} \underline{E^{10}Q^{14}}_{\mathsf{LT}} + L^{12}L^{12} - f_1'M^8R^{15},$$

$$0 \overset{40}{\equiv} Q^{14}U^{17} + 6\,L^{12}V^{19} + 5\,\underline{E^{10}X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{41}{\equiv} -6\,Q^{14}L^{12}V^{19} - Q^{14}Q^{14}U^{17} + 5\,\underline{X^{21}L^{12}L^{12}}_{\mathsf{LT}} - 5\,f_1'M^8R^{15}X^{21}.$$

Theorem. (M. 2008) *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra UDS^4_4 of jet polynomials $\mathsf{P}\big(j^4 f_1, j^4 f_2, j^4 f_3, j^4 f_4\big)$ invariant by reparametrization and invariant under the unipotent action is generated by the* **16** *mutually independent bi-invariants defined above :*

$$\boxed{\begin{array}{cccccccc} W^{10}, & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & D^6, & D^8, & N^{10}, \\ M^8, & E^{10}, & L^{12}, & Q^{14}, & R^{15}, & U^{17}, & V^{19}, & X^{21}, \end{array}}$$

whose restriction to $\{f_1' = 0\}$ has a reduced gröbnerized ideal of relations, for the Lexicographic ordering, which consists of the **41** *syzygies written above. Furthermore, any bi-invariant of weight m writes uniquely in the finite polynomial form :*

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \, \big(W^{10}\big)^p \sum_{\substack{(a,\dots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41}) \\ 3a+\cdots+21n=m-o-10p}} \mathsf{coeff}_{a,\dots,n,o,p} \cdot$$
$$\cdot \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h$$
$$\big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n,$$

with coefficients $\mathsf{coeff}_{a,\dots,n,o,p}$ subjected to no restriction, where $\square_1$, ..., $\square_{41}$ denote the quadrants in $\mathbb{N}^{14}$ having vertex at the leading terms of the **41** *syzygies in question.*

III-4 – Asymptotics of Euler characteristic

Theorem. *On a hypersurface $X^4 \subset \mathbb{P}^5(\mathbb{C})$ of dimension $n = 4$, the graduate m-th part $\mathscr{E}^{DS}_{4,m}T^*_X$ of the complete Demailly-Semple bundle possesses the Schur decomposition :*

$$\mathscr{E}^{DS}_{4,m}T^*_X = \bigoplus_{\substack{(a,b,\ldots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41})\\ o+3a+\cdots+21n+10p=m}} \Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n+p \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n+p \\ d+e+f+h+i+j+2k+2l+2m'+2n+p \\ p \end{pmatrix} T^*_X,$$

where the **41** *subsets $\square_i$, $i = 1, 2, \ldots, 41$ of $\mathbb{N}^{14} \ni (a, b, \ldots, l, m', n)$ are defined by :*

$\{a \geqslant 1,\, c \geqslant 1\}$, $\{a \geqslant 1,\, e \geqslant 1\}$, $\{a \geqslant 1,\, f \geqslant 1\}$, $\{a \geqslant 1,\, i \geqslant 1\}$,
$\{a \geqslant 1,\, j \geqslant 1\}$, $\{a \geqslant 1,\, k \geqslant 1\}$, $\{a \geqslant 1,\, l \geqslant 1\}$, $\{a \geqslant 1,\, m' \geqslant 1\}$,
$\{a \geqslant 1,\, n \geqslant 1\}$, $\{b \geqslant 1,\, e \geqslant 1\}$, $\{b \geqslant 1,\, f \geqslant 1\}$, $\{b \geqslant 1,\, i \geqslant 1\}$,
$\{b \geqslant 1,\, j \geqslant 1\}$, $\{b \geqslant 1,\, k \geqslant 1\}$, $\{b \geqslant 1,\, l \geqslant 1\}$, $\{b \geqslant 1,\, m' \geqslant 1\}$,
$\{b \geqslant 1,\, n \geqslant 1\}$, $\{c \geqslant 1,\, k \geqslant 1\}$, $\{c \geqslant 1,\, l \geqslant 1\}$, $\{c \geqslant 1,\, m' \geqslant 1\}$,
$\{c \geqslant 1,\, n \geqslant 1\}$, $\{d \geqslant 1,\, f \geqslant 1\}$, $\{d \geqslant 1,\, i \geqslant 1\}$, $\{d \geqslant 1,\, j \geqslant 1\}$,
$\{d \geqslant 1,\, m \geqslant 1\}$, $\{d \geqslant 1,\, n \geqslant 1\}$, $\{e \geqslant 1,\, i \geqslant 1\}$, $\{e \geqslant 1,\, j \geqslant 1\}$,
$\{e \geqslant 1,\, m' \geqslant 1\}$, $\{e \geqslant 1,\, n \geqslant 1\}$, $\{d \geqslant 1,\, g \geqslant 1,\, k \geqslant 1\}$,
$\{e \geqslant 1,\, g \geqslant 1,\, k \geqslant 1\}$, $\{f \geqslant 1,\, g \geqslant 1,\, k \geqslant 1\}$, $\{g \geqslant 1,\, k \geqslant 2\}$,
$\{h \geqslant 1,\, j \geqslant 1\}$, $\{h \geqslant 1,\, n \geqslant 1\}$, $\{i \geqslant 2,\, n \geqslant 1\}$,
$\{f \geqslant 1,\, h \geqslant 1,\, m' \geqslant 1\}$, $\{f \geqslant 1,\, i \geqslant 1,\, m' \geqslant 1\}$, $\{f \geqslant 1,\, i \geqslant 1,\, n \geqslant 1\}$,
$\{f \geqslant 1,\, l \geqslant 1,\, n \geqslant 1\}$.

• **Approximation of the 41 subsets :** Keep only the families of sums of Schur bundles that contribute in $\mathrm{O}(m^{16})$ to the final characteristic.

• **24 families (with multiplicities) of sums of Schur bundles :**

$$\mathsf{A},\quad \mathsf{B},\quad \mathsf{C},\quad \ldots,\quad \mathsf{W},\quad \mathsf{X}.$$

defined as follows :

$$\underline{\mathsf{A}:}\quad 2\cdot \bigoplus_{m=o+14j+15k+17l+19m+21n+10p} \Gamma\begin{pmatrix} o+4j+3k+3l+4m+5n+p \\ 2j+2k+3l+3m+3n+p \\ j+2k+2l+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{B}:}\quad \bigoplus_{m=o+12i+14j+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+3i+4j+3k+3l+4m+p \\ 2i+2j+2k+3l+3m+p \\ i+j+2k+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{C}:}\quad \bigoplus_{m=o+10h+12i+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+2h+3i+3k+3l+4m+p \\ 2h+2i+2k+3l+3m+p \\ h+i+2k+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{D}:}\quad 4\cdot \bigoplus_{m=o+8g+14j+17l+19m+21n+10p} \Gamma\begin{pmatrix} o+2g+4j+3l+4m+5n+p \\ 2g+2j+3l+3m+3n+p \\ j+2l+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{E}:}\quad 2\cdot \bigoplus_{m=o+8g+12i+14j+17l+19m+10p} \Gamma\begin{pmatrix} o+2g+3i+4j+3l+4m+p \\ 2g+2i+2j+3l+3m+p \\ i+j+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

F : $\mathbf{2}\cdot \bigoplus_{m=o+8g+10h+12i+17l+19m+10p} \Gamma \begin{pmatrix} o+2g+2h+3i+3l+4m+p \\ 2g+2h+2i+3l+3m+p \\ h+i+2l+2m+p \\ p \end{pmatrix} T^*_X,$

G : $\bigoplus_{m=o+10f+14j+15k+19m+21n+10p} \Gamma \begin{pmatrix} o+3f+4j+3k+4m+5n+p \\ f+2j+2k+3m+3n+p \\ f+j+2k+2m+2n+p \\ p \end{pmatrix} T^*_X,$

H : $\bigoplus_{m=o+10f+14j+15k+17l+19m+10p} \Gamma \begin{pmatrix} o+3f+4j+3k+3l+4m+p \\ f+2j+2k+3l+3m+p \\ f+j+2k+2l+2m+p \\ p \end{pmatrix} T^*_X,$

I : $\bigoplus_{m=o+10f+12i+14j+15k+17l+10p} \Gamma \begin{pmatrix} o+3f+3i+4j+3k+3l+p \\ f+2i+2j+2k+3l+p \\ f+i+j+2k+2l+p \\ p \end{pmatrix} T^*_X,$

J : $\bigoplus_{m=o+10f+10h+12i+15k+17l+10p} \Gamma \begin{pmatrix} o+3f+2h+3i+3k+3l+p \\ f+2h+2i+2k+3l+p \\ f+h+i+2k+2l+p \\ p \end{pmatrix} T^*_X,$

K : $\bigoplus_{m=o+10f+8g+14j+19m+21n+10p} \Gamma \begin{pmatrix} o+3f+2g+4j+4m+5n+p \\ f+2g+2j+3m+3n+p \\ f+j+2m+2n+p \\ p \end{pmatrix} T^*_X,$

L : $\bigoplus_{m=o+10f+8g+14j+17l+19m+10p} \Gamma \begin{pmatrix} o+3f+2g+4j+3l+4m+p \\ f+2g+2j+3l+3m+p \\ f+j+2l+2m+p \\ p \end{pmatrix} T^*_X,$

M :
$$\bigoplus_{m=o+10f+8g+12i+14j+17l+10p} \Gamma\begin{pmatrix} o+3f+2g+3i+4j+3l+p \\ f+2g+2i+2j+3l+p \\ f+i+j+2l+p \\ p \end{pmatrix} T_X^*,$$

N :
$$\bigoplus_{m=o+10f+8g+10h+12i+17l+10p} \Gamma\begin{pmatrix} o+3f+2g+2h+3i+3l+p \\ f+2g+2h+2i+3l+p \\ f+h+i+2l+p \\ p \end{pmatrix} T_X^*,$$

O :
$$\bigoplus_{m=o+8e+10f+10h+15k+17l+10p} \Gamma\begin{pmatrix} o+2e+3f+2h+3k+3l+p \\ e+f+2h+2k+3l+p \\ e+f+h+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

P :
$$\bigoplus_{m=o+8e+10f+8g+10h+17l+10p} \Gamma\begin{pmatrix} o+2e+3f+2g+2h+3l+p \\ e+f+2g+2h+3l+p \\ e+f+h+2l+p \\ p \end{pmatrix} T_X^*,$$

Q :
$$\bigoplus_{m=o+6d+8e+10h+15k+17l+10p} \Gamma\begin{pmatrix} o+d+2e+2h+3k+3l+p \\ d+e+2h+2k+3l+p \\ d+e+h+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

R :
$$\bigoplus_{m=o+6d+8e+8g+10h+17l+10p} \Gamma\begin{pmatrix} o+d+2e+2g+2h+3l+p \\ d+e+2g+2h+3l+p \\ d+e+h+2l+p \\ p \end{pmatrix} T_X^*,$$

S :
$$\bigoplus_{m=o+7c+10f+8g+12i+14j+10p} \Gamma\begin{pmatrix} o+3c+3f+2g+3i+4j+p \\ c+f+2g+2i+2j+p \\ f+i+j+p \\ p \end{pmatrix} T_X^*,$$

T :
$$\bigoplus_{m=o+7c+10f+8g+10h+12i+10p} \Gamma \begin{pmatrix} o+3c+3f+2g+2h+3i+p \\ c+f+2g+2h+2i+p \\ f+h+i+p \\ p \end{pmatrix} T_X^*,$$

U :
$$\bigoplus_{m=o+7c+8e+10f+8g+10h+10p} \Gamma \begin{pmatrix} o+3c+2e+3f+2g+2h+p \\ c+e+f+2g+2h+p \\ e+f+h+p \\ p \end{pmatrix} T_X^*,$$

V :
$$\bigoplus_{m=o+7c+6d+8e+8g+10h+10p} \Gamma \begin{pmatrix} o+3c+d+2e+2g+2h+p \\ c+d+e+2g+2h+p \\ d+e+h+p \\ p \end{pmatrix} T_X^*,$$

W :
$$\bigoplus_{m=o+5b+7c+6d+8g+10h+10p} \Gamma \begin{pmatrix} o+2b+3c+d+2g+2h+p \\ b+c+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*,$$

X :
$$\bigoplus_{m=o+3a+5b+6d+8g+10h+10p} \Gamma \begin{pmatrix} o+a+2b+d+2g+2h+p \\ a+b+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*.$$

- **Computations on Maple 12 :** $\sim$ 50 minutes.

Theorem. *If $X^4 \subset \mathbb{P}^5(\mathbb{C})$ is a projective algebraic hypersurface of degree d, then as $m \to \infty$, one has the asymptotic for the Euler characteristic :*

$$\begin{aligned}\chi\big(X,\, \mathsf{E}^4_{4,m}T^*_X\big) = \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \cdot \\ \cdot \big(500485111357970342562 35\, d^4 - \\ - 617060662250595525598878 6\, d^3 - \\ - 928886901354141153880624704\, d + \\ + 1411704752502476621473639 41\, d^2 + \\ + 1624908955061039283976041114\big) \\ + \mathrm{O}\big(m^{15}\big).\end{aligned}$$

Moreover, the coefficient of m^{16}, a (factorized) polynomial of degree 5 with respect to d, takes positive values as soon as $d \geqslant 96$.

- **Euler characteristic :**

$$\chi = h^0 - h^1 + h^2 - h^3 + h^4$$

- **Trivial minoration :**

$$h^0 \geqslant \chi - h^2 - h^4$$

- **Vanishing theorem :**

$$h^4 = 0\,.$$

- **Majoration of h^2 :**

Theorem. [DMR 2008] *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$. Then :*

$$
\begin{aligned}
&h^2\big(X, \Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} T_X^*\big) \\
&\qquad \leqslant \frac{1}{80}\, d\,(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4) \\
&\qquad \cdot \big(\lambda_1+\lambda_2+\lambda_3+\lambda_4\big)^2 \big[5\lambda_2\lambda_1 d^2 + 132\lambda_2\lambda_1 d + 132\lambda_1\lambda_3 d + 5\lambda_2\lambda_3 d^2 \\
&\qquad + 132\lambda_2\lambda_4 d + 5\lambda_2 d^2\lambda_4 + 132\lambda_1\lambda_4 d + 5\lambda_3\lambda_4 d^2 + 5\lambda_1\lambda_3 d^2 \\
&\qquad + 132\lambda_3\lambda_4 d + 132\lambda_2\lambda_3 d + 1308\lambda_2\lambda_1 + 648\lambda_2^2 + 648\lambda_3^2 \\
&\qquad + 72\lambda_3^2 d + 648\lambda_1^2 + 72\lambda_1^2 d + 1308\lambda_1\lambda_4 + 5\lambda_1 d^2\lambda_4 + 1308\lambda_2\lambda_4 \\
&\qquad + 1308\lambda_2\lambda_3 + 648\lambda_4^2 + 72\lambda_2^2 d + 1308\lambda_1\lambda_3 + 72\lambda_4^2 d + 1308\lambda_3\lambda_4\big] \\
&\qquad + O\big(|\lambda|^9\big).
\end{aligned}
$$

Theorem. [DMR 2008] *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$ and let A be any ample line bundle over X. Then :*

$$
\begin{aligned}
&h^0(X, E_{4,m} T_X^* \otimes \mathscr{O}(-A)) \\
&\qquad \geqslant \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \\
&\qquad\quad \cdot \big[-867659678949860838548185438614 \\
&\qquad\quad - 934880693607607850940593792 16\, d \\
&\qquad\quad - 1369327265177339103292331439\, d^2 \\
&\qquad\quad - 617060662250595525598878 6\, d^3 \\
&\qquad\quad + 500485111357970342562 35\, d^4\big] \\
&\qquad\qquad + O\big(m^{15}\big).
\end{aligned}
$$

*In particular, if $d \geqslant$ **259** then $E_{4,m} T_X^* \otimes \mathscr{O}(-A)$ admits non trivial sections for m large, and every entire curve $f : \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.*

IV – Algebraic Morse inequalities

• **Strategy :** Avoid full algebra of invariants by reparametrization.

• **Same objective :** Construct global sections of jet bundles that will canalize all entire curves :

$$\mathsf{P}\big(j^k f\big) \equiv 0\,.$$

• **Significant obstacle :** [ROUSSEAU, 2006] In dimension 3, the jet order must be $\geqslant\ 3$. More generally, **Brückmann-Rackwitz vanishing theorem** yields :

Corollary. [DIVERIO, 2008] *For every* $k \leqslant \dim X - 1$ *and every ample line bundle* $A \to X$ *:*

$$0 = H^0\big(X,\, \mathscr{E}_{k,m}^{GG \text{ or } DS} \otimes A^{-1}\big).$$

• **Demailly tower for** $k = n = 3$ **:**

$$\begin{array}{llllll}
9 & X_3 & & & \mathscr{O}_{X_3}(1) & u_3 \\
 & \downarrow{\scriptstyle \pi_{2,3}} & & & & \\
7 & X_2 & V_2 & \mathsf{c}_1^2,\ \mathsf{c}_2^2,\ \mathsf{c}_3^2 & \mathscr{O}_{X_2}(1) & u_2 \\
 & \downarrow{\scriptstyle \pi_{1,2}} & & & & \\
5 & X_1 & V_1 & \mathsf{c}_1^1,\ \mathsf{c}_2^1,\ \mathsf{c}_3^1 & \mathscr{O}_{X_1}(1) & u_1 \\
 & \downarrow{\scriptstyle \pi_{0,1}} & & & & \\
3 & X & T_X & \mathsf{c}_1,\ \mathsf{c}_2,\ \mathsf{c}_3 & & h
\end{array}\,.$$

• **Geometrical construction :** Let X be a complex manifold of dimension n. Let V be a subbundle of T_X of rank $r \geqslant 1$, without any integrability condition.

• **Define :** $X_1 := \mathbb{P}(V)$, and a lifted subbundle $V_1 \subset T_{X_1}$ by :

$$V_{1,(x,[v])} := \left\{ \xi \in T_{X_1,(x,[v])} \colon \pi_* \xi \in \mathbb{C} v \right\},$$

where $\pi \colon X_1 \to X$ is the natural projection. The rank of V_1 still equals r.

• **Observation :** If $f : (\mathbb{C}, 0) \to (X, x)$ is a germ of holomorphic curve tangent to the subbundle : $f'(\zeta) \in V_{f(\zeta)}$, then it lifts to X_1 and is tangent to the lifted bundle :

$$f'_{[1]}(\zeta) \in V_{1, f_{[1]}(\zeta)} \,.$$

• **Induction :** Obtain a sequence of manifolds + subbundles :

$$(X_k, V_k) \quad (X_{k-1}, V_{k-1}) \quad \cdots (X_1, V_1) \quad (X, V)$$

to which entire curves lift :

$$f_{[k]}(\zeta) \in V_{k, f_{[k]}(\zeta)} \,.$$

• **Projections :** $\pi_k \colon X_k \to X$

• **Tautological line bundles :**

$$\mathscr{O}_{X_k}(-1) \quad \text{and its dual} \quad \mathscr{O}_{X_k}(1) \,.$$

• **Absolute case (the only interesting one) :** Take :

$$V := T_X.$$

• **Demailly tower for $k = n$:**

$$
\begin{array}{cccc}
n^2 & X_n & & \boxed{\mathscr{O}_{X_n}(1)} \\
 & \downarrow{\scriptstyle \pi_{n-1,n}} & & \\
n^2 - n + 1 & X_{n-1} & V_{n-1} & \mathscr{O}_{X_{n-1}}(1) \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
\bullet & \bullet & \bullet & \bullet \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
(\ell+1)n - \ell & X_\ell & V_\ell & \mathscr{O}_{X_\ell}(1) \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
\bullet & \bullet & \bullet & \bullet \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
3n - 2 & X_2 & V_2 & \mathscr{O}_{X_2}(1) \\
 & \downarrow{\scriptstyle \pi_{1,2}} & & \\
2n - 1 & X_1 & V_1 & \mathscr{O}_{X_1}(1) \\
 & \downarrow{\scriptstyle \pi_{0,1}} & & \\
n & X & T_X &
\end{array}
.
$$

• **Fundamental fact :** (DEMAILLY 1997) : The direct image :

$$\pi_{k*}(\mathscr{O}_{X_k}(m))$$

identifies to a vector bundle over X :

$$
\begin{array}{ccc}
\mathscr{O}_{X_k}(m) & \longrightarrow & \mathscr{E}^{DS}_{k,m} T^*_X \\
\downarrow & & \downarrow \\
X_k & \xrightarrow{\pi_k} & X
\end{array}
.
$$

• **Description in local coordinates :** A local section of $\pi_{k*}(\mathscr{O}_{X_k}(m))$ is an algebraic differential operator :

$$P = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} R_\alpha(z)(f')^{\alpha_1}\dots(f^{(k)})^{\alpha_k}$$

which is invariant under any local reparametrization $\phi\colon (\mathbb{C},0)\to(\mathbb{C},0)$:

$$P\big((f\circ\phi)',\dots,(f\circ\phi)^{(k)}\big) = \phi'(0)^m P\big(f',\dots,f^{(k)}\big)\,.$$

• **Observation :** This bundle $\mathscr{E}^{DS}_{k,m}T^*_X$ of invariant jets is a subbundle of the bundle of (plain) Green-Griffiths jets $\mathscr{E}^{GG}_{k,m}T^*_X$.

• **Fundamental isomorphism :** If X is a projective manifold, with an ample line bundle A on X, one has :

$$\boxed{H^0\big(X,\mathscr{E}^{DS}_{k,m}\otimes A^{-1}\big)\simeq H^0\big(X_k,\mathscr{O}_{X_k}(m)\otimes\pi_k^*A^{-1}\big)}.$$

• **Interest :** To understand cohomology, line bundles are easier to handle :

$$\boxed{\mathscr{O}_{X_k}(m)\quad\text{is a line bundle on}\quad X_k}$$

• **Now assume $k=n$ and use Morse inequalities :**

Theorem. [TRAPANI, SIU, DEMAILLY] *Let $\mathscr{L} \to X$ be a holomorphic line bundle on a compact Kähler manifold of dimension n which may be written as a certain difference between two line bundles $\mathscr{F}$ and $\mathscr{G}$ that are numerically effective : $\mathscr{L} = \mathscr{F} \otimes \mathscr{G}^{-1}$. If :*

$$\mathscr{F}^n - n\mathscr{F}^{n-1} \cdot \mathscr{G} > 0\,,$$

then for any holomorphic vector bundle $\mathscr{E} \to X$, the multi-tensored bundle $\mathscr{L}^{\otimes m} \otimes \mathscr{E}$ possesses non-zero global sections, asymptotically as $m \gg 1$.

• **Strategy :** [DEMAILLY, TRAPANI, DIVERIO] Find a subbundle of the line bundle $\mathscr{O}_{X_n}(m) \longrightarrow X_n$ which can be decomposed as a certain difference between two nef line bundles.

• **Definition :** Let $\pi_{j,n} \colon X_n \to X_j$ be the projection from level n to level j in Demailly's tower. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, consider the **line bundle** on X_n :

$$\mathscr{O}_{X_n}(\mathbf{a}) := \pi_{1,n}^* \mathscr{O}_{X_1}(a_1) \otimes \pi_{2,n}^* \mathscr{O}_{X_2}(a_2) \otimes \cdots \otimes \mathscr{O}_{X_n}(a_n)\,.$$

Proposition. [DEMAILLY] *Let :* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ *and let* $m := a_1 + \cdots + a_n$*. Then one has a sheaf injection :*

$$(\pi_{0,n})_* \mathscr{O}_{X_n}(\mathbf{a}) \hookrightarrow \mathscr{O}\big(\mathscr{E}^{DS}_{n,m}\big) .$$

Moreover, $\mathscr{O}_{X_n}(\mathbf{a})$ *is relatively numerically effective on* X *as soon as :*

$$a_1 \geqslant 3a_2, \ldots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1 .$$

Corollary. [DIVERIO] *Let* $X^n \subset \mathbb{P}^{n+1}$ *be a smooth projective algebraic hypersurface and let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ *satisfying again :*

$$a_1 \geqslant 3a_2, \ldots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1 .$$

Then :

$$\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi^*_{0,n} \mathscr{O}_X(l)$$

is fully numerically effective as soon as :

$$l \geqslant 2|\mathbf{a}|$$

where $|\mathbf{a}| = a_1 + \cdots + a_n$*.*

• **Consequently :** We have two numerically effective bundles over X_n, firstly :

$$\mathscr{F} := \mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi^*_{0,n} \mathscr{O}_X(2|\mathbf{a}|)$$

and secondly, trivially :

$$\mathscr{G} := \pi^*_{0,n} \mathscr{O}_X\big(2|\mathbf{a}|\big) .$$

• **Application :** Express $\mathscr{O}_{X_n}(\mathbf{a})$ artificially as a difference between two nef line bundles :

$$\begin{aligned}\mathscr{O}_{X_n}(\mathbf{a}) &= \mathscr{L} = \mathscr{F}\otimes\mathscr{G}^{-1}\\ &= \big(\mathscr{O}_{X_n}(\mathbf{a})\otimes\pi_{0,n}^*\mathscr{O}_X(2|\mathbf{a}|)\big)\otimes\big(\pi_{0,n}^*\mathscr{O}_X(-2|\mathbf{a}|)\big).\end{aligned}$$

• **Demailly tower for $k = n = 3$:**

$$\begin{array}{llllll}
9 & X_3 & & & \mathscr{O}_{X_3}(1) & u_3 \\
 & \downarrow{\scriptstyle \pi_{2,3}} & & & & \\
7 & X_2 & V_2 & \mathsf{c}_1^2,\ \mathsf{c}_2^2,\ \mathsf{c}_3^2 & \mathscr{O}_{X_2}(1) & u_2 \\
 & \downarrow{\scriptstyle \pi_{1,2}} & & & & \\
5 & X_1 & V_1 & \mathsf{c}_1^1,\ \mathsf{c}_2^1,\ \mathsf{c}_3^1 & \mathscr{O}_{X_1}(1) & u_1 \\
 & \downarrow{\scriptstyle \pi_{0,1}} & & & & \\
3 & X & T_X & \mathsf{c}_1,\ \mathsf{c}_2,\ \mathsf{c}_3 & & h
\end{array}.$$

• **First Chern classes :** For any level ℓ satisfying $1 \leqslant \ell \leqslant n$, denote :

$$u_\ell := \mathsf{ch}_1\big(\mathscr{O}_{X_\ell}(1)\big)\,.$$

so that $\big(u_\ell\big)^k$ is a (k,k)-form on X_ℓ.

• **Second Chern classes :** Introduce the (j,j)-forms :

$$\mathsf{c}_j^\ell := \mathsf{ch}_j(V_\ell)$$

on X_ℓ.

• **Expression of the intersection product in terms of Chern classes :**

$$\big(a_1u_1+\cdots+a_nu_n+2|\mathbf{a}|h\big)^{n^2}-$$
$$-\,n^2\big(a_1u_1+\cdots+a_nu_n+2|\mathbf{a}|h\big)^{n^2-1}\cdot\big(2|\mathbf{a}|h\big).$$

Lemma. *After eliminations, fiber-integrations, and annihilations, this intersection product :*

$$\big(a_1u_1+\cdots+a_nu_n+2|\mathbf{a}|h\big)^{n^2}-$$
$$-\,n^2\big(a_1u_1+\cdots+a_nu_n+2|\mathbf{a}|h\big)^{n^2-1}\cdot\big(2|\mathbf{a}|h\big).$$

becomes a certain polynomial :

$$p_{n+1,\mathbf{a}}d^{n+1}+p_{n,\mathbf{a}}d^n+\cdots+p_{1,\mathbf{a}}d$$

in terms of $d=\deg X$ *having coefficients :*

$$p_{k,\mathbf{a}}\in\mathbb{Z}\big[a_1,\ldots,a_n\big],$$

difficult to compute explicitly.

• **Dimension of** X_n **:**

$$\dim\,X_n=n+n(n-1)=n^2\,.$$

• **Estimate on the degree :** If the coefficient $\mathsf{p}_{n+1,\mathbf{a}}$ of the dominant term d^{n+1} is positive, then for every sufficiently large degree, say for :

$$\deg X=d\geqslant d_n,$$

this polynomial takes only positive values, and there exist global sections of the (sub)bundle of jets.

- **Simplest choice of weights :**

$$a_1 = 1, \quad a_2 = 2, \quad a_3 = 2 \cdot 3, \quad \cdots, \quad a_n = 2 \cdot 3^{n-2}.$$

- **In dimension** $n = 3$ **:**

$$333162\, d^4 - 21628710\, d^3 - 460474830\, d^2 - 466509222\, d\,;$$

which is positive for all $d = \deg X \geqslant \mathbf{82}$.

- **In dimension** $n = 4$ **:**

$$170114889178454 4\, d^5 - 39934769846141376 0\, d^4 -$$
$$- 502967681502861425 76\, d^3 - 583578200119254857568\, d^2 -$$
$$- 646476679639160501760\, d\,,$$

which is positive for all $d \geqslant \mathbf{329}$.

- **In dimension** $n = 5$ **:** [Diverio, 2008]

$$\mathsf{P}_{54,18,6,2,1}(d) = 829705552526846689513237554474 24\, d^6 -$$
$$- 690923576923829601983160082796154 24\, d^5 -$$
$$- 375919573131846296972181088319559277 44\, d^4 -$$
$$- 216114449751608047695560783767127869958 4\, d^3 -$$
$$- 207679317231737411175485558372431638061 44\, d^2 -$$
$$- 237364617790381662461159583045518710563 84\, d,$$

which is positive for all $d \geqslant \mathbf{1222}$.

- **In dimension** $n = 6$ **:** [M., 2008] Computation using what follows, distributed on 15 computers : $d \geqslant \mathbf{4352}$.

V – Intersection product

- **First family of relations :**

(rel$_1$)

$$\left[\begin{array}{l}
\mathsf{C}_j^{n-1} = \mathsf{C}_j^{n-2} + \lambda_{j,1}\,\mathsf{C}_{j-1}^{n-2}\,u_{n-1} + \lambda_{j,2}\,\mathsf{C}_{j-2}^{n-2}\,(u_{n-1})^2 + \cdots + \lambda_{j,j}\,(u_{n-1})^j \\
\mathsf{C}_j^{n-2} = \mathsf{C}_j^{n-3} + \lambda_{j,1}\,\mathsf{C}_{j-1}^{n-3}\,u_{n-2} + \lambda_{j,2}\,\mathsf{C}_{j-2}^{n-3}\,(u_{n-2})^2 + \cdots + \lambda_{j,j}\,(u_{n-2})^j \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\quad \mathsf{C}_j^{2} = \mathsf{C}_j^{1} + \lambda_{j,1}\,\mathsf{C}_{j-1}^{1}\,u_{2} + \lambda_{j,2}\,\mathsf{C}_{j-2}^{1}\,(u_{2})^2 + \cdots + \lambda_{j,j}\,(u_{2})^j \\
\quad \mathsf{C}_j^{1} = \mathsf{c}_j + \lambda_{j,1}\,\mathsf{c}_{j-1}\,u_{1} + \lambda_{j,2}\,\mathsf{c}_{j-2}\,(u_{1})^2 + \cdots + \lambda_{j,j}\,(u_{1})^j,
\end{array}\right.$$

where $j = 1, 2, \ldots, n$ is arbitrary and where the coefficients $\lambda_{j,j-k}$ ($1 \leqslant j \leqslant n$, $0 \leqslant k \leqslant j$), independent of ℓ, are differences of binomial numbers :

$$\lambda_{j,j-k} := \frac{(n-k)!}{(j-k)!\,(n-j)!} - \frac{(n-k)!}{(j-k-1)!(n-j+1)!}\,.$$

- **Second family of relations :**

(rel$_2$)

$$\left[\begin{array}{l}
(u_n)^n = -\mathsf{C}_1^{n-1}\,(u_n)^{n-1} - \mathsf{C}_2^{n-1}\,(u_n)^{n-2} - \cdots - \mathsf{C}_{n-1}^{n-1}\,u_n - \mathsf{C}_n^{n-1} \\
(u_{n-1})^n = -\mathsf{C}_1^{n-2}\,(u_{n-1})^{n-1} - \mathsf{C}_2^{n-2}\,(u_{n-1})^{n-2} - \cdots - \mathsf{C}_{n-1}^{n-2}\,u_{n-1} - \mathsf{C}_n^{n-2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\quad (u_2)^n = -\mathsf{C}_1^{1}\,(u_2)^{n-1} - \mathsf{C}_2^{1}\,(u_2)^{n-2} - \cdots - \mathsf{C}_{n-1}^{1}\,u_2 - \mathsf{C}_n^{1} \\
\quad (u_1)^n = -\mathsf{c}_1\,(u_1)^{n-1} - \mathsf{c}_2\,(u_1)^{n-2} - \cdots - \mathsf{c}_{n-1} u_n - \mathsf{c}_n.
\end{array}\right.$$

- **Chern classes in terms of the degree :**

$$h := \mathsf{c}_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big) \qquad\qquad h^n = \int_X h^n = d.$$

- **Ground level :** $\ell = 0$, Chern classes (small c) :

$$\mathsf{c}_j := \mathsf{c}_j\big(T_X\big)\,.$$

- **Third family of relations :**

(c-d)

$$\left[\begin{array}{l}
\mathsf{c}_1 = -h\big(d-n-2\big) \\
\mathsf{c}_2 = h^2\big(d^2 - \frac{(n+2)!}{(n+1)!\;1!}\,d + \frac{(n+2)!}{n!\;2!}\big) \\
\mathsf{c}_3 = -h^3\big(d^3 - \frac{(n+2)!}{(n+1)!\;1!}\,d^2 + \frac{(n+2)!}{n!\;2!}\,d - \frac{(n+2)!}{(n-1)!\;3!}\big) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathsf{c}_n = (-1)^n\,h^n\big(d^n - \frac{(n+2)!}{(n+1)!\;1!}\,d^{n-1} + \cdots + (-1)^n\,\frac{(n+2)!}{2!\;n!}\big).
\end{array}\right.$$

Elimination problem

- **Three processes of elimination :**

"vanishing for degree-form reasons"

"fiber-integration"

"replacement"

Five main ideas

(~ 30 pages of proof)

- ☐ Performing terminal inequalities within algebra.
- ☐ Introducing Jacobi-Trudy determinants.
- ☐ Highlighting the central monomial $u_1^n \cdots u_n^n$.
- ☐ Minorating effectively $p_{n+1,\mathbf{a}}$.
- ☐ Majorating effectively the other coefficients $p_{k,\mathbf{a}}$.

Theorem. *For an appropriate choice of weights* $a_1(n), \ldots, a_n(n)$ *that are explicit in terms of* n*, the integer coefficients of the considered intersection product*

$$\mathscr{F}^{n^2} - n^2 \mathscr{F}^{n^2-1} \cdot \mathscr{G}$$

namely of the polynomial :

$$\mathsf{p}_{n+1,\mathbf{a}}\, d^{n+1} + \sum_{0 \leqslant k \leqslant n} \mathsf{p}_{k,\mathbf{a}}\, d^k$$

satisfy the effective inequalities :

$$\mathsf{p}_{n+1,\mathbf{a}} \geqslant n^{n^{n+4}} \quad \text{and} \quad \mathsf{p}_{k,\mathbf{a}} \leqslant n^{(n+1)^{n+5}}.$$

• **In summary :** We have constructed just **at least one** not-identically zero differential equation :

$$\mathsf{P} \in H^0\big(X,\, \mathscr{E}^{DS}_{n,m} T^*_X \otimes A^{-1}\big)$$

with the property (AHLFORS-GRAUERT-GREEN-GRIFFITHS-DEMAILLY) that every nonconstant entire holomorphic curve $f : X \to \mathbb{C}$ must satisfy :

$$\mathsf{P}\big(j^n f\big) \equiv 0\,.$$

• **Question :** How to conclude that $f(\mathbb{C})$ is contained in some proper algebraic subset $Y \subsetneqq X$?

VII – Siu's beautiful strategy

• **Universal hypersurface :** Define, in a system of homogeneous coordinates :

$$[Z] = [Z_0 : Z_1 : \cdots : Z_n : Z_{n+1}] \in \mathbb{P}^{n+1}$$

$$[A] = \big[(A_\alpha)_{\alpha\in\mathbb{N}^{n+2},\,|\alpha|=d}\big] \in \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\;d!}-1},$$

the **universal hypersurface** of degree d :

$$\mathscr{X} \quad : \qquad 0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+2}\\ |\alpha|=d}} A_{\alpha_0,\ldots,\alpha_{n+1}}\, Z_0^{\alpha_0}\cdots Z_{n+1}^{\alpha_{n+1}}$$

as the zero-locus of the general homogenous polynomial of degree d. Set $N_d^n := \frac{(n+1+d)!}{(n+1)!\;d!} - 1$.

• **Double projection :**

$$\begin{tikzcd} & \mathscr{X} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathbb{P}^{n+1} & & \mathbb{P}^{N_d^n}. \end{tikzcd}$$

• **Inhomogeneous coordinates :** In the chart $\{Z_0 \neq 0\} \times \{A_{0d0\cdots0} \neq 0\}$, write :

$$\mathscr{X}_0 \quad : \qquad 0 = z_1^d + \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\,\alpha_1<d}} a_\alpha\, z^\alpha\,.$$

• **Principal hypothesis :** Entire holomorphic map valued in a fixed projective hypersurface :

$$0 \equiv f_1(\zeta)^d + \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leqslant d,\, \alpha_1 < d}} a_\alpha\, f(\zeta)^\alpha\,,$$

where the coefficients a_α do not depend on ζ.

• **Coordinates in the space of vertical n-jets :**

$$\Big(z_i,\, a_\alpha,\, z'_{j_1},\, z''_{j_2}, \ldots, z^{(n)}_{j_n}\Big) \in$$
$$\in \mathbb{C}^{n+1} \times \mathbb{C}^{N_d^n} \times \underbrace{\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \cdots \times \mathbb{C}^{n+1}}_{n \text{ times}}.$$

• **Chain rule :** At order $\kappa = 4$:

$$0 = \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leqslant d,\, a_{d0\cdots 0} = 1}} a_\alpha\, z^\alpha$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z'_{j_1} \Big)$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1} \partial z_{j_2}}\, z'_{j_1} z'_{j_2} \Big)$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z'''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1} \partial z_{j_2}}\, 3\, z'_{j_1} z''_{j_2} + \sum_{j_1, j_2, j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3}}\, z'_{j_1} z'_{j_2} z'_{j_3} \Big)$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z''''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1} \partial z_{j_2}} \big(4\, z'_{j_1} z'''_{j_2} + 3\, z''_{j_1} z''_{j_2}\big) +$$
$$+ \sum_{j_1, j_2, j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3}}\, 6\, z'_{j_1} z'_{j_2} z''_{j_3} + \sum_{j_1, j_2, j_3, j_4} \frac{\partial^4(z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3} \partial z_{j_4}}\, z'_{j_1} z'_{j_2} z'_{j_3} z'_{j_4} \Big).$$

• **Total differentiation operator :**

$$\mathsf{D}(\bullet) := \sum_{\lambda\in\mathbb{N}} \sum_{k=1}^{n+1} \frac{\partial(\bullet)}{\partial z_k^{(\lambda)}} \cdot z_k^{(\lambda+1)},$$

• **General rewriting :**

$$0 = \sum_{\alpha} a_\alpha z^\alpha = \mathsf{D}\Big(\sum_{\alpha} a_\alpha z^\alpha\Big) = \cdots = \mathsf{D}^n\Big(\sum_{\alpha} a_\alpha z^\alpha\Big).$$

Lemma. *The* $(n+1)$ *defining equations of* $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ *write as follows :*

$$0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\, a_{d0\cdots 0}=1}} a_\alpha \sum_{e=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_e\leqslant\kappa} \sum_{\mu_1\geqslant 1,\ldots,\mu_e\geqslant 1} \sum_{\mu_1\lambda_1+\cdots+\mu_e\lambda_e=\kappa} \frac{\kappa!}{(\lambda_1!)^{\mu_1}\mu_1!\cdots(\lambda_e!)^{\mu_e}\mu_e!}$$

$$\sum_{j_1^1,\ldots,j_{\mu_1}^1=1}^{n+1} \cdots \sum_{j_1^e,\ldots,j_{\mu_e}^e=1}^{n+1} \frac{\partial^{\mu_1+\cdots+\mu_e}\big(z^\alpha\big)}{\partial z_{j_1^1}\cdots\partial z_{j_{\mu_1}^1}\cdots\partial z_{j_1^e}\cdots\partial z_{j_{\mu_e}^e}} z_{j_1^1}^{(\lambda_1)}\cdots z_{j_{\mu_1}^1}^{(\lambda_1)}\cdots z_{j_1^e}^{(\lambda_e)}\cdots z_{j_{\mu_e}^e}^{(\lambda_e)},$$

where $\kappa = 0, 1, 2, 3, 4, \ldots, n.$

• **Generation by global sections :** An arbitrary vector field defined in the ambient space :

$$\mathbb{C}^{n+1} \times \mathbb{C}^{N_d^n} \times \mathbb{C}^{n(n+1)}$$

writes under the general form :

$$\mathsf{T} = \sum_{i=1}^{n+1} \mathsf{Z}_i \frac{\partial}{\partial z_i} + \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\, \alpha_1<d}} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha} + \sum_{k=1}^{n+1} \mathsf{Z}'_k \frac{\partial}{\partial z'_k} + \sum_{k=1}^{n+1} \mathsf{Z}''_k \frac{\partial}{\partial z''_k} + \cdots + \sum_{k=1}^{n+1} \mathsf{Z}_k^{(n)} \frac{\partial}{\partial z_k^{(n)}}.$$

Theorem. [M., 2009] *Let* $\widetilde{\Sigma}$ *be the closure, in* $J^n_{\mathsf{vert}}(\mathscr{X})$*, of the closed algebraic subset of affine vertical jets* $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ *which is defined by the annihilation of all the first order jets :*

$$\widetilde{\Sigma}_0 := \Big\{ \big(z_i, a_\alpha, z'_{j_1}, \ldots, z^{(n)}_{j_n}\big) : \ z'_1 = z'_2 = \cdots = z'_{n+1} = 0 \Big\}.$$

Then the following two properties hold true :

- $J^n_{\mathsf{vert}}(\mathscr{X})\backslash\Sigma$ *is smooth of pure codimension equal to* $n+1$ *at every point, namely of dimension equal to :*

$$\begin{aligned} j^d_n &:= n+1+N^n_d + n(n+1) - (n+1) \\ &= \tfrac{(n+1+d)!}{(n+1)!\,d!} + n(n+1). \end{aligned}$$

- *The twisted tangent bundle :*

$$T_{J^n_{\mathrm{vert}}(\mathscr{X})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(n^2+2n\big) \otimes \mathscr{O}_{\mathbb{P}^{N^n_d}}(1)$$

is generated by its global sections on $J^n_{\mathsf{vert}}(\mathscr{X})\backslash\widetilde{\Sigma}$*, that is to say : at every point* $p^{[n]} \in J^n_{\mathsf{vert}}(\mathscr{X})\backslash\widetilde{\Sigma}$ *which does not belong to* $\widetilde{\Sigma}$*, one may find* j^d_n glo-bal *sections* $\mathsf{T}_1, \ldots, \mathsf{T}_{j^d_n}$ *on* X *of this twisted tangent bundle such that :*

$$\mathbb{C}\mathsf{T}_1(p^{[n]}) \oplus \cdots \oplus \mathbb{C}\mathsf{T}_{j^d_n}(p^{[n]}) = T_{J^n_{\mathsf{vert}}(\mathscr{X}),\, p^{[n]}}.$$

VII-1 – Algebraic degeneracy

Theorem. [DMR 2009] *Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface of arbitrary dimension $n \geqslant 2$. Then there exists a* **noneffective** *positive integer :*

$$d_n \gg 1,$$

such that, if X is generic of degree $\deg X \geqslant d_n$, then there exists a proper *algebraic subvariety :*

$$Y \subsetneqq X,$$

such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ entirely contained in Y.

• **From above :** [DIVERIO, 2008] For a jet order $k = n$ equal to the dimension, there exists $d_n \gg 1$ such that the two isomorphic spaces of sections :

$$H^0\big(X_n, \mathscr{O}_{X_n}(m) \otimes \pi_{0,n}^* A^{-1}\big) \simeq H^0\big(X, E_{n,m} T_X^* \otimes A^{-1}\big) \neq 0\,,$$

are *nonvoid* when $d \geqslant d_n$, provided $m \geqslant m_{d,n}$ is sufficiently large.

• **Canonical bundle :**

$$K_X \simeq \mathscr{O}_X(d-n-2)\,.$$

It will play the rôle of the ample line bundle A.

• **Continuity argument :** For $\delta > 0$ sufficiently small :

$$H^0\big(X_n, \mathscr{O}_{X_n}(m) \otimes \pi_{0,n}^* K_X^{-\delta m}\big) \simeq H^0\big(X, E_{n,m} T_X^* \otimes K_X^{-\delta m}\big) \neq 0\,.$$

• **Slices of the universal hypersurface :**

$$X_s := \mathscr{X}\big|_s, \qquad s \in \mathbb{P}^{N_d^n}.$$

• **Beautiful idea of Siu (2002) :** Holomorphic family of jet differentials not identically zero :

$$P = \big\{P|_s \in H^0\big(X_s,\, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big)\big\}.$$

• **Semi-continuity of cohomology :** [HARTSHORNE] The parameters s range outside a certain (uncontrolled) exceptional algebraic subvariety of the parameter space $\mathbb{P}^{N_d^n}$.

• **Fix s_0 outside this exceptional set**.

• **Nonconstant entire curve** $f\colon \mathbb{C} \to X_{s_0}$.

• **Define the zero-set locus :**

$$Y_{s_0} := \big\{x \in X_{s_0}\colon P|_{s_0}(x) = 0\big\}$$

of the non-identically zero section $P|_{s_0}$ of the vector bundle $E_{n,m}T^*_{X_{s_0}} \otimes K_{X_{s_0}}^{-\delta m}$.

Lemma. *Then Y_{s_0} is a proper algebraic subset of X which contains all nonconstant entire holomorphic curves :*

$$f(\mathbb{C}) \subset Y_{s_0}.$$

• **Existence of at least one differential equation :**

$$P|_{s_0}\big(j^n f(\zeta)\big) \equiv 0.$$

- **By contradiction :** There exists $\zeta_0 \in \mathbb{C}$ such that :

$$f(\zeta_0) \notin Y_{s_0} \qquad \text{et} \quad f'(\zeta_0) \neq 0\,.$$

- **In local coordinates :**

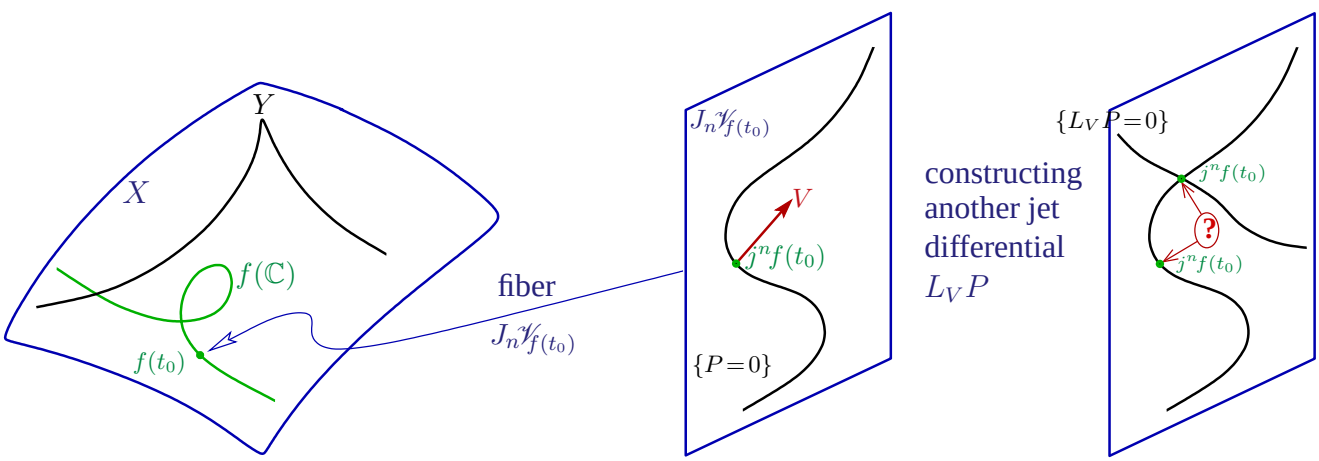
$$P = \sum_{|i_1|+\cdots+n|i_n|=m} q_{i_1,\ldots,i_n}(s,z)\,(z')^{i_1}\cdots(z^{(n)})^{i_n}$$

$$Y_{s_0} = \left\{z \in X_{s_0} \colon q_{i_1,\ldots,i_n}(s_0,z) = 0, \ \ \forall\, i_1,\ldots,i_n\right\}.$$

- **Relative polynomialness :** With respect to the jets.

- **Differentiate by a vector field :**

$$(\bullet) \otimes \mathscr{O}_{X_{s_0}}(n^2+2n).$$



- **Differentiate by p vector fields :**

$$(\bullet) \otimes \mathscr{O}_{X_{s_0}}(p(n^2+2n))$$

- **Global section in :**

$$H^0\big(X_{s_0}, E_{n,m}T^*_{X_{s_0}} \otimes \mathscr{O}_{X_{s_0}}(-\delta m(d-n-2)+p(n^2+2n))\big).$$

- **Still insure the inverse of an ample line bundle :**

$$-\delta m(d-n-2) + p(n^2+2n) < 0. \qquad \square$$

Effective algebraic degeneracy

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I. Statement of the main result

II. Bundles of (invariant) jets

III. Explicit algebras of invariant jets

IV. Algebraic Morse inequalities

V. Intersection product

VI. Elimination

VII. Siu's beautiful strategy

« Algebraic varieties and hyperbolicity : geometric and arithmetic aspects »

Strasbourg, Thursday 28 mai 2009

Organized by Gianluca Pacienza and Erwan Rousseau

I – Statement of the main result

- **Complex projective algebraic hypersurface :**

$$X^n \subset \mathbb{P}^{n+1}(\mathbb{C}) \,.$$

- **Canonical line bundle :**

$$K_X := \Lambda^n T^*_X .$$

- **General type :** As $m \to \infty$:

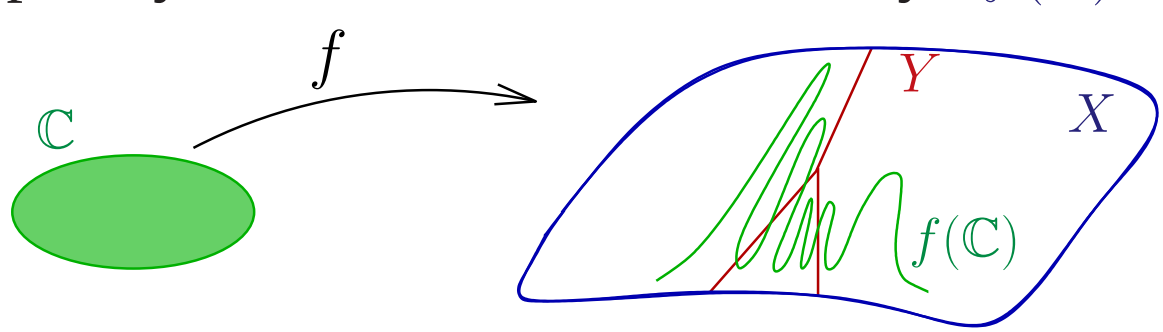
$$h^0\big(X,\, K_X^{\otimes m}\big) \sim c \cdot m^{\dim X} \,.$$

for a certain constant $c > 0$

- **Equivalent characterization :**

$$\boxed{\deg X \geqslant \dim X + 3} \,.$$

- **(Strong) Conjecture of Green-Griffiths (1979) :** If the projective algebraic hypersurface $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is generic of degree $d \geqslant n+3$, then there exists a proper algebraic subvariety $Y \subset X$ such that every nonconstant entire holomorphic curve $f : \mathbb{C} \to X$ is necessarily completely contained inside Y, namely : $f(\mathbb{C}) \subset Y$.



• **Kobayashi hyperbolicity conjecture (1970) :** A complex projective algebraic $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is **hyperbolic**, namely [Brody] : « every entire holomorphic curve $f : \mathbb{C} \to X$ is necessarily **constant** », whenever

$$\deg X \geqslant 2n+1\,,$$

provided X is generic.

• **Siu 2002, 2004 :** There exists $d_n \gg 1$ such that the generic hypersurfaces $X^n \subset \mathbb{P}^{n+1}$ of degree :

$$\deg X \geqslant d_n$$

are Kobayashi-hyperbolic.

• **Today's goal :** Realize this, using works/ideas of BLOCH, KOBAYASHI, BRODY, GREEN, GRIFFITHS, CLEMENS, EIN, VOISIN, MCQUILLAN, SIU, TRAPANI, CAMPANA, DEMAILLY, EL GOUL, DETHLOFF, ROUSSEAU, PAŬN, DIVERIO, M.

• **Some previous last decade results :**

• **Dimension 2 :** $X^2 \subset \mathbb{P}^3(\mathbb{C})$: Green-Griffiths + Kobayashi :

- ☐ MCQUILLAN, 1999 : $d \geqslant$ **36**.
- ☐ DEMAILLY-EL GOUL, 2000 : $d \geqslant$ **21**.
- ☐ PAŬN, 2008 : $d \geqslant$ **18**.

• **Dimension 3 :** $X^3 \subset \mathbb{P}^4(\mathbb{C})$: Algebraic degeneracy :

- ☐ ROUSSEAU 2007 : $d \geqslant$ **593**.

Theorem. [DMR, 5 feb. 2008] *If* $X \subset \mathbb{P}^{n+1}$ *is a generic complex projective algebraic hypersurface, there exists a proper algebraic subvariety* $Y \subsetneq X$ *such that* $f(\mathbb{C}) \subset Y$ *for every nonconstant entire holomorphic curve :*

- *for* $\dim X = \mathbf{4}$*, whenever* $\deg X \geqslant \mathbf{3203}$;
- *for* $\dim X = \mathbf{5}$*, whenever* $\deg X \geqslant \mathbf{35355}$;
- *for* $\dim X = \mathbf{6}$*, whenever* $\deg X \geqslant \mathbf{172925}$.

Theorem. [Diverio-M.-Rousseau, 17 nov. 2008] *In arbitrary dimension* $n \geqslant 2$ *with* $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *generic, strong algebraic degeneracy of nonconstant entire holomorphic curves holds whenever :*

$$\deg X \geqslant n^{(n+1)^{(n+5)}}.$$

II – Bundles of invariant jets

- **Germ of holomorphic curve :**

$$f\colon (\mathbb{C},0) \longrightarrow (\mathbb{C}^n,\,0) \simeq (X,x).$$

- **Algebraic differential operator of order k :**

$$\mathsf{P}\big(f',f'',\dots,f^{(k)}\big) = \sum_{\alpha_1,\alpha_2,\dots,\alpha_k\in\mathbb{N}^n} \mathsf{p}_{\alpha_1\alpha_2\dots\alpha_k}(f)\,(f')^{\alpha_1}(f'')^{\alpha_2}\dots(f^{(k)})^{\alpha_k},$$

the sum being *finite*, where the $\mathsf{p}_{\alpha_1\alpha_2\dots\alpha_k}(z)$ are holomorphic functions, and where :

$$(f^{(i)})^{\alpha_i} = (f_1^{(i)})^{\alpha_{i,1}}\dots(f_n^{(i)})^{\alpha_{i,n}}.$$

- **Definition :** Denote $\mathscr{E}_{k,m}^{GG}$ the bundle (introduced by Green-Griffiths) whose sections are differential operators of order k that are homogeneous of weight :

$$m = |\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k|\,.$$

- **Structure :** The graduate m-th part $\mathscr{E}_{k,m}^{GG}$ of this bundle $\mathscr{E}_k^{GG}$ is vectorial :

$$Gr^\bullet\mathscr{E}_{k,m}^{GG} = \bigoplus_{l_1+2l_2+\dots+kl_k=m} S^{l_1}T_X^*\otimes\dots\otimes S^{l_k}T_X^*\,.$$

• **Demailly-Semple invariant jets :** In local coordinates $z = (z_1, \dots, z_n)$ centered in $x \in X$, a differential operator :

$$\mathsf{P} = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} \mathsf{p}_\alpha(f)(f')^{\alpha_1} \dots (f^{(k)})^{\alpha_k}$$

is said to be **invariant under local reparametrizations** $\phi\colon (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$ if :

$$\mathsf{P}\big((f \circ \phi)', \dots, (f \circ \phi)^{(k)}\big) = \phi'(0)^m P\big(f', \dots, f^{(k)}\big) .$$

• **Structure :** Make a subbundle $\mathscr{E}_{k,m}^{DS}$ of $\mathscr{E}_{k,m}^{GG}$.

Theorem. [Bloch, Ahlfors, Green-Griffiths, Demailly, Siu] *Let X be a complex projective algebraic hypersurface, let A be an ample line bundle on X — just take $A = \mathscr{O}_X(1)$ — and let :*

$$\mathsf{P} \in H^0\big(X, \mathscr{E}_{k,m}^{GG \text{ or } DS} \otimes A^{-1}\big)$$

be a global section. Then every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ satisfies the corresponding differential equation :

$$\mathsf{P}\big(f', \dots, f^{(k)}\big) \equiv 0.$$

• **Extremely difficult problem :** Understand the condition of invariancy in the definition of Demailly invariant jets $\mathscr{E}_{k,m}^{DS}$.

Recent works concerned

- Joël M. *Jets de Demailly-Semple d'ordres 4 et 5 en dimension 2*, Int. J. Contemp. Math. Sciences, **3** (2008) no. 18, 861–933.

- Joël M. *An algorithm to generate all polynomials in the k-jet of a holomorphic disc $D \to \mathbb{C}^n$ that are invariant under source reparametrization*, arxiv.org/abs/0808.3547/, 103 pages.

- Joël M. *Low pole order frames on vertical jets of the universal hypersurface*, Ann. Inst. Fourier (Grenoble), to appear, 31 pages.

- Simone Diverio, Joël M. and Erwan Rousseau, *Effective algebraic degeneracy*, arxiv.org/abs/0811.2346/, 47 pages.

First main goal

Explain Green-Griffiths algebraic degeneracy for $\dim X = \mathbf{4}$ *whenever* $\deg X \geqslant \mathbf{3203}$.

Second main goal

Explain Green-Griffiths algebraic degeneracy for $\dim X = \mathbf{n}$ *whenever* $\deg X \geqslant n^{(n+1)^{n+5}}$.

III – Explicit invariant jets

• **(Refined) jets of de Demailly-Semple : Invariancy under reparametrization** is supposed :

$$\text{set } \; g := f \circ \phi$$

$$\text{require that } \; \mathsf{P}\big(j^k g\big) = (\phi')^m \, P\big(j^k f\big)$$

for every $(\mathbb{D}, 0) \xrightarrow{\phi} (\mathbb{D}, 0)$, where $m \geqslant 1$ is called the **weight** of P.

• **Examples :**

- ☐ Jets of order **1** :

$$\mathsf{E}_1^n = \mathbb{C}\big[f_1', f_2', \ldots, f_n'\big].$$

- ☐ Jets of order **2** :

$$f_i' \qquad \text{and the :} \qquad \begin{vmatrix} f_{i_1}' & f_{i_2}' \\ f_{i_1}'' & f_{i_2}'' \end{vmatrix} = \begin{vmatrix} f_{i_1}' & f_2' \\ f_1'' + \phi'' f_{i_1}' & f_2'' + \phi'' f_{i_2}' \end{vmatrix}$$

• **Concretely :** Set $g_i := f_i \circ \phi$ and compute :

$$\begin{aligned}
g_i' &= \phi' f_i', \\
g_i'' &= \phi'' f_i' + \phi'^2 f_i'', \\
g_i''' &= \phi''' f_i' + 3\,\phi'' \phi' f_i'' + \phi'^3 f_i''', \\
g_i'''' &= \phi'''' f_i' + 4\,\phi''' \phi' f_i'' + 3\,\phi''^2 f_i'' + 6\,\phi'' \phi'^2 f_i''' + \phi'^4 f_i'''', \\
g_i''''' &:= \phi''''' f_i' + 5\,\phi'''' \phi' f_i'' + 10\,\phi''' \phi'' f_i'' + 15\,\phi''^2 \phi' f_i''' + \\
&\qquad + 10\,\phi''' \phi'^2 f_i''' + 10\,\phi'' \phi'^3 f_i'''' + \phi'^5 f_i'''''.
\end{aligned}$$

$$\mathsf{P}\big(g', g'', g''', g'''', g'''''\big) = \big(\phi'\big)^m \, \mathsf{P}\big(f', f'', f''', f'''', f'''''\big).$$

• **Known algebraic descriptions :**

☐ $n = 2$, $k = 3$: DEMAILLY (unpublished) ; ROUSSEAU. **5**.

☐ $n = 3$, $k = 3$: ROUSSEAU. **16**.

☐ $n = 2$, $k = 4$: DEMAILLY-EL GOUL (unpublished) ; M. **9**.

☐ $n = \mathbf{2}$, $k = \mathbf{5}$: M. **56**.

☐ $n = \mathbf{4}$, $k = \mathbf{4}$: M. **2835**.

Green-Griffiths and Kobayashi conjectures

• **For any effective application to entire holomorphic curves, it is necessary to solve all the following questions :** (not only the first one)

• **Question 1 :** Is the algebra of Demailly-Semple invariants finitely generated **?**

• **Question 2 :** Is the ideal of relations between them generated by specific regular processes **?**

• **Question 3 :** Is the algebra Cohen-Macaulay ? If it is, describe an effective set of primary invariants.

• **Question 4 :** If it is not Cohen-Macaulay, describe an effective Gröbner basis for the ideal of relations between the invariants.

• **Accessible in the future :** $n = \mathbf{5}$, $k = \mathbf{5}$.

III-1 – An inappropriate algorithm

• **Perform the cross-product between two invariants :**
Suppose we know two invariants : P of weight m and Q of weight n :

$$\mathsf{P}\big(j^k g\big) = \big(\phi'\big)^m \, \mathsf{P}\big(j^k f\big),$$
$$\mathsf{Q}\big(j^\tau g\big) = \big(\phi'\big)^n \, \mathsf{Q}\big(j^\tau f\big),$$

where we have set $g := f \circ \phi$.

• **Differentiate :** with respect to the variable $\zeta \in \mathbb{C}$:

$$\mathsf{P}' = m\,\phi''\,\big(\phi'\big)^{m-1}\,\mathsf{P} + \big(\phi'\big)^{m-1}\,\mathsf{P}'$$
$$\mathsf{Q}' = n\,\phi''\,\big(\phi'\big)^{n-1}\,\mathsf{Q} + \big(\phi'\big)^{n-1}\,\mathsf{Q}'.$$

• **One should eliminate the second derivative ϕ''.**

Observation. Every pair of invariants yields automatically a new invariant :

$$\boxed{\big[\mathsf{P},\,\mathsf{Q}\big] := n\,\mathsf{P}'\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{Q}'},$$

which is visibly skew-symmetric in P and Q.

• **Three generating families of relations :**

($\mathscr{J}ac$) $\boxed{0 \equiv \big[\big[\mathsf{P},\,\mathsf{Q}\big],\,\mathsf{R}\big] + \big[\big[\mathsf{R},\,\mathsf{P}\big],\,\mathsf{Q}\big] + \big[\big[\mathsf{Q},\,\mathsf{R}\big],\,\mathsf{P}\big]}$.

($\mathscr{P}lck_1$) $\boxed{0 \equiv m\,\mathsf{P}\,\big[\mathsf{Q},\,\mathsf{R}\big] + o\,\mathsf{R}\,\big[\mathsf{P},\,\mathsf{Q}\big] + n\,\mathsf{Q}\,\big[\mathsf{R},\,\mathsf{P}\big]}$.

($\mathscr{P}lck_2$)

$$\boxed{0 \equiv \big[\mathsf{P},\,\mathsf{Q}\big]\cdot\big[\mathsf{R},\,\mathsf{S}\big] + \big[\mathsf{S},\,\mathsf{P}\big]\cdot\big[\mathsf{R},\,\mathsf{Q}\big] + \big[\mathsf{Q},\,\mathsf{S}\big]\cdot\big[\mathsf{R},\,\mathsf{P}\big]}.$$

III-2 – The appropriate algorithm

Theorem. [M., 2008] *Construction of complete algorithm which generates all jet polynomials invariant under reparametrization and all the relations between them, in arbitrary dimension n and for jets of arbitrary order k.*

• **First illustration :** Dimension $n = 3$ and jet of order $k = 3$ [Rousseau 2006 ; M. 2007] : $\boxed{\textbf{4}\text{ bi-invariants}}$.

• **9 jet variables :**

$$\left(f_1', f_2', f_3', f_1'', f_2'', f_3'', f_1''', f_2''', f_3'''\right).$$

• **Double trick of reparametrizing by f_1^{-1} and of requiring unipotent invariance yields an initial form :**

$$\boxed{\mathsf{P}\big(j^3 f\big) = \sum\nolimits_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}_a\Big(\Lambda^3,\, \Lambda^5,\, D^6\Big).}$$

• **Explicit expressions :** Each P_a depends upon the following basic bi-invariants :

$$\Lambda^3 := \begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix} =: \Delta_{1,2}^{',''},$$

$$\Lambda^5 := \Delta_{1,2}^{','''} f_1' - 3\,\Delta_{1,2}^{',''} f_1'',$$

$$D^6 := \begin{vmatrix} f_1' & f_2' & f_3'' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix} =: \Delta^{','','''}.$$

• **Observation : these 3 bi-invariants are algebraically independent, even after setting $f_1' = 0$:**

$$\Lambda^3\big|_0 = -f_2' f_1'',$$
$$\Lambda^5\big|_0 = f_2'\, f_1''\, f_1'',$$
$$D^6\big|_0 = \begin{vmatrix} 0 & f_2' & f_3'' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}.$$

Theorem. *Then in the sum :*

$$\mathsf{P}\big(j^3 f\big) = \sum\nolimits_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(\Lambda^3,\, \Lambda^5,\, D^6\Big),$$

there are in fact no negative powers of f_1', so that :

$$\mathsf{UDS}_3^3 = \mathbb{C}\big[f_1',\, \Lambda^3,\, \Lambda^5,\, D^6\big]\,.$$

Proof. Ortherwise, get by chasing the denominators :

$$(f_1')^{-a_{\min}}\, \mathsf{P}\big(j^3 f\big) = \mathsf{P}_{a_{\min}}\big(\Lambda^3,\, \Lambda^5,\, D^6\big) + f_1' \times \text{remainder}\,.$$

Then set $f_1' = 0$ hence get :

$$0 \equiv \mathsf{P}_{a_{\min}}\big(\Lambda^3\big|_0,\, \Lambda^5\big|_0,\, D^6\big|_0\big),$$

whence $\mathsf{P}_{-a_{\min}} = 0$ by the algebraic independence of $\Lambda^3\big|_0$, $\Lambda^5\big|_0$, $D^6\big|_0$, which visibly contradicts the definition of $-a_{\min}$. $\square$

• **Anticipated reinterpretation :**

$$\text{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ D^6\big|_0\Big) = \{0\}\,.$$

Corollary. [Rousseau 2006 ; M. 2007] *By polarizing these* **4** *bi-invariants, one obtains that the full algebra* DS^3_3 *of invariants under reparametrization is generated by the* **16** *bi-invariants :*

$$f'_i,\ \Lambda^3_{i,j},\ \Lambda^5_{i,j;\,k},\ D^5.$$

• **Second illustration :** dimension $n = 4$ and jet order $k = 4$.

• **16 jet variables :**

$$\left(f'_1, f'_2, f'_3, f'_4, f''_1, f''_2, f''_3, f''_4, f'''_1, f'''_2, f'''_3, f'''_4, f''''_1, f''''_2, f''''_3, f''''_4\right).$$

• **Same double trick and some computations provide an initial representation :**

$$\mathsf{P}\big(j^4 f\big) = \sum\nolimits_{-\frac{3}{4}m \leqslant a \leqslant m} (f'_1)^a\, \mathsf{P}_a\Big(\Lambda^3,\ \Lambda^5,\ \Lambda^7,\ D^6,\ D^8,\ N^{10},\ W^{10}\Big).$$

• **Explicit expressions :** Each P_a depends upon the following basic bi-invariants :

$$\Lambda^7 := \big(\Delta^{\prime,\,''''}_{1,2} + 4\,\Delta^{\prime\prime,\,'''}_{1,2}\big)\, f'_1 f'_1 - \\ - 10\,\Delta^{\prime,\,'''}_{1,2}\, f'_1 f''_1 + 15\,\Delta^{\prime,\,''}_{1,2}\, f''_1 f''_1,$$

$$D^8 := f'_1 \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \\ f''''_1 & f''''_2 & f''''_3 \end{vmatrix} - 6\, f''_1 \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \\ f'''_1 & f'''_2 & f'''_3 \end{vmatrix},$$

$$N^{10} := \Delta^{\prime,\prime\prime\prime,\prime\prime\prime\prime}_{1,2,3}\, f_1' f_1' - 3\,\Delta^{\prime,\prime\prime,\prime\prime\prime\prime}_{1,2,3}\, f_1' f_1'' + \\ + 4\,\Delta^{\prime,\prime\prime,\prime\prime\prime}_{1,2,3}\, f_1' f_1''' + 3\,\Delta^{\prime,\prime\prime,\prime\prime\prime}_{1,2,3}\, f_1'' f_1'',$$

and finally, the **Wronskian** :

$$W^{10} := \begin{vmatrix} f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ f_1''' & f_2''' & f_3''' & f_4''' \\ f_1'''' & f_2'''' & f_3'''' & f_4'''' \end{vmatrix}.$$

- **Starting point for the algorithm :**

$$\boxed{\mathsf{P}\big(j^5 f\big) = \sum\nolimits_{-\frac{3}{4}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(\Lambda^3,\, \Lambda^5,\, \Lambda^7,\, D^6,\, D^8,\, N^{10},\, W^{10}\Big)}$$

- **Presence of negative powers of f_1' !**

□ **Compute the Ideal of Relations of these seven bi-invariants restricted to $\{f_1' = 0\}$:**

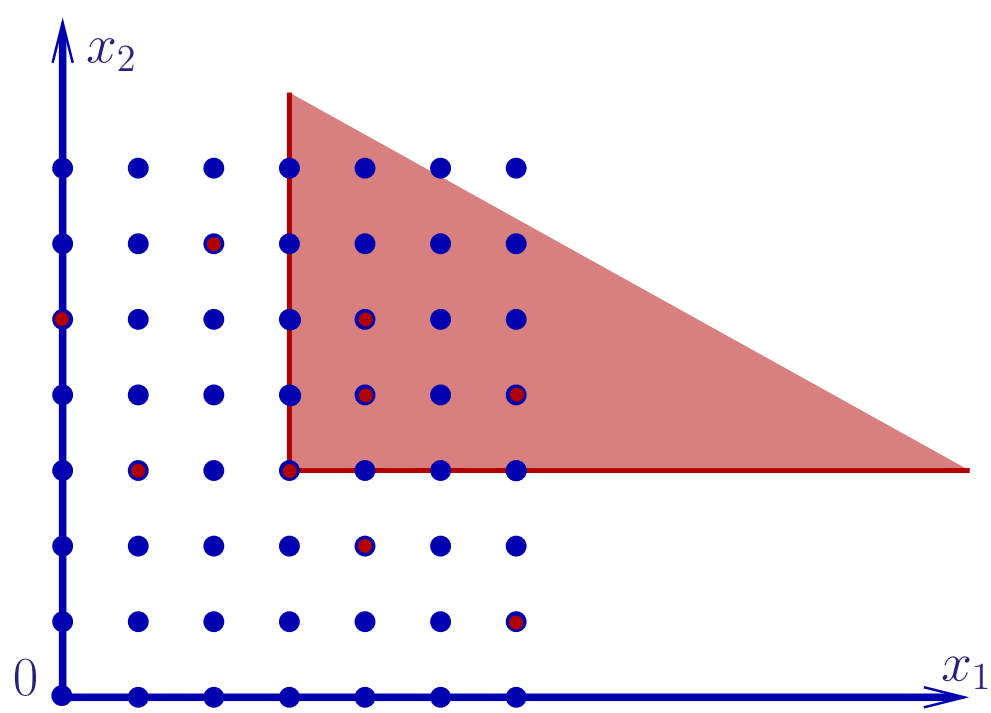
$$\text{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ D^6\big|_0,\ D^8\big|_0,\ N^{10}\big|_0,\ W^{10}\big|_0\Big),$$

namely a generating set of the ideal of all polynomials in seven variables that give zero, identically, after substituting these four restricted invariants.

Get the six relations valuable for $f_1' = 0$:

$$
\begin{aligned}
0 &\overset{a}{\equiv} 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7\Big|_0,\\
0 &\overset{b}{\equiv} 2\,\Lambda^5 D^6 - \Lambda^3 D^8\Big|_0,\\
0 &\overset{c}{\equiv} \Lambda^7 D^6 - 5\,\Lambda^3 N^{10}\Big|_0,\\
0 &\overset{d}{\equiv} \Lambda^5 D^8 - 6\,\Lambda^3 N^{10}\Big|_0,\\
0 &\overset{e}{\equiv} \Lambda^7 D^8 - 10\,\Lambda^5 N^{10}\Big|_0,\\
0 &\overset{f}{\equiv} D^8 D^8 - 12\,D^6 N^{10}\Big|_0.
\end{aligned}
$$

Gröbner bases ; Dickson lemma ; S-polynomials



□ **So without setting $f_1' = 0$, there should be three remainders that are a multiple of f_1' :**

$$0 \overset{a}{\equiv} 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7 + f_1' \times \text{something},$$
$$0 \overset{b}{\equiv} 2\,\Lambda^5 D^6 - \Lambda^3 D^8 + f_1' \times \text{something},$$
$$0 \overset{c}{\equiv} \Lambda^7 D^6 - 5\,\Lambda^3 N^{10} + f_1' \times \text{something},$$
$$0 \overset{d}{\equiv} \Lambda^5 D^8 - 6\,\Lambda^3 N^{10} + f_1' \times \text{something},$$
$$0 \overset{e}{\equiv} \Lambda^7 D^8 - 10\,\Lambda^5 N^{10} + f_1' \times \text{something},$$
$$0 \overset{f}{\equiv} D^8 D^8 - 12\,D^6 N^{10} + f_1' \times \text{something}.$$

□ **Each something necessarily also is a bi-invariant.**

□ **Find the maximal power of f_1' which factors each something.**

- **Get the six completed expressions :**

$$0 \overset{a}{\equiv} 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7 + f_1' f_1' M^8,$$
$$0 \overset{b}{\equiv} 2\,\Lambda^5 D^6 - \Lambda^3 D^8 + \tfrac{1}{3}\, f_1' E^{10},$$
$$0 \overset{c}{\equiv} \Lambda^7 D^6 - 5\,\Lambda^3 N^{10} + f_1' L^{12},$$
$$0 \overset{d}{\equiv} \Lambda^5 D^8 - 6\,\Lambda^3 N^{10} + f_1' L^{12},$$
$$0 \overset{e}{\equiv} \Lambda^7 D^8 - 10\,\Lambda^5 N^{10} - f_1' Q^{14},$$
$$0 \overset{f}{\equiv} D^8 D^8 - 12\,D^6 N^{10} - f_1' R^{15}.$$

- **Test whether or not the obtained bi-invariants :**

M^8 E^{10} L^{12} Q^{14} R^{15}

belong or do not belong to the algebra generated by the previously known bi-invariants :

f_1' Λ^3 Λ^5 Λ^7 D^6 D^8 N^{10} W^{10}

- **Here : none of the above 13 bi-invariants is equal to a polynomial with respect to the 12 remaining ones.**

- **Compute the explicit expressions :**

$$M^8 := \frac{-5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7}{f_1' f_1'}$$
$$= 3\,\Delta_{1,2}^{',''''}\,\Delta_{1,2}^{',''} + 12\,\Delta_{1,2}^{'','''}\,\Delta_{1,2}^{',''} - 5\,\Delta_{1,2}^{','''}\,\Delta_{1,2}^{','''},$$

$$E^{10} := \frac{-6\,\Lambda^5\,D^6 + 3\,\Lambda^3\,D^8}{f_1'}$$
$$= 3\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2} - 6\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\,,$$

$$L^{12} := \frac{-\Lambda^7 D^6 + 5\,\Lambda^3 N^{10}}{f_1'}$$
$$= -\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''''}_{1,2}\;f_1' - 4\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{'',\,'''}_{1,2}\;f_1' + 5\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1' + 10\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\,f_1'' -$$
$$- 15\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'' + 20\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'''\,,$$

$$Q^{14} := \frac{\Lambda^7 D^8 - 10\,\Lambda^5 N^{10}}{f_1'}$$
$$= -10\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1' f_1' + \Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,''''}_{1,2}\;f_1' f_1' + 4\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{'',\,'''}_{1,2}\;f_1' f_1' +$$
$$+ 20\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1' f_1'' + 30\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1' f_1'' - 6\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,''''}_{1,2}\;f_1' f_1'' -$$
$$- 24\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{'',\,'''}_{1,2}\;f_1' f_1'' - 40\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1' f_1''' - 75\,\Delta^{',\,'',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'' f_1'' +$$
$$+ 30\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'''}_{1,2}\;f_1'' f_1'' + 120\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,''}_{1,2}\,f_1'' f_1'''\,,$$

$$R^{15} := \Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;f_1' - 12\,\Delta^{',\,''',\,'''',\,''''}_{1,2,3}\;\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;f_1' + 24\,\Delta^{',\,'',\,''',\,''''}_{1,2,3}\;\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;f_1'' -$$
$$- 48\,\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;\Delta^{',\,'',\,''',\,'''}_{1,2,3}\;f_1'''\,,$$

□ **Then restart the process with the 12 bi-invariants.**

$$\text{Ideal-Rel}\left(\Lambda^3\big|_0,\;\Lambda^5\big|_0,\;\Lambda^7\big|_0,\;D^6\big|_0,\;D^8\big|_0,\;N^{10}\big|_0,\;M^8\big|_0,\;E^{10}\big|_0,\;L^{12}\big|_0,\;Q^{14}\big|_0,\;R^{15}\big|_0,\;W^{10}\big|_0\right)$$

□ **Compute the Ideal of Relations of these bi-invariants restricted to $\{f_1' = 0\}$:**

$$0 \overset{g}{\equiv} 4\,D^8Q^{14} - 5\,\Lambda^7R^{15} - f_1'X^{21},$$

$$0 \overset{h}{\equiv} 24\,D^6Q^{14} - 25\,\Lambda^5R^{15} + f_1'V^{19},$$

$$0 \overset{i}{\equiv} L^{12}L^{12} + E^{10}Q^{14} - f_1'M^8R^{15},$$

$$0 \overset{j}{\equiv} 8\,N^{10}L^{12} + \Lambda^7R^{15} + f_1'X^{21},$$

$$0 \overset{k}{\equiv} 4\,D^8L^{12} + 5\,\Lambda^5R^{15} - f_1'V^{19},$$

$$0 \overset{l}{\equiv} 8\,D^6L^{12} + 5\,\Lambda^3R^{15} - \tfrac{1}{3}\,f_1'U^{17},$$

$$0 \overset{m}{\equiv} \Lambda^7L^{12} + \Lambda^5Q^{14} - 2\,f_1'M^8N^{10},$$

$$0 \overset{n}{\equiv} 5\,\Lambda^5L^{12} + 3\,\Lambda^3Q^{14} - f_1'D^8M^8,$$

$$0 \overset{o}{\equiv} 8\,N^{10}E^{10} + \Lambda^5\,R^{15} - f_1'V^{19},$$

$$0 \overset{p}{\equiv} 4\,D^8E^{10} + 3\,\Lambda^3R^{15} - f_1'U^{17},$$

$$0 \overset{q}{\equiv} 5\,\Lambda^7E^{10} + 3\,\Lambda^3Q^{14} - 6f_1'D^8M^8,$$

$$0 \overset{r}{\equiv} 5\,\Lambda^5E^{10} - 3\,\Lambda^3L^{12} - 6\,f_1'D^6M^8,$$

$$0 \overset{s}{\equiv} 8\,\Lambda^5N^{10}Q^{14} - \Lambda^7\Lambda^7R^{15} + f_1'Q^{14}Q^{14} + 4\,f_1'N^{10}N^{10}M^8,$$

$$0 \overset{t}{\equiv} 24\,\Lambda^3N^{10}Q^{14} - 5\,\Lambda^5\Lambda^7R^{15} - 5\,f_1'L^{12}Q^{14} + 2\,f_1'M^8D^8N^{10}.$$

□ **Get 3 new bi-invariants :**

$$U^{17}, \qquad V^{19}, \qquad X^{21},$$

that are defined explicitly by :

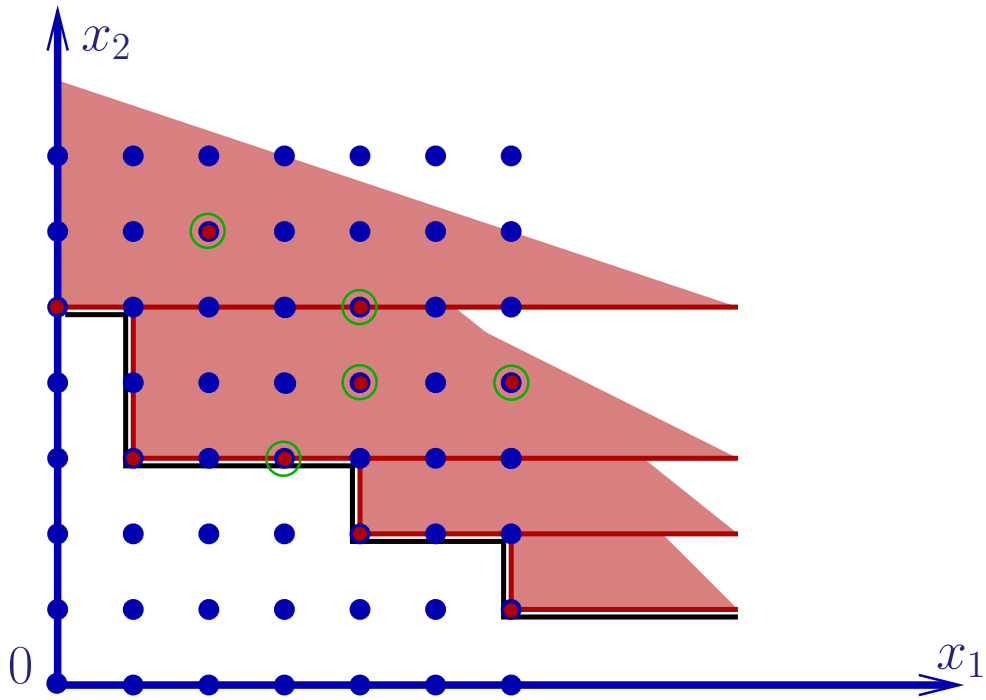
$$
\begin{aligned}
U^{17} &= \frac{4\,D^8E^{10}+3\,\Lambda^3R^{15}}{f_1'}\\
&= 15\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2} - 36\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2} -\\
&\quad - 24\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2} + 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,,
\end{aligned}
$$

$$
\begin{aligned}
V^{19} &= \frac{8\,N^{10}E^{10}+\Lambda^5R^{15}}{f_1'}\\
&= 24\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1' - 60\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1' +\\
&\quad + \Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1' - 75\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1'' +\\
&\quad + 36\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1'' + 168\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'' -\\
&\quad - 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'' + 96\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''' -\\
&\quad - 240\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1''',
\end{aligned}
$$

$$
\begin{aligned}
X^{21} &= \frac{4\,D^8Q^{14}-5\,\Lambda^7R^{15}}{f_1'}\\
&= -40\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1' - 4\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1' -\\
&\quad - 4\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1' + 60\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1' +\\
&\quad + 240\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1' + 130\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1'' +\\
&\quad + 120\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1'f_1'' - 168\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1'' -\\
&\quad - 668\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1'' - 360\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1'' -\\
&\quad - 160\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1'f_1''' + 240\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1'f_1''' +\\
&\quad + 960\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1'f_1''' - 375\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''f_1'' +\\
&\quad + 840\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1''f_1'' + 180\,\Delta^{',\,''',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''f_1'' +\\
&\quad + 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''''}_{1,2}\,f_1''f_1'' + 144\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{'',\,'''}_{1,2}\,f_1''f_1'' -\\
&\quad - 1440\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,'''}_{1,2}\,f_1''f_1''' + 480\,\Delta^{',\,'',\,''''}_{1,2,3}\,\Delta^{',\,'',\,'''}_{1,2,3}\,\Delta^{',\,''}_{1,2}\,f_1''f_1'''.
\end{aligned}
$$

□ **Termination of the algorithm : Use plain lexicographic ordering of the 14 bi-invariants :**

$$\Lambda^3 > \Lambda^5 > \Lambda^7 > D^6 > D^8 > N^{10} > M^8 > E^{10} > L^{12} > \\ > Q^{14} > R^{15} > U^{17} > V^{19} > X^{21},$$



□ **Obtain 41 completed syzygies between these 14 bi-invariants :**

$$0 \overset{1}{\equiv} -5\,\Lambda^5\Lambda^5 + 3\,\underline{\Lambda^3\Lambda^7}_{\mathsf{LT}} - f_1' f_1' M^8,$$

$$0 \overset{2}{\equiv} -2\,\Lambda^5 D^6 + \underline{\Lambda^3 D^8}_{\mathsf{LT}} - \tfrac{1}{3} f_1'\, E^{10},$$

$$0 \overset{3}{\equiv} -\Lambda^7 D^6 + 5\,\underline{\Lambda^3 N^{10}}_{\mathsf{LT}} - f_1' L^{12},$$

$$0 \overset{4}{\equiv} -5\,\Lambda^5 E^{10} + 3\,\underline{\Lambda^3 L^{12}}_{\mathsf{LT}} + 6\, f_1' D^6 M^8,$$

$$0 \overset{5}{\equiv} 5\,\Lambda^7 E^{10} + 3\,\underline{\Lambda^3 Q^{14}}_{\mathsf{LT}} - 6\, f_1' D^8 M^8,$$

$$0 \overset{6}{\equiv} 4\, D^8 E^{10} + 3\,\underline{\Lambda^3 R^{15}}_{\mathsf{LT}} - f_1' U^{17},$$

$$0 \overset{7}{\equiv} -36\, D^6 D^6 M^8 - 5\, E^{10} E^{10} + 3\, \underline{\Lambda^3 U^{17}}_{\mathsf{LT}} + 0,$$
$$0 \overset{8}{\equiv} -5\, E^{10} L^{12} - 6\, D^6 D^8 M^8 + 3\, \underline{\Lambda^3 V^{19}}_{\mathsf{LT}} + 0,$$
$$0 \overset{9}{\equiv} 5\, L^{12} L^{12} + 3\, \underline{\Lambda^3 X^{21}}_{\mathsf{LT}} + M^8 D^8 D^8 + 0,$$
$$0 \overset{10}{\equiv} -6\, \Lambda^7 D^6 + 5\, \underline{\Lambda^5 D^8}_{\mathsf{LT}} - f_1' L^{12},$$
$$0 \overset{11}{\equiv} -\Lambda^7 D^8 + 10\, \underline{\Lambda^5 N^{10}}_{\mathsf{LT}} + f_1' Q^{14},$$
$$0 \overset{12}{\equiv} \underline{\Lambda^5 L^{12}}_{\mathsf{LT}} - \Lambda^7 E^{10} + f_1' D^8 M^8,$$

$$0 \overset{13}{\equiv} \Lambda^7 L^{12} + \underline{\Lambda^5 Q^{14}}_{\mathsf{LT}} - 2\, f_1' M^8 N^{10},$$
$$0 \overset{14}{\equiv} 8\, N^{10} E^{10} + \underline{\Lambda^5 R^{15}}_{\mathsf{LT}} - f_1' V^{19},$$
$$0 \overset{15}{\equiv} \underline{\Lambda^5 U^{17}}_{\mathsf{LT}} - E^{10} L^{12} - 6\, D^6 D^8 M^8 + 0,$$
$$0 \overset{16}{\equiv} \underline{\Lambda^5 V^{19}}_{\mathsf{LT}} - M^8 D^8 D^8 - L^{12} L^{12} + f_1' M^8 R^{15},$$
$$0 \overset{17}{\equiv} \underline{\Lambda^5 X^{21}}_{\mathsf{LT}} - L^{12} Q^{14} + 2\, D^8 N^{10} M^8 + 0,$$
$$0 \overset{18}{\equiv} 8\, N^{10} L^{12} + \underline{\Lambda^7 R^{15}}_{\mathsf{LT}} + f_1' X^{21},$$

$$0 \overset{19}{\equiv} -L^{12} L^{12} + \underline{\Lambda^7 U^{17}}_{\mathsf{LT}} - 5\, M^8 D^8 D^8 + 0,$$
$$0 \overset{20}{\equiv} L^{12} Q^{14} + \underline{\Lambda^7 V^{19}}_{\mathsf{LT}} - 10\, D^8 M^8 N^{10} + 0,$$

$$0 \overset{21}{\equiv} 20\, N^{10} N^{10} M^8 + Q^{14} Q^{14} + \underline{\Lambda^7 X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{22}{\equiv} 6\, \underline{D^6 M^8 R^{15}}_{\mathsf{LT}} + L^{12} U^{17} - E^{10} V^{19} + 0,$$
$$0 \overset{23}{\equiv} 5\, \underline{D^8 M^8 R^{15}}_{\mathsf{LT}} - Q^{14} U^{17} - L^{12} V^{19} + 0,$$
$$0 \overset{24}{\equiv} 10\, \underline{N^{10} M^8 R^{15}}_{\mathsf{LT}} - Q^{14} V^{19} + L^{12} X^{21} + 0,$$

$$0 \overset{25}{\equiv} 5\,\underline{M^8R^{15}R^{15}}_{\mathsf{LT}} + V^{19}V^{19} + U^{17}X^{21} + 0,$$

$$0 \overset{26}{\equiv} -D^8D^8 + 12\,\underline{D^6N^{10}}_{\mathsf{LT}} + f_1'R^{15},$$

$$0 \overset{27}{\equiv} -5\,D^8E^{10} + 6\,\underline{D^6L^{12}}_{\mathsf{LT}} + f_1'U^{17},$$

$$0 \overset{28}{\equiv} 3\,\underline{D^6Q^{14}}_{\mathsf{LT}} + 25\,N^{10}E^{10} - 3\,f_1'V^{19},$$

$$0 \overset{29}{\equiv} 5\,E^{10}R^{15} - D^8U^{17} + 6\,\underline{D^6V^{19}}_{\mathsf{LT}} + 0,$$

$$0 \overset{30}{\equiv} -3\,L^{12}R^{15} + N^{10}U^{17} + 3\,\underline{D^6X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{31}{\equiv} -10\,N^{10}E^{10} + \underline{D^8L^{12}}_{\mathsf{LT}} + f_1'V^{19},$$

$$0 \overset{32}{\equiv} \underline{D^8Q^{14}}_{\mathsf{LT}} + 10\,N^{10}L^{12} + f_1'X^{21},$$

$$0 \overset{33}{\equiv} -2\,N^{10}U^{17} + \underline{D^8V^{19}}_{\mathsf{LT}} + L^{12}R^{15} + 0,$$

$$0 \overset{34}{\equiv} Q^{14}R^{15} + 2\,N^{10}V^{19} + \underline{D^8X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{35}{\equiv} -2\,L^{12}N^{10}U^{17} + R^{15}L^{12}L^{12} + 10\,\underline{V^{19}N^{10}E^{10}}_{\mathsf{LT}} - f_1'V^{19}V^{19},$$

$$0 \overset{36}{\equiv} 2\,N^{10}U^{17}Q^{14} - R^{15}L^{12}Q^{14} + 10\,\underline{V^{19}N^{10}L^{12}}_{\mathsf{LT}} + f_1'V^{19}X^{21},$$

$$0 \overset{37}{\equiv} 10\,\underline{N^{10}L^{12}X^{21}}_{\mathsf{LT}} - R^{15}Q^{14}Q^{14} - 2\,Q^{14}N^{10}V^{19} + f_1'X^{21}X^{21},$$

$$0 \overset{38}{\equiv} 2\,\underline{N^{10}U^{17}X^{21}}_{\mathsf{LT}} - X^{21}L^{12}R^{15} + V^{19}Q^{14}R^{15} + 2\,N^{10}V^{19}V^{19} + 0,$$

$$0 \overset{39}{\equiv} \underline{E^{10}Q^{14}}_{\mathsf{LT}} + L^{12}L^{12} - f_1'M^8R^{15},$$

$$0 \overset{40}{\equiv} Q^{14}U^{17} + 6\,L^{12}V^{19} + 5\,\underline{E^{10}X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{41}{\equiv} -6\,Q^{14}L^{12}V^{19} - Q^{14}Q^{14}U^{17} + 5\,\underline{X^{21}L^{12}L^{12}}_{\mathsf{LT}} - 5\,f_1'M^8R^{15}X^{21}.$$

Theorem. (M. 2008) *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra UDS^4_4 of jet polynomials $\mathsf{P}\big(j^4 f_1, j^4 f_2, j^4 f_3, j^4 f_4\big)$ invariant by reparametrization and invariant under the unipotent action is generated by the* **16** *mutually independent bi-invariants defined above :*

$$\boxed{\begin{array}{cccccccc} W^{10}, & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & D^6, & D^8, & N^{10}, \\ M^8, & E^{10}, & L^{12}, & Q^{14}, & R^{15}, & U^{17}, & V^{19}, & X^{21}, \end{array}}$$

whose restriction to $\{f_1' = 0\}$ has a reduced gröbnerized ideal of relations, for the Lexicographic ordering, which consists of the **41** *syzygies written above. Furthermore, any bi-invariant of weight m writes uniquely in the finite polynomial form :*

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \, \big(W^{10}\big)^p \sum_{\substack{(a,\dots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41}) \\ 3a+\cdots+21n=m-o-10p}} \mathsf{coeff}_{a,\dots,n,o,p} \cdot$$
$$\cdot \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h$$
$$\big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n,$$

with coefficients $\mathsf{coeff}_{a,\dots,n,o,p}$ subjected to no restriction, where $\square_1$, …, $\square_{41}$ denote the quadrants in $\mathbb{N}^{14}$ having vertex at the leading terms of the **41** *syzygies in question.*

Corollary *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra DS^4_4 of jet polynomials $\mathsf{P}\big(j^4 f\big)$ invariant*

by reparametrization is generated by the polarizations :

$$\boxed{\begin{array}{c} W^{10}, \quad f'_i, \quad \Lambda^3_{[i,j]}, \quad \Lambda^5_{[i,j];\,\alpha}, \quad \Lambda^7_{[i,j];\,\alpha,\beta}, \quad D^6_{[i,j,k]}, \\ D^8_{[i,j,k];\,\alpha}, \quad N^{10}_{[i,j,k];\,\alpha,\beta}, \quad M^8_{[i,j],[k,l]}, \quad E^{10}_{[i,j,k],[p,q]}, \quad L^{12}_{[i,j,k],[p,q];\,\alpha}, \\ Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta}, \quad R^{15}_{[i,j,k],[p,q,r];\,\alpha}, \quad U^{17}_{[i,j,k],[p,q,r],[s,t]}, \\ V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha}, \quad X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta}, \end{array}}$$

of the **16** *bi-invariants* W^{10}, f'_1, Λ^3, Λ^5, Λ^7, D^6, D^8, N^{10}, M^8, E^{10}, L^{12}, Q^{14}, R^{15}, U^{17}, V^{19}, X^{21} *generating the algebra* UDS^4_4 *of bi-invariants ; these polarized invariants are skew-symmetric with respect to each collection of bracketed indices* $[i,j,k]$, $[p,q,r]$, $[s,t]$, *and they are* explicitly *represented in terms of* Δ*-determinants by the following complete explicit formulas :*

$$W^{10}_{1,2,3,4},$$

$$f'_i,$$

$$\Lambda^3_{[i,j]} := \Delta^{',''}_{i,j},$$

$$\Lambda^5_{[i,j];\,\alpha} := \Delta^{','''}_{i,j}\, f'_\alpha - 3\,\Delta^{',''}_{i,j}\, f''_\alpha,$$

$$\Lambda^7_{[i,j];\,\alpha,\beta} := \Delta^{',''''}_{i,j}\, f'_\alpha f'_\beta + 4\,\Delta^{'','''}_{i,j}\, f'_\alpha f'_\beta - 5\Delta^{','''}_{i,j}\,\big(f'_\alpha f''_\beta + f''_\alpha f'_\beta\big) + \\ + 15\,\Delta^{',''}_{i,j}\, f''_\alpha f''_\beta,$$

$$D^6_{[i,j,k]} := \Delta^{','','''}_{i,j,k},$$

$$
\begin{aligned}
D^{8}_{[i,j,k];\,\alpha} &:= \Delta^{',\,'',\,''''}_{i,j,k}\, f'_\alpha - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\, f''_\alpha,\\
N^{10}_{[i,j,k];\,\alpha,\beta} &:= \Delta^{',\,''',\,''''}_{i,j,k}\, f'_\alpha f'_\beta - \frac{3}{2}\,\Delta^{',\,'',\,''''}_{i,j,k}\,\left(f'_\alpha f''_\beta + f''_\alpha f'_\beta\right) +\\
&\quad + 2\,\Delta^{',\,'',\,'''}_{i,j,k}\,\left(f'_\alpha f'''_\beta + f'''_\alpha f'_\beta\right) + 3\,\Delta^{',\,'',\,'''}_{i,j,k}\, f''_\alpha f''_\beta,\\
M^{8}_{[i,j],[k,l]} &:= 3\,\Delta^{',\,''''}_{i,j}\,\Delta^{',\,''}_{k,l} + 12\,\Delta^{'',\,'''}_{i,j}\,\Delta^{',\,''}_{k,l} -\\
&\quad - 5\,\Delta^{',\,'''}_{i,j}\,\Delta^{',\,'''}_{k,l}\,,\\
E^{10}_{[i,j,k],[p,q]} &:= 3\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''}_{l,m} - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{l,m},\\
L^{12}_{[i,j,k],[l,m];\,\alpha} &:= 5\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha - 15\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f''_\alpha - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,''''}_{p,q}\, f'_\alpha -\\
&\quad - 24\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{'',\,'''}_{p,q}\, f'_\alpha + 30\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha,\\
Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta} &:= -10\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha f'_\beta + \Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''''}_{p,q}\, f'_\alpha f'_\beta +\\
&\quad + 4\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{'',\,'''}_{p,q}\, f'_\alpha f'_\beta + +20\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha f''_\beta +\\
&\quad + 30\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f'_\alpha f''_\beta - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,''''}_{p,q}\, f'_\alpha f''_\beta -\\
&\quad - 24\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{'',\,'''}_{p,q}\, f'_\alpha f''_\beta - 40\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha f'''_\beta -\\
&\quad - 75\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f''_\alpha f''_\beta + 30\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f''_\alpha f''_\beta +\\
&\quad + 120\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f''_\alpha f'''_\beta,\\
R^{15}_{[i,j,k],[p,q,r];\,\alpha} &:= \Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\, f'_\alpha - 12\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\, f'_\alpha +\\
&\quad + 24\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\, f''_\alpha - 48\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\, f'''_\alpha,\\
U^{17}_{[i,j,k],[p,q,r],[s,t]} &:= 15\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,''}_{s,t} - 36\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,''}_{s,t} -\\
&\quad - 24\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,'''}_{s,t} + 144\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{'',\,'''}_{s,t}\,,\\
V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha} &:= 24\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,''}_{s,t}\, f'_\alpha - 60\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,'''}_{s,t}\, f'_\alpha +\\
&\quad + \Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,'''}_{s,t}\, f'_\alpha - 75\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,''}_{s,t}\, f''_\alpha +\\
&\quad + 36\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,''}_{s,t}\, f''_\alpha + 168\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,'''}_{s,t}\, f'_\alpha
\end{aligned}
$$

$$- 144\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{''}_{s,t}\,f''_\alpha + 96\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'''_\alpha -$$
$$- 240\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'''_\alpha,$$

$$X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta} :=$$
$$:= -40\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f'_\beta - 4\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f'_\beta -$$
$$- 4\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{''}_{s,t}\,f'_\alpha f'_\beta + 60\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f'_\beta +$$
$$+ 240\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{''}_{s,t}\,f'_\alpha f'_\beta + 130\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f''_\beta +$$
$$+ 120\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f'_\alpha f''_\beta - 168\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f''_\beta -$$
$$- 668\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{''}_{s,t}\,f'_\alpha f''_\beta - 360\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f''_\beta -$$
$$- 160\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f'''_\beta + 240\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f'_\alpha f'''_\beta +$$
$$+ 960\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{''}_{s,t}\,f'_\alpha f'''_\beta - 375\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f''_\alpha f''_\beta +$$
$$+ 840\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f''_\alpha f''_\beta + 180\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f''_\alpha f''_\beta +$$
$$+ 144\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f''_\alpha f''_\beta + 144\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{''}_{s,t}\,f''_\alpha f''_\beta -$$
$$- 1440\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f''_\alpha f'''_\beta + 480\,\Delta^{',''}_{i,j,k}\,\Delta^{',''}_{p,q,r}\,\Delta^{'}_{s,t}\,f''_\alpha f'''_\beta,$$

where the roman indices satisfy $1 \leqslant i < j < k \leqslant 4$*, where* $1 \leqslant p < q < r \leqslant 4$*, where* $1 \leqslant s < r \leqslant 4$ *and where the two greek indices* α, β *satisfy* $1 \leqslant \alpha, \beta \leqslant 4$ *without restriction and finally the* **total number** *of these invariants generating the Demailly-Semple algebra* DS^4_4 *equals :*

$$1 + 4 + 6 + 24 + 96 + 4 + 16 + 64 +$$
$$+ 36 + 24 + 96 + 384 + 64 + 96 + 384 + 1536 = \boxed{\mathbf{2835}}\,.$$

III-4 – Asymptotics of Euler characteristic

Theorem. *On a hypersurface $X^4 \subset \mathbb{P}^5(\mathbb{C})$ of dimension $n = 4$, the graduate m-th part $\mathscr{E}^{DS}_{4,m}T^*_X$ of the complete Demailly-Semple bundle possesses the Schur decomposition :*

$$\mathscr{E}^{DS}_{4,m}T^*_X = \bigoplus_{\substack{(a,b,\ldots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41})\\ o+3a+\cdots+21n+10p=m}} \Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n+p \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n+p \\ d+e+f+h+i+j+2k+2l+2m'+2n+p \\ p \end{pmatrix} T^*_X,$$

where the **41** *subsets $\square_i$, $i = 1, 2, \ldots, 41$ of $\mathbb{N}^{14} \ni (a, b, \ldots, l, m', n)$ are defined by :*

$\{a \geqslant 1,\, c \geqslant 1\}$, $\{a \geqslant 1,\, e \geqslant 1\}$, $\{a \geqslant 1,\, f \geqslant 1\}$, $\{a \geqslant 1,\, i \geqslant 1\}$,
$\{a \geqslant 1,\, j \geqslant 1\}$, $\{a \geqslant 1,\, k \geqslant 1\}$, $\{a \geqslant 1,\, l \geqslant 1\}$, $\{a \geqslant 1,\, m' \geqslant 1\}$,
$\{a \geqslant 1,\, n \geqslant 1\}$, $\{b \geqslant 1,\, e \geqslant 1\}$, $\{b \geqslant 1,\, f \geqslant 1\}$, $\{b \geqslant 1,\, i \geqslant 1\}$,
$\{b \geqslant 1,\, j \geqslant 1\}$, $\{b \geqslant 1,\, k \geqslant 1\}$, $\{b \geqslant 1,\, l \geqslant 1\}$, $\{b \geqslant 1,\, m' \geqslant 1\}$,
$\{b \geqslant 1,\, n \geqslant 1\}$, $\{c \geqslant 1,\, k \geqslant 1\}$, $\{c \geqslant 1,\, l \geqslant 1\}$, $\{c \geqslant 1,\, m' \geqslant 1\}$,
$\{c \geqslant 1,\, n \geqslant 1\}$, $\{d \geqslant 1,\, f \geqslant 1\}$, $\{d \geqslant 1,\, i \geqslant 1\}$, $\{d \geqslant 1,\, j \geqslant 1\}$,
$\{d \geqslant 1,\, m \geqslant 1\}$, $\{d \geqslant 1,\, n \geqslant 1\}$, $\{e \geqslant 1,\, i \geqslant 1\}$, $\{e \geqslant 1,\, j \geqslant 1\}$,
$\{e \geqslant 1,\, m' \geqslant 1\}$, $\{e \geqslant 1,\, n \geqslant 1\}$, $\{d \geqslant 1,\, g \geqslant 1,\, k \geqslant 1\}$,
$\{e \geqslant 1,\, g \geqslant 1,\, k \geqslant 1\}$, $\{f \geqslant 1,\, g \geqslant 1,\, k \geqslant 1\}$, $\{g \geqslant 1,\, k \geqslant 2\}$,
$\{h \geqslant 1,\, j \geqslant 1\}$, $\{h \geqslant 1,\, n \geqslant 1\}$, $\{i \geqslant 2,\, n \geqslant 1\}$,
$\{f \geqslant 1,\, h \geqslant 1,\, m' \geqslant 1\}$, $\{f \geqslant 1,\, i \geqslant 1,\, m' \geqslant 1\}$, $\{f \geqslant 1,\, i \geqslant 1,\, n \geqslant 1\}$,
$\{f \geqslant 1,\, l \geqslant 1,\, n \geqslant 1\}$.

• **Approximation of the 41 subsets :** Keep only the families of sums of Schur bundles that contribute in $\mathrm{O}(m^{16})$ to the final characteristic.

• **24 families (with multiplicities) of sums of Schur bundles :**

$$\mathsf{A},\quad \mathsf{B},\quad \mathsf{C},\quad \ldots,\quad \mathsf{W},\quad \mathsf{X}.$$

defined as follows :

$$\underline{\mathsf{A}:}\quad 2\cdot \bigoplus_{m=o+14j+15k+17l+19m+21n+10p} \Gamma\begin{pmatrix} o+4j+3k+3l+4m+5n+p \\ 2j+2k+3l+3m+3n+p \\ j+2k+2l+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{B}:}\quad \bigoplus_{m=o+12i+14j+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+3i+4j+3k+3l+4m+p \\ 2i+2j+2k+3l+3m+p \\ i+j+2k+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{C}:}\quad \bigoplus_{m=o+10h+12i+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+2h+3i+3k+3l+4m+p \\ 2h+2i+2k+3l+3m+p \\ h+i+2k+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{D}:}\quad 4\cdot \bigoplus_{m=o+8g+14j+17l+19m+21n+10p} \Gamma\begin{pmatrix} o+2g+4j+3l+4m+5n+p \\ 2g+2j+3l+3m+3n+p \\ j+2l+2m+2n+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{E}:}\quad 2\cdot \bigoplus_{m=o+8g+12i+14j+17l+19m+10p} \Gamma\begin{pmatrix} o+2g+3i+4j+3l+4m+p \\ 2g+2i+2j+3l+3m+p \\ i+j+2l+2m+p \\ p \end{pmatrix} T_X^*,$$

F : $\mathbf{2}\cdot \bigoplus_{m=o+8g+10h+12i+17l+19m+10p} \Gamma \begin{pmatrix} o+2g+2h+3i+3l+4m+p \\ 2g+2h+2i+3l+3m+p \\ h+i+2l+2m+p \\ p \end{pmatrix} T^*_X,$

G : $\bigoplus_{m=o+10f+14j+15k+19m+21n+10p} \Gamma \begin{pmatrix} o+3f+4j+3k+4m+5n+p \\ f+2j+2k+3m+3n+p \\ f+j+2k+2m+2n+p \\ p \end{pmatrix} T^*_X,$

H : $\bigoplus_{m=o+10f+14j+15k+17l+19m+10p} \Gamma \begin{pmatrix} o+3f+4j+3k+3l+4m+p \\ f+2j+2k+3l+3m+p \\ f+j+2k+2l+2m+p \\ p \end{pmatrix} T^*_X,$

I : $\bigoplus_{m=o+10f+12i+14j+15k+17l+10p} \Gamma \begin{pmatrix} o+3f+3i+4j+3k+3l+p \\ f+2i+2j+2k+3l+p \\ f+i+j+2k+2l+p \\ p \end{pmatrix} T^*_X,$

J : $\bigoplus_{m=o+10f+10h+12i+15k+17l+10p} \Gamma \begin{pmatrix} o+3f+2h+3i+3k+3l+p \\ f+2h+2i+2k+3l+p \\ f+h+i+2k+2l+p \\ p \end{pmatrix} T^*_X,$

K : $\bigoplus_{m=o+10f+8g+14j+19m+21n+10p} \Gamma \begin{pmatrix} o+3f+2g+4j+4m+5n+p \\ f+2g+2j+3m+3n+p \\ f+j+2m+2n+p \\ p \end{pmatrix} T^*_X,$

L : $\bigoplus_{m=o+10f+8g+14j+17l+19m+10p} \Gamma \begin{pmatrix} o+3f+2g+4j+3l+4m+p \\ f+2g+2j+3l+3m+p \\ f+j+2l+2m+p \\ p \end{pmatrix} T^*_X,$

M :
$$\bigoplus_{m=o+10f+8g+12i+14j+17l+10p} \Gamma\begin{pmatrix} o+3f+2g+3i+4j+3l+p \\ f+2g+2i+2j+3l+p \\ f+i+j+2l+p \\ p \end{pmatrix} T_X^*,$$

N :
$$\bigoplus_{m=o+10f+8g+10h+12i+17l+10p} \Gamma\begin{pmatrix} o+3f+2g+2h+3i+3l+p \\ f+2g+2h+2i+3l+p \\ f+h+i+2l+p \\ p \end{pmatrix} T_X^*,$$

O :
$$\bigoplus_{m=o+8e+10f+10h+15k+17l+10p} \Gamma\begin{pmatrix} o+2e+3f+2h+3k+3l+p \\ e+f+2h+2k+3l+p \\ e+f+h+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

P :
$$\bigoplus_{m=o+8e+10f+8g+10h+17l+10p} \Gamma\begin{pmatrix} o+2e+3f+2g+2h+3l+p \\ e+f+2g+2h+3l+p \\ e+f+h+2l+p \\ p \end{pmatrix} T_X^*,$$

Q :
$$\bigoplus_{m=o+6d+8e+10h+15k+17l+10p} \Gamma\begin{pmatrix} o+d+2e+2h+3k+3l+p \\ d+e+2h+2k+3l+p \\ d+e+h+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

R :
$$\bigoplus_{m=o+6d+8e+8g+10h+17l+10p} \Gamma\begin{pmatrix} o+d+2e+2g+2h+3l+p \\ d+e+2g+2h+3l+p \\ d+e+h+2l+p \\ p \end{pmatrix} T_X^*,$$

S :
$$\bigoplus_{m=o+7c+10f+8g+12i+14j+10p} \Gamma\begin{pmatrix} o+3c+3f+2g+3i+4j+p \\ c+f+2g+2i+2j+p \\ f+i+j+p \\ p \end{pmatrix} T_X^*,$$

T :
$$\bigoplus_{m=o+7c+10f+8g+10h+12i+10p} \Gamma \begin{pmatrix} o+3c+3f+2g+2h+3i+p \\ c+f+2g+2h+2i+p \\ f+h+i+p \\ p \end{pmatrix} T_X^*,$$

U :
$$\bigoplus_{m=o+7c+8e+10f+8g+10h+10p} \Gamma \begin{pmatrix} o+3c+2e+3f+2g+2h+p \\ c+e+f+2g+2h+p \\ e+f+h+p \\ p \end{pmatrix} T_X^*,$$

V :
$$\bigoplus_{m=o+7c+6d+8e+8g+10h+10p} \Gamma \begin{pmatrix} o+3c+d+2e+2g+2h+p \\ c+d+e+2g+2h+p \\ d+e+h+p \\ p \end{pmatrix} T_X^*,$$

W :
$$\bigoplus_{m=o+5b+7c+6d+8g+10h+10p} \Gamma \begin{pmatrix} o+2b+3c+d+2g+2h+p \\ b+c+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*,$$

X :
$$\bigoplus_{m=o+3a+5b+6d+8g+10h+10p} \Gamma \begin{pmatrix} o+a+2b+d+2g+2h+p \\ a+b+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*.$$

- **Computations on Maple 12 :** $\sim$ 50 minutes.

Theorem. *If $X^4 \subset \mathbb{P}^5(\mathbb{C})$ is a projective algebraic hypersurface of degree d, then as $m \to \infty$, one has the asymptotic for the Euler characteristic :*

$$\begin{aligned}\chi\big(X,\, \mathsf{E}^4_{4,m}T^*_X\big) = \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \cdot \\ \cdot \big(500485111357970342562 35\, d^4 - \\ - 617060662250595525598878 6\, d^3 - \\ - 928886901354141153880624704\, d + \\ + 1411704752502476621473639 41\, d^2 + \\ + 1624908955061039283976041114\big) \\ + \mathrm{O}\big(m^{15}\big).\end{aligned}$$

Moreover, the coefficient of m^{16}, a (factorized) polynomial of degree 5 with respect to d, takes positive values as soon as $d \geqslant 96$.

- **Euler characteristic :**

$$\chi = h^0 - h^1 + h^2 - h^3 + h^4$$

- **Trivial minoration :**

$$h^0 \geqslant \chi - h^2 - h^4$$

- **Vanishing theorem :**

$$h^4 = 0\,.$$

- **Majoration of h^2 :**

Theorem. [DMR 2008] *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$. Then :*

$$
\begin{aligned}
&h^2\big(X, \Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} T_X^*\big) \\
&\qquad \leqslant \frac{1}{80}\, d\,(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4) \\
&\qquad \cdot \big(\lambda_1+\lambda_2+\lambda_3+\lambda_4\big)^2 \big[5\lambda_2\lambda_1 d^2 + 132\lambda_2\lambda_1 d + 132\lambda_1\lambda_3 d + 5\lambda_2\lambda_3 d^2 \\
&\qquad + 132\lambda_2\lambda_4 d + 5\lambda_2 d^2\lambda_4 + 132\lambda_1\lambda_4 d + 5\lambda_3\lambda_4 d^2 + 5\lambda_1\lambda_3 d^2 \\
&\qquad + 132\lambda_3\lambda_4 d + 132\lambda_2\lambda_3 d + 1308\lambda_2\lambda_1 + 648\lambda_2^2 + 648\lambda_3^2 \\
&\qquad + 72\lambda_3^2 d + 648\lambda_1^2 + 72\lambda_1^2 d + 1308\lambda_1\lambda_4 + 5\lambda_1 d^2\lambda_4 + 1308\lambda_2\lambda_4 \\
&\qquad + 1308\lambda_2\lambda_3 + 648\lambda_4^2 + 72\lambda_2^2 d + 1308\lambda_1\lambda_3 + 72\lambda_4^2 d + 1308\lambda_3\lambda_4\big] \\
&\qquad + O\big(|\lambda|^9\big).
\end{aligned}
$$

Theorem. [DMR 2008] *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$ and let A be any ample line bundle over X. Then :*

$$
\begin{aligned}
&h^0(X, E_{4,m} T_X^* \otimes \mathscr{O}(-A)) \\
&\qquad \geqslant \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \\
&\qquad\quad \cdot \big[-867659678949860838548185438614 \\
&\qquad\quad - 934880693607607850940593792 16\, d \\
&\qquad\quad - 1369327265177339103292331439\, d^2 \\
&\qquad\quad - 617060662250595525598878 6\, d^3 \\
&\qquad\quad + 500485111357970342562 35\, d^4\big] \\
&\qquad\qquad + O\big(m^{15}\big).
\end{aligned}
$$

*In particular, if $d \geqslant$ **259** then $E_{4,m} T_X^* \otimes \mathscr{O}(-A)$ admits non trivial sections for m large, and every entire curve $f : \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.*

IV – Algebraic Morse inequalities

• **Strategy : Avoid full algebra of invariants by reparametrization**.

• **Same objective :** Construct global sections of jet bundles that will canalize all entire curves.

• **Significant obstacle :** [ROUSSEAU, 2006] In dimension 3, the jet order must be $\geqslant 3$. More generally, **Brückmann-Rackwitz vanishing theorem** yields :

Corollary. [DIVERIO, 2008] *For every $k \leqslant \dim X - 1$ and every ample line bundle $A \to X$:*

$$\begin{aligned} 0 &= H^0\big(X,\, \mathscr{E}_{k,m}^{GG} \otimes A^{-1}\big) \\ &= H^0\big(X,\, \mathscr{E}_{k,m}^{DS} \otimes A^{-1}\big)\,. \end{aligned}$$

• **So assume $k = n$ and use Morse inequalities :**

Theorem. [TRAPANI, SIU, DEMAILLY] *Let $L \to X$ be a holomorphic line bundle on a compact Kähler manifold of dimension n which may be written as a certain difference between two line bundles $\mathscr{F}$ and $\mathscr{G}$ that are numerically effective : $\mathscr{L} = \mathscr{F} \otimes \mathscr{G}^{-1}$. If :*

$$\mathscr{F}^n - n\mathscr{F}^{n-1} \cdot \mathscr{G} > 0\,,$$

then for any holomorphic vector bundle $\mathscr{E} \to X$, the multi-tensored bundle $\mathscr{L}^{\otimes m} \otimes \mathscr{E}$ possesses non-zero global sections, asymptotically as $m \gg 1$.

- **Demailly tower for $k = n$:**

$$
\begin{array}{cccc}
n^2 & X_n & & \boxed{\mathscr{O}_{X_n}(1)} \\
 & \downarrow{\scriptstyle \pi_{n-1,n}} & & \\
n^2 - n + 1 & X_{n-1} & V_{n-1} & \mathscr{O}_{X_{n-1}}(1) \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
\bullet & \bullet & \bullet & \bullet \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
(\ell+1)n - \ell & X_\ell & V_\ell & \mathscr{O}_{X_\ell}(1) \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
\bullet & \bullet & \bullet & \bullet \\
 & \downarrow{\scriptstyle \pi_\bullet} & & \\
3n - 2 & X_2 & V_2 & \mathscr{O}_{X_2}(1) \\
 & \downarrow{\scriptstyle \pi_{1,2}} & & \\
2n - 1 & X_1 & V_1 & \mathscr{O}_{X_1}(1) \\
 & \downarrow{\scriptstyle \pi_{0,1}} & & \\
n & X & T_X &
\end{array}
.
$$

- **Direct image formula :**

$$(\pi_{0,n})_* \mathscr{O}_{X_n}(m) = \mathscr{O}\big(\mathscr{E}^{DS}_{n,m} T^*_X\big).$$

$$
\begin{array}{ccc}
\mathscr{O}_{X_n}(m) & \longrightarrow & \mathscr{E}^{DS}_{n,m} T^*_X \\
\downarrow & & \downarrow \\
X_n & \xrightarrow{\pi_n} & X
\end{array}
.
$$

- **Strategy :** [Demailly, Trapani, Diverio] Find a subbundle of the line bundle $\mathscr{O}_{X_n}(m) \longrightarrow X_n$ which

can be decomposed as a certain difference between two nef line bundles.

• **Definition :** Let $\pi_{j,n}\colon X_n \to X_j$ be the projection from level n to level j in Demailly's tower. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, consider the **line bundle** on X_n :

$$\mathscr{O}_{X_n}(\mathbf{a}) := \pi_{1,n}^*\mathscr{O}_{X_1}(a_1) \otimes \pi_{2,n}^*\mathscr{O}_{X_2}(a_2) \otimes \cdots \otimes \mathscr{O}_{X_n}(a_n)\,.$$

Proposition. [DEMAILLY] *Let :* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ *and let* $m := a_1 + \cdots + a_n$. *Then one has a sheaf injection :*

$$(\pi_{0,n})_*\mathscr{O}_{X_n}(\mathbf{a}) \hookrightarrow \mathscr{O}\big(\mathscr{E}_{n,m}\big)\,.$$

Moreover, $\mathscr{O}_{X_n}(\mathbf{a})$ *is relatively numerically effective on* X *as soon as :*

$$a_1 \geqslant 3a_2, \ldots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1\,.$$

Corollary. [DIVERIO] *Let* $X^n \subset \mathbb{P}^{n+1}$ *be a smooth projective algebraic hypersurface and let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ *satisfying again :*

$$a_1 \geqslant 3a_2, \ldots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1\,.$$

Then :

$$\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^*\mathscr{O}_X(l)$$

is fully numerically effective as soon as :

$$l \geqslant 2|\mathbf{a}|$$

where $|\mathbf{a}| = a_1 + \cdots + a_n$.

• **Proof :** Based on the fact that $T^*_X \otimes \mathscr{O}_X(2)$ is generated by its global sections. $\square$

• **Consequently :** We have to numerically effective bundles over X_n, firstly :

$$\mathscr{F} := \mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi^*_{0,n}\mathscr{O}_X(2|\mathbf{a}|)$$

and secondly, trivially :

$$\mathscr{G} := \pi^*_{0,n}\mathscr{O}_X\big(2|\mathbf{a}|\big)\,.$$

• **Application :** Express $\mathscr{O}_{X_n}(\mathbf{a})$ artificially as a difference between two nef line bundles :

$$\begin{aligned}\mathscr{O}_{X_n}(\mathbf{a}) &= \mathscr{L} = \mathscr{F}\otimes\mathscr{G}^{-1}\\ &= \big(\mathscr{O}_{X_n}(\mathbf{a})\otimes\pi^*_{0,n}\mathscr{O}_X(2|\mathbf{a}|)\big)\otimes\big(\pi^*_{0,n}\mathscr{O}_X(-2|\mathbf{a}|)\big).\end{aligned}$$

• **Dimension :**

$$\dim\, X_n = n + n(n-1) = n^2\,.$$

• **Algebraic Morse inequalities :** Existence of sections as soon as the intersection product :

$$\mathscr{F}^{n^2} - n^2\,\mathscr{F}^{n^2-1}\cdot\mathscr{G} > 0$$

is positive.

• **Concrete meaning :** This intersection product can be expressed **in principle** as a certain polynomial having

integer coefficients in terms of the degree d of the hypersurface X :

$$\mathsf{P}_{a_1,\dots,a_n}(d) = \mathsf{p}_{n+1,\mathbf{a}}\, d^{n+1} + \sum_{0\leqslant k\leqslant n} \mathsf{p}_{k,\mathbf{a}}\, d^k > 0\,.$$

• **Estimate on the degree :** If the coefficient $\mathsf{p}_{n+1,\mathbf{a}}$ of the dominant term d^{n+1} is positive, then for every sufficiently large degree, say for :

$$\deg X = d \geqslant d_n,$$

this polynomial takes only positive values, and there exist global sections of the bundle of jets.

• **Major difficulty :** To really access to the integer coefficients of this polynomial.

• **In dimension** $n = 3$ **:**

$$333162\, d^4 - 21628710\, d^3 - 460474830\, d^2 - 466509222\, d\,;$$

which is positive for all $d = \deg X \geqslant \mathbf{82}$.

• **In dimension** $n = 4$ **:**

$$\begin{aligned}&170114889178454 4\, d^5 - 39934769846141 3760\, d^4 - \\ &- 5029676815028614 2576\, d^3 - 583578200119254857568\, d^2 - \\ &- 6464766796391605017 60\, d\,,\end{aligned}$$

which is positive for all $d \geqslant \mathbf{329}$.

• **In dimension** $n = 5$ **:** [DIVERIO, 2008]

$$\begin{aligned}\mathsf{P}_{54,18,6,2,1}(d) = {} & 829705552526846689513237554474 24\, d^6 - \\ & - 69092357692382960198316008279615424\, d^5 - \\ & - 3759195731318462969721810883195592774 4\, d^4 - \\ & - 2161144497516080476955607837671278699584\, d^3 - \\ & - 2076793172317374111754855583724316380614 4\, d^2 - \\ & - 2373646177903816624611595830455187105638 4\, d,\end{aligned}$$

which is positive for all $d \geqslant \mathbf{1222}$.

• **In dimension** $n = 6$ **:** [M., 2008] Computation using what follows, distributed on 15 computers : $d \geqslant \mathbf{4352}$.

Thesis. *On the algebraic side, invariants by reparametrization are* **intrinsically complex**. *Whichever strategy is devised to skirt, to go round, or to bypass the complexity happens to fail somehow unavoidably.* **Reaching effectivity** *is the very main difficulty of the subject, nowadays, after the groundbreaking works of Siu and of Demailly.*

• **Question :** Then how to compute — at least partially — the intersection product ?

V – Intersection product

- **Summary :**

  ☐ **Principal goal :** Compute in terms of $d = \deg X$ the intersection product :

$$\mathscr{F}^{n^2} - n^2 \mathscr{F}^{n^2-1} \cdot \mathscr{G} > 0 \,.$$

  ☐ **At first :** Exhibit some of the difficulties.

- **Demailly tower for $k = n = 3$:**

$$\begin{array}{llllll}
9 & X_3 & & & \mathscr{O}_{X_3}(1) & u_3 \\
 & \downarrow{\scriptstyle \pi_{2,3}} & & & & \\
7 & X_2 & V_2 & \mathsf{c}_1^2,\ \mathsf{c}_2^2,\ \mathsf{c}_3^2 & \mathscr{O}_{X_2}(1) & u_2 \\
 & \downarrow{\scriptstyle \pi_{1,2}} & & & & \\
5 & X_1 & V_1 & \mathsf{c}_1^1,\ \mathsf{c}_2^1,\ \mathsf{c}_3^1 & \mathscr{O}_{X_1}(1) & u_1 \\
 & \downarrow{\scriptstyle \pi_{0,1}} & & & & \\
3 & X & T_X & \mathsf{c}_1,\ \mathsf{c}_2,\ \mathsf{c}_3 & & h
\end{array} .$$

- **Intersection product to be estimated for $k = n = 3$:**

$$\int_{X_3} \big(a_1u_1 + a_2u_2 + a_3u_3 + 2|\mathbf{a}|h\big)^9 - \\ - 9\big(a_1u_1 + a_2u_2 + a_3u_3 + 2|\mathbf{a}|h\big)^8\big(2|\mathbf{a}|h\big).$$

- **Chern classes :**

$$\begin{array}{llll}
 & & & u_3 = \mathsf{ch}_1\big(\mathscr{O}_{X_3}(1)\big) \\
\mathsf{c}_1^2 = \mathsf{ch}_1(V_2) & \mathsf{c}_2^2 = \mathsf{ch}_2(V_2) & \mathsf{c}_3^2 = \mathsf{ch}_3(V_2) & u_2 = \mathsf{ch}_1\big(\mathscr{O}_{X_2}(1)\big) \\
\mathsf{c}_1^1 = \mathsf{ch}_1(V_1) & \mathsf{c}_2^1 = \mathsf{ch}_2(V_1) & \mathsf{c}_3^1 = \mathsf{ch}_3(V_1) & u_1 = \mathsf{ch}_1\big(\mathscr{O}_{X_2}(1)\big) \\
\mathsf{c}_1 = \mathsf{ch}_1(T_X) & \mathsf{c}_2 = \mathsf{ch}_2(T_X) & \mathsf{c}_3 = \mathsf{ch}_3(T_X) & h = \mathsf{ch}_1\big(\mathscr{O}_{\mathbb{P}^4}(1)\big).
\end{array}$$

- **Here :** C_j^ℓ is a (j,j)-form on X_ℓ and u_ℓ is a $(1,1)$-form, also living on X_ℓ.

- **Convention :** Suppress the symbols $\pi^*_{\ell,\kappa}$.

- **Example :** Write shortly :

$$a_1 u_1 + a_2 u_2 + a_3 u_3 \,,$$

instead of the clumsier expression :

$$a_1 \pi^*_{1,3}(u_1) + a_2 \pi^*_{2,3}(u_2) + a_3 u_3 \,.$$

- **Write all relations between these Chern classes :**

$$\left[\begin{array}{l} u_3^3 = -\mathsf{C}_1^2\, u_3^2 - \mathsf{C}_2^2\, u_3 - \mathsf{C}_3^2 \\ u_2^3 = -\mathsf{C}_1^1\, u_2^2 - \mathsf{C}_2^1\, u_2 - \mathsf{C}_3^1 \\ u_1^3 = -\mathsf{c}_1\, u_1^2 - \mathsf{c}_2\, u_1 - \mathsf{c}_3 \end{array}\right.$$

$$\left[\begin{array}{lll} \mathsf{C}_1^2 = 2u_2 + \mathsf{C}_1^1 & \mathsf{C}_2^2 = u_2\mathsf{C}_1^1 + \mathsf{C}_2^1 & \mathsf{C}_3^2 = \mathsf{C}_3^1 - \mathsf{C}_1^1 u_2^2 - 2u_2^3 \\ \mathsf{C}_1^1 = 2u_2 + \mathsf{c}_1 & \mathsf{C}_2^1 = u_2\mathsf{c}_1 + \mathsf{c}_2 & \mathsf{C}_3^1 = \mathsf{c}_3 - \mathsf{c}_1 u_2^2 - 2u_2^3 \end{array}\right.$$

$$\left[\begin{array}{l} \mathsf{c}_1 = -h(d-5) \\ \mathsf{c}_2 = h^2\big(d^2 - 5d + 10\big) \\ \mathsf{c}_3 = -h^3\big(d^3 - 5d^2 + 10d - 10\big) \end{array}\right.$$

$$h^3 = \int_X h^3 = d.$$

- **Three processes of elimination :**

"replacement"

"fiber-integration"

"vanishing for degree-form reasons"

• **Simplify the task to be achieved :** Try at first to perform the elimination computations just with :

$$\big(a_1u_1 + a_2u_2 + a_3u_3\big)^9 =$$
$$= \sum_{i_1+i_2+i_3=9} \frac{9!}{i_1!\,i_2!\,i_3!}\, a_1^{i_1}a_2^{i_2}a_3^{i_3}\, u_1^{i_1}u_2^{i_2}u_3^{i_3}.$$

• **Decompose in parts the problem :** Express the integration of every such $(9,9)$-form :

$$\int_{X_3} u_1^{i_1}u_2^{i_2}u_3^{i_3}$$

in terms of $d = \deg X$.

• **Fiber-integration :** All fibers of the projections in the tower are $\simeq \mathbb{P}^2(\mathbb{C})$, and the Fubini theorem yields :

$$\int_{\text{fiber}} u_\ell^2 = \int_{\mathbb{P}^2} u_\ell^2 = 1\,,$$

so we can simplify any u_ℓ^2 :

for any $i_1 + i_2 + 2 = 9$, get : $\quad u_1^{i_1}u_2^{i_2}\underline{u_3^2}_{\int} = u_1^{i_1}u_2^{i_2}$

for any $i_1 + 2 = 7$, get : $\quad u_1^{i_1}\underline{u_2^2}_{\int} = u_1^{i_1}.$

- **Vanishing for degree-form reasons : "$\underline{u_1^{i_1} u_2^{i_2}}_{\circ}$" :**

☐ initial homogeneity : $i_1 + i_2 + i_3 = 9$

☐ $u_1^{i_1} u_2^{i_2} u_3^{i_3}$ is a $(9,9)$-form on X_3, of $\dim_{\mathbb{C}} = 9$

☐ $u_1^{i_1} u_2^{i_2}$ is a $(i_1 + i_2, i_1 + i_2)$-form on X_2, of $\dim = 7$

hence it vanishes whenever : $i_1 + i_2 \geqslant 8$

or equivalently whenever : $i_3 \leqslant 1$

☐ $u_1^{i_1}$ is a (i_1, i_1)-form on X_1 of $\dim = 5$

hence it vanishes whenever : $i_1 \geqslant 6$

or equivalently whenever : $i_2 + i_3 \leqslant 3$

- **Starting triangle of u-monomials to be reduced :**

u_3^9	$u_1 u_3^8$	$u_1^2 u_3^7$	$u_1^3 u_3^6$	$u_1^4 u_3^5$	$u_1^5 u_3^4$	$\underline{u_1^6}_{\circ} u_3^3$	$\underline{u_1^7}_{\circ} \underline{u_3^2}_{\int}$	$\underline{u_1^8}_{\circ} u_3$	$\underline{u_1^9}_{\circ}$
$u_2 u_3^8$	$u_1 u_2 u_3^7$	$u_1^2 u_2 u_3^6$	$u_1^3 u_2 u_3^5$	$u_1^4 u_2 u_3^4$	$u_1^5 u_2 u_3^3$	$\underline{u_1^6}_{\circ} u_2 \underline{u_3^2}_{\int}$	$\underline{u_1^7 u_2}_{\circ} u_3$	$\underline{u_1^8 u_2}_{\circ}$	
$u_2^2 u_3^7$	$u_1 u_2^2 u_3^6$	$u_1^2 u_2^2 u_3^5$	$u_1^3 u_2^2 u_3^4$	$u_1^4 u_2^2 u_3^3$	$u_1^5 \underline{u_2^2}_{\int} \underline{u_3^2}_{\int}$	$\underline{u_1^6 u_2^2}_{\circ} u_3$	$\underline{u_1^7}_{\circ} \underline{u_2^2}_{\int}$		
$u_2^3 u_3^6$	$u_1 u_2^3 u_3^5$	$u_1^2 u_2^3 u_3^4$	$\boxed{u_1^3 u_2^3 u_3^3}$	$u_1^4 u_2^3 \underline{u_3^2}_{\int}$	$\underline{u_1^5 u_2^3}_{\circ} u_3$	$\underline{u_1^6 u_2^3}_{\circ}$			
$u_2^4 u_3^5$	$u_1 u_2^4 u_3^4$	$u_1^2 u_2^4 u_3^3$	$u_1^3 u_2^4 \underline{u_3^2}_{\int}$	$\underline{u_1^4 u_2^4}_{\circ} u_3$	$\underline{u_1^5 u_2^4}_{\circ}$				
$u_2^5 u_3^4$	$u_1 u_2^5 u_3^3$	$u_1^2 u_2^5 \underline{u_3^2}_{\int}$	$\underline{u_1^3 u_2^5}_{\circ} u_3$	$\underline{u_1^4 u_2^5}_{\circ}$					
$u_2^6 u_3^3$	$u_1 u_2^6 \underline{u_3^2}_{\int}$	$\underline{u_1^2 u_2^6}_{\circ} u_3$	$\underline{u_1^3 u_2^6}_{\circ}$						
$u_2^7 \underline{u_3^2}_{\int}$	$\underline{u_1 u_2^7}_{\circ} u_3$	$\underline{u_1^2 u_2^7}_{\circ}$							
$\underline{u_2^8}_{\circ} u_3$	$\underline{u_1 u_2^8}_{\circ}$								
$\underline{u_2^9}_{\circ}$									

- **Try at first to reduce, in terms of the Chern classes c_1, c_2, c_3, the nonzero lower diagonal :**

$$u_1^5 \underline{u_2^2}_{\int} \longrightarrow u_1^4 u_2^3 \longrightarrow u_1^3 u_2^4 \longrightarrow u_1^2 u_2^5 \longrightarrow u_1 u_2^6 \longrightarrow u_2^7.$$

Lemma. *One has :*

$$u_1^5 = -\begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 1 & \mathsf{c}_1 \end{vmatrix}.$$

Proof. Indeed, multiply by u_1^2 the fundamental relation :

$$\begin{aligned} u_1^2 u_1^3 &= u_1^2\big(- \mathsf{c}_1 u_1^2 - \mathsf{c}_2 u_1 - \mathsf{c}_3\big) \\ &= -\mathsf{c}_1 u_1^4 - \mathsf{c}_2 u_1^3 - \mathsf{c}_3 \underline{u_1^2}_{\int} \\ &= -\mathsf{c}_1 u_1^4 - \mathsf{c}_2 u_1^3 - \mathsf{c}_3. \end{aligned}$$

Need to reduce $\mathsf{c}_1 u_1^4$, so analogously :

$$\begin{aligned} \mathsf{c}_1 u_1 u_1^3 &= \mathsf{c}_1 u_1\big(- \mathsf{c}_1 u_1^2 - \mathsf{c}_2 u_1 - \mathsf{c}_3\big) \\ &= -\mathsf{c}_1\mathsf{c}_1 u_1^3 - \mathsf{c}_1\mathsf{c}_2 \underline{u_1^2}_{\int} - \underline{\mathsf{c}_1\mathsf{c}_3}_{\circ} u_1 \\ &= -\mathsf{c}_1\mathsf{c}_1 u_1^3 - \mathsf{c}_2\mathsf{c}_1. \end{aligned}$$

Again, need to reduce $\mathsf{c}_1\mathsf{c}_1 u_1^3$, so analogously :

$$\begin{aligned} \mathsf{c}_1\mathsf{c}_1 u_1^3 &= \mathsf{c}_1\mathsf{c}_1\big(- \mathsf{c}_1 u_1^2 - \mathsf{c}_2 u_1 - \mathsf{c}_3\big) \\ &= -\mathsf{c}_1\mathsf{c}_1\mathsf{c}_1 \underline{u_1^2}_{\int} - \underline{\mathsf{c}_1\mathsf{c}_1\mathsf{c}_2}_{\circ} u_1^2 - \underline{\mathsf{c}_1\mathsf{c}_1\mathsf{c}_3}_{\circ} u_1 \\ &= -\mathsf{c}_1\mathsf{c}_1\mathsf{c}_1 \end{aligned}$$

The same reduction holds for $-\mathsf{c}_2 u_1^3$. Then we sum everything and we reconstitute the pleasant determinant. $\square$

Lemma. *A similar argument yields :*

$$u_1^4 u_2^3 = -u_1^4 \cdot \left| \mathsf{C}_1^1 \right|$$

$$u_1^3 u_2^4 = u_1^3 \cdot \begin{vmatrix} \mathsf{C}_1^1 & \mathsf{C}_2^1 \\ 1 & \mathsf{C}_1^1 \end{vmatrix}$$

$$u_1^2 u_2^5 = -u_1^2 \cdot \begin{vmatrix} \mathsf{C}_1^1 & \mathsf{C}_2^1 & \mathsf{C}_3^1 \\ 1 & \mathsf{C}_1^1 & \mathsf{C}_2^1 \\ 0 & 1 & \mathsf{C}_1^1 \end{vmatrix}$$

$$u_1 u_2^6 = u_1 \cdot \begin{vmatrix} \mathsf{C}_1^1 & \mathsf{C}_2^1 & \mathsf{C}_3^1 & 0 \\ 1 & \mathsf{C}_1^1 & \mathsf{C}_2^1 & \mathsf{C}_3^1 \\ 0 & 1 & \mathsf{C}_1^1 & \mathsf{C}_2^1 \\ 0 & 0 & 1 & \mathsf{C}_1^1 \end{vmatrix}$$

$$u_2^7 = - \begin{vmatrix} \mathsf{C}_1^1 & \mathsf{C}_2^1 & \mathsf{C}_3^1 & 0 & 0 \\ 1 & \mathsf{C}_1^1 & \mathsf{C}_2^1 & \mathsf{C}_3^1 & 0 \\ 0 & 1 & \mathsf{C}_1^1 & \mathsf{C}_2^1 & \mathsf{C}_3^1 \\ 0 & 0 & 1 & \mathsf{C}_1^1 & \mathsf{C}_2^1 \\ 0 & 0 & 0 & 1 & \mathsf{C}_1^1 \end{vmatrix}.$$

• **Continuing the reduction :** We must now insert the following relations inside these determinants :

$$\mathsf{C}_1^1 = 2u_2 + \mathsf{c}_1 \qquad \mathsf{C}_2^1 = u_2\mathsf{c}_1 + \mathsf{c}_2 \qquad \mathsf{C}_3^1 = \mathsf{c}_3 - \mathsf{c}_1 u_2^2 - 2u_2^3 .$$

- **First, easiest computation :**

$$
\begin{aligned}
-u_1^4 \cdot \left| \mathsf{C}_1^1 \right| &= -u_1^4 \big(\mathsf{c}_1 + 2u_1 \big) \\
&= -u_1^4 \cdot \mathsf{c}_1 - 2u_1^5 \\
&= -\mathsf{c}_1 \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 \\ 1 & \mathsf{c}_1 \end{vmatrix} + 2 \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 1 & \mathsf{c}_1 \end{vmatrix}
\end{aligned}
$$

- **Second computation :**

$$
\begin{aligned}
u_1^3 \cdot \begin{vmatrix} \mathsf{C}_1^1 & \mathsf{C}_2^1 \\ 1 & \mathsf{C}_1^1 \end{vmatrix} &= u_1^3 \cdot \begin{vmatrix} \mathsf{c}_1 + 2u_1 & \mathsf{c}_2 + u_1\mathsf{c}_1 \\ 1 & \mathsf{c}_1 + 2u_1 \end{vmatrix} \\
&= u_1^3 \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 \\ 1 & \mathsf{c}_1 \end{vmatrix} + u_1^3 \begin{vmatrix} \mathsf{c}_1 & u_1\mathsf{c}_1 \\ 1 & 2u_1 \end{vmatrix} \\
&\quad + u_1^3 \begin{vmatrix} 2u_1 & \mathsf{c}_2 \\ 0 & \mathsf{c}_1 \end{vmatrix} + u_1^3 \begin{vmatrix} 2u_1 & u_1\mathsf{c}_1 \\ 0 & 2u_1 \end{vmatrix} \\
&= -|\mathsf{c}_1| \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 \\ 1 & \mathsf{c}_1 \end{vmatrix} + u_1^3 \begin{vmatrix} \mathsf{c}_1 & -u_1\mathsf{c}_1 \\ 1 & 0 \end{vmatrix} + 2u_1^4\mathsf{c}_1 + 4u_1^5 \\
&= -\mathsf{c}_1 \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 \\ 1 & \mathsf{c}_1 \end{vmatrix} + 3u_1^4\mathsf{c}_1 + 4u_1^5 \\
&= -\mathsf{c}_1 \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 \\ 1 & \mathsf{c}_1 \end{vmatrix} + 3\,\mathsf{c}_1 \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 \\ 1 & \mathsf{c}_1 \end{vmatrix} - 4 \begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 1 & \mathsf{c}_1 \end{vmatrix}
\end{aligned}
$$

- **Third computation :** Skipped ! All computations are too intertwined and interdependent. Even finding the leading d^4-coefficient of each monomial is very difficult.

- [DIVERIO 2008A] : GP PARI code. Unreach $n = 6$.
- **Idea : consider at least the central monomial :**

$$\boxed{u_1^3 u_2^3 u_3^3}$$

Lemma. [DIVERIO 2008B] *Easily generalizable observation :*

$$\begin{aligned}\mathsf{coeff}_{d^4}\left[u_1^3 u_2^3 u_3^3\right] &= \mathsf{coeff}_{d^4}\left[(-1)^3\, \mathsf{c}_1^3\right] \\ &= +1.\end{aligned}$$

Proof. We start from the fundamental relation :

$$u_3^3 = -\mathsf{C}_1^2 u_3^2 - \mathsf{C}_2^2 u_3 - \mathsf{C}_3^2\,.$$

So we may compute :

$$\begin{aligned}\mathsf{coeff}_{d^4}\left[u_1^3 u_2^3 u_3^3\right] &= \mathsf{coeff}_{d^4}\left[-\mathsf{C}_1^2 u_1^3 u_2^3 \underline{u_3^2}_{\int} - \underline{\mathsf{C}_2^2 u_1^3 u_2^3}_{\circ}\, u_3 - \underline{\mathsf{C}_3^3 u_1^3 u_2^3}_{\circ}\right] \\ &= -\mathsf{coeff}_{d^4}\left[\mathsf{C}_1^2 u_1^3 u_2^3\right] \\ &= -\mathsf{coeff}_{d^4}\left[\mathsf{c}_1 u_1^3 u_2^3 + 2\underline{u_1^4 u_2^3}_{\circ} + 2\underline{u_1^3 u_2^4}_{\circ}\right] \\ &= -\mathsf{coeff}_{d^4}\left[\mathsf{c}_1 u_1^3 u_2^3\right],\end{aligned}$$

because a very easy induction yields in advance :

$$\mathsf{C}_1^2 = \mathsf{c}_1 + 2u_1 + 2u_2\,,$$

and because the Brückmann-Rackwitz vanishing theorem entails (without specific computations) that :

$$0 = \mathsf{coeff}_{d^4}\left[u_1^{i_1} u_2^{i_2}\right] \qquad \text{whenever } \; i_1 + i_2 = 7.$$

Continuing similar computations gives at the end :

$$\begin{aligned}\text{result wanted} &= (-1)^2 \mathsf{coeff}_{d^4}\left[\mathsf{c}_1^2 u_1^3\right] \\ &= (-1)^3 \mathsf{coeff}_{d^4}\left[\mathsf{c}_1^3\right] \\ &= (-1)^3 \mathsf{coeff}_{d^4}\left[(-1)^3 d(d-5)^3\right] = +1. \qquad \square\end{aligned}$$

Proposition. [IBIDEM] *In arbitrary dimension* $n \geqslant 2$ *:*

$$\begin{aligned}\mathsf{coeff}_{d^{n+1}}\big[u_1^n u_2^n \cdots u_n^n\big] &= \mathsf{coeff}_{d^{n+1}}\big[(-1)^n\, \mathsf{c}_1^n\big] \\ &= +1.\end{aligned}$$

• **Interpretation :** So we are sure that at least one monomial contributes to the leading d^{n+1}-coefficient of the intersection product :

$$\begin{aligned}\mathsf{coeff}_{d^{n+1}}\big[\text{Intersection product}\big] &= \mathsf{coeff}_{d^{n+1}}\big[(a_1u_1 + \cdots + a_nu_n)^{n^2}\big] \\ &= \text{polynomial in } \mathbb{Z}\big[a_1, \ldots, a_n\big],\end{aligned}$$

and this polynomial is **nonzero**, because it contains :

$$\frac{n^2!}{n! \cdots n!}\, a_1^n \cdots a_n^n\, \mathsf{coeff}_{d^{n+1}}\big[u_1^n \cdots u_n^n\big] = \frac{n^2!}{n! \cdots n!}\, a_1^n \cdots a_n^n\,.$$

• **Tricky idea** [TRAPANI-DIVERIO] **:** Since the relative nef cone defined by :

$$a_1 \geqslant 3a_2, \ldots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1\,.$$

is open, there are weights $(a_1, \ldots, a_n)$ in this cone with :

$$\mathsf{coeff}_{d^{n+1}}\big[\text{Intersection product}\big] \neq 0.$$

But the considered line bundle :

$$\mathscr{O}_{X_1}(a_1) \otimes \mathscr{O}_{X_2}(a_2) \otimes \cdots \otimes \mathscr{O}_{X_n}(a_n)\,.$$

is known to be nef, so *this nonzero coefficient is* > 0.

Theorem. [DIVERIO, MATH. ANN., MAIN RESULT] *There exist* d_n *noneffective such that for* $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *with* $\deg X \geqslant d_n$ *and for* $m \gg 1$ *large :*

$$H^0\big(X,\, \mathscr{E}_{n,m}^{DS} T_X^*\big) \neq 0.$$

VI – Elimination

- **Demailly tower :** Fix any level ℓ satisfying $1 \leqslant \ell \leqslant n$.

- **First notation :** For $\ell = 1, 2, \ldots, n$, denote :

$$u_\ell := \mathsf{ch}_1\big(\mathscr{O}_{X_\ell}(1)\big)\,.$$

so that $\big(u_\ell\big)^k$ is a (k,k)-form on X_ℓ.

- **Second notation :** Introduce the (j,j)-forms on X_ℓ :

$$\mathsf{C}_j^\ell := \mathsf{ch}_j(V_\ell).$$

- **First family of relations :**

(rel_1)

$$\boxed{\begin{aligned}\mathsf{C}_j^\ell &= \mathsf{C}_j^{\ell-1} + \lambda_{j,1}\,\mathsf{C}_{j-1}^{\ell-1}\,u_\ell + \lambda_{j,2}\,\mathsf{C}_{j-2}^{\ell-1}(u_\ell)^2 + \cdots + \lambda_{j,j}\,(u_\ell)^j \\ &= \sum_{k=0}^{j} \lambda_{j,j-k}\,\mathsf{C}_k^{\ell-1}\,(u_\ell)^{j-k}\end{aligned}}\,,$$

where the coefficients $\lambda_{j,j-k}$ ($1 \leqslant j \leqslant n$, $0 \leqslant k \leqslant j$), independent of ℓ, are differences of binomial numbers :

$$\lambda_{j,j-k} := \frac{(n-k)!}{(j-k)!\,(n-j)!} - \frac{(n-k)!}{(j-k-1)!(n-j+1)!}\,.$$

Lemma. *These integer coefficients $\lambda_{j,j-k} \in \mathbb{Z}$ satisfy the simple majoration :*

$$\big|\lambda_{j,j-k}\big| \leqslant 2^n$$

expressed in terms of the dimension n only.

- **Expanded rewriting :**

(rel$_1$)

$$\left[\begin{array}{l}
\mathsf{C}_j^{n-1} = \mathsf{C}_j^{n-2} + \lambda_{j,1}\,\mathsf{C}_{j-1}^{n-2}\,u_{n-1} + \lambda_{j,2}\,\mathsf{C}_{j-2}^{n-2}\,(u_{n-1})^2 + \cdots + \lambda_{j,j}\,(u_{n-1})^j \\
\mathsf{C}_j^{n-2} = \mathsf{C}_j^{n-3} + \lambda_{j,1}\,\mathsf{C}_{j-1}^{n-3}\,u_{n-2} + \lambda_{j,2}\,\mathsf{C}_{j-2}^{n-3}\,(u_{n-2})^2 + \cdots + \lambda_{j,j}\,(u_{n-2})^j \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\quad \mathsf{C}_j^{2} = \mathsf{C}_j^{1} + \lambda_{j,1}\,\mathsf{C}_{j-1}^{1}\,u_{2} + \lambda_{j,2}\,\mathsf{C}_{j-2}^{1}\,(u_{2})^2 + \cdots + \lambda_{j,j}\,(u_{2})^j \\
\quad \mathsf{C}_j^{1} = \mathsf{c}_j + \lambda_{j,1}\,\mathsf{c}_{j-1}\,u_{1} + \lambda_{j,2}\,\mathsf{c}_{j-2}\,(u_{1})^2 + \cdots + \lambda_{j,j}\,(u_{1})^j,
\end{array}\right.$$

where $j = 1, 2, \ldots, n$ is arbitrary.

- **Ground level :** $\ell = 0$, Chern classes (small c) :

$$\mathsf{c}_j := \mathsf{c}_j\big(T_X\big)\,.$$

- **Indicial memory :**
  - ☐ Letter ℓ : level of the tower.
  - ☐ Letter j : (bi)degree of Chern classes.

- **Second family of relations :**

(rel$_2$)

$$\boxed{(u_\ell)^n = -\mathsf{C}_1^{\ell-1}(u_\ell)^{n-1} - \mathsf{C}_2^{\ell-1}(u_\ell)^{n-2} - \cdots - \mathsf{C}_{n-1}^{\ell-1}\,u_\ell - \mathsf{C}_n^{\ell-1}}\,.$$

- **Expanded rewriting :**

(rel$_2$)

$$\left[\begin{array}{l}
(u_n)^n = -\mathsf{C}_1^{n-1}\,(u_n)^{n-1} - \mathsf{C}_2^{n-1}\,(u_n)^{n-2} - \cdots - \mathsf{C}_{n-1}^{n-1}\,u_n - \mathsf{C}_n^{n-1} \\
(u_{n-1})^n = -\mathsf{C}_1^{n-2}\,(u_{n-1})^{n-1} - \mathsf{C}_2^{n-2}\,(u_{n-1})^{n-2} - \cdots - \mathsf{C}_{n-1}^{n-2}\,u_{n-1} - \mathsf{C}_n^{n-2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(u_2)^n = -\mathsf{C}_1^{1}\,(u_2)^{n-1} - \mathsf{C}_2^{1}\,(u_2)^{n-2} - \cdots - \mathsf{C}_{n-1}^{1}\,u_2 - \mathsf{C}_n^{1} \\
(u_1)^n = -\mathsf{c}_1\,(u_1)^{n-1} - \mathsf{c}_2\,(u_1)^{n-2} - \cdots - \mathsf{c}_{n-1}u_n - \mathsf{c}_n.
\end{array}\right.$$

- **Chern classes in terms of the degree :**

$$h := \mathsf{c}_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big) \qquad h^n = \int_X h^n = d.$$

(c-d)

$$\left[\begin{array}{l}
\mathsf{c}_1 = -h\big(d-n-2\big) \\
\mathsf{c}_2 = h^2\big(d^2 - \frac{(n+2)!}{(n+1)!\ 1!}\, d + \frac{(n+2)!}{n!\ 2!}\big) \\
\mathsf{c}_3 = -h^3\big(d^3 - \frac{(n+2)!}{(n+1)!\ 1!}\, d^2 + \frac{(n+2)!}{n!\ 2!}\, d - \frac{(n+2)!}{(n-1)!\ 3!}\big) \\
\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mathsf{c}_n = (-1)^n\, h^n\big(d^n - \frac{(n+2)!}{(n+1)!\ 1!}\, d^{n-1} + \cdots + (-1)^n\, \frac{(n+2)!}{2!\ n!}\big).
\end{array}\right.$$

Thesis. *(Mysterious) displacement of the difficulties toward pure (uncontrollable) algebra.*

- **Line bundle on X_n :**

$$\mathscr{O}_{X_n}(\mathbf{a}) := \pi_{1,n}^*\mathscr{O}_{X_1}(a_1) \otimes \pi_{2,n}^*\mathscr{O}_{X_2}(a_2) \otimes \cdots \otimes \mathscr{O}_{X_n}(a_n)\,.$$

- **Estimate numerically :**

$$\boxed{\mathscr{F}^{n^2} - n^2\mathscr{F}^{n^2-1}\,.\,\mathscr{G}}$$

$$\mathscr{F} := \mathscr{O}_{X_n}(\mathbf{a}) \otimes \mathscr{O}_X\big(2|\mathbf{a}|\big)$$

$$\mathscr{G} := \mathscr{O}_X\big(2|\mathbf{a}|\big).$$

- **Weight :** $\mathbf{a} = (a_1, \ldots, a_n)$ such that :

$$a_1 \geqslant 3a_2, \ldots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1\,.$$

- **Expression in terms of bundles :**

$$\big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2} - \\ - n^2\big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2-1} \cdot \big(\mathscr{O}_X(2|\mathbf{a}|)\big)\,.$$

- **Expression in terms of Chern classes :**

$$\big(a_1u_1 + \cdots + a_nu_n + 2|\mathbf{a}|h\big)^{n^2} + \\ + n^2\big(a_1u_1 + \cdots + a_nu_n + 2|\mathbf{a}|h\big)^{n^2-1} \cdot \big(2|\mathbf{a}|h\big).$$

- **Moreover, neglect h : Model case :**

$$\big(a_1u_1 + a_2u_2 + \cdots + a_nu_n\big)^{n^2}\,.$$

- **Multinomial expansion :**

$$\sum_{i_1+i_2+\cdots+i_n=n^2} \frac{n^2!}{i_1!\,i_2!\,\cdots\,i_n!}\, a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n}\, u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n}\,.$$

- **General monomial :**

$$\boxed{u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n}} \qquad \text{with} \qquad n^2 = i_1 + i_2 + \cdots + i_n\,.$$

Lemma. *After eliminations, fiber-integrations, and annihilations,* $(a_1u_1 + \cdots + a_nu_n)^{n^2}$ *becomes a certain polynomial :*

$$p_{n+1}d^{n+1} + p_nd^n + \cdots + p_1d$$

in terms of $d = \deg X$ *having coefficients* $p_k \in \mathbb{Z}\big[a_1, \ldots, a_n\big]$*, difficult to compute explicitly.*

Elimination problem

- **Three processes of elimination :**

"vanishing for degree-form reasons"

"fiber-integration"

"replacement"

(rel$_2$)

$$(u_\ell)^n = -\mathsf{C}_1^{\ell-1}(u_\ell)^{n-1} - \mathsf{C}_2^{\ell-1}(u_\ell)^{n-2} - \cdots - \mathsf{C}_{n-1}^{\ell-1}\, u_\ell - \mathsf{C}_n^{\ell-1}$$

(rel$_1$)

$$\mathsf{C}_j^\ell = \mathsf{C}_j^{\ell-1} + \lambda_{j,1}\, \mathsf{C}_{j-1}^{\ell-1}\, u_\ell + \lambda_{j,2}\, \mathsf{C}_{j-2}^{\ell-1}(u_\ell)^2 + \cdots + \lambda_{j,j}\, (u_\ell)^j$$

(c-d)

$$\mathsf{c}_j = (-1)^j\, h^j \left(d^j - \frac{(n+2)!}{1!\ (n+1)!}\, d^{j-1} + \cdots + (-1)^j \frac{(n+2)!}{j!\ (n+2-j)!} \right)$$

- **Main goal : know something about :**

$$\left(a_1 u_1 + \cdots + a_n u_n\right)^{n^2} = p_{n+1,\mathbf{a}} d^{n+1} + \sum_{k=1}^{n} p_{k,\mathbf{a}} d^k \,.$$

Five main ideas

(~ 30 pages of proof)

- ☐ Performing terminal inequalities within algebra.
- ☐ Introducing Jacobi-Trudy determinants.
- ☐ Highlighting the central monomial $u_1^n \cdots u_n^n$.
- ☐ Minorating effectively $p_{n+1,\mathbf{a}}$.
- ☐ Majorating effectively the other coefficients $p_{k,\mathbf{a}}$.

VI-1 – A minoration and n majorations

Lemma. *Suppose in advance that we know one minoration and n minorations :*

$$p_{n+1} \geqslant \mathsf{G}(n) \qquad \textit{and} \qquad \big|p_k\big| \leqslant \mathsf{E}(n)\,.$$

Then the intersection product takes only positive values for all degrees :

$$d \geqslant 1 + \mathsf{Ent}\Big[\frac{n\,\mathsf{E}(n)}{\mathsf{G}(n)}\Big] =: d_n\,.$$

• **Choice of $a_1, \ldots, a_n$:** Explicit in terms of n (below).

• **Ordering the u-monomials :** Declare that the monomial $u_1^{i_1} \cdots u_n^{i_n}$ is **smaller**, for the **reverse lexicographic ordering**, than another monomial $u_1^{j_1} \cdots u_n^{j_n}$, again of course with $j_1 + \cdots + j_n = n^2$, if :

$$\begin{cases} \quad i_n > j_n \\ \text{or if } \; i_n = j_n \;\text{ but }\; i_{n-1} > j_{n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{or if } \; i_n = j_n, \ldots, i_3 = j_3 \;\text{ but }\; i_2 > j_2. \end{cases}$$

• **Equivalent language :** The multiindices themselves are ordered in this way :

$$(i_1, \ldots, i_n) <_{\mathsf{revlex}} (j_1, \ldots, j_n)\,.$$

• **Crucial control of the d^{n+1}-coefficients :**

Proposition. *The coefficient of d^{n+1} in any monomial $u_1^{i_1}\cdots u_n^{i_n}$ which is larger than $u_1^n\cdots u_n^n$ is zero :*

$$\mathsf{coeff}_{d^{n+1}}\left[u_1^{i_1}\cdots u_n^{i_n}\right]=0$$

for any $\quad (i_1,\ldots,i_n)>_{\mathsf{revlex}}(n,\ldots,n).$

- **Recall :**

$$+1=\mathsf{coeff}_{d^{n+1}}\left[u_1^n\cdots u_n^n\right].$$

- **Expansion :**

$$\left(a_1u_1+\cdots+a_nu_n\right)^{n^2}=\sum_{i_1+\cdots+i_n=n^2}\frac{n^2!}{i_1!\,\cdots\,i_n!}\,a_1^{i_1}\cdots a_n^{i_n}\,u_1^{i_1}\cdots u_n^{i_n}.$$

- **The strategy :** Choose appropriately the weights so as to confer the highest importance to the central monomial $a_1^n\cdots a_n^n\,u_1^n\cdots u_n^n$, the other monomials $a_1^{i_1}\cdots a_n^{i_n}\,u_1^{i_1}\cdots u_n^{i_n}$ being very small in comparison.

$$\sum_{\substack{i_1+\cdots+i_n=n^2\\(i_1,\ldots,i_n)\neq(n,\ldots,n)}}\mathsf{coeff}_{d^{n+1}}\left[\tfrac{n^2!}{n!\cdots n!}\,a_1^{i_1}\cdots a_n^{i_n}\,u_1^{i_1}\cdots u_n^{i_n}\right]=$$

$$=\sum_{(i_1,\ldots,i_n)<_{\mathsf{revlex}}(n,\ldots,n)}\mathsf{coeff}_{d^{n+1}}\left[\tfrac{n^2!}{n!\cdots n!}\,a_1^{i_1}\cdots a_n^{i_n}\,u_1^{i_1}\cdots u_n^{i_n}\right]\ll$$

$$\ll\mathsf{coeff}_{d^{n+1}}\left[a_1^n\cdots a_n^n\,u_1^n\cdots u_n^n\right].$$

Proposition. *For any real number $\mu \gg 1$ arbitrary large, there are tower weights $a_1, a_2, \ldots, a_{n-1}, a_n$ with $a_n = 1$ satisfying the nef-cone inequalities such that, for all nonnegative exponents $(i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n = n^2$:*

$$a_1^{i_1} \cdots a_n^{i_n} \leqslant \frac{1}{\mu} a_1^n \cdots a_n^n$$

whenever $\quad (i_1, \ldots, i_n) <_{\text{revlex}} (n, \ldots, n).$

In fact, one may choose :

$$\left[\begin{array}{l}
a_1(n) = \mu^{(n+1)^{n-2}} \\
a_2(n) = \mu^{(n+1)^{n-2}-1} \\
a_3(n) = \mu^{(n+1)\left[(n+1)^{n-3}-1\right]} \\
\cdots\cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n-3}(n) = \mu^{(n+1)^{n-5}\left[(n+1)^{3}-1\right]} \\
a_{n-2}(n) = \mu^{(n+1)^{n-4}\left[(n+1)^{2}-1\right]} \\
a_{n-1}(n) = \mu^{(n+1)^{n-3}\left[(n+1)^{1}-1\right]} \\
a_n(n) = 1.
\end{array}\right.$$

Observe that then $a_1(n) \gg a_2(n) \gg \cdots \gg a_{n-1}(n)$, where $a_{n-1}(n)$ is already quite large.

- **Majorating :**

$$\max_{i_1+\cdots+i_n=n^2} \left|\mathsf{coeff}_{d^{n+1}}\left[u_1^{i_1} \cdots u_n^{i_n}\right]\right| \leqslant \mathsf{C}_{n+1}(n)\,.$$

$$\max_{i_1+\cdots+i_n=n^2} \left|\mathsf{coeff}_{d^k}\left[u_1^{i_1}\cdots u_n^{i_n}\right]\right| \leqslant \mathsf{D}_k(n)\,.$$

- **Choosing $\mu = \mu(n)$ effectively in terms of n**

Proposition. *By choosing :*

$$\mu := \mu(n) := 4\,n^{2n-2}\,\mathsf{C}_{n+1}(n),$$

one insures that :

$$\mathsf{coeff}_{d^{n+1}}\left[(a_1u_1+\cdots+a_nu_n)^{n^2}\right] \geqslant \tfrac{1}{2}\,n^{n^2-2n}\,\mu^{\frac{1}{2}\,n^n}.$$

and that, for any exponent k with $1 \leqslant k \leqslant n$:

$$\left|\mathsf{coeff}_{d^k}\left[\text{same}\right]\right| \leqslant 6\,n^{2n-1}\cdot(n+1)^{n^2}\cdot 2^{n^2}\,\mu^{(n+1)^n}\cdot\mathsf{D}_k(n).$$

- **Estimate of $\mathsf{C}_{n+1}(n)$ and of the $\mathsf{D}_k(n)$:**

Theorem. *With $n \geqslant 2$, for any $i_1,\ldots,i_n \in \mathbb{N}$ with $i_1+\cdots+i_n = n^2$, we have the following uniform effective upper bound holds :*

$$\left|\mathsf{coeff}_{d^k}\left[u_1^{i_1}\cdots u_n^{i_n}\right]\right| \leqslant n^{5n^4} =: \mathsf{D}_k(n)$$

$$\left|\mathsf{coeff}_{d^{n+1}}\left[u_1^{i_1}\cdots u_n^{i_n}\right]\right| \leqslant n^{5n^4} =: \mathsf{D}_{n+1}(n).$$

- **Enough for the final estimates.**

- **Summarized ideas of proof.**

- **Jacobi-Trudy determinants :** At any level ℓ with $0 \leqslant \ell \leqslant n-1$ and for any J with $0 \leqslant J \leqslant n+\ell(n-$

$1) = \dim X_\ell$, we define the corresponding *Jacobi-Trudy determinant* :

$$\mathscr{C}_J^\ell := \begin{vmatrix} \mathsf{C}_1^\ell & \mathsf{C}_2^\ell & \mathsf{C}_3^\ell & \cdots & \mathsf{C}_J^\ell \\ 1 & \mathsf{C}_1^\ell & \mathsf{C}_2^\ell & \cdots & \mathsf{C}_{J-1}^\ell \\ 0 & 1 & \mathsf{C}_1^\ell & \cdots & \mathsf{C}_{J-1}^\ell \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \mathsf{C}_1^\ell \end{vmatrix}.$$

Lemma. *For any ℓ with $1 \leqslant \ell \leqslant n$, given any (K,K)-form $\Omega_K^{\ell-1}$ at level $\ell-1$ and any integer i_ℓ with $i_\ell \geqslant n-1$ and $i_\ell + K = \dim X_\ell$, the reduction of $\Omega_K^{\ell-1} u_\ell^{i_\ell}$ down to level $\ell-1$ precisely reads :*

$$\begin{aligned} \Omega_K^{\ell-1} u_\ell^{i_\ell} &= (-1)^{i_\ell-n+1}\, \Omega_K^{\ell-1} \begin{vmatrix} \mathsf{C}_1^{\ell-1} & \mathsf{C}_2^{\ell-1} & \cdots & \mathsf{C}_{i_\ell-n+1}^{\ell-1} \\ 1 & \mathsf{C}_1^{\ell-1} & \cdots & \mathsf{C}_{i_\ell-n}^{\ell-1} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \mathsf{C}_1^{\ell-1} \end{vmatrix} \\ &= (-1)^{i_\ell-n+1}\, \Omega_K^{\ell-1}\, \mathscr{C}_{i_\ell-n+1}^{\ell-1}. \end{aligned}$$

Lemma. *At an arbitrary level ℓ with $1 \leqslant \ell \leqslant n-1$, consider the Jacobi-Trudy determinant $\mathscr{C}_J^\ell$ of an arbitrary size $J \times J$ with $1 \leqslant J \leqslant \dim X_\ell$ and furthermore, let Ω_K^ℓ be any (K,K)-form on X_ℓ whose degree K satisfies $K + J = \dim X_\ell = n + \ell(n-1)$. Then the reduction of $\Omega_K^\ell \mathscr{C}_J^\ell$ down to level $\ell-1$ relies upon the following formulas :*

$$\Omega_K^\ell \mathscr{C}_J^\ell = \Omega_K^\ell \big[\mathscr{C}_J^{\ell-1} + \mathscr{C}_0^\ell \mathsf{A}_J^\ell + \mathscr{C}_1^\ell \mathsf{A}_{J-1}^\ell + \cdots + \mathscr{C}_{J-1}^\ell \mathsf{A}_1^\ell\big],$$

in which, for any k with $1 \leqslant k \leqslant J$, one has set :

$$\mathsf{A}_k^\ell := \mathsf{X}_1^\ell \mathscr{C}_{k-1}^{\ell-1} - \mathsf{X}_2^\ell \mathscr{C}_{k-2}^{\ell-1} + \cdots + (-1)^{k-1} \mathsf{X}_k^\ell \mathscr{C}_0^{\ell-1},$$

where the X-terms here gather all the terms after $\mathsf{C}_j^{[\ell-1]}$ in a convenient rewriting of the fundamental relation under the following form :

$$\mathsf{C}_j^\ell = \mathsf{C}_j^{\ell-1} + \underbrace{\lambda_{j,1}\, \mathsf{C}_{j-1}^{\ell-1} u_\ell + \lambda_{j,2}\, \mathsf{C}_{j-2}^{\ell-1} u_\ell^2 + \cdots + \lambda_{j,j}\, u_\ell^j}_{\stackrel{\text{def}}{=} \mathsf{X}_j^\ell},$$

with the convention that $\mathsf{X}_j^\ell = 0$ for any $j \geqslant n+1$.

• **Incompleteness :** As J varies, the formulas given by this lemma :

$$\mathscr{C}_J^\ell = \mathscr{C}_J^{\ell-1} + \mathscr{C}_0^\ell \mathsf{A}_J^\ell + \mathscr{C}_1^\ell \mathsf{A}_{J-1}^\ell + \cdots + \mathscr{C}_{J-1}^\ell \mathsf{A}_1^\ell\,,$$

are still imperfect, for their right-hand sides still involve Jacobi-Trudy determinants at the level ℓ. So necessarily, we must perform further reductions.

Lemma. *For any J with $0 \leqslant J \leqslant \dim X_\ell$ and any ℓ with $1 \leqslant \ell \leqslant n$, one has :*

$$\mathscr{C}_J^\ell = \sum_{j=0}^{J} \mathscr{C}_{J-j}^{\ell-1} \Bigg(\sum_{\nu=1}^{j} \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \Bigg),$$

with the convention that for $j=0$, the empty sum in parentheses equals 1.

• **Denote :** (with of course $\Sigma_0^\ell(\mathsf{A}) = 1$)

$$\Sigma_j^\ell(\mathsf{A}) := \sum_{\nu=1}^{j} \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\ldots,k_\nu\geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell .$$

• **Induction formulas :** These $\Sigma_j^\ell(\mathsf{A})$ satisfy useful induction formulas :

$$\begin{aligned}
\Sigma_j^\ell(\mathsf{A}) &= \mathsf{A}_j^\ell + \sum_{\nu=2}^{j} \sum_{\substack{k_1+k_2+\cdots+k_\nu=j \\ k_1,k_2,\ldots,k_\nu\geqslant 1}} \mathsf{A}_{k_1}^\ell \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \\
&= \mathsf{A}_j^\ell + \sum_{\nu=2}^{j} \Bigg(\mathsf{A}_1^\ell \sum_{\substack{k_2+\cdots+k_\nu=j-1 \\ k_2,\ldots,k_\nu\geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \mathsf{A}_2^\ell \sum_{\substack{k_2,\ldots,k_\nu=j-2 \\ k_2,\ldots,k_\nu\geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \\
&\qquad + \cdots + \mathsf{A}_{j-1}^\ell \sum_{\substack{k_2+\cdots+k_\nu=1 \\ k_1,\ldots,k_\nu\geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \Bigg) \\
&= \mathsf{A}_j^\ell + \mathsf{A}_1^\ell \sum_{\nu=2}^{j-1} \sum_{\substack{k_2+\cdots+k_\nu=j-1 \\ k_2,\ldots,k_\nu\geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \mathsf{A}_2^\ell \sum_{\nu=2}^{j-2} \sum_{\substack{k_2+\cdots+k_\nu=j-2 \\ k_2,\ldots,k_\nu\geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \\
&\qquad + \cdots + \mathsf{A}_{j-1}^\ell \sum_{\nu=2}^{2} \sum_{\substack{k_2=1 \\ k_2\geqslant 1}} \mathsf{A}_{k_2}^\ell \\
&= \mathsf{A}_j^\ell \Sigma_0^\ell(\mathsf{A}) + \mathsf{A}_1^\ell\, \Sigma_{j-1}^\ell(\mathsf{A}) + \mathsf{A}_2^\ell \Sigma_{j-2}^\ell(\mathsf{A}) + \cdots + \mathsf{A}_{j-1}^\ell \Sigma_1^\ell(\mathsf{A}).
\end{aligned}$$

• **Reduction :** The reduction process transforms a general monomial of the form $h^l u_1^{i_1} \cdots u_n^{i_n}$ with $l + i_1 + \cdots + i_n = n^2$ into a polynomial $\mathscr{R}\big(h^l u_1^{i_1} \cdots u_n^{i_n}\big)$ of degree $\leqslant n+1$ in d, where the symbol "$\mathscr{R}$" stands for "*reduction*".

• **Beyond uncontrollable Algebra using Analysis :** From now on, complete explicit algebraic computations will not be conducted anymore, and instead, to tame their complexity, *inequalities* will be dealt with.

• **Upper reduction operator :** For our majoration purposes, we now introduce an important *upper reduction operator* $\mathscr{R}^+$ which by definition, at each computational step of the reduction process, while going down in the Demailly's tower, always replaces any incoming sign "$-$" by a sign "$+$". Accordingly, for any two monomials $h^l u_1^{i_1} \cdots u_n^{i_n}$ and $h^{l'} u_1^{i'_1} \cdots u_n^{i'_n}$, we shall say that :

$$\mathscr{R}^+\big(h^l u_1^{i_1} \cdots u_n^{i_n}\big) \leqslant_{\mathscr{R}^+} \big(h^{l'} u_1^{i'_1} \cdots u_n^{i'_n}\big),$$

and write more briefly :

$$h^l u_1^{i_1} \cdots u_n^{i_n} \leqslant_{\mathscr{R}^+} h^{l'} u_1^{i'_1} \cdots u_n^{i'_n},$$

if the corresponding two (upper) reduced polynomials $\sum_{k=0}^{n+1} \mathsf{p}_k \cdot d^k$ and $\sum_{k=0}^{n+1} \mathsf{p}'_k \cdot d^k$ have all their coefficients satisfying :

$$\big(0 \leqslant\big)\, \mathsf{p}_k \leqslant \mathsf{p}'_k \qquad \text{for every } \ k = 0, 1, \ldots, n+1.$$

Then obviously the absolute values of the coefficients of the reduction are smaller than the (nonnegative) coefficients of the upper reduction :

$$\big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| \leqslant \mathsf{coeff}_{d^k}\big[\mathscr{R}^+\big(h^l u_1^{i_1} \cdots u_n^{i_n}\big)\big]\,.$$

Lemma. *For any* $\lambda_1, \lambda_2, \dots, \lambda_n$ *with* $n = \lambda_1 + 2\lambda_2 + \dots + n\lambda_n$*, one has :*

$$c_1^{\lambda_1} \big(\mathscr{C}_2^0\big)^{\lambda_2} \cdots \big(\mathscr{C}_n^0\big)^{\lambda_n} \leqslant_{\mathscr{R}^+} \mathscr{C}_n^0 \,.$$

Lemma. *For any two* J_1*,* J_2 *with* $0 \leqslant J_1, J_2 \leqslant \dim X_\ell$ *satisfying in addition* $J_1 + J_2 \leqslant \dim X_\ell$*, and for any* j_1 *with* $0 \leqslant j_1 \leqslant n$ *satisfying in addition* $j_1 + J_2 \leqslant \dim X_\ell$*, one has the two majorations :*

$$\mathscr{R}^+\big(\Omega_K^\ell \cdot \mathscr{C}_{J_1}^\ell \cdot \mathscr{C}_{J_2}^\ell\big) \leqslant \mathscr{R}^+\big(\Omega_K^\ell \cdot \mathscr{C}_{J_1+J_2}^\ell\big) \quad \textit{and} \quad \mathscr{R}^+\big(\Omega_K^\ell \cdot \mathsf{C}_{j_1}^\ell \cdot \mathscr{C}_{J_2}^\ell\big) \leqslant \mathscr{R}^+\big(\Omega_K^\ell \cdot \mathscr{C}_{j_1+J_2}^\ell\big),$$

where Ω_K^ℓ *is any* (K, K)*-form living on* X_ℓ *completing to* $\dim X_\ell$ *the degree, namely with* $K + J_1 + J_2$ *and with* $K + j_1 + J_2$ *both equal to* $\dim X_\ell$*.*

Lemma : *The coefficients* $\lambda_{j,j-k} = \frac{(n-k)!}{(j-k)!\,(n-j)!} - \frac{(n-k)!}{(j-k-1)!(n-j+1)!}$ *appearing in the relations* (c-d) *satisfy the uniform majoration :*

$$\big|\lambda_{j,j-k}\big| \leqslant 2^n =: \lambda$$

expressed in terms of the dimension n *only.*

• **Majorating uniformly :** In the subsequent majorations, while applying the upper majoration operator $\mathscr{R}^+$, we also replace any incoming $\lambda_{j,j-k}$ by this majorant $\lambda = 2^n$. As a result, we define a generalized upper majoration operator "$\mathscr{R}_\lambda^+$" which both replaces any minus sign by a plus sign and any $\lambda_{j,j-k}$ by $\lambda = 2^n$.

Lemma : *For all* $k = 1, 2, \ldots, n$*, one has the* $\mathscr{R}_\lambda^+$*-majorations :*

$$\mathsf{A}_k^\ell \leqslant_{\mathscr{R}_\lambda^+} k\lambda\big(\mathscr{C}_{k-1}^{\ell-1} u_\ell + \mathscr{C}_{k-2}^{\ell-1} u_\ell^2 + \cdots + u_\ell^k\big).$$

• **Majorate conveniently the** $\Sigma_j^\ell(\mathsf{A})$ **polynomials :** Introduce the majorant :

$$\Theta_k^\ell := \mathscr{C}_{k-1}^{\ell-1} u_\ell + \mathscr{C}_{k-2}^{\ell-1} u_\ell^2 + \cdots + \mathscr{C}_1^{\ell-1} u_\ell^{k-1} + u_\ell^k\,,$$

and let us keep in mind that the lemma just proved provided the majorations :

$$\mathsf{A}_k^\ell \leqslant_{\mathscr{R}_\lambda^+} k\lambda\,\Theta_k^\ell\,.$$

Lemma : *For any* $k_1, k_2, \ldots, k_\nu$ *with* $k_1, k_2, \ldots, k_\nu \geqslant 1$ *whose sum* $k_1 + k_2 + \cdots + k_\nu = j$ *equals* j*, one has the majoration :*

$$\Theta_{k_1}^\ell \Theta_{k_2}^\ell \cdots \Theta_{k_\nu}^\ell \leqslant_{\mathscr{R}_\lambda^+} k_1 k_2 \cdots k_\nu\,\Theta_{k_1+k_2+\cdots+k_\nu}^\ell.$$

• **At last :** State the main useful majoration proposition :

Proposition. *At any level ℓ with $1 \leqslant \ell \leqslant n-1$, consider the Jacobi-Trudy determinant $\mathscr{C}_J^\ell$ of an arbitrary size $J \times J$ with $1 \leqslant J \leqslant \dim X_\ell$ and furthermore, let Ω_K^ℓ be any (K,K)-form on X_ℓ the degree K of which satisfies $K + J = \dim X_\ell = n + \ell(n-1)$. Then the upper reduction $\mathscr{R}_\lambda^+(\bullet)$ of $\Omega_K^\ell \mathscr{C}_J^\ell$ in which any incoming $\lambda_{j,j-k}$ is replaced by $\lambda = 2^n \geqslant \left|\lambda_{j,j-k}\right|$ enjoys the following majoration in the right-hand side of which, notably, all the appearing Jacobi-Trudy determinants live at level $\ell - 1$:*

$$\Omega_K^\ell \mathscr{C}_J^\ell \leqslant_{\mathscr{R}_\lambda^+} J \cdot 2^J \cdot J^{2J} \cdot 2^{nJ} \cdot \Omega_K^\ell \left[\mathscr{C}_J^{\ell-1} + \mathscr{C}_{J-1}^{\ell-1} u_\ell + \cdots + \mathscr{C}_1^{\ell-1} u_\ell^{J-1} + u_\ell^J\right] .$$

- **Final steps of the proof of $\mathsf{D}_k(n) \leqslant n^{5n^4}$:**

- **Elementary majorations :**

$$\begin{aligned} J \cdot 2^J \cdot J^{2J} \cdot 2^{nJ} &= 2^{(n+1)J} \cdot J^{2J+1} \\ &\leqslant 2^{n^3+1} \left(n^2 - n + 1\right)^{2n^2-2n+3} \\ &\leqslant 2^{n^3} \left(n^2\right)^{2n^2}. \end{aligned}$$

- **Define :**

$$\mathsf{N} := 2^{n^3} n^{4n^2}.$$

- **Perform a uniform upper majoration of an arbitrary monomial $u_1^{i_1} \cdots u_n^{i_n}$ with $i_1 + \cdots + i_n = n^2$ down**

to level $\ell = 0$:

$$
\begin{aligned}
u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n} &= u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \mathscr{C}_{i_n-n+1}^{n-1} \\
&\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^{i_{n-1}} \big[\mathscr{C}_{i_n-n+1}^{n-2} + \mathscr{C}_{i_n-n}^{n-2} u_{n-1} \\
&\qquad + \cdots + \mathscr{C}_1^{n-2} u_{n-1}^{i_n-n} + u_{n-1}^{i_n-n+1} \big] \qquad \text{[Lemma]} \\
&\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\underline{\mathscr{C}_{i_n-n+1}^{n-2} u_{n-1}^{i_{n-1}} + \cdots}_{\circ} \\
&\qquad + \mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} \underline{u_{n-1}^{n-1}}_{\int} + \cdots + u_{n-1}^{i_{n-1}+i_n-n+1} \big] \\
&\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} + \mathscr{C}_{i_{n-1}+i_n-2n+1}^{n-2} u_{n-1}^{n} \\
&\qquad + \cdots + u_{n-1}^{i_{n-1}+i_n-n+1} \big] \\
&\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} + \mathscr{C}_{i_{n-1}+i_n-2n+1}^{n-2} \mathscr{C}_1^{n-2} \\
&\qquad + \cdots + \mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} \big] \qquad \text{[Lemma]} \\
&\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N}\, n^2 \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} \qquad \text{[Lemma]} \\
&\leqslant_{\mathscr{R}_\lambda^+} \big(\mathsf{N}\, n^2\big)^2 \cdot u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} \mathscr{C}_{i_{n-2}+i_{n-1}+i_n-3n+3}^{n-3} \qquad \text{[induction]} \\
&\leqslant_{\mathscr{R}_\lambda^+} \big(\mathsf{N}\, n^2\big)^3 \cdot u_1^{i_1} \cdots u_{n-4}^{i_{n-4}} \mathscr{C}_{i_{n-3}+i_{n-2}+i_{n-1}+i_n-4n+4}^{n-4} \qquad \text{[induction]}.
\end{aligned}
$$

- **Clear induction down to level $\ell = 1$ yields :**

$$
\begin{aligned}
u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n} &\leqslant_{\mathscr{R}_\lambda^+} \big(\mathsf{N}\, n^2\big)^{n-2} \cdot u_1^{i_1} \mathscr{C}_{i_2+\cdots+i_n-(n-1)n+n-1}^{1} \\
&\leqslant_{\mathscr{R}_\lambda^+} \big(\mathsf{N}\, n^2\big)^{n-2} \cdot \mathsf{N} \cdot \big[\underline{\mathscr{C}_{2n-1-i_1}^{0} + \cdots}_{\circ} + \\
&\qquad + \mathscr{C}_n^0 \underline{u_1^{n-1}}_{\int} + \cdots + u_1^{2n-1} \big] \\
&\leqslant_{\mathscr{R}_\lambda^+} \big(\mathsf{N}\, n^2\big)^{n-1} \mathscr{C}_n^0.
\end{aligned}
$$

Lemma. *The $n \times n$ Jacobi-Trudy determinant $\mathscr{C}_0^n$ enjoys the majoration :*

$$
\mathscr{C}_n^0 \leqslant_{\mathscr{R}_d^+} 2^{n^2+2n}\, n!\, n^n \big[d^{n+1} + d^n + \cdots + d \big].
$$

• **Application :** Applying this lemma to the last obtained inequality :

$$u_1^{i_1} \cdots u_n^{i_n} \leqslant_{\mathscr{R}_\lambda^+} (Nn^2)^{n-1}\, 2^{n^2+2n}\, n!\, n^n \cdot \left[d^{n+1} + d^n + \cdots + 1\right],$$

obtain the announced bound n^{5n^4} :

$$\begin{aligned}\left|\mathsf{coeff}_{d^k}\left[u_1^{i_1} \cdots u_n^{i_n}\right]\right| &\leqslant \left(2^{n^3}\, n^{4n^2}\, n^2\right)^{n-1} 2^{n^2+2n}\, n!\, n^n \\ &\leqslant 2^{n^4-n^3+n^2+2n}\, n^{4n^3-4n^2+2n-2}\, n^n\, n^n \\ &\leqslant n^{5n^4}. \quad \square\end{aligned}$$

VII – Siu's beautiful strategy

• **Universal hypersurface :** Define, in a system of homogeneous coordinates :

$$[Z] = [Z_0 : Z_1 : \cdots : Z_n : Z_{n+1}] \in \mathbb{P}^{n+1}$$

$$[A] = \big[(A_\alpha)_{\alpha\in\mathbb{N}^{n+2},\,|\alpha|=d}\big] \in \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\;d!}-1},$$

the **universal hypersurface** of degree d :

$$\mathscr{X} \quad : \qquad 0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+2}\\ |\alpha|=d}} A_{\alpha_0,\ldots,\alpha_{n+1}}\, Z_0^{\alpha_0}\cdots Z_{n+1}^{\alpha_{n+1}}$$

as the zero-locus of the general homogenous polynomial of degree d. Set $N_d^n := \frac{(n+1+d)!}{(n+1)!\;d!} - 1$.

• **Double projection :**

$$\begin{tikzcd} & \mathscr{X} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathbb{P}^{n+1} & & \mathbb{P}^{N_d^n}. \end{tikzcd}$$

• **Inhomogeneous coordinates :** In the chart $\{Z_0 \neq 0\} \times \{A_{0d0\cdots0} \neq 0\}$, write :

$$\mathscr{X}_0 \quad : \qquad 0 = z_1^d + \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\,\alpha_1<d}} a_\alpha\, z^\alpha\,.$$

• **Principal hypothesis :** Entire holomorphic map valued in a fixed projective hypersurface :

$$0 \equiv f_1(\zeta)^d + \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leqslant d,\, \alpha_1 < d}} a_\alpha\, f(\zeta)^\alpha\,,$$

where the coefficients a_α do not depend on ζ.

• **Coordinates in the space of vertical n-jets :**

$$\Big(z_i,\, a_\alpha,\, z'_{j_1},\, z''_{j_2}, \ldots, z^{(n)}_{j_n}\Big) \in$$
$$\in \mathbb{C}^{n+1} \times \mathbb{C}^{N_d^n} \times \underbrace{\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \cdots \times \mathbb{C}^{n+1}}_{n \text{ times}}.$$

• **Chain rule :** At order $\kappa = 4$:

$$0 = \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leqslant d,\, a_{d0\cdots 0} = 1}} a_\alpha\, z^\alpha$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z'_{j_1} \Big)$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1} \partial z_{j_2}}\, z'_{j_1} z'_{j_2} \Big)$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z'''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1} \partial z_{j_2}}\, 3\, z'_{j_1} z''_{j_2} + \sum_{j_1, j_2, j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3}}\, z'_{j_1} z'_{j_2} z'_{j_3} \Big)$$

$$0 = \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z''''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1} \partial z_{j_2}} \big(4\, z'_{j_1} z'''_{j_2} + 3\, z''_{j_1} z''_{j_2}\big) +$$
$$+ \sum_{j_1, j_2, j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3}}\, 6\, z'_{j_1} z'_{j_2} z''_{j_3} + \sum_{j_1, j_2, j_3, j_4} \frac{\partial^4(z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3} \partial z_{j_4}}\, z'_{j_1} z'_{j_2} z'_{j_3} z'_{j_4} \Big).$$

• **Total differentiation operator :**

$$\mathsf{D}(\bullet) := \sum_{\lambda\in\mathbb{N}} \sum_{k=1}^{n+1} \frac{\partial(\bullet)}{\partial z_k^{(\lambda)}} \cdot z_k^{(\lambda+1)},$$

• **General rewriting :**

$$0 = \sum_{\alpha} a_\alpha z^\alpha = \mathsf{D}\Big(\sum_{\alpha} a_\alpha z^\alpha\Big) = \cdots = \mathsf{D}^n\Big(\sum_{\alpha} a_\alpha z^\alpha\Big).$$

Lemma. *The* $(n+1)$ *defining equations of* $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ *write as follows :*

$$0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\, a_{d0\cdots 0}=1}} a_\alpha \sum_{e=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_e\leqslant\kappa} \sum_{\mu_1\geqslant 1,\ldots,\mu_e\geqslant 1} \sum_{\mu_1\lambda_1+\cdots+\mu_e\lambda_e=\kappa} \frac{\kappa!}{(\lambda_1!)^{\mu_1}\mu_1!\cdots(\lambda_e!)^{\mu_e}\mu_e!}$$

$$\sum_{j_1^1,\ldots,j_{\mu_1}^1=1}^{n+1} \cdots \sum_{j_1^e,\ldots,j_{\mu_e}^e=1}^{n+1} \frac{\partial^{\mu_1+\cdots+\mu_e}\big(z^\alpha\big)}{\partial z_{j_1^1}\cdots\partial z_{j_{\mu_1}^1}\cdots\partial z_{j_1^e}\cdots\partial z_{j_{\mu_e}^e}} z_{j_1^1}^{(\lambda_1)}\cdots z_{j_{\mu_1}^1}^{(\lambda_1)}\cdots z_{j_1^e}^{(\lambda_e)}\cdots z_{j_{\mu_e}^e}^{(\lambda_e)},$$

où $\kappa = 0, 1, 2, 3, 4, \ldots, n.$

• **Generation by global sections :** An arbitrary vector field defined in the ambient space :

$$\mathbb{C}^{n+1} \times \mathbb{C}^{N_d^n} \times \mathbb{C}^{n(n+1)}$$

writes under the general form :

$$\mathsf{T} = \sum_{i=1}^{n+1} \mathsf{Z}_i \frac{\partial}{\partial z_i} + \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\, \alpha_1<d}} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha} + \sum_{k=1}^{n+1} \mathsf{Z}'_k \frac{\partial}{\partial z'_k} + \sum_{k=1}^{n+1} \mathsf{Z}''_k \frac{\partial}{\partial z''_k} + \cdots + \sum_{k=1}^{n+1} \mathsf{Z}_k^{(n)} \frac{\partial}{\partial z_k^{(n)}}.$$

Theorem. [M., 2009] *Let* $\widetilde{\Sigma}$ *be the closure, in* $J^n_{\mathsf{vert}}(\mathscr{X})$*, of the closed algebraic subset of affine vertical jets* $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ *which is defined by the annihilation of all the first order jets :*

$$\widetilde{\Sigma}_0 := \Big\{ \big(z_i, a_\alpha, z'_{j_1}, \ldots, z^{(n)}_{j_n}\big) : \ z'_1 = z'_2 = \cdots = z'_{n+1} = 0 \Big\}.$$

Then the following two properties hold true :

- $J^n_{\mathsf{vert}}(\mathscr{X})\backslash\Sigma$ *is smooth of pure codimension equal to* $n+1$ *at every point, namely of dimension equal to :*

$$\begin{aligned} j^d_n &:= n+1+N^n_d + n(n+1) - (n+1) \\ &= \tfrac{(n+1+d)!}{(n+1)!\,d!} + n(n+1). \end{aligned}$$

- *The twisted tangent bundle :*

$$T_{J^n_{\mathrm{vert}}(\mathscr{X})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(n^2+2n\big) \otimes \mathscr{O}_{\mathbb{P}^{N^n_d}}(1)$$

is generated by its global sections on $J^n_{\mathsf{vert}}(\mathscr{X})\backslash\widetilde{\Sigma}$*, that is to say : at every point* $p^{[n]} \in J^n_{\mathsf{vert}}(\mathscr{X})\backslash\widetilde{\Sigma}$ *which does not belong to* $\widetilde{\Sigma}$*, one may find* j^d_n glo-bal *sections* $\mathsf{T}_1, \ldots, \mathsf{T}_{j^d_n}$ *on* X *of this twisted tangent bundle such that :*

$$\mathbb{C}\mathsf{T}_1(p^{[n]}) \oplus \cdots \oplus \mathbb{C}\mathsf{T}_{j^d_n}(p^{[n]}) = T_{J^n_{\mathsf{vert}}(\mathscr{X}),\, p^{[n]}}.$$

VII-1 – Algebraic degeneracy

Theorem. [DMR 2009] *Let $X \subset \mathbb{P}^{n+1}$ be a projective hypersurface of arbitrary dimension $n \geqslant 2$. Then there exists a* **noneffective** *positive integer :*

$$d_n \gg 1,$$

such that, if X is generic of degree $\deg X \geqslant d_n$, then there exists a proper *algebraic subvariety :*

$$Y \subsetneqq X,$$

such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ entirely contained in Y.

• **From above :** [DIVERIO, 2008] For a jet order $k = n$ equal to the dimension, there exists $d_n \gg 1$ such that the two isomorphic spaces of sections :

$$H^0\big(X_n, \mathscr{O}_{X_n}(m) \otimes \pi_{0,n}^* A^{-1}\big) \simeq H^0\big(X, E_{n,m} T_X^* \otimes A^{-1}\big) \neq 0\,,$$

are *nonvoid* when $d \geqslant d_n$, provided $m \geqslant m_{d,n}$ is sufficiently large.

• **Canonical bundle :**

$$K_X \simeq \mathscr{O}_X(d-n-2)\,.$$

It will play the rôle of the ample line bundle A.

• **Continuity argument :** For $\delta > 0$ sufficiently small :

$$H^0\big(X_n, \mathscr{O}_{X_n}(m) \otimes \pi_{0,n}^* K_X^{-\delta m}\big) \simeq H^0\big(X, E_{n,m} T_X^* \otimes K_X^{-\delta m}\big) \neq 0\,.$$

• **Slices of the universal hypersurface :**

$$X_s := \mathscr{X}\big|_s, \qquad s \in \mathbb{P}^{N_d^n}.$$

• **Beautiful idea of Siu (2002) :** Holomorphic family of jet differentials not identically zero :

$$P = \big\{P|_s \in H^0\big(X_s,\, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big)\big\}.$$

• **Semi-continuity of cohomology :** [HARTSHORNE] The parameters s range outside a certain (uncontrolled) exceptional algebraic subvariety of the parameter space $\mathbb{P}^{N_d^n}$.

• **Fix s_0 outside this exceptional set**.

• **Nonconstant entire curve** $f\colon \mathbb{C} \to X_{s_0}$.

• **Define the zero-set locus :**

$$Y_{s_0} := \big\{x \in X_{s_0}\colon P|_{s_0}(x) = 0\big\}$$

of the non-identically zero section $P|_{s_0}$ of the vector bundle $E_{n,m}T^*_{X_{s_0}} \otimes K_{X_{s_0}}^{-\delta m}$.

Lemma. *Then Y_{s_0} is a proper algebraic subset of X which contains all nonconstant entire holomorphic curves :*

$$f(\mathbb{C}) \subset Y_{s_0}.$$

• **Existence of at least one differential equation :**

$$P|_{s_0}\big(j^n f(\zeta)\big) \equiv 0.$$

- **By contradiction :** There exists $\zeta_0 \in \mathbb{C}$ such that :

$$f(\zeta_0) \notin Y_{s_0} \qquad \text{et} \quad f'(\zeta_0) \neq 0\,.$$

- **In local coordinates :**

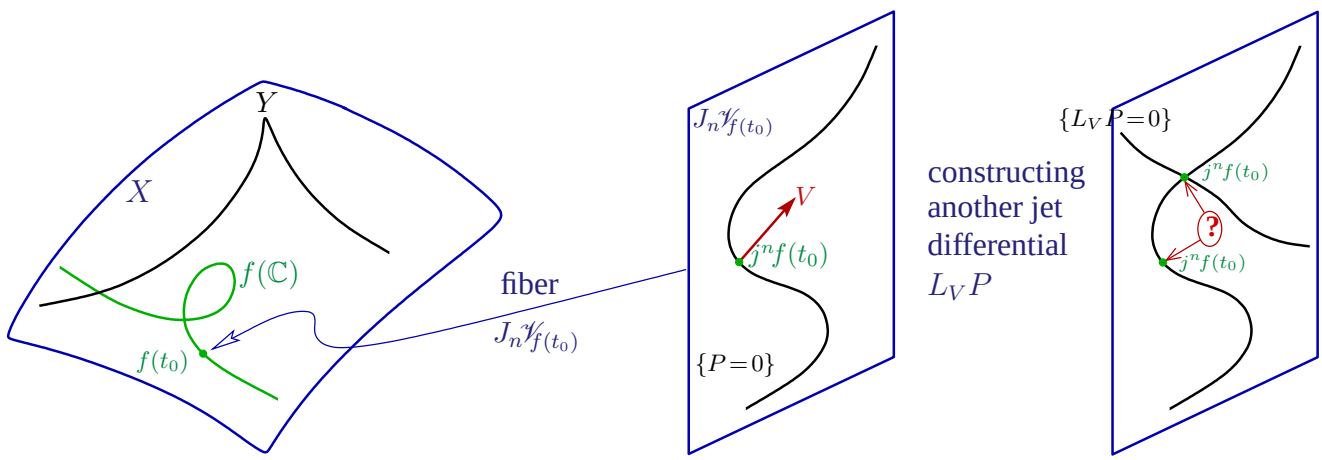
$$P = \sum_{|i_1|+\cdots+n|i_n|=m} q_{i_1,\ldots,i_n}(s,z)\,(z')^{i_1}\cdots(z^{(n)})^{i_n}$$

$$Y_{s_0} = \left\{z \in X_{s_0} \colon q_{i_1,\ldots,i_n}(s_0,z) = 0, \ \ \forall\, i_1,\ldots,i_n\right\}.$$

- **Relative polynomialness :** With respect to the jets.

- **Differentiate by a vector field :**

$$(\bullet) \otimes \mathscr{O}_{X_{s_0}}(n^2+2n).$$



- **Differentiate by p vector fields :**

$$(\bullet) \otimes \mathscr{O}_{X_{s_0}}(p(n^2+2n))$$

- **Global section in :**

$$H^0\big(X_{s_0}, E_{n,m}T^*_{X_{s_0}} \otimes \mathscr{O}_{X_{s_0}}(-\delta m(d-n-2) + p(n^2+2n))\big).$$

- **Still insure the inverse of an ample line bundle :**

$$-\delta m(d-n-2) + p(n^2+2n) < 0. \quad \square$$

Équations différentielles algébriques pour les courbes holomorphes entières dans les hypersurfaces projectives complexes de degré optimal

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I. Conjecture de Green-Griffiths

II. L'« enfer algébrique » des jets invariants

III. Retour aux jets de Green-Griffiths

IV. Décomposition en fibrés de Schur

V. Caractéristique d'Euler-Poincaré

VI. Comportements asymptotiques

VII. Spéculations prospectives

Séminaire « Analyse et Géométrie Complexe »

Université de Paris 6, le 10 novembre 2009

I – Conjecture de Green-Griffiths

- **Hypersurface projective complexe algébrique :**

$$X = \big\{ \big[z_0 \colon z_1 \colon \ldots \colon z_n \colon z_{n+1}\big] \in \mathbb{P}^{n+1} \colon P(z_0, z_1, \ldots, z_n, z_{n+1}) = 0 \big\}.$$

- **Fibré canonique :**

$$K_X := \Lambda^n T^*_X.$$

- X **de type général :** Lorsque $m \to \infty$:

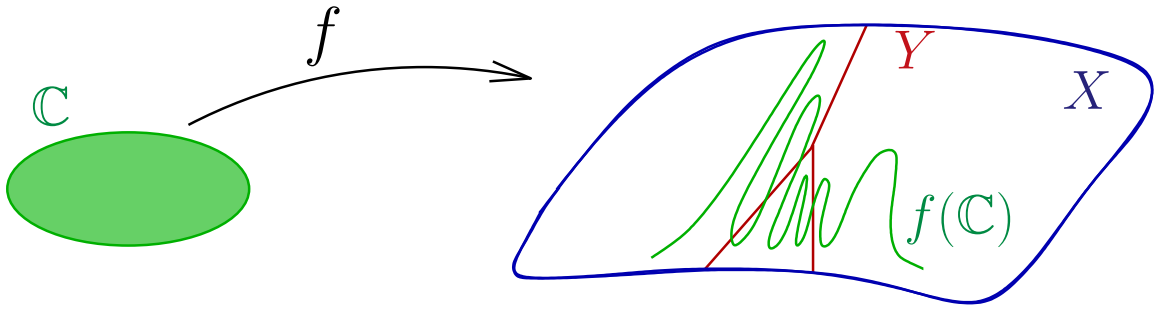
$$h^0\big(X,\, K_X^{\otimes m}\big) \sim c \cdot m^{\dim X},$$

pour une certaine constante $c > 0$.

- **Caractérisation équivalente :** degré optimal :

$$\boxed{\deg X \geqslant \dim X + 3}.$$

- **Conjecture de Green-Griffiths forte (1979) :** Si l'hypersurface projective algébrique $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ est générique de degré $\geqslant n+3$, alors il existe un sous-ensemble algébrique propre $Y \subset X$ tel que toute courbe holomorphe entière non constante $f : \mathbb{C} \to X$ est nécessairement intégralement contenue dans Y, à savoir : $f(\mathbb{C}) \subset Y$.



• **Degré optimal :** $\boxed{\deg X \geqslant n+3}$, ce qui correspond ainsi à demander que X soit **de type général**.

• **Conjecture d'hyperbolicité de Kobayashi (1970) :** Une hypersurface projective lisse $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ est **hyperbolique** : « toute courbe holomorphe entière $f : \mathbb{C} \to X$ est nécessairement **constante** », lorsque

$$\boxed{\deg X \geqslant 2n+1},$$

pourvu que X soit générique.

• **Siu 2002, 2004 :** Il existe $d_n \gg 1$ tel que les hypersurfaces génériques $X^n \subset \mathbb{P}^{n+1}$ de degré :

$$\deg X \geqslant d_n$$

sont hyperboliques.

• **Demailly à Luminy en juin 2006 : grands degrés !**

• **Dimension 2 :** $X^2 \subset \mathbb{P}^3(\mathbb{C})$: G.-G. + Kobayashi :

- ☐ Siu-Yeung 1996 : $d \geqslant \mathbf{10^{13}}$.
- ☐ McQuillan, 1999 : $d \geqslant \mathbf{36}$.
- ☐ Demailly-El Goul, 2000 : $d \geqslant \mathbf{21}$.
- ☐ Paŭn, 2008 : $d \geqslant \mathbf{18}$.

• **Dimension 3 :** $X^3 \subset \mathbb{P}^4(\mathbb{C})$:

- ☐ Rousseau 2007 : $d \geqslant \mathbf{593}$, Green-Griffiths.
- ☐ Diverio-Trapani 2009 : $d \geqslant \mathbf{593}$, Kobayashi.

Théorème. (DIVERIO-M.-ROUSSEAU, nov. 2008) *Si* $X \subset \mathbb{P}^{n+1}$ *est une hypersurface projective algébrique* générique, *il existe un sous-ensemble algébrique propre* $Y \subsetneqq X$ *tel que* $f(\mathbb{C}) \subset Y$ *pour toute courbe holomorphe entière non constante :*

- *pour* $\dim X = \mathbf{4}$, *lorsque* $\deg X \geqslant \mathbf{3203}$;
- *pour* $\dim X = \mathbf{5}$, *lorsque* $\deg X \geqslant \mathbf{35355}$;
- *pour* $\dim X = \mathbf{6}$, *lorsque* $\deg X \geqslant \mathbf{172925}$.

Théorème. (DIVERIO-M.-ROUSSEAU) *En dimension quelconque* $n \geqslant 2$ *avec* $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *générique, même conclusion lorsque :*

$$\deg X \geqslant \mathbf{2^{n^5}}.$$

Stratégies (à éviter !)

- ☐ Jets simples de Green-Griffiths (trop simples ?) **Non !**
- ☐ Jets raffinés de Demailly (incompréhensibles !)
- ☐ Inégalités de Morse-Demailly-Trapani-Siu (insuffisantes pour atteindre la borne optimale !)
- ☐ Fibrés de jets sur les variétés projectives (lieu-base non contrôlé $\Rightarrow$ Kobayashi inacessible !)

Théorème. (Aujourd'hui) *Équations différentielles en degré optimal* $\deg X \geqslant n+3$ *(avec hypothèse).*

Certitude : revenir aux jets de Green-Griffiths

Structure générale des démonstrations

Étape 1 : Les courbes holomorphes entières $f : \mathbb{C} \to X$ satisfont des équations différentielles algébriques.

Étape 2 : Une pléthore de telles équations différentielles implique la dégénérescence algébrique $f(\mathbb{C}) \subset Y \subsetneqq X$.

Théorème. [NOMBREUX AUTEURS] *Toute courbe holomorphe entière non constante $f : \mathbb{C} \to X$ satisfait des équations différentielles algébriques globales lorsque $d = \deg X$ est assez grand.*

- **Dimension 2 :** [GREEN-GRIFFITHS 1980] optimal :

$$\deg X \geqslant \mathbf{5}.$$

- **Dimension 3 :**
  - ☐ [ROUSSEAU 2006] $\deg X \geqslant \mathbf{97}$.
  - ☐ [DIVERIO 2008] $\deg X \geqslant \mathbf{74}$.
  - ☐ [M., 30 MAI 2009] $\deg X \geqslant \mathbf{34}$.
- **Dimension 4 :**
  - ☐ [DIVERIO 2008] $\deg X \geqslant \mathbf{298}$.
  - ☐ [D.M.R. 2009] $\deg X \geqslant \mathbf{259}$.
- **Dimension 5 :**
  - ☐ [DIVERIO 2008] $\deg X \geqslant \mathbf{1222}$.
- **Dimension 6 :**
  - ☐ [D.M.R. 2009] $\deg X \geqslant \mathbf{4352}$.

Très loin de : $\deg X \geqslant n + 3 = \mathbf{6},\ \mathbf{7},\ \mathbf{8},\ \mathbf{9},\ \ldots$

II – L'« enfer algébrique » des jets invariants

• **Notations :**

$$\boxed{n = \text{dimension}}$$

$$\boxed{d = \text{degré}}$$

$$\boxed{\kappa = \text{ordre des jets}}$$

• **Deux fibré de jets de courbes holomorphes :**

□ **Green-Griffiths :**

$$\mathscr{E}^{GG}_{\kappa,m} T^*_X.$$

□ **Demailly-Semple :**

$$\mathscr{E}^{DS}_{\kappa,m} T^*_X.$$

• **Obstacle significatif :** [ROUSSEAU, 2006] En dimension **3**, l'ordre des jets doit être $\geqslant$ **3**.

• **Plus généralement :** Un théorème d'annulation dû à Brückmann et Rackwitz donne facilement :

Corollaire. [DIVERIO, 2008] *Pour tout fibré en droites ample $A \to X$ et tout $\kappa \leqslant \dim X - 1$, on a :*

$$\begin{aligned} 0 &= H^0\big(X,\, \mathscr{E}^{GG}_{\kappa,m} \otimes A^{-1}\big) \\ &= H^0\big(X,\, \mathscr{E}^{DS}_{\kappa,m} \otimes A^{-1}\big)\,. \end{aligned}$$

• **Donc il faut supposer toujours $\kappa \geqslant \dim X$.**

• **Définition :** Dans une carte locale, considérer le jet d'ordre κ :

$$j^\kappa f := \big(f_1', \dots, f_n', f_1'', \dots, f_n'', \dots\dots, f_1^{(\kappa)}, \dots, f_n^{(\kappa)}\big)$$

d'un disque holomorphe :

$$f = (f_1, f_2, \dots, f_n) : \mathbb{D} \longrightarrow X^n.$$

• **Descriptions connues des jets de Demailly-Semple :**

□ $n = 2$, $\kappa = 3$: DEMAILLY (non publié) ; ROUSSEAU : **5** invariants fondamentaux.

□ $n = 3$, $\kappa = 3$: ROUSSEAU : **16**.

□ $n = 2$, $\kappa = 4$: DEMAILLY-EL GOUL (non publié) ; M. : **9**.

□ $n = \mathbf{2}$, $\kappa = \mathbf{5}$: M. : **56**.

□ $n = \mathbf{4}$, $\kappa = \mathbf{4}$: M. : **2835**.

• **Algorithme général :**

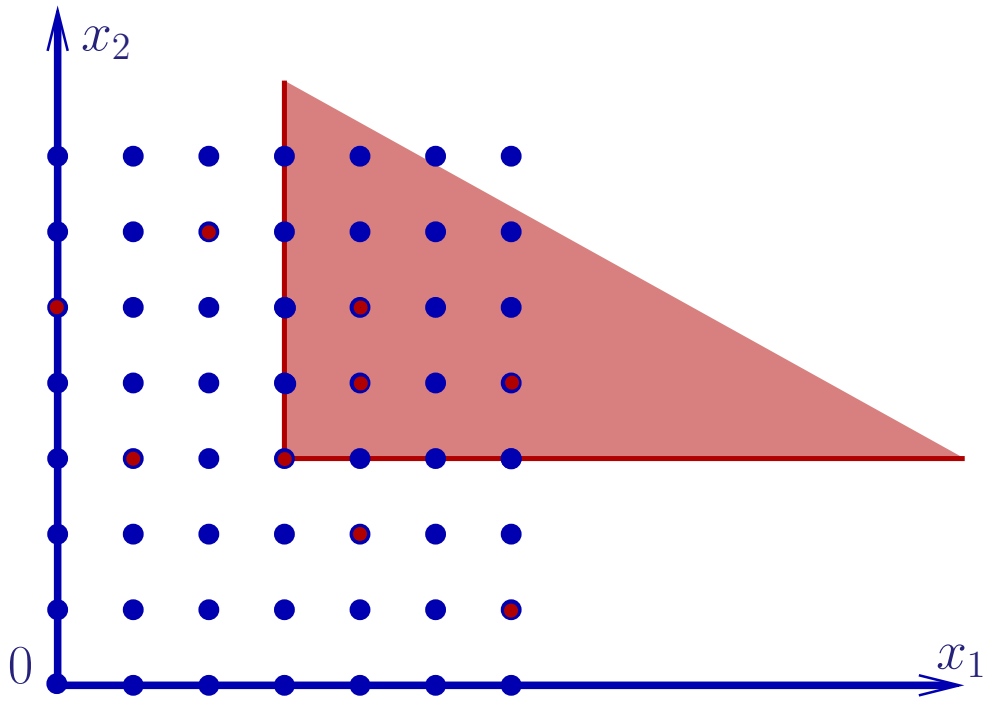
Théorème. [M., 2008] *Construction d'un algorithme complet qui engendre tous les polynômes invariants par reparamétrisation, en dimension arbitraire n et pour des jets d'ordre quelconque κ.*

• Joël M. *Jets de Demailly-Semple d'ordres 4 et 5 en dimension 2*, Int. J. Contemp. Math. Sciences, **3** (2008) no. 18, 861–933.

• Joël M. *An algorithm to generate all polynomials in the k-jet of a holomorphic disc $D \to \mathbb{C}^n$ that are invariant under source reparametrization*, arxiv.org/abs/0808.3547/, 103 pages.

• **Syzygies :** Les relations entre invariants sont engendrées en même temps.

• **Bases de Gröbner ; Polytopes de Newton :**



• **Obtenir 16 bi-invariants mutuellement indépendants :**

W^{10},	f_1',	Λ^3,	Λ^5,	Λ^7,	D^6,	D^8,	N^{10},
M^8,	E^{10},	L^{12},	Q^{14},	R^{15},	U^{17},	V^{19},	X^{21}

• **Ordre lexicographique :**

$$\Lambda^3 > \Lambda^5 > \Lambda^7 > D^6 > D^8 > N^{10} > M^8 > E^{10} > L^{12} >$$
$$> Q^{14} > R^{15} > U^{17} > V^{19} > X^{21}.$$

• **Expressions explicites :**

$$\Lambda^3 := \begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix} =: \Delta_{1,2}^{',''},$$

$$\Lambda^5 := \Delta_{1,2}^{','''} f_1' - 3\,\Delta_{1,2}^{',''} f_1'',$$

$$D^6 := \begin{vmatrix} f_1' & f_2' & f_3'' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix} =: \Delta^{',''',''''} .$$

$$\Lambda^7 := \left(\Delta_{1,2}^{',''''} + 4\,\Delta_{1,2}^{'',''''}\right) f_1' f_1' - \\ - 10\,\Delta_{1,2}^{','''} f_1' f_1'' + 15\,\Delta_{1,2}^{',''} f_1'' f_1'',$$

$$D^8 := f_1' \begin{vmatrix} f_1' & f_2' & f_3'' \\ f_1'' & f_2'' & f_3'' \\ f_1'''' & f_2'''' & f_3'''' \end{vmatrix} - 6\, f_1'' \begin{vmatrix} f_1' & f_2' & f_3'' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix},$$

$$N^{10} := \Delta_{1,2,3}^{',''',''''} f_1' f_1' - 3\,\Delta_{1,2,3}^{','',''''} f_1' f_1'' + \\ + 4\,\Delta_{1,2,3}^{','','''} f_1' f_1''' + 3\,\Delta_{1,2,3}^{','','''} f_1'' f_1'',$$

$$M^8 := \frac{-5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7}{f_1' f_1'} \\ = 3\,\Delta_{1,2}^{',''''}\,\Delta_{1,2}^{',''} + 12\,\Delta_{1,2}^{'','''}\,\Delta_{1,2}^{',''} - 5\,\Delta_{1,2}^{','''}\,\Delta_{1,2}^{','''},$$

$$E^{10} := \frac{-6\,\Lambda^5\,D^6 + 3\,\Lambda^3\,D^8}{f_1'} \\ = 3\,\Delta_{1,2,3}^{','',''''}\,\Delta_{1,2}^{',''} - 6\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{','''},$$

$$L^{12} := \frac{-\Lambda^7 D^6 + 5\,\Lambda^3 N^{10}}{f_1'} \\ = -\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{',''''}\, f_1' - 4\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{'','''}\, f_1' + 5\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{',''}\, f_1' + 10\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{','''}\, f_1'' - \\ - 15\,\Delta_{1,2,3}^{','',''''}\,\Delta_{1,2}^{',''}\, f_1'' + 20\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{',''}\, f_1''',$$

$$
\begin{aligned}
Q^{14} &:= \frac{\Lambda^7 D^8 - 10\,\Lambda^5 N^{10}}{f_1'} \\
&= -10\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{','''}\,f_1' f_1' + \Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''''}\,f_1' f_1' + 4\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{'''}\,f_1' f_1' + \\
&\quad + 20\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{'''}\,f_1' f_1'' + 30\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''}\,f_1' f_1'' - 6\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''''}\,f_1' f_1'' - \\
&\quad - 24\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{'''}\,f_1' f_1'' - 40\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{'''}\,f_1' f_1''' - 75\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''}\,f_1'' f_1'' + \\
&\quad + 30\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{'''}\,f_1'' f_1'' + 120\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''}\,f_1'' f_1''', \\
R^{15} &:= \Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,f_1' - 12\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,f_1' + 24\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,f_1'' - \\
&\quad - 48\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2,3}^{',''','''}\,f_1''', \\
U^{17} &= \frac{4\,D^8 E^{10} + 3\,\Lambda^3 R^{15}}{f_1'} \\
&= 15\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''} - 36\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''} - \\
&\quad - 24\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{'''} + 144\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''''}, \\
V^{19} &= \frac{8\,N^{10} E^{10} + \Lambda^5 R^{15}}{f_1'} \\
&= 24\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''}\,f_1' - 60\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{'''}\,f_1' + \\
&\quad + \Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{'''}\,f_1' - 75\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''}\,f_1'' + \\
&\quad + 36\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''}\,f_1'' + 168\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{'''}\,f_1'' - \\
&\quad - 144\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''''}\,f_1'' + 96\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''}\,f_1''' - \\
&\quad - 240\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{'''}\,f_1''', \\
X^{21} &= \frac{4\,D^8 Q^{14} - 5\,\Lambda^7 R^{15}}{f_1'} \\
&= -40\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{'''}\,f_1' f_1' - 4\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''''}\,f_1' f_1' - \\
&\quad - 4\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''''}\,f_1' f_1' + 60\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''''}\,f_1' f_1' + \\
&\quad + 240\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''''}\,f_1' f_1' + 130\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{'''}\,f_1' f_1'' + \\
&\quad + 120\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{''}\,f_1' f_1'' - 168\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2,3}^{',''','''}\,\Delta_{1,2}^{''''}\,f_1' f_1'' -
\end{aligned}
$$

$$
\begin{aligned}
&- 668\,\Delta^{',''}_{1,2,3}\,\Delta^{',''''}_{1,2,3}\,\Delta^{'','''}_{1,2}\,f_1'f_1'' - 360\,\Delta^{','''}_{1,2,3}\,\Delta^{',''''}_{1,2,3}\,\Delta^{','''}_{1,2}\,f_1'f_1''-\\
&- 160\,\Delta^{',''}_{1,2,3}\,\Delta^{',''''}_{1,2,3}\,\Delta^{','''}_{1,2}\,f_1'f_1''' + 240\,\Delta^{',''}_{1,2,3}\,\Delta^{','''}_{1,2,3}\,\Delta^{',''''}_{1,2}\,f_1'f_1'''+\\
&+ 960\,\Delta^{',''}_{1,2,3}\,\Delta^{','''}_{1,2,3}\,\Delta^{'','''}_{1,2}\,f_1'f_1''' - 375\,\Delta^{',''}_{1,2,3}\,\Delta^{',''''}_{1,2,3}\,\Delta^{',''}_{1,2}\,f_1''f_1''+\\
&+ 840\,\Delta^{',''}_{1,2,3}\,\Delta^{',''''}_{1,2,3}\,\Delta^{','''}_{1,2}\,f_1''f_1'' + 180\,\Delta^{','''}_{1,2,3}\,\Delta^{',''''}_{1,2,3}\,\Delta^{',''}_{1,2}\,f_1''f_1''+\\
&+ 144\,\Delta^{',''}_{1,2,3}\,\Delta^{','''}_{1,2,3}\,\Delta^{',''''}_{1,2}\,f_1''f_1'' + 144\,\Delta^{',''}_{1,2,3}\,\Delta^{','''}_{1,2,3}\,\Delta^{'','''}_{1,2}\,f_1''f_1''-\\
&- 1440\,\Delta^{',''}_{1,2,3}\,\Delta^{','''}_{1,2,3}\,\Delta^{','''}_{1,2}\,f_1''f_1''' + 480\,\Delta^{',''}_{1,2,3}\,\Delta^{',''''}_{1,2,3}\,\Delta^{',''}_{1,2}\,f_1''f_1'''.
\end{aligned}
$$

- **41 syzygies complétées entre 16 bi-invariants :**

$$
\begin{aligned}
0 &\overset{1}{\equiv} -5\,\Lambda^5\Lambda^5 + 3\,\underline{\Lambda^3\Lambda^7}_{\mathsf{LT}} - f_1'f_1'M^8,\\
0 &\overset{2}{\equiv} -2\,\Lambda^5 D^6 + \underline{\Lambda^3 D^8}_{\mathsf{LT}} - \tfrac{1}{3}f_1'\,E^{10},\\
0 &\overset{3}{\equiv} -\Lambda^7 D^6 + 5\,\underline{\Lambda^3 N^{10}}_{\mathsf{LT}} - f_1'L^{12},\\
0 &\overset{4}{\equiv} -5\,\Lambda^5 E^{10} + 3\,\underline{\Lambda^3 L^{12}}_{\mathsf{LT}} + 6\,f_1'D^6M^8,\\
0 &\overset{5}{\equiv} 5\,\Lambda^7 E^{10} + 3\,\underline{\Lambda^3 Q^{14}}_{\mathsf{LT}} - 6\,f_1'D^8M^8,\\
0 &\overset{6}{\equiv} 4\,D^8E^{10} + 3\,\underline{\Lambda^3 R^{15}}_{\mathsf{LT}} - f_1'U^{17},
\end{aligned}
$$

$$
\begin{aligned}
0 &\overset{7}{\equiv} -36\,D^6D^6M^8 - 5\,E^{10}E^{10} + 3\,\underline{\Lambda^3 U^{17}}_{\mathsf{LT}} + 0,\\
0 &\overset{8}{\equiv} -5\,E^{10}L^{12} - 6\,D^6D^8M^8 + 3\,\underline{\Lambda^3 V^{19}}_{\mathsf{LT}} + 0,\\
0 &\overset{9}{\equiv} 5\,L^{12}L^{12} + 3\,\underline{\Lambda^3 X^{21}}_{\mathsf{LT}} + M^8D^8D^8 + 0,\\
0 &\overset{10}{\equiv} -6\,\Lambda^7 D^6 + 5\,\underline{\Lambda^5 D^8}_{\mathsf{LT}} - f_1'L^{12},\\
0 &\overset{11}{\equiv} -\Lambda^7 D^8 + 10\,\underline{\Lambda^5 N^{10}}_{\mathsf{LT}} + f_1'Q^{14},\\
0 &\overset{12}{\equiv} \underline{\Lambda^5 L^{12}}_{\mathsf{LT}} - \Lambda^7 E^{10} + f_1'D^8M^8,
\end{aligned}
$$

$$0 \overset{13}{\equiv} \Lambda^7 L^{12} + \underline{\Lambda^5 Q^{14}}_{\mathsf{LT}} - 2\, f_1' M^8 N^{10},$$
$$0 \overset{14}{\equiv} 8\, N^{10} E^{10} + \underline{\Lambda^5 R^{15}}_{\mathsf{LT}} - f_1' V^{19},$$
$$0 \overset{15}{\equiv} \underline{\Lambda^5 U^{17}}_{\mathsf{LT}} - E^{10} L^{12} - 6\, D^6 D^8 M^8 + 0,$$
$$0 \overset{16}{\equiv} \underline{\Lambda^5 V^{19}}_{\mathsf{LT}} - M^8 D^8 D^8 - L^{12} L^{12} + f_1' M^8 R^{15},$$
$$0 \overset{17}{\equiv} \underline{\Lambda^5 X^{21}}_{\mathsf{LT}} - L^{12} Q^{14} + 2\, D^8 N^{10} M^8 + 0,$$
$$0 \overset{18}{\equiv} 8\, N^{10} L^{12} + \underline{\Lambda^7 R^{15}}_{\mathsf{LT}} + f_1' X^{21},$$

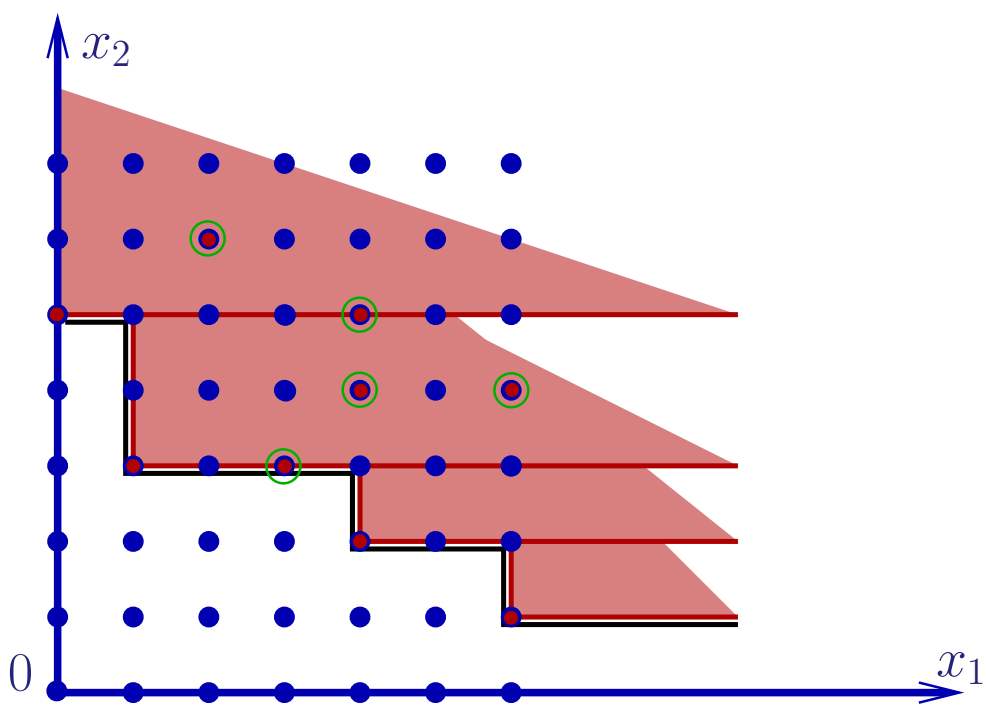
$$0 \overset{19}{\equiv} -L^{12} L^{12} + \underline{\Lambda^7 U^{17}}_{\mathsf{LT}} + f_1', -5\, M^8 D^8 D^8 + 0,$$
$$0 \overset{20}{\equiv} L^{12} Q^{14} + \underline{\Lambda^7 V^{19}}_{\mathsf{LT}} - 10\, D^8 M^8 N^{10} + 0,$$

$$0 \overset{21}{\equiv} 20\, N^{10} N^{10} M^8 + Q^{14} Q^{14} + \underline{\Lambda^7 X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{22}{\equiv} 6\, \underline{D^6 M^8 R^{15}}_{\mathsf{LT}} + L^{12} U^{17} - E^{10} V^{19} + 0,$$
$$0 \overset{23}{\equiv} 5\, \underline{D^8 M^8 R^{15}}_{\mathsf{LT}} - Q^{14} U^{17} - L^{12} V^{19} + 0,$$
$$0 \overset{24}{\equiv} 10\, \underline{N^{10} M^8 R^{15}}_{\mathsf{LT}} - Q^{14} V^{19} + L^{12} X^{21} + 0,$$

$$0 \overset{25}{\equiv} 5\, \underline{M^8 R^{15} R^{15}}_{\mathsf{LT}} + V^{19} V^{19} + U^{17} X^{21} + 0,$$
$$0 \overset{26}{\equiv} -D^8 D^8 + 12\, \underline{D^6 N^{10}}_{\mathsf{LT}} + f_1' R^{15},$$
$$0 \overset{27}{\equiv} -5\, D^8 E^{10} + 6\, \underline{D^6 L^{12}}_{\mathsf{LT}} + f_1' U^{17},$$
$$0 \overset{28}{\equiv} 3\, \underline{D^6 Q^{14}}_{\mathsf{LT}} + 25\, N^{10} E^{10} - 3\, f_1' V^{19},$$
$$0 \overset{29}{\equiv} 5\, E^{10} R^{15} - D^8 U^{17} + 6\, \underline{D^6 V^{19}}_{\mathsf{LT}} + 0,$$
$$0 \overset{30}{\equiv} -3\, L^{12} R^{15} + N^{10} U^{17} + 3\, \underline{D^6 X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{31}{\equiv} -10\,N^{10}E^{10} + \underline{D^8L^{12}}_{\mathsf{LT}} + f_1'V^{19},$$
$$0 \overset{32}{\equiv} \underline{D^8Q^{14}}_{\mathsf{LT}} + 10\,N^{10}L^{12} + f_1'X^{21},$$
$$0 \overset{33}{\equiv} -2\,N^{10}U^{17} + \underline{D^8V^{19}}_{\mathsf{LT}} + L^{12}R^{15} + 0,$$
$$0 \overset{34}{\equiv} Q^{14}R^{15} + 2\,N^{10}V^{19} + \underline{D^8X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{35}{\equiv} -2\,L^{12}N^{10}U^{17} + R^{15}L^{12}L^{12} + 10\,\underline{V^{19}N^{10}E^{10}}_{\mathsf{LT}} - f_1'V^{19}V^{19},$$
$$0 \overset{36}{\equiv} 2\,N^{10}U^{17}Q^{14} - R^{15}L^{12}Q^{14} + 10\,\underline{V^{19}N^{10}L^{12}}_{\mathsf{LT}} + f_1'V^{19}X^{21},$$
$$0 \overset{37}{\equiv} 10\,\underline{N^{10}L^{12}X^{21}}_{\mathsf{LT}} - R^{15}Q^{14}Q^{14} - 2\,Q^{14}N^{10}V^{19} + f_1'X^{21}X^{21},$$
$$0 \overset{38}{\equiv} 2\,\underline{N^{10}U^{17}X^{21}}_{\mathsf{LT}} - X^{21}L^{12}R^{15} + V^{19}Q^{14}R^{15} + 2\,N^{10}V^{19}V^{19} + 0,$$
$$0 \overset{39}{\equiv} \underline{E^{10}Q^{14}}_{\mathsf{LT}} + L^{12}L^{12} - f_1'M^8R^{15},$$
$$0 \overset{40}{\equiv} Q^{14}U^{17} + 6\,L^{12}V^{19} + 5\,\underline{E^{10}X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{41}{\equiv} -6\,Q^{14}L^{12}V^{19} - Q^{14}Q^{14}U^{17} + 5\,\underline{X^{21}L^{12}L^{12}}_{\mathsf{LT}} - 5\,f_1'M^8R^{15}X^{21}.$$

- **Dimension 16 avec 41 sommets.**



2835

THÉORÈME (M. 2008) *En dimension $n = 4$ pour les jets d'ordre $\kappa = 4$, l'algèbre UE_4^4 de polynômes de jets $\mathsf{P}\big(j^4 f_1, j^4 f_2, j^4 f_3, j^4 f_4\big)$ invariants par reparamétrisation et invariants sous l'action unipotente est engendrée par* **16** *bi-invariants mutuellement indépendants :*

$$\boxed{\begin{array}{cccccccc} W^{10}, & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & D^6, & D^8, & N^{10}, \\ M^8, & E^{10}, & L^{12}, & Q^{14}, & R^{15}, & U^{17}, & V^{19}, & X^{21}, \end{array}}$$

dont la restriction à $\{f_1' = 0\}$ possède un idéal de relations, pour l'ordre purement lexicographique, qui est constitué des **41** *syzygies écrites ci-dessus.*

De plus, tout bi-invariant de poids m s'écrit de manière unique sous la forme polynomiale suivante :

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \, \big(W^{10}\big)^p \sum_{\substack{(a,\ldots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41}) \\ 3a+\cdots+21n=m-o-10p}} \mathsf{coeff}_{a,\ldots,n,o,p} \cdot$$
$$\cdot \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h$$
$$\big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n,$$

avec des coefficients $\mathsf{coeff}_{a,\ldots,n,o,p}$ quelconques, où $\square_1$, …, $\square_{41}$ désignent les quadrants de $\mathbb{N}^{14}$ ayant des sommets correspondant aux puissances des **41** *monômes de tête en question.*

Par conséquent, en dimension $n = 4$ pour les jets d'ordre $\kappa = 4$, l'algèbre $\mathscr{E}_4^4$ des polynômes de

jets $\mathsf{P}\big(j^4 f\big)$ *invariants par reparamétrisation est engendrée par les polarisations :*

$$\boxed{\begin{array}{c} W^{10}, \quad f'_i, \quad \Lambda^3_{[i,j]}, \quad \Lambda^5_{[i,j];\,\alpha}, \quad \Lambda^7_{[i,j];\,\alpha,\beta}, \quad D^6_{[i,j,k]}, \\ D^8_{[i,j,k];\,\alpha}, \quad N^{10}_{[i,j,k];\,\alpha,\beta}, \quad M^8_{[i,j],[k,l]}, \quad E^{10}_{[i,j,k],[p,q]}, \quad L^{12}_{[i,j,k],[p,q];\,\alpha}, \\ Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta}, \quad R^{15}_{[i,j,k],[p,q,r];\,\alpha}, \quad U^{17}_{[i,j,k],[p,q,r],[s,t]}, \\ V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha}, \quad X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta}, \end{array}}$$

de ces 16 *bi-invariants* W^{10}, f'_1, Λ^3, Λ^5, Λ^7, D^6, D^8, N^{10}, M^8, E^{10}, L^{12}, Q^{14}, R^{15}, U^{17}, V^{19}, X^{21} *; ces invariants polarisés sont anti-symétriques par rapport à chaque collection d'indices entre crochets* $[i,j,k]$, $[p,q,r]$, $[s,t]$, *et ils sont* **explicitement** *représentés en termes de* Δ*-déterminants par les formules complètes suivantes :*

$$\begin{aligned} &W^{10}_{1,2,3,4}, \\ &\quad f'_i, \\ &\quad \Lambda^3_{[i,j]} := \Delta^{',''}_{i,j}, \end{aligned}$$

$$\Lambda^5_{[i,j];\,\alpha} := \Delta^{','''}_{i,j}\, f'_\alpha - 3\,\Delta^{',''}_{i,j}\, f''_\alpha,$$

$$\begin{aligned} \Lambda^7_{[i,j];\,\alpha,\beta} := \Delta^{',''''}_{i,j}\, f'_\alpha f'_\beta + 4\,\Delta^{'','''}_{i,j}\, f'_\alpha f'_\beta - 5\Delta^{','''}_{i,j}\, \big(f'_\alpha f''_\beta + f''_\alpha f'_\beta\big) + \\ + 15\,\Delta^{',''}_{i,j}\, f''_\alpha f''_\beta, \end{aligned}$$

$$D^6_{[i,j,k]} := \Delta^{','','''}_{i,j,k},$$

$$
\begin{aligned}
D^{8}_{[i,j,k];\,\alpha} &:= \Delta^{',\,'',\,''''}_{i,j,k}\, f'_\alpha - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\, f''_\alpha,\\
N^{10}_{[i,j,k];\,\alpha,\beta} &:= \Delta^{',\,''',\,''''}_{i,j,k}\, f'_\alpha f'_\beta - \frac{3}{2}\,\Delta^{',\,'',\,''''}_{i,j,k}\,\left(f'_\alpha f''_\beta + f''_\alpha f'_\beta\right) +\\
&\quad + 2\,\Delta^{',\,'',\,'''}_{i,j,k}\,\left(f'_\alpha f'''_\beta + f'''_\alpha f'_\beta\right) + 3\,\Delta^{',\,'',\,'''}_{i,j,k}\, f''_\alpha f''_\beta,
\end{aligned}
$$

$$
\begin{aligned}
M^{8}_{[i,j],[k,l]} &:= 3\,\Delta^{',\,''''}_{i,j}\,\Delta^{',\,''}_{k,l} + 12\,\Delta^{'',\,'''}_{i,j}\,\Delta^{',\,''}_{k,l} -\\
&\quad - 5\,\Delta^{',\,'''}_{i,j}\,\Delta^{',\,'''}_{k,l}\,,\\
E^{10}_{[i,j,k],[p,q]} &:= 3\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''}_{l,m} - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{l,m},\\
L^{12}_{[i,j,k],[l,m];\,\alpha} &:= 5\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha - 15\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f''_\alpha - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,''''}_{p,q}\, f'_\alpha -\\
&\quad - 24\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{'',\,'''}_{p,q}\, f'_\alpha + 30\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha,
\end{aligned}
$$

$$
\begin{aligned}
Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta} &:= -10\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha f'_\beta + \Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''''}_{p,q}\, f'_\alpha f'_\beta +\\
&\quad + 4\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{'',\,'''}_{p,q}\, f'_\alpha f'_\beta + +20\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha f''_\beta +\\
&\quad + 30\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f'_\alpha f''_\beta - 6\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,''''}_{p,q}\, f'_\alpha f''_\beta -\\
&\quad - 24\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{'',\,'''}_{p,q}\, f'_\alpha f''_\beta - 40\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f'_\alpha f'''_\beta -\\
&\quad - 75\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f''_\alpha f''_\beta + 30\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'''}_{p,q}\, f''_\alpha f''_\beta +\\
&\quad + 120\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,''}_{p,q}\, f''_\alpha f'''_\beta\,,\\
R^{15}_{[i,j,k],[p,q,r];\,\alpha} &:= \Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\, f'_\alpha - 12\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\, f'_\alpha +\\
&\quad + 24\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\, f''_\alpha - 48\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\, f'''_\alpha\,,\\
U^{17}_{[i,j,k],[p,q,r],[s,t]} &:= 15\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,''}_{s,t} - 36\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,''}_{s,t} -\\
&\quad - 24\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,'''}_{s,t} + 144\,\Delta^{',\,'',\,'''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{'',\,'''}_{s,t}\,,\\
V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha} &:= 24\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,''}_{s,t}\, f'_\alpha - 60\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,'''}_{s,t}\, f'_\alpha +\\
&\quad + \Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,'''}_{s,t}\, f'_\alpha - 75\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,''''}_{p,q,r}\,\Delta^{',\,''}_{s,t}\, f''_\alpha +\\
&\quad + 36\,\Delta^{',\,''',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,''}_{s,t}\, f''_\alpha + 168\,\Delta^{',\,'',\,''''}_{i,j,k}\,\Delta^{',\,'',\,'''}_{p,q,r}\,\Delta^{',\,'''}_{s,t}\, f''_\alpha -
\end{aligned}
$$

$$- 144\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{''}_{s,t}\, f''_\alpha + 96\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'''_\alpha -$$
$$- 240\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'''_\alpha,$$

$$X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta} :=$$
$$:= -40\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f'_\beta - 4\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f'_\beta -$$
$$- 4\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{''}_{s,t}\, f'_\alpha f'_\beta + 60\, \Delta^{'}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f'_\beta +$$
$$+ 240\, \Delta^{'}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{''}_{s,t}\, f'_\alpha f'_\beta + 130\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f''_\beta +$$
$$+ 120\, \Delta^{'}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f''_\beta - 168\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f''_\beta -$$
$$- 668\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{''}_{s,t}\, f'_\alpha f''_\beta - 360\, \Delta^{'}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f''_\beta -$$
$$- 160\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f'''_\beta + 240\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f'_\alpha f'''_\beta +$$
$$+ 960\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{''}_{s,t}\, f'_\alpha f'''_\beta - 375\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f''_\alpha f''_\beta +$$
$$+ 840\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f''_\alpha f''_\beta + 180\, \Delta^{'}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f''_\alpha f''_\beta +$$
$$+ 144\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f''_\alpha f''_\beta + 144\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{''}_{s,t}\, f''_\alpha f''_\beta -$$
$$- 1440\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f''_\alpha f'''_\beta + 480\, \Delta^{',''}_{i,j,k}\, \Delta^{',''}_{p,q,r}\, \Delta^{'}_{s,t}\, f''_\alpha f'''_\beta,$$

où les indices romains satisfont $1 \leqslant i < j < k \leqslant 4$*, où* $1 \leqslant p < q < r \leqslant 4$*, où* $1 \leqslant s < r \leqslant 4$ *et où les indices grecs* α, β *satisfont sans restriction* $1 \leqslant \alpha, \beta \leqslant 4$*, d'où en définitive, le* **nombre total** *d'invariants qui engendrent l'algèbre de Demailly-Semple* E^4_4 *est égal à :*

$$1 + 4 + 6 + 24 + 96 + 4 + 16 + 64 +$$
$$+ 36 + 24 + 96 + 384 + 64 + 96 + 384 + 1536 = \boxed{\mathbf{2835}}.$$

Spéculation intermédiaire

• **Pour toute application effective aux courbes holomorphes entières, il est nécessaire de résoudre toutes les questions suivantes, et pas seulement la première :**

• **Question 1 :** L'algèbre des invariants de Demailly-Semple est-elle finiment engendrée **?**

• **Question 2 :** L'idéal des relations entre les invariants est-il engendré par un procédé régulier spécifique dont la combinatoire est dominable **?**

• **Question 3 :** L'algèbre est-elle de Cohen-Macaulay **?** Si tel est le cas, peut-on décrire de manière effective une base d'invariants primaires **?**

• **Question 4 :** Si l'algèbre des invariants n'est pas de Cohen-Macaulay, peut-on décrire une base de Gröbner effective pour l'idéal des relations entre tous les invariants **?**

• **Semblerait néanmoins accessible à un algébriste encore plus courageux :** $n = \mathbf{5}$, $\kappa = \mathbf{5}$.

Les jets de Green-Griffiths sont combinatoirement très réguliers

On peut (enfin !) faire monter les jets à l'infini

III – Retour aux jets de Green-Griffiths

• **Immédiatement :** Différentier par rapport à $\zeta \in \mathbb{C}$ la contrainte initiale $f : \mathbb{C} \to X$:

$$P\big(f_1(\zeta), \dots, f_n(\zeta), f_{n+1}(\zeta)\big) \equiv 0$$

où — on l'aura compris — :

$$P\big(z_1, \dots, z_n, z_{n+1}\big) = 0$$

désigne l'équation polynomiale de $X^n \subset \mathbb{C}^{n+1}$.

$\kappa =$ un tel ordre arbitraire de dérivation

• **Jet de Green-Griffiths :** Dans une carte locale centrée en un point $x \in X$, on considère les disques holomorphes passant par x :

$$f : (\mathbb{D}, 0) \to (X, x) \simeq (\mathbb{C}^n, 0)$$

qui possèdent bien sûr n composantes :

$$\big(f_1(\zeta), f_2(\zeta), \dots, f_n(\zeta)\big),$$

et au-dessus du point x, on considère les polynômes dans les variables de jets $f_i^{(\lambda)}$ qui sont du type suivant :

$$\sum_{m = |a_1| + 2|a_2| + \cdots + \kappa |a_\kappa|} \mathsf{coeff} \cdot (f')^{a_1} (f'')^{a_2} \dots (f^{(\kappa)})^{a_\kappa},$$

et qui sont homogènes d'un certain poids fixé m par rapport à la dilatation de jets :

$$\delta \cdot \big(f'_{i_1}, f''_{i_2}, \dots, f^{(\kappa)}_{i_\kappa}\big) := \big(\delta^1 f'_{i_1}, \delta^2 f''_{i_2}, \dots, \delta^\kappa f^{(\kappa)}_{i_\kappa}\big).$$

• **Pour mémoire :**

$$\boxed{m = \text{poids} = \text{nombre total (fixé) de « primes »}}$$

• **Structure vectorielle/non vectorielle :** Bien que le fibré des jets $J^\kappa(\mathbb{D},\, X)$ de courbes holomorphes ne soit pas un fibré vectoriel dès que $\kappa \geqslant 2$, le fibré des jets de Green-Griffiths $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ **est un fibré vectoriel**, car l'espace *vectoriel* des polynômes de poids m en le jet $j^\kappa f$ est stable par tout changement de coordonnées $j^\kappa f \mapsto j^\kappa\big(\Psi \circ f\big)$ induit par un changement de carte Ψ sur X.

Lemme. (Green-Griffiths 1980) *Ces données ponctuelles s'organisent en un fibré vectoriel holomorphe :*

$$\mathscr{E}^{GG}_{\kappa,m} T^*_X$$

sur $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$. *Ce fibré possède une filtration naturelle, et le fibré gradué associé :*

$$\mathsf{Gr}^\bullet\big(\mathscr{E}^{GG}_{\kappa,m} T^*_X\big)$$

est un fibré vectoriel holomorphe qui est isomorphe à la somme directe :

$$\boxed{\bigoplus_{\ell_1 + 2\ell_2 + \cdots + \kappa\ell_\kappa = m} \mathrm{Sym}^{\ell_1} T^*_X \otimes \mathrm{Sym}^{\ell_2} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X}$$

• **Inégalités cohomologiques :** Pour tout $i = 1, 2, \ldots, n$, on a :

$$h^i\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) \leqslant h^i\big(X,\, \mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X\big).$$

Théorème. [BLOCH, AHLFORS, GREEN-GRIFFITHS, DEMAILLY, SIU] *Soit X une hypersurface projective algébrique complexe, soit A un fibré en droites ample sur X — prendre* e.g. $A = \mathscr{O}_X(1)$ — *et soit :*

$$\mathsf{P} \in H^0\big(X, \mathscr{E}_{\kappa,m}^{GG \text{ ou } \cancel{DS}} \otimes A^{-1}\big)$$

une section globale. Alors toute courbe holomorphe entière $f\colon \mathbb{C} \to X$ non constante satisfait l'équation différentielle correspondante :

$$\mathsf{P}\big(f', \dots, f^{(\kappa)}\big) \equiv 0.$$

• **But :** $\boxed{\text{Construire des sections de } \mathscr{E}_{\kappa,m}^{GG} T_X^* \otimes A^{-1}}$

• **Caractéristique d'Euler-Poincaré du fibré des jets de Green-Griffiths :**

$$\chi\big(X, \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) = \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n \big((\kappa+1)n-1\big)!\; n!} \rightsquigarrow \text{[facteur innocent]}$$

$$\Big\{(-\mathsf{c}_1)^n (\log \kappa)^n + \mathrm{O}\big((\log \kappa)^{n-1}\big)\Big\} + \mathrm{O}\big(m^{(\kappa+1)n-2}\big),$$

où $\mathsf{c}_1 = \mathsf{c}_1(T_X)$ est la première classe de Chern de T_X.

• **Ré-expression en fonction du degré :**

$$\begin{aligned}\chi_{\kappa,m} &= \text{Constante}_{n,\kappa,m}\, (-\mathsf{c}_1)^n + \text{reste négligeable} \\ &= \text{Constante}_{n,\kappa,m}\, d\,(d-n-2)^n + \cdots\end{aligned}$$

• **Type général :** Ainsi, pourvu seulement que X soit de type général : $d \geqslant n+3$, on a $\chi_{\kappa,m} \to \infty$ avec κ et m.

• **Stratégie naturelle mais difficile :** On veut construire des sections globales de $\mathscr{E}^{GG}_{\kappa,m}T^*_X$, à savoir on veut que :

$$\underbrace{h^0\big(X,\,\mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \dim H^0\big(X,\,\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)}_{>0}.$$

• **Définition de la caractéristique :**

$$\chi = h^0 - h^1 + h^2 - h^3 + h^4 - \cdots + (-1)^n h^n.$$

• **Minoration triviale :**

$$\begin{aligned} h^0 &= \chi + h^1 - h^2 + h^3 - h^4 + \cdots - (-1)^n\, h^n \\ &\geqslant \chi \qquad\quad - h^2 \qquad\quad - h^4 \qquad\quad - \cdots \end{aligned}.$$

• **Il suffirait de majorer les cohomologies paires :** Si $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ est de type général, *i.e.* de degré $d \geqslant n+3$, alors les quantités :

$$h^{2i}_{\kappa,m} := \dim H^{2i}\big(X,\,\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)$$

croîtraient moins rapidement que la caractéristique :

$$\chi_{\kappa,m} \gg h^{2i}_{\kappa,m} \qquad \text{pour } \kappa \text{ et } m \text{ grands.}$$

• **Dimension 2 :** BOGOMOLOV : annulation du h^2.

Théorème. [GREEN-GRIFFITHS 1980] *Sur une surface $X^2 \subset \mathbb{P}^3(\mathbb{C})$, toutes les courbes holomorphes entières $f\colon \mathbb{C} \to X$ satisfont des équations différentielles en degré optimal :*

$$\deg X \geqslant \mathbf{5}\,.$$

• **Dimension 3 :** [ROUSSEAU 2006] Le h^2 ne s'annule en général pas.

• **Fait général (voir ci-dessous) :** Les fibrés de jets de Green-Griffiths et de Demailly-Semple peuvent être décomposés comme certaines sommes de fibrés de Schur.

Théorème. [ROUSSEAU 2006] *Sur une hypersurface projective algébrique* $X^3 \subset \mathbb{P}^4(\mathbb{C})$*, pour les jets de Demailly-Semple d'ordre* $\kappa = 3$ *:*

$$\mathsf{Gr}^\bullet \mathscr{E}_{3,m}^{DS} T_X^* = \bigoplus_{a+3b+5c+6d=m} \mathscr{S}^{(a+b+2c+d,\, b+c+d,\, d)}\, T_X^*.$$

on peut majorer :

$$h^2\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ell_3)} T_X^*\big) \leqslant d(d+13)\, \frac{3(\ell_1+\ell_2+\ell_3)^3}{2}\, (\ell_1-\ell_2)(\ell_1-\ell_3)(\ell_2-\ell_3) + \mathrm{O}\big(|\ell|^5\big),$$

de telle sorte qu'il existe des différentielles de jets dans $\mathscr{E}_{3,m}^{DS} T_X^*$ *pour* m *grand lorsque* $\deg X \geqslant \mathbf{97}$*.*

Théorème. [M. avril 2008 / D.M.R. novembre 2008] *Analogue complet avec les jets de Demailly-Semple pour* $n = 4$ *et* $\kappa = 4$ *lorsque :*

$$\deg X \geqslant \mathbf{259}\,.$$

Théorème. [M. 10 Juin 2009] *Dimension* **3** *avec les jets de Green-Griffiths d'ordre* $\kappa \sim \infty$ *lorsque :*

$$\deg X \geqslant \mathbf{34}\,.$$

IV – Décomposition en fibrés de Schur

• **Fibrés de Schur :** Soit $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ une hypersurface projective algébrique lisse. Classiquement, on a les fibrés vectoriels holomorphes suivants sur X :

- ☐ T^*_X ;
- ☐ $\Lambda^k T^*_X$ (théorie de Hodge) ;
- ☐ $K_X := \Lambda^n T^*_X$ le fibré canonique ;
- ☐ $K_X^{\otimes m}$ ses puissances tensorielles (plurigenres) ;
- ☐ $\mathsf{Sym}^k T^*_X$ (k-genre cotangentiel) ;

Ce sont tous des cas particuliers des **fibrés de Schur** :

$$\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)} T^*_X,$$

qui sont paramétrés par n entiers décroissants :

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0,$$

et l'on retrouve notamment :

$$\Lambda^k T^*_X = \mathscr{S}^{(1,\dots,1,0,\dots,0)} T^*_X \qquad \text{avec } k \text{ fois } 1;$$

$$\mathrm{Sym}^k T^*_X = \mathscr{S}^{(k,0,\dots,0)} T^*_X.$$

• **Brève définition :** En fait, les fibrés de Schur apparaissent quand on décompose en représentations irréductibles de $\mathsf{GL}_n(\mathbb{C})$ les puissances tensorielles :

$$\underbrace{T^*_X \otimes \cdots \otimes T^*_X}_{r \text{ fois}} = \bigoplus_{(\ell)} \left[\mathscr{S}^{(\ell_1,\dots,\ell_n)} T^*_X\right]^{\oplus N_{(\ell)}},$$

avec $\ell_1 \geqslant \cdots \geqslant \ell_n$ où $N(\ell)$ est une certaine multiplicité.

• **Théorie des représentations :** Toute représentation (action) de $\mathsf{GL}_n(\mathbb{C})$ peut s'écrire comme une certaine somme directe de représentations de Schur, lesquelles constituent la liste de **toutes** les représentations **irréductibles** possibles de $\mathsf{GL}_n(\mathbb{C})$.

• **Fait général offert à la géométrie complexe : Tout fibré vectoriel holomorphe** E sur X, sur les fibres duquel on peut faire agir $\mathsf{GL}_n(\mathbb{C})$, doit en principe se **décomposer comme une certaine somme directe de fibrés de Schur**, qui s'avèrent ainsi être les briques élémentaires avec lesquelles on peut reconstituer tout fibré dans l'anneau de Grothendieck, notamment : $\mathsf{Gr}\mathscr{E}^{GG}_{\kappa,m}T^*_X$.

Théorème. [M., 10 Juin 2009] *Supposons que pour tout* $i = 1, 2, \ldots, n$, *on peut majorer les dimensions cohomologiques :*

$$h^i = \dim H^i\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n)}T^*_X\big)$$

par une formule du type général :

$$h^i \leqslant \mathrm{const}_n \cdot \big[1 + d + d^2 + \cdots + d^{n+1}\big] \cdot \sum_{\substack{\alpha_1+\cdots+\alpha_{n-1}+\alpha_n=\frac{n(n+1)}{2} \\ \alpha_n \leqslant n-1}} (\ell_1-\ell_2)^{\alpha_1}\cdots(\ell_{n-1}-\ell_n)^{\alpha_{n-1}}(\ell_n)^{\alpha_n}.$$

Alors toutes les courbes holomorphes entières non constantes $f\colon \mathbb{C} \to X$ *satisfont des équations différentielles algébriques globales en degré optimal :*

$$\deg X \geqslant \mathbf{n+3}\,.$$

Résumé / Interlude explicatif

□ **But principal :** Constuire des équations différentielles sur les hypersurfaces projectives X de type général en degré optimal $\deg X \geqslant \mathbf{n+3}$.

□ **Décomposer** le fibré des jets de Green-Griffiths comme une certaine somme directe de **fibrés de Schur**, au moins de manière asymptotique en un sens à préciser ultérieurement :

$$\mathscr{E}^{GG}_{\kappa,m} T^*_X = \bigoplus_{\text{certains}} \mathscr{S}^{(\ell_1,\dots,\ell_n)} T^*_X .$$

□ **Admettre** temporairement l'hypothèse qui s'avérera naturelle — **Conjecture** ou **problème ouvert** — que pour $i = 1, 2, \dots, n$, on a :

$(*)$

$$\left[\begin{array}{l} h^i\big(X, \mathscr{S}^{(\ell_1,\dots,\ell_n)} T^*_X\big) \leqslant \text{const.}\big[1 + d + d^2 + \cdots + d^{n+1}\big] \cdot \\ \qquad \cdot \displaystyle\sum_{\substack{\alpha_1+\cdots+\alpha_{n-1}+\alpha_n = \frac{n(n+1)}{2} \\ \alpha_n \leqslant n-1}} (\ell_1 - \ell_2)^{\alpha_1} \cdots (\ell_{n-1} - \ell_n)^{\alpha_{n-1}} (\ell_n)^{\alpha_n} . \end{array}\right.$$

□ **Déduire** par sommation de ces inégalités des majorants pour les $h^{2i}\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big)$.

□ **Gagner** par les asymptotiques la positivité :

$$h^0\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X \otimes A^{-1}\big) > 0 ,$$

pour des ordre de jets κ très grands, pourvu que $m \gg 1$.

Suite sur la décomposition en fibrés de Schur

• **Gradué du fibré de Green-Griffiths :**

$$\mathsf{Gr}^\bullet \mathsf{E}^{GG}_{\kappa,m} T^*_X = \bigoplus_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} \mathrm{Sym}^{\ell_1} T^*_X \otimes \mathrm{Sym}^{\ell_2} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X .$$

• **Question :** Comment, donc, obtenir la décomposition de ce fibré vectoriel en somme directe de fibrés de Schur, pour $\kappa \to \infty$ **?**

• **Rappel :**

$$\mathrm{Sym}^\ell T^*_X = \mathscr{S}^{(\ell,0,\ldots,0)} T^*_X .$$

• **Deux moyens d'obtenir une telle décomposition :**

- ☐ formule de Pieri ;
- ☐ vecteurs de plus haut poids.

• **Itérer une infinité de fois la formule :**

$$\mathscr{S}^{(t_1,\ldots,t_n)} T^*_X \otimes \mathscr{S}^{(\ell,0,\ldots,0)} T^*_X = \\ = \sum_{\substack{s_1+\cdots+s_n=\ell+t_1+\cdots+t_n \\ s_1\geqslant t_1\geqslant s_2\geqslant t_2\geqslant\cdots\geqslant s_n\geqslant t_n\geqslant 0}} \mathscr{S}^{(s_1,\ldots,s_n)} T^*_X .$$

• **Fait connu en combinatoire :** Nombres de Kostka ; formules non closes ; aspects asymptotiques connus **?**

• **Approche théorie des invariants :** accessible sur le plan asymptotique.

• **Matrice unipotente générale :**

$$\mathsf{u} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_{21} & 1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & 1 & \cdots & 0 \\ .. & .. & .. & \cdots & .. \\ u_{n1} & u_{n2} & u_{n3} & \cdots & 1 \end{pmatrix}.$$

• **Action sur les variables de jets :** Pour tout niveau de jets λ avec $1 \leqslant \lambda \leqslant \kappa$, définir :

$$g^{(\lambda)} := \mathsf{u} \cdot f^{(\lambda)},$$

c'est-à-dire avec tous les indices :

$$\begin{aligned} g_1^{(\lambda)} &:= f_1^{(\lambda)} \\ g_2^{(\lambda)} &:= f_2^{(\lambda)} + u_{21}\, f_1^{(\lambda)} \\ g_3^{(\lambda)} &:= f_3^{(\lambda)} + u_{32}\, f_2^{(\lambda)} + u_{31}\, f_1^{(\lambda)} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_n^{(\lambda)} &= f_n^{(\lambda)} + u_{n,n-1}\, f_{n-1}^{(\lambda)} + \cdots + u_{n1}\, f_1^{(\lambda)}. \end{aligned}$$

• **Fait général de la théorie des représentations de** $\mathsf{GL}_n(\mathbb{C})$ **:** Les 'Vecteurs' de 'plus haut poids' sont ceux qui sont invariants par cette action unipotente, c'est-à-dire, ce sont les polynômes de jets $\mathsf{P}\big(j^\kappa f\big)$ qui satisfont :

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(\mathsf{u} \cdot j^\kappa f\big) \equiv \mathsf{P}\big(j^\kappa f\big),$$

pour toute matrice unipotente $\mathsf{u} \in \mathsf{U}_n(\mathbb{C})$.

Théorème. [19ÈME SIÈCLE] *L'algèbre des polynômes de jets invariants par cette action unipotente est engendrée par la collection de tous les invariants fondamentaux que sont les déterminants :*

$$\left|f_1^{(\lambda_1)}\right|, \qquad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} \end{vmatrix}, \qquad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} & f_3^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} & f_3^{(\lambda_2)} \\ f_1^{(\lambda_3)} & f_2^{(\lambda_3)} & f_3^{(\lambda_3)} \end{vmatrix},$$

$$\cdots\cdots \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} & \cdots & f_n^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} & \cdots & f_n^{(\lambda_2)} \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(\lambda_n)} & f_2^{(\lambda_n)} & \cdots & f_n^{(\lambda_n)} \end{vmatrix} =: \Delta_{1,2,\ldots,n}^{\lambda_1,\lambda_2,\ldots,\lambda_n}.$$

où $1 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n \leqslant \kappa$ *sont arbitraires.*

• **Lien avec fibrés de Schur :** (voir ci-dessous)

Unique Δ – monôme $\Longleftrightarrow$ Unique fibré de Schur.

• **Syzygies (ou relations) :** Toutefois, il y a des relations quadratiques **plückériennes** entre tous ces déterminants, par exemple les suivantes, connues au 16ème siècle :

$$0 \equiv \Delta_1^{\lambda_1} \cdot \Delta_{1,2}^{\lambda_2,\lambda_3} + \Delta_1^{\lambda_3} \cdot \Delta_{1,2}^{\lambda_1,\lambda_2} + \Delta_1^{\lambda_2} \cdot \Delta_{1,2}^{\lambda_3,\lambda_1}$$

$$0 \equiv \Delta_{1,2}^{\lambda_1,\lambda_2}\, \Delta_{1,2}^{\lambda_3,\lambda_4} + \Delta_{1,2}^{\lambda_1,\lambda_4}\, \Delta_{1,2}^{\lambda_2,\lambda_3} + \Delta_{1,2}^{\lambda_1,\lambda_3}\, \Delta_{1,2}^{\lambda_4,\lambda_2}$$

• **Syzygies générales :** Connues, on peut les écrires, et elles forment d'emblée une base de Gröbner.

• **Rappel :** Avec les invariants de Demailly-Semple, on ne voit rien et on ne comprend rien à la combinatoire.

Théorème. [19ÈME SIÈCLE] *Pour tous déterminants* $\Delta^{\lambda_1,\ldots,\lambda_i}_{1,\ldots,i}$ *et* $\Delta^{\mu_1,\ldots,\mu_j}_{1,\ldots,j}$ *avec* $i \geqslant j$ *satisfaisant :*

- *lorsque* $i > j$*, il existe un indice* $t \in \{1,\ldots,j\}$ *tel que :*

$$\lambda_1 \leqslant \mu_1, \quad \ldots\ldots, \quad \lambda_{t-1} \leqslant \mu_{t-1}, \quad \textit{mais :} \quad \lambda_t > \mu_t;$$

- *lorsque* $i = j$*, il existe deux indices* $s \in \{1,\ldots,j\}$ *et* $t \in \{1,\ldots,j\}$ *avec* $t \geqslant s+1$ *tels que :*

$$\lambda_1 = \mu_1, \ \ldots, \ \lambda_{s-1} = \mu_{s-1}, \ \lambda_s < \mu_s,$$
$$\lambda_{s+1} \leqslant \mu_{s+1}, \ \ldots, \ \lambda_{t-1} \leqslant \mu_{t-1}, \ \textit{mais :} \ \lambda_t > \mu_t;$$

la relation quadratique générale est satisfaite identiquement dans l'anneau de base $\mathbb{C}[f'_{i_1}, f''_{i_2}, \ldots, f^{(\kappa)}_{i_\kappa}]$ *:*

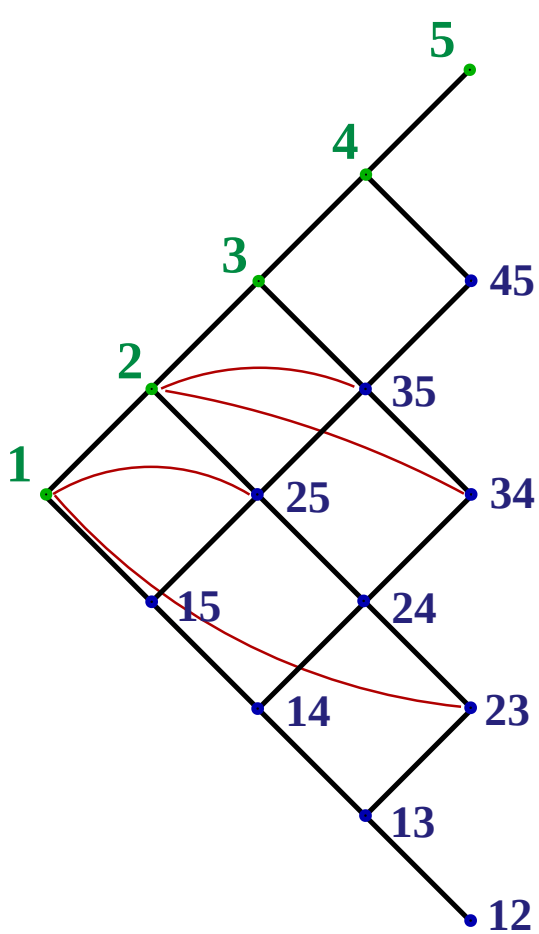
$$\boxed{0 \equiv \sum_{\pi \in \mathfrak{S}_{i+1}} \sum_{\substack{\pi(\lambda_t) < \cdots < \pi(\lambda_i) \\ \pi(\mu_1) < \cdots < \pi(\mu_t)}} \mathrm{sgn}(\pi) \cdot \Delta^{\lambda_1,\ldots,\lambda_{t-1},\pi(\lambda_t),\pi(\lambda_{t+1}),\ldots,\pi(\lambda_{j-1}),\pi(\lambda_j),\ldots,\pi(\lambda_i)}_{1,\ldots,t-1,t,t+1,\ldots,j-1,j,\ldots,i} \cdot \Delta^{\pi(\mu_1),\pi(\mu_2),\ldots,\pi(\mu_t),\mu_{t+1},\mu_{t+2},\ldots,\mu_j}_{1,2,\ldots,t,t+1,t+2,\ldots,j}}.$$

Théorème. [19ÈME SIÈCLE] *L'idéal des relations* Id-rel(Δ) *entre tous les* Δ*-déterminants est engendré par ces relations Plückériennes. De plus, elles constituent une base de Gröbner pour un ordre monomial naturel. Enfin, l'idéal monomial des* termes de tête *est constitué des paires incomparables dans un diagramme de Hasse.*

- **Exemple :** $n = 2$, $\kappa = 5$:

$$\Delta_1^{\mathbf{1}},\ \Delta_1^{\mathbf{2}},\ \Delta_1^{\mathbf{3}},\ \Delta_1^{\mathbf{4}},\ \Delta_1^{\mathbf{5}},$$

$$\begin{array}{llll}
\Delta_{1,2}^{\mathbf{1,2}}, & \Delta_{1,2}^{\mathbf{1,3}}, & \Delta_{1,2}^{\mathbf{1,4}}, & \Delta_{1,2}^{\mathbf{1,5}}, \\
 & \Delta_{1,2}^{\mathbf{2,3}}, & \Delta_{1,2}^{\mathbf{2,4}}, & \Delta_{1,2}^{\mathbf{2,5}}, \\
 & & \Delta_{1,2}^{\mathbf{3,4}}, & \Delta_{1,2}^{\mathbf{3,5}}, \\
 & & & \Delta_{1,2}^{\mathbf{4,5}}.
\end{array}$$



Paires incomparables

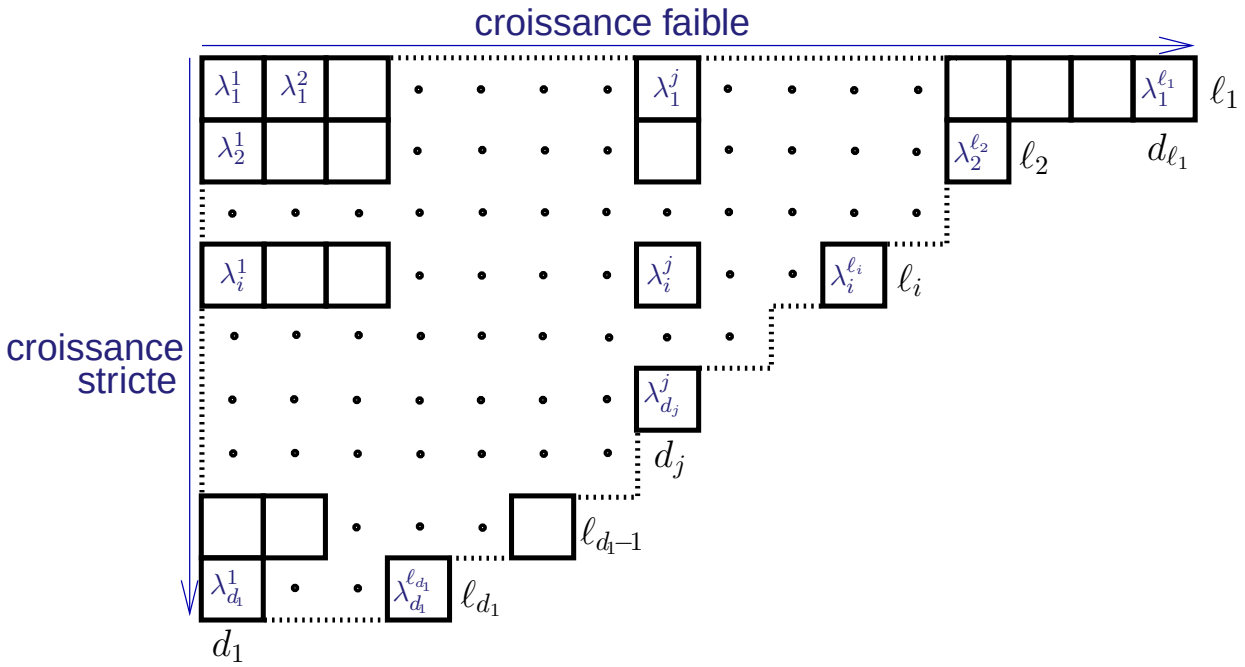
Théorème. [19ÈME SIÈCLE] *L'espace vectoriel quotient :*

tous les Δ-polynômes / modulo leurs relations

possède une base sur $\mathbb{C}$ qui est constituée de tous les Δ-monômes :

$$\prod_{1\leqslant j\leqslant \ell_{d_1}} \Delta_{1,\dots,d_1}^{\lambda_1^j,\dots,\lambda_{d_1}^j} \prod_{\ell_{d_1}+1\leqslant j\leqslant \ell_{d_1-1}} \Delta_{1,\dots,d_1-1}^{\lambda_1^j,\dots,\lambda_{d_1-1}^j} \cdots \prod_{\ell_2+1\leqslant j\leqslant \ell_1} \Delta_1^{\lambda_1^j}$$

tels que la collection des indices supérieurs (λ_i^j) forme un tableau de Young semi-standard :



• **Décomposition exacte de $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$ en fibrés de Schur.** Afin d'appliquer cette information combinatoire à notre problème, nous pouvons aussi représenter un Δ-monôme général sous la forme plus concise :

$$\prod_{d_1\geqslant i\geqslant 1} \prod_{1+\ell_{i+1}\leqslant j\leqslant \ell_i} \Delta_{1,\dots,i}^{\lambda_1^j,\dots,\lambda_i^j}.$$

• **Vecteur propre pour l'action diagonale :** Pour toute matrice $\mathsf{e} = \mathrm{diag}\big(\mathsf{e}_1, \ldots, \mathsf{e}_n\big)$, on a :

$$\mathsf{e} \cdot \Delta_{1,2,\ldots,i}^{\lambda_1^j, \lambda_2^j, \ldots, \lambda_i^j} = \mathsf{e}_1\, \mathsf{e}_2 \cdots \mathsf{e}_i\, \Delta_{1,2,\ldots,i}^{\lambda_1^j, \lambda_2^j, \ldots, \lambda_i^j},$$

• **En effet :** Multiplication uniforme des colonnes :

$$\mathsf{e} \cdot \Delta_{1,2,\ldots,i}^{\lambda_1^j, \lambda_2^j, \ldots, \lambda_i^j} = \begin{vmatrix} \mathsf{e}_1 f_1^{\lambda_1^j} & \mathsf{e}_2 f_2^{\lambda_1^j} & \cdots & \mathsf{e}_i f_i^{\lambda_1^j} \\ \mathsf{e}_1 f_1^{\lambda_2^j} & \mathsf{e}_2 f_2^{\lambda_2^j} & \cdots & \mathsf{e}_i f_i^{\lambda_2^j} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ \mathsf{e}_1 f_1^{\lambda_i^j} & \mathsf{e}_2 f_2^{\lambda_i^j} & \cdots & \mathsf{e}_i f_i^{\lambda_i^j} \end{vmatrix}.$$

• **Action sur un Δ-monôme général :**

$$\begin{aligned} \mathsf{e} \cdot \Bigg(\prod_{d_1 \geqslant i \geqslant 1} \; \prod_{1+\ell_{i+1} \leqslant j \leqslant \ell_i} \Delta_{1,\ldots,i}^{\lambda_1^j, \ldots, \lambda_i^j} \Bigg) &= \mathsf{e} \cdot \big(\Delta\text{-monôme général}\big) \\ &= \prod_{d_1 \geqslant i \geqslant 1} \; \prod_{1+\ell_{i+1} \leqslant j \leqslant \ell_i} \mathsf{e}_1 \cdots \mathsf{e}_i \cdot \big(\text{mêm } \Delta\text{-monôme}\big) \\ &= \prod_{d_1 \geqslant i \geqslant 1} \big(\mathsf{e}_1 \cdots \mathsf{e}_i\big)^{\ell_i - \ell_{i+1}} \cdot \big(\text{même } \Delta\text{-monôme}\big) \\ &= (\mathsf{e}_1)^{\ell_1} (\mathsf{e}_2)^{\ell_2} \cdots (\mathsf{e}_n)^{\ell_n} \cdot \big(\text{même } \Delta\text{-monôme}\big). \end{aligned}$$

• **Conséquence intéressante :**

$$\boxed{\text{Unique } \Delta\text{-monôme semi-standard} \longleftrightarrow \text{Unique fibré de Schur}}.$$

Théorème. (M.) *Le fibré gradué* $\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X$ *associé au fibré* $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ *des* κ*-jets de poids* m *de Green-Griffiths s'identifie à la somme directe suivante de fibrés de Schur :*

$$\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X = \bigoplus_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} \left(\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T^*_X \right)^{M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n}},$$

avec des multiplicités $M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n} \in \mathbb{N}$ *égales au nombre de fois qu'un diagramme de Young* $\mathsf{YD}_{(\ell_1,\ldots,\ell_n)}$ *dont les lignes ont les longueurs* $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0$ *peut être rempli par des entiers strictement positifs* $\lambda^j_i \leqslant \kappa$ *placés à la* i*-ème ligne et à la* j*-ème colonne de manière à constituer un tableau semi-standard, avec la contrainte supplémentaire que la somme de tous ces entiers :*

$$\begin{aligned} m = {} & \lambda^1_1 + \cdots + \lambda^{\ell_n}_1 + \cdots + \lambda^{\ell_2}_1 + \cdots + \lambda^{\ell_1}_1 \\ & + \lambda^1_2 + \cdots + \lambda^{\ell_n}_2 + \cdots + \lambda^{\ell_2}_2 \\ & + \cdots\cdots\cdots\cdots\cdots\cdots + \\ & + \lambda^1_n + \cdots + \lambda^{\ell_n}_n \end{aligned}$$

soit égale au degré d'homogénéité prescrit m.

Inaccessible avec les jets de Demailly-Semple :
Le plus haut niveau de jets plafonne à : $n = \kappa = 4$

V – Caractéristique d'Euler-Poincaré

$$\boxed{\chi\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) = \sum_{\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant 0} M^{\kappa,m}_{\ell_1,\ldots,\ell_n} \cdot \chi\big(X,\, \mathscr{S}^{(\ell_1,\ldots,\ell_n)} T^*_X\big)}$$

- **Classes de Chern :** $\mathsf{c}_k := \mathsf{c}_k(T_X) = (-1)^k \mathsf{c}_k$.
- $X^2 \subset \mathbb{P}^3(\mathbb{C})$ **de dimension 2 :**

$$\chi\big(X,\, \mathscr{S}^{(\ell_1,\ell_2)} T^*_X\big) = \frac{\mathsf{c}_1^2 - \mathsf{c}_2}{0!\; 3!} \begin{vmatrix} \ell_1^3 & \ell_2^3 \\ 1 & 1 \end{vmatrix} + \frac{\mathsf{c}_2}{1!\, 2!} \begin{vmatrix} \ell_1^2 & \ell_2^2 \\ \ell_1 & \ell_2 \end{vmatrix} + \\ + \mathsf{O}\big(|\ell|^2\big);$$

- **Partitions de 2 :**

$$\begin{aligned} 2 &= 2 + 0 \\ &= 1 + 1. \end{aligned}$$

- $X^3 \subset \mathbb{P}^4(\mathbb{C})$ **de dimension 3 :**

$$\chi\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ell_3)} T^*_X\big) = \\ = \frac{\mathsf{c}_1^3 - 2\,\mathsf{c}_1\mathsf{c}_2 + \mathsf{c}_3}{0!\; 1!\; 5!} \begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 \\ \ell_1 & \ell_2 & \ell_3 \\ 1 & 1 & 1 \end{vmatrix} + \frac{\mathsf{c}_1\mathsf{c}_2 - \mathsf{c}_3}{0!\; 2!\; 4!} \begin{vmatrix} \ell_1^4 & \ell_2^4 & \ell_3^4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ 1 & 1 & 1 \end{vmatrix} + \\ + \frac{\mathsf{c}_3}{1!\; 2!\; 3!} \begin{vmatrix} \ell_1^3 & \ell_2^3 & \ell_3^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ \ell_1 & \ell_2 & \ell_3 \end{vmatrix} + \mathsf{O}\big(|\ell|^5\big).$$

- **Partitions de 3 :**

$$\begin{aligned} 3 &= 3 + 0 + 0 \\ &= 2 + 1 + 0 \\ &= 1 + 1 + 1. \end{aligned}$$

- $X^4 \subset \mathbb{P}^5(\mathbb{C})$ **de dimension 4 :**

$$\chi\big(X, \mathscr{S}^{(\ell_1,\ell_2,\ell_3,\ell_4)} T_X^*\big) =$$

$$= \frac{\mathsf{c}_1^4 - 3\,\mathsf{c}_1^2\mathsf{c}_2 + \mathsf{c}_2^2 + 2\,\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 1!\ 2!\ 7!} \begin{vmatrix} \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^1 & \ell_2^1 & \ell_3^1 & \ell_4^1 \\ 1 & 1 & 1 & 1 \end{vmatrix} +$$

$$+ \frac{\mathsf{c}_1^2\mathsf{c}_2 - \mathsf{c}_2^2 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_4}{0!\ 1!\ 3!\ 6!} \begin{vmatrix} \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \frac{-\mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_2^2}{0!\ 1!\ 4!\ 5!} \begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} +$$

$$+ \frac{\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 2!\ 3!\ 5!} \begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \frac{\mathsf{c}_4}{1!\ 2!\ 3!\ 4!} \begin{vmatrix} \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{vmatrix} + \mathsf{O}\big(|\ell|^9\big).$$

- **Partitions de 4 :**

$$\begin{aligned} 4 &= 4 + 0 + 0 + 0 \\ &= 3 + 1 + 0 + 0 \\ &= 2 + 2 + 0 + 0 \\ &= 2 + 1 + 1 + 0 \\ &= 1 + 1 + 1 + 1. \end{aligned}$$

Théorème. *Les termes de plus haut degré par rapport à* $|\ell| = \max_{1\leqslant i\leqslant n} \ell_i$ *dans la caractéristique d'Euler-Poincaré du fibré de Schur* $\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}\, T^*_X$ *sont homogènes d'ordre* $\mathsf{O}\big(|\ell|^{\frac{n(n+1)}{2}}\big)$ *et ils sont donnés par une somme de déterminants indexée par toutes les partitions* $(\nu_1,\ldots,\nu_n)$ *de* n *:*

$$(-1)^n\,\chi\Big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}\, T^*_X\Big) =$$

$$= \sum_{\nu \text{ partition de } n} \frac{\mathsf{C}_{\nu^c}}{(\nu_1+n-1)!\cdots\nu_n!} \begin{vmatrix} {\ell'_1}^{\nu_1+n-1} & {\ell'_2}^{\nu_1+n-1} & \cdots & {\ell'_n}^{\nu_1+n-1} \\ {\ell'_1}^{\nu_2+n-2} & {\ell'_2}^{\nu_2+n-2} & \cdots & {\ell'_n}^{\nu_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell'_1}^{\nu_n} & {\ell'_2}^{\nu_n} & \cdots & {\ell'_n}^{\nu_n} \end{vmatrix} +$$

$$+\mathsf{O}\Big(|\ell|^{\frac{n(n+1)}{2}-1}\Big),$$

où $\ell'_i := \ell_i + n - i$*, avec des coefficients* C_{ν^c} *qui sont exprimés en termes des classes de Chern* $\mathsf{c}_k = \mathsf{c}_k\big(T_X\big)$ *de* T_X *au moyen des expressions déterminantales de Giambelli, qui dépendent de la partition* conjuguée ν^c *:*

$$\mathsf{C}_{\nu^c} = \mathsf{C}_{(\nu^c_1,\ldots,\nu^c_n)} = \begin{vmatrix} \mathsf{c}_{\nu^c_1} & \mathsf{c}_{\nu^c_1+1} & \mathsf{c}_{\nu^c_1+2} & \cdots & \mathsf{c}_{\nu^c_1+n-1} \\ \mathsf{c}_{\nu^c_2-1} & \mathsf{c}_{\nu^c_2} & \mathsf{c}_{\nu^c_2+1} & \cdots & \mathsf{c}_{\nu^c_2+n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{c}_{\nu^c_n-n+1} & \mathsf{c}_{\nu^c_n-n+2} & \mathsf{c}_{\nu^c_n-n+3} & \cdots & \mathsf{c}_{\nu^c_n} \end{vmatrix},$$

avec la convention que $\mathsf{c}_k := 0$ *pour* $k<0$ *ou* $k>n$*, et que* $\mathsf{c}_0 := 1$*.*

- **Remplacer les** ℓ'_i **par les** ℓ_i **:** Ne change rien, la différences est en $\mathsf{O}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$.

VI – Comportements asymptotiques

• **Objectif :** Étudier la cohomologie de :

$$\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X = \bigoplus_{\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant 0} \Big(\mathscr{S}^{(\ell_1,\ldots,\ell_n)} T^*_X \Big)^{M^{\kappa,m}_{\ell_1,\ldots,\ell_n}},$$

au moins asymptotiquement lorsque $m \to \infty$ et $\kappa \to \infty$.

• **Rappel :** on en déduit immédiatement :

$$\chi\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) = \sum_{\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant 0} M^{\kappa,m}_{\ell_1,\ldots,\ell_n} \cdot \chi\big(X,\, \mathscr{S}^{(\ell_1,\ldots,\ell_n)} T^*_X\big).$$

• **Cohérence des formules :** Retrouver d'abord la formule de caractéristique :

$$\chi\left(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\right) = \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n \big((\kappa+1)n-1\big)!\; n!} \cdot \Big\{ (-\mathsf{c}_1)^n (\log \kappa)^n + \mathrm{O}\big((\log \kappa)^{n-1}\big) \Big\} + \mathrm{O}\big(m^{(\kappa+1)n-2}\big),$$

qui avait été calculée en 1979 par Green-Griffiths, **sans** décomposer $\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X$ en fibrés de Schur.

• **Se débarasser tout d'abord des restes :**

Théorème.

$$\sum_{\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant 0} M^{\kappa,m}_{\ell_1,\ldots,\ell_n} \cdot \mathsf{O}\Big(|\ell|^{\frac{n(n+1)}{2}-1}\Big) = \mathsf{O}\big(m^{(\kappa+1)n-2}\big).$$

• **Commentaire :** La démonstration de cet énoncé auxiliaire est non triviale, car les multiplicités $M^{\kappa,m}_{\ell_1,\ldots,\ell_n}$ ne sont pas réellement calculables explicitement.

- **Convention :** Ne plus écrire les restes en $\mathsf{O}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$.

- **Développement des déterminants de Giambelli :**

$$\chi\big(X,\, \mathscr{S}^{\ell_1,\ldots,\ell_n}T_X^*\big) \equiv$$
$$\equiv \sum \mathsf{Poly}_{\mathbb{Q}}\big(\mathsf{c}_1,\ldots,\mathsf{c}_n\big)\cdot$$
$$\cdot \sum_{\beta_1+\beta_2+\cdots+\beta_{n-1}+\beta_n=\frac{n(n+1)}{2}} \ell_1^{\beta_1}\ell_2^{\beta_2}\cdots\ell_{n-1}^{\beta_{n-1}}\ell_n^{\beta_n}.$$

- **Réécrire artificiellement**

$$\ell_1^{\beta_1}\ell_2^{\beta_2}\cdots\ell_{n-1}^{\beta_{n-1}}\ell_n^{\beta_n} = \big(\ell_1-\ell_2+\ell_2-\ell_3+\cdots+\ell_{n-1}-\ell_n+\ell_n\big)^{\beta_1}\cdot$$
$$\cdot\big(\ell_2-\ell_3+\cdots+\ell_{n-1}-\ell_n+\ell_n\big)^{\beta_2}\cdot$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdot\big(\ell_{n-1}-\ell_n+\ell_n\big)^{\beta_{n-1}}\cdot$$
$$\cdot\big(\ell_n\big)^{\beta_n}.$$

- **Ré-exprimer :**

$$\chi\big(X,\, \mathscr{S}^{(\ell_1,\ldots,\ell_{n-1},\ell_n)}T_X^*\big) \equiv \sum \widetilde{\mathsf{Poly}}_{\mathbb{Q}}\big(\mathsf{c}_1,\ldots,\mathsf{c}_n\big)\cdot$$
$$\cdot \sum_{\substack{\alpha_1+\cdots+\alpha_{n-1}+\alpha_n=\frac{n(n+1)}{2}\\ \alpha_n\leqslant n}} \big(\ell_1-\ell_2\big)^{\alpha_1}\cdots\cdots\big(\ell_{n-1}-\ell_n\big)^{\alpha_{n-1}}\big(\ell_n\big)^{\alpha_n}.$$

- **Traiter séparément chaque ℓ-monôme :** Calculer :

$$\boxed{\sum_{\ell_1\geqslant\cdots\geqslant\ell_n\geqslant 0} M^{\kappa,m}_{\ell_1,\ldots,\ell_n}\cdot\big(\ell_1-\ell_2\big)^{\alpha_1}\cdots\cdots\big(\ell_{n-1}-\ell_n\big)^{\alpha_{n-1}}\big(\ell_n\big)^{\alpha_n}}.$$

- **Observation :** Toutes ces sommes sont maintenant purement numériques !

Théorème. *Supposons que l'ordre des jets* $\kappa \geqslant n$ *est au moins égal à la dimension, et soient* $\alpha_1, \dots, \alpha_{n-1}, \alpha_n$ *des entiers positifs ou nuls satisfaisant :*

$$\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n = \frac{n(n+1)}{2} \qquad \textit{et :} \quad \alpha_n \leqslant n-1.$$

Alors on a l'estimation logarithmique :

$$\sum_{\substack{\mathsf{YT\,semi-standard} \\ \mathsf{poids}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha_{n-1}} \big(\ell_n(\mathsf{YT})\big)^{\alpha_n} =$$

$$= \sum_{\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant 0} M^{\kappa,m}_{\ell_1,\dots,\ell_n} \cdot \big(\ell_1 - \ell_2\big)^{\alpha_1} \cdots \big(\ell_{n-1} - \ell_n\big)^{\alpha_{n-1}} \big(\ell_n\big)^{\alpha_n}$$

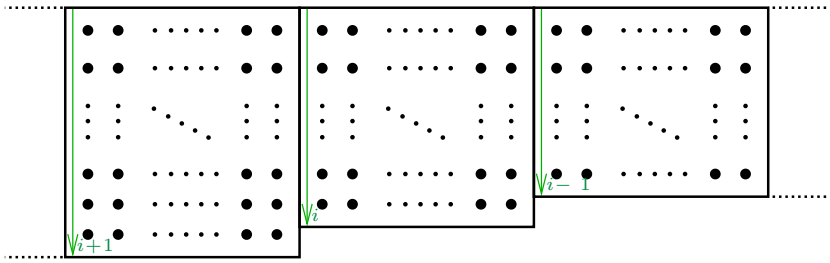
$$\leqslant \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!} \Big\{ \mathsf{const}_n \big(\log \kappa\big)^{\alpha_n} + \mathsf{O}_n \big(\log \kappa\big)^{\alpha_n - 1} \Big\} + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).$$

- **Difficulté principale :** Les multiplicités $M^{\kappa,m}_{\ell_1,\dots,\ell_n}$!

$$(*) \qquad \begin{aligned} &\sum_{\substack{\mathsf{YT\,semi-standard} \\ \mathsf{poids}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha_{n-1}} \big(\ell_n(\mathsf{YT})\big)^{\alpha_n} \\ &= \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big) + \\ &+ \frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!} \sum_{\mu^i_l \in \nabla_{n,\kappa}} \frac{N^{\kappa}_{\mu^1_1}}{\kappa \cdots \mu^1_1} \frac{N^{\mu^1_1,\kappa}_{\mu^2_1,\mu^2_2}}{(\kappa+\mu^1_1)\cdots(\mu^2_2+\mu^2_1)} \cdots \\ &\qquad \cdots \frac{N^{\mu^{n-1}_1,\dots,\mu^{n-1}_{n-1},\kappa}_{\mu^n_1,\dots,\mu^n_{n-1},\mu^n_n}}{(\kappa+\mu^{n-1}_{n-1}+\cdots+\mu^{n-1}_1)\cdots(\mu^n_n+\mu^n_{n-1}+\cdots+\mu^n_1)} \cdot \\ &\alpha_1! \cdots \alpha_n! \cdot \sum_{q^1_0+\cdots+q^1_{\tau^1}=\alpha_1} \cdots \sum_{q^n_0+\cdots+q^n_{\tau^n}=\alpha_n} \\ &\Big(\prod_{0 \leqslant s^1 \leqslant \tau^1} \frac{1}{\big(s^1+\mu^1_1\big)^{q^1_{s^1}}} \cdots \prod_{0 \leqslant s^n \leqslant \tau^n} \frac{1}{\big(s^n+\mu^n_1+\cdots+\mu^n_n\big)^{q^n_{s^n}}} \Big) \end{aligned} .$$

- **Sous-multiplicités :** $N^{\mu^{i-1}_1,\dots,\mu^{i-1}_{i-1},\kappa}_{\mu^i_1,\dots,\mu^{i-1}_i,\mu^i_i}$.

- **Visionnement des blocs de Young décroissants :**



- **Colonnes-extrémités d'un bloc semi-standard :**

$$\boxed{\begin{matrix} \lambda_1^{1+\ell_{i+1}} & \cdots & \lambda_1^{\ell_i} \\ \lambda_2^{1+\ell_{i+1}} & \cdots & \lambda_2^{\ell_i} \\ .. & .. & .. \\ \lambda_i^{1+\ell_{i+1}} & \cdots & \lambda_i^{\ell_i} \end{matrix}} = \boxed{\begin{bmatrix} \mu_1^j \\ \mu_2^j \\ .. \\ \mu_i^j \end{bmatrix}^{a_{\mu_1^j,\mu_2^j,\ldots,\mu_i^j}} \cdots \begin{bmatrix} \lambda_1^j \\ \lambda_2^j \\ .. \\ \lambda_i^j \end{bmatrix}^{a_{\lambda_1^j,\lambda_2^j,\ldots,\lambda_i^j}} \cdots \begin{bmatrix} \nu_1^j \\ \nu_2^j \\ .. \\ \nu_i^j \end{bmatrix}^{a_{\nu_1^j,\nu_2^j,\ldots,\nu_i^j}}}.$$

- **Colonnes distinctes :**

$$\begin{bmatrix} \mu_1^j \\ \mu_2^j \\ .. \\ \mu_i^j \end{bmatrix} < \cdots < \begin{bmatrix} \lambda_1^j \\ \lambda_2^j \\ .. \\ \lambda_i^j \end{bmatrix} < \cdots < \begin{bmatrix} \nu_1^j \\ \nu_2^j \\ .. \\ \nu_i^j \end{bmatrix},$$

- **Synthèse diagrammatique :**

$$\boxed{\begin{bmatrix} \mu_1^{d_1} \\ \mu_2^{d_1} \\ \mu_3^{d_1} \\ . \\ . \\ \mu_{d_1-1}^{d_1} \\ \mu_{d_1}^{d_1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1} \\ \nu_2^{d_1} \\ \nu_3^{d_1} \\ . \\ . \\ \nu_{d_1-1}^{d_1} \\ \nu_{d_1}^{d_1} \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^{d_1-1} \\ \mu_2^{d_1-1} \\ \mu_3^{d_1-1} \\ . \\ . \\ \mu_{d_1-1}^{d_1-1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1-1} \\ \nu_2^{d_1-1} \\ \nu_3^{d_1-1} \\ . \\ . \\ \nu_{d_1-1}^{d_1-1} \end{bmatrix}^*} \cdots \boxed{\begin{bmatrix} \mu_1^3 \\ \mu_2^3 \\ \mu_3^3 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^3 \\ \nu_2^3 \\ \nu_3^3 \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^2 \\ \mu_2^2 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^2 \\ \nu_2^2 \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^1 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^1 \end{bmatrix}^*}.$$

- **Visionner le voisinage entre blocs généraux :**

$$\cdots \boxed{\begin{bmatrix} \mu_1^{i+1} \\ \mu_2^{i+1} \\ \cdot \\ \cdot \\ \mu_i^{i+1} \\ \mu_{i+1}^{i+1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{i+1} \\ \nu_2^{i+1} \\ \cdot \\ \cdot \\ \nu_i^{i+1} \\ \nu_{i+1}^{i+1} \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^{i} \\ \mu_2^{i} \\ \cdot \\ \cdot \\ \mu_i^{i} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{i} \\ \nu_2^{i} \\ \cdot \\ \cdot \\ \nu_i^{i} \end{bmatrix}^*} \cdots ,$$

- **Chemins de longueur maximale :**

$$\underbrace{\boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ n-1 \\ n \end{bmatrix}^* \cdots \begin{bmatrix} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ n-1 \\ \kappa \end{bmatrix}^*}}_{\kappa-n+1} \underbrace{\boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ n-1 \end{bmatrix}^* \cdots \begin{bmatrix} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ \kappa \end{bmatrix}^*}}_{\kappa-n+2} \cdots \underbrace{\boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^* \cdots \begin{bmatrix} 1 \\ 2 \\ \kappa \end{bmatrix}^*}}_{\kappa-2} \underbrace{\boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}^* \cdots \begin{bmatrix} 1 \\ \kappa \end{bmatrix}^*}}_{\kappa-1} \underbrace{\boxed{\begin{bmatrix} 1 \end{bmatrix}^* \cdots \begin{bmatrix} \kappa \end{bmatrix}^*}}_{\kappa} .$$

- **Wronskien n-dimensionnel :**

$$\Delta_{1,2,3,\ldots,n-1,n}^{1,2,3,\ldots,n-1,n} = \begin{vmatrix} f_1' & f_2' & f_3' & \cdots & f_{n-1}' & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_{n-1}'' & f_n'' \\ f_1''' & f_2''' & f_3''' & \cdots & f_{n-1}''' & f_n''' \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_{n-1}^{(n-1)} & f_n^{(n-1)} \\ f_1^{(n)} & f_2^{(n)} & f_3^{(n)} & \cdots & f_{n-1}^{(n)} & f_n^{(n)} \end{vmatrix}$$

- **Multiplicités et longueurs :**

$$\begin{cases} \ell_n = a_{1,2,3,\ldots,n-1,n} + \cdots + a_{1,2,3,\ldots,n-1,\kappa} \\ \ell_{n-1} - \ell_n = a_{1,2,3,\ldots,n-1} + \cdots + a_{1,2,3,\ldots,\kappa} \\ \cdot\cdot \qquad \cdot\cdot = \cdots\cdots\cdots\cdots\cdots\cdots \\ \ell_3 - \ell_4 = a_{1,2,3} + \cdots + a_{1,2,\kappa} \\ \ell_2 - \ell_3 = a_{1,2} + \cdots + a_{1,\kappa} \\ \ell_1 - \ell_2 = a_1 + \cdots + a_\kappa. \end{cases}$$

• Contrainte de poids :

$$\begin{aligned} m = {} & \big[1+2+3+\cdots+n-1+n\big]a_{1,2,3,\ldots,n-1,n} + \cdots + \big[1+2+3+\cdots+n-1+\kappa\big]a_{1,2,3,\ldots,n-1,\kappa} + \\ & + \big[1+2+3+\cdots+n-1\big]a_{1,2,3,\ldots,n-1} + \cdots + \big[1+2+3+\cdots+\kappa\big]a_{1,2,3,\ldots,\kappa} + \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ & + \big[1+2+3\big]a_{1,2,3} + \cdots + \big[1+2+\kappa\big]a_{1,2,\kappa} + \\ & + \big[1+2\big]a_{1,2} + \cdots + \big[1+\kappa\big]a_{1,\kappa} + \\ & + \big[1\big]a_1 + \cdots + \big[\kappa\big]a_\kappa; \end{aligned}$$

• Nombre maximal de colonnes distinctes :

$$(*) \qquad \underbrace{\boxed{\begin{bmatrix} \mu_1^{d_1} \\ \mu_2^{d_1} \\ \mu_3^{d_1} \\ \cdot \\ \cdot \\ \mu_{d_1-1}^{d_1} \\ \mu_{d_1}^{d_1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1} \\ \nu_2^{d_1} \\ \nu_3^{d_1} \\ \cdot \\ \cdot \\ \nu_{d_1-1}^{d_1} \\ \nu_{d_1}^{d_1} \end{bmatrix}^*}}_{\#\leqslant D_{d_1}} \underbrace{\boxed{\begin{bmatrix} \mu_1^{d_1-1} \\ \mu_2^{d_1-1} \\ \mu_3^{d_1-1} \\ \cdot \\ \cdot \\ \mu_{d_1-1}^{d_1-1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1-1} \\ \nu_2^{d_1-1} \\ \nu_3^{d_1-1} \\ \cdot \\ \cdot \\ \nu_{d_1-1}^{d_1-1} \end{bmatrix}^*}}_{\#\leqslant D_{d_1-1}} \cdots \underbrace{\boxed{\begin{bmatrix} \mu_1^3 \\ \mu_2^3 \\ \mu_3^3 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^3 \\ \nu_2^3 \\ \nu_3^3 \end{bmatrix}^*}}_{\#\leqslant D_3} \underbrace{\boxed{\begin{bmatrix} \mu_1^2 \\ \mu_2^2 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^2 \\ \nu_2^2 \end{bmatrix}^*}}_{\#\leqslant D_2} \underbrace{\boxed{\begin{bmatrix} \mu_1^1 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^1 \end{bmatrix}^*}}_{\#\leqslant D_1} .$$

Lemme. *Le nombre total de colonnes distinctes deux à deux dans un tableau semi-standard de profondeur $\leqslant n$ rempli par des entiers $\lambda_i^j \leqslant \kappa$ est toujours $\leqslant n\kappa - \frac{n(n+1)}{2}$.*

• **Chemins entre colonnes :**

$$\gamma^i\colon\ \{0,1,2,\dots,\tau^i\} \longrightarrow \text{colonnes décroissantes} \in \{1,\dots,\kappa\}^i$$

• **Inégalités strictes :**

$$\begin{bmatrix}\mu_1^i=\gamma_1^i(0)\\ \mu_2^i=\gamma_2^i(0)\\ ..\\ \mu_i^i=\gamma_i^i(0)\end{bmatrix}^* < \begin{bmatrix}\gamma_1^i(1)\\ \gamma_2^i(1)\\ .\\ \gamma_i^i(1)\end{bmatrix}^* < \cdots < \begin{bmatrix}\gamma_1^i(s^i)\\ \gamma_2^i(s^i)\\ .\\ \gamma_i^i(s^i)\end{bmatrix}^* < \cdots < \begin{bmatrix}\gamma_1^i(\tau^i)=\nu_1^i\\ \gamma_2^i(\tau^i)=\nu_2^i\\ ..\\ \gamma_i^i(\tau^i)=\nu_i^i\end{bmatrix}^* .$$

• **Bloc associé :**

$$\mathsf{block}^i\big(\gamma^i\big) := \boxed{\begin{bmatrix}\gamma_1^i(0)\\ \gamma_2^i(0)\\ .\\ \gamma_i^i(0)\end{bmatrix}^* \begin{bmatrix}\gamma_1^i(1)\\ \gamma_2^i(1)\\ .\\ \gamma_i^i(1)\end{bmatrix}^* \cdots \begin{bmatrix}\gamma_1^i(s^i)\\ \gamma_2^i(s^i)\\ .\\ \gamma_i^i(s^i)\end{bmatrix}^* \cdots \begin{bmatrix}\gamma_1^i(\tau^i)\\ \gamma_2^i(\tau^i)\\ .\\ \gamma_i^i(\tau^i)\end{bmatrix}^*}$$

• **Familles asymptotiquement négligeables de Δ-monômes :**

Proposition. *Pour tous entiers $\alpha_1,\dots,\alpha_{n-1},\alpha_n$ dont la somme est égale à $\frac{n(n+1)}{2}$, la contribution de :*

$$\sum_{\mathsf{YT}\in\mathscr{N}\mathscr{G}_{\kappa,m}} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n} \leqslant$$
$$\leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{(\kappa+1)n-2}$$

est asymptotiquement négligeable en comparaison à $m^{(\kappa+1)n-1}$.

• **Familles significatives de Δ-monômes.** Ainsi, il reste à étudier la collection de toutes les familles de tableaux

semi-standard :

$$\mathscr{YT}^{\max}_{\kappa,m} := \bigcup_{\substack{\mu^i_l,\nu^i_l,\tau^i,\gamma^i(s^i) \\ \sum_{i=1}^n (1+\tau^i) = n\kappa - \frac{n(n-1)}{2}}} \mathscr{YT}_{\kappa,m}\big(\mu^i_l,\nu^i_l,\tau^i,\gamma^i(s^i)\big)$$

dont le nombre de colonnes distinctes deux à deux est maximal, égal à $n\kappa - \frac{n(n-1)}{2}$. En voici une description :

Proposition. *Le nombre D de colonnes deux à deux distinctes dans un tableau semi-standard tel que représenté dans la « synthèse diagrammatique » ci-dessus est toujours $\leqslant n\kappa - \frac{n(n-1)}{2}$. De plus, un tableau semi-standard donné atteint le nombre maximal :*

$$D = n\kappa - \frac{n(n-1)}{2}$$

de colonnes distinctes deux à deux si et seulement si les conditions suivantes sont toutes satisfaites :

- *la profondeur du tableau est maximale : $d_1 = n$;*
- *des blocks non vides de toute profondeur $i = 1, 2, 3, \ldots, n-1, n$ existent, de telle sorte que le nombre de ces block est maximal, égal à n ;*
- *la $*$-colonne extrême-gauche du tableau correspond au Wronskien $\Delta^{1,2,3,\ldots,n-1,n}_{1,2,3,\ldots,n-1,n}$ reproduit un certain nombre $* \geqslant 1$ de fois ;*
- *la composante inférieure droite de chaque block est maximale :*

$$\nu^1_1 = \nu^2_2 = \nu^3_3 = \cdots = \nu^{n-1}_{n-1} = \nu^n_n = \kappa;$$

$$\underbrace{\boxed{\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\\n\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^{n-1}\\\mu_2^{n-1}\\\mu_3^{n-1}\\\cdot\\\cdot\\\mu_{n-1}^{n-1}\\\kappa\end{bmatrix}^*}}_{1+\tau^n}\ \underbrace{\boxed{\begin{bmatrix}\mu_1^{n-1}\\\mu_2^{n-1}\\\mu_3^{n-1}\\\cdot\\\cdot\\\mu_{n-1}^{n-1}\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^{n-2}\\\mu_2^{n-2}\\\mu_3^{n-2}\\\cdot\\\cdot\\\kappa\end{bmatrix}^*}}_{1+\tau^{n-1}} \cdots \underbrace{\boxed{\begin{bmatrix}\mu_1^3\\\mu_2^3\\\mu_3^3\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^2\\\mu_2^2\\\kappa\end{bmatrix}^*}}_{1+\tau^3}\ \underbrace{\boxed{\begin{bmatrix}\mu_1^2\\\mu_2^2\end{bmatrix}^* \cdots \begin{bmatrix}\mu_1^1\\\kappa\end{bmatrix}^*}}_{1+\tau^2}\ \underbrace{\boxed{\begin{bmatrix}\mu_1^1\end{bmatrix}^* \cdots \begin{bmatrix}\kappa\end{bmatrix}^*}}_{1+\tau^1}.$$

- *les entrées de toute paire de colonnes bordantes (sauf la dernière de la plus longue des deux colonnes, égale à κ) sont égales :*

$$\cdots \boxed{\begin{bmatrix}\mu_1^{i+1}\\\mu_2^{i+1}\\\cdot\\\cdot\\\mu_{i-1}^{i+1}\\\mu_i^{i+1}\\\mu_{i+1}^{i+1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{i+1}=\mu_1^i\\\nu_2^{i+1}=\mu_2^i\\\cdot\\\cdot\\\nu_{i-1}^{i+1}=\mu_{i-1}^i\\\nu_i^{i+1}=\mu_i^i\\\kappa\end{bmatrix}^*} \boxed{\begin{bmatrix}\mu_1^i\\\mu_2^i\\\cdot\\\cdot\\\mu_{i-1}^i\\\mu_i^i\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^i=\mu_1^{i-1}\\\nu_2^i=\mu_2^{i-1}\\\cdot\\\cdot\\\nu_{i-1}^i=\mu_{i-1}^{i-1}\\\kappa\end{bmatrix}^*} \boxed{\begin{bmatrix}\mu_1^{i-1}\\\mu_2^{i-1}\\\cdot\\\cdot\\\mu_{i-1}^{i-1}\end{bmatrix}^* \cdots \begin{bmatrix}\nu_1^{i-1}=\mu_1^{i-2}\\\nu_2^{i-1}=\mu_2^{i-2}\\\cdot\\\cdot\\\kappa\end{bmatrix}^*} \cdots,$$

- *le nombre de colonnes distinctes dans chaque bloc de profondeur i, pour $i = 1, 2, 3, \ldots, n-1, n$, est maximale, égal à :*

$$\begin{aligned}1+\tau^i :=& 1+(\mu_1^{i-1}-\mu_1^i)+(\mu_2^{i-1}-\mu_2^i)+\cdots+(\mu_{i-1}^{i-1}-\mu_{i-1}^i)+(\kappa-\mu_i^i)\\ =& 1+\kappa+\sum_{l=1}^{i-1}\mu_l^{i-1}-\sum_{l=1}^{i}\mu_l^i,\end{aligned}$$

de telle sorte que le nombre total de colonne distinctes deux à deux est donc effectivement égal à :

$$\begin{aligned}(1+\tau^1)+(1+\tau^2)+(1+\tau^3)+\cdots+(1+\tau^{n-1})+(1+\tau^n) &= n+n\kappa-\sum_{l=1}^{n}\mu_l^n\\ &= n\kappa-\tfrac{n(n-1)}{2}.\end{aligned}$$

Grande endurance requise dans les démonstrations

VII – Spéculations prospectives

• **Apparition de poly-logarithmes et de constantes d'Euler généralisées.**

« **Mer intérieur** » de calculs fins possibles.

• **Décider d'une « charnière critique » entre l'Algèbre et l'Analyse :** En quel endroit remplacer les calculs algébriques explicites complets par des majorations d'analyste ?

• **Grande liberté de choix pour l'« effort en calcul » !**

• **Décroissance conjecturale des dimensions cohomologiques :** En fait, puisque la dualité de Serre et un théorème d'annulation connu assurent que la contribution à la caractéristique de la dernière dimension cohomologique :

$$h^n\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) = \dim H^n\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big)$$

s'annule, on pourrait même s'attendre à ce que les h^i satisfassent de **meilleures inégalités**, lorsque i augmente de 1 à n, que celles qui sont conjecturées.

Fin des méditations métaphysiques pour aujourd'hui