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(Tentative title)

## THEORIE

DER

# TRANSFORMATIONSGRUPPEN 

DRITTER UND LETZTER ABSCHNITT

UNTER MITWIRKUNG

VON
Prof. Dr. FRIEDRICH ENGEL

BEARBEITET

VON

## SOPHUS LIE,

PROFESSOR DER GEOMETRIE AND DER UNIVERSITÄT LEIPZIG
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## Chapter H

## Invariant Families of Infinitesimal Transformations

## A Priori Linear Dependence Relations

We study in this chapter the general linear combination:

$$
e_{1} X_{1}+\cdots+e_{q} X_{q}
$$

of $q \geqslant 1$ given arbitrary local infinitesimal transformations:

$$
X_{k}=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial}{\partial x_{i}} \quad(k=1 \cdots q)
$$

having analytic coefficients $\xi_{k i}(x)$ and which we assume to be independent of each other. When one introduces new variables $x_{i}^{\prime}=\varphi_{i}\left(x_{1}, \ldots, x_{n}\right)$ in place of the $x_{l}$, every transformation $X_{k}$ of our family receives another form, but it may sometimes happen under certain circumstances that the complete family in its wholeness remains unchanged, namely that there are functions $e_{k}^{\prime}=e_{k}^{\prime}\left(e_{1}, \ldots, e_{q}\right)$ such that:

$$
\varphi_{*}\left(e_{1} X_{1}+\cdots+e_{q} X_{q}\right)=e_{1}^{\prime}(e) X_{1}^{\prime}+\cdots+e_{q}^{\prime}(e) X_{q}^{\prime},
$$

where, as usual, the $X_{k}^{\prime}=\sum_{i=1}^{n} \xi_{k i}\left(x^{\prime}\right) \frac{\partial}{\partial x_{i}^{\prime}}$ denote the same vector fields, viewed in the target space $x_{1}^{\prime}, \ldots, x_{n}^{\prime}$. Without loss of generality because we work locally, it is assumed implicitly that the diffeomorphism $\varphi$ is close to the identity map.
Definition. The family $e_{1} X_{1}+\cdots+e_{q} X_{q}$ of infinitesimal transformations is said to remain invariant after the introduction of the new variables $x^{\prime}=\varphi(x)$ if there are functions $e_{k}^{\prime}=e_{k}^{\prime}\left(e_{1}, \ldots, e_{q}\right)$ depending on $\varphi$ such that:

$$
\begin{equation*}
\varphi_{*}\left(e_{1} X_{1}+\cdots+e_{q} X_{q}\right)=e_{1}^{\prime}(e) X_{1}^{\prime}+\cdots+e_{q}^{\prime}(e) X_{q}^{\prime} \tag{1}
\end{equation*}
$$

alternately, one says that the family admits the transformation which is represented by the concerned change of variables.

Proposition. Then the functions $e_{k}^{\prime}(e)$ in question necessarily are linear:

$$
e_{k}^{\prime}=\sum_{j=1}^{q} \rho_{k j} \cdot e_{j} \quad(k=1 \cdots q),
$$

with the constant matrix $\left(\rho_{k j}\right)_{1 \leqslant k \leqslant q}^{1 \leqslant j \leqslant q}$ being invertible: $e_{k}=\sum_{j=1}^{q} \widetilde{\rho}_{k j} \cdot e_{j}^{\prime}$.

Proof. Indeed, through the change of coordinates $x^{\prime}=\varphi(x)$, if we write that the vector fields $X_{k}$ are transferred to:

$$
\varphi_{*}\left(X_{k}\right)=\sum_{i=1}^{n} X_{k}\left(x_{i}^{\prime}\right) \frac{\partial}{\partial x_{i}^{\prime}}=: \sum_{i=1}^{n} \eta_{k i}\left(x_{1}^{\prime}, \ldots, x_{n}^{\prime}\right) \frac{\partial}{\partial x_{i}^{\prime}} \quad(k=1 \cdots q)
$$

with their coefficients $\eta_{k i}=\eta_{k i}\left(x^{\prime}\right)$ being expressed in terms of the target coordinates, and if we substitute the resulting expression into (1), we get the following linear relations:

$$
\begin{equation*}
\sum_{k=1}^{q} e_{k}^{\prime} \cdot \xi_{k i}\left(x^{\prime}\right)=\sum_{k=1}^{q} e_{k} \cdot \eta_{k i}\left(x^{\prime}\right) \quad(i=1 \cdots n) \tag{1’}
\end{equation*}
$$

The idea is to substitute here for $x^{\prime}$ exactly the same number $q$ of different systems of fixed values:

$$
x_{1}^{(1)}, \ldots, x_{n}^{(1)}, x_{1}^{(2)}, \ldots, x_{n}^{(2)}, \ldots \ldots, x_{1}^{(q)}, \ldots, x_{n}^{(q)}
$$

that are mutually in general position and considered will be considered as constant. In fact, according to the proposition on p. ??, or equivalently, according to the assertion formulated just below the long matrix located on p. ??, the linear independence of $X_{1}, \ldots, X_{q}$ insures that for most such $q$ points, the long $q \times q n$ matrix in question:

$$
\left(\begin{array}{cccccccccc}
\xi_{11}^{(1)} & \cdots & \xi_{1 n}^{(1)} & \xi_{11}^{(2)} & \cdots & \xi_{1 n}^{(2)} & \cdots \cdots & \xi_{11}^{(q)} & \cdots & \xi_{1 n}^{(q)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \cdots & \cdots & \cdots & \cdots \\
\xi_{q 1}^{(1)} & \cdots & \xi_{q n}^{(1)} & \xi_{q 1}^{(2)} & \cdots & \xi_{q n}^{(2)} & \cdots \cdots & \xi_{q 1}^{(q)} & \cdots & \xi_{q n}^{(q)}
\end{array}\right)
$$

has rank equal to $q$, where we have set $\xi_{k i}^{(\nu)}:=\xi_{x i}\left(x^{(\nu)}\right)$. Consequently, while considering the values of $\xi_{k i}\left(x^{(\nu)}\right)$ and of $\eta_{k i}\left(x^{(\nu)}\right)$ as constant, the linear system above is solvable with respect to the unknowns $e_{k}^{\prime}$ and we obtain:

$$
e_{k}^{\prime}=\sum_{j=1}^{q} \rho_{k j} \cdot e_{j} \quad(k=1 \cdots q)
$$

for some constants $\rho_{k j}$. In addition, we claim that the determinant of the matrix $\left(\rho_{k j}\right)_{1 \leqslant k \leqslant q}^{1 \leqslant j \leqslant q}$ is in fact nonzero. Indeed, the linear independence of $X_{1}, \ldots, X_{q}$ being obviously equivalent to the linear independence of $\varphi_{*}\left(X_{1}\right), \ldots, \varphi_{*}\left(X_{q}\right)$, the other corresponding long matrix:

$$
\left(\begin{array}{cccccccccc}
\eta_{11}^{(1)} & \cdots & \eta_{1 n}^{(1)} & \eta_{11}^{(2)} & \cdots & \eta_{1 n}^{(2)} & \cdots \cdots & \eta_{11}^{(q)} & \cdots & \eta_{1 n}^{(q)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \cdots & \cdots & \cdots & \cdots \\
\eta_{q 1}^{(1)} & \cdots & \eta_{q n}^{(1)} & \eta_{q 1}^{(2)} & \cdots & \eta_{q n}^{(2)} & \cdots \cdots & \eta_{q 1}^{(q)} & \cdots & \eta_{q n}^{(q)}
\end{array}\right)
$$

then also has rank equal to $q$ and we therefore can also solve symmetrically:

$$
e_{k}=\sum_{j=1}^{q} \widetilde{\rho}_{k j} \cdot e_{j}^{\prime} \quad(k=1 \cdots q),
$$

with coefficients $\widetilde{\rho}_{k j}$ which necessarily coincide with the elements of the inverse matrix.

## Families Invariant through One-Term Subgroups

For an important application to the study of the adjoint group in Chap. L, we now want to study families $e_{1} X_{1}+\cdots+e_{q} X_{q}$ that are invariant when the transition from $x$ to a new variable $x^{\prime}$ is performed by an arbitrary transformation of some one-term group $x^{\prime}=\exp (t Y)(x)$, where $Y$ is any (local, analytic) vector field. Here, we can just translate the presentation of [25], pp. 249-253, since it does not demand any adaptation.

Under which conditions does the family $\sum e_{k} X_{k} f$ remain invariant through every transformation $x_{i}^{\prime}=f_{i}\left(x_{1}, \ldots, x_{n}, t\right)$ of the one-term group $Y f$, that is to say, under which conditions does a relation:

$$
\sum_{k=1}^{q} e_{k} \cdot X_{k} f=\sum_{k=1}^{q} e_{k}^{\prime} \cdot X_{k}^{\prime} f,
$$

hold for all systems of values $e_{1}, \ldots e_{q}$, t, in which the $e_{k}^{\prime}$, aside from the $e_{j}$, yet only depend upon $t$ ?

When, in order to introduce new variables in $X_{k} f$, we apply the general transformation:

$$
x_{i}^{\prime}=x_{i}+t \cdot Y x_{i}+\cdots \quad \quad(i=1 \cdots n)
$$

of the one-term group $Y f$, we obtain according to Chap. 8, p. 141, formula (5) [here: lemma on $p$. ??]:

$$
X_{k} f=X_{k}^{\prime} f+t\left(X_{k}^{\prime} Y^{\prime} f-Y^{\prime} X_{k}^{\prime} f\right)+\cdots ;
$$

hence also inversely:

$$
\begin{equation*}
X_{k}^{\prime} f=X_{k} f+t\left[Y, X_{k}\right]+\cdots, \tag{3}
\end{equation*}
$$

which is convenient for what follows.
Now, if every infinitesimal transformation $X_{k} f+t\left[Y, X_{k}\right]+\cdots$ shall belong to the family $e_{1} X_{1} f+\cdots+e_{q} X_{q} f$, and in fact so for every value of $t$, then obviously every infinitesimal transformation $\left[Y, X_{k}\right]$ must also be contained in this family. As a result, certain necessary conditions for the invariance of our family would be found, conditions which amount to
the fact that $q$ relations of the form:

$$
\begin{equation*}
\left[Y, X_{k}\right]=\sum_{j=1}^{q} g_{k j} \cdot X_{j} f \quad(k=1 \cdots q) \tag{4}
\end{equation*}
$$

should hold, in which the $g_{k j}$ denote absolute constants.
If the family of the infinitesimal transformations:

$$
e_{1} \cdot X_{1} f+\cdots+e_{q} \cdot X_{q} f
$$

is constituted so that for every $k$, a relation of the form (4) holds true, then we want to say that the family admits the infinitesimal transformation $Y f$. By this fixing of terminology, we can state as follows the result just obtained:

If the family of the infinitesimal transformations:

$$
e_{1} \cdot X_{1} f+\cdots+e_{q} \cdot X_{q} f
$$

admits all transformations of the one-term group $Y f$, then it also admits the infinitesimal transformation $Y f$.

But the converse too holds true, as we will now show.
We want to suppose that the family of the transformations $\sum e_{k} X_{k} f$ admits the infinitesimal transformation $Y f$, hence that relations of the form (4) hold true. If now the family $\sum e_{k} X_{k} f$ shall simultaneously admit all finite transformations of the one-term group $Y f$, then it must be possible to determine $e_{1}^{\prime}, \ldots, e_{q}^{\prime}$ as functions of $e_{1}, \ldots, e_{q}$ in such a way that the equation:

$$
\sum_{k=1}^{q} e_{k}^{\prime} \cdot X_{k}^{\prime} f=\sum_{k=1}^{q} e_{k} \cdot X_{k} f
$$

is identically satisfied, as soon as one introduces the variable $x$ in place of $x^{\prime}$ in the $X_{k}^{\prime} f$. Consequently, if $X_{k}^{\prime} f$ takes the form:

$$
X_{k}^{\prime} f=\sum_{i=1}^{n} \zeta_{k i}\left(x_{1}, \ldots, x_{n}, t\right) \frac{\partial}{\partial x_{i}}
$$

after the introduction of the $x$, then the $e_{k}^{\prime}$ must be determined so that the expression:

$$
\sum_{k=1}^{q} e_{k}^{\prime} \cdot X_{k}^{\prime} f=\sum_{k=1}^{q} \sum_{i=1}^{n} e_{k}^{\prime} \cdot \zeta_{k i}\left(x_{1}, \ldots, x_{n}, t\right) \frac{\partial f}{\partial x_{i}}
$$

is free of $t$, hence so that the differential quotient:

$$
\frac{\partial}{\partial t} \sum_{k=1}^{q} e_{k}^{\prime} \cdot X_{k}^{\prime} f=\sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}} \cdot \frac{\partial}{\partial t} \sum_{k=1}^{q} e_{k}^{\prime} \zeta_{k i}\left(x_{1}, \ldots, x_{n}, t\right)
$$

vanishes [indeed, differentiation with respect to $t$ of $e_{1} X_{1}+\cdots+e_{r} X_{r}$ yields $\left.0 \equiv \frac{\partial}{\partial t} \sum_{k=1}^{q} e_{k} X_{k}\right]$; but at the same time, the $e$ must still also satisfy the initial condition: $e_{k}^{\prime}=e_{k}$ for $t=0$.

In order to be able to show that under the assumptions made there really are functions $e^{\prime}$ of the required constitution, we must at first calculate the differential quotient:

$$
\frac{\partial}{\partial t} X_{k}^{\prime} f=\sum_{i=1}^{n} \frac{\partial \zeta_{k i}\left(x_{1}, \ldots, x_{n}, t\right)}{\partial t} \frac{\partial f}{\partial x_{i}}
$$

for this, we shall take an indirect route.
Above, we saw that $X_{k}^{\prime} f$ can be expressed in the following way in terms of $x_{1}, \ldots, x_{n}$ and $t$ :

$$
X_{k}^{\prime} f=X_{k} f+t\left[Y, X_{k}\right]+\cdots,
$$

when the independent variables $x^{\prime}$ entering the $X_{k}^{\prime}$ are determined by the equations $x_{i}^{\prime}=f_{i}\left(x_{1}, \ldots, x_{n}, t\right)$ of the one-term group $Y f$. So the desired differential quotient obtains by differentiation of the infinite power series in $t$ lying in the right-hand side, or differently enunciated: it is the coefficient of $\tau_{1}$ in the expansion of the expression:

$$
X_{k} f+(t+\tau)\left[Y, X_{k}\right]+\cdots=\sum_{i=1}^{n} \xi_{k i}\left(x_{1}^{\prime \prime}, \ldots, x_{n}^{\prime \prime}\right) \frac{\partial f}{\partial x_{i}^{\prime \prime}}=X_{k}^{\prime \prime} f
$$

with respect to powers of $\tau$. Here, the $x^{\prime \prime}$ mean the quantities:

$$
x_{i}^{\prime \prime}=f_{i}\left(x_{1}, \ldots, x_{n}, t+\tau\right) .
$$

However, the expansion coefficient [Entwickelungscoefficient] discussed just above appears at first as an infinite series of powers of $t$; but there is no difficulty to find a finite closed expression for it.

As we know, the transition from the variables $x$ to the variables $x_{i}^{\prime}=f_{i}\left(x_{1}, \ldots, x_{n}, t\right)$ occurs through a transformation of the one-term group $Y f$, and to be precise, through a transformation with the parameter $t$. One comes from the $x$ to the $x_{i}^{\prime \prime}=f_{i}\left(x_{1}, \ldots, x_{n}, t+\tau\right)$ through a transformation of the same group, namely through the transformation with the parameter $t+\tau$. But this transformation can be substituted for the succession of two transformations, of which the first possesses the parameter $t$, and the second the parameter $\tau$; consequently, the transition from the $x^{\prime}$ to
|the $x^{\prime \prime}$ is likewise got through a transformation of the one-term group $Y f$, namely through the transformation whose paramter is $\tau$ :

$$
x_{i}^{\prime \prime}=f_{i}\left(x_{1}^{\prime}, \ldots, x_{n}^{\prime}, \tau\right)
$$

From this, we conclude that the series expansion of $X_{k}^{\prime \prime} f$ with respect to powers of $\tau$ reads:

$$
X_{k}^{\prime \prime} f=X_{k}^{\prime} f+\tau\left[Y^{\prime}, X_{k}^{\prime}\right]+\cdots
$$

As a result, we have found a finite closed expression for the expansion coefficient mentioned a short while ago; the sought differential quotient $\frac{\partial\left(X_{k}^{\prime} f\right)}{\partial t}$ is hence:

$$
\begin{equation*}
\frac{\partial}{\partial t} X_{k}^{\prime} f=\left[Y^{\prime}, X_{k}^{\prime}\right]=Y^{\prime} X_{k}^{\prime} f-X_{k}^{\prime} Y^{\prime} f \tag{5}
\end{equation*}
$$

Naturally, this formula holds generally, whatever also one can choose as the two infinitesimal transformations $X_{k} f$ and $Y f$. However, in our specific case, $X_{1} f, \ldots, X_{q} f, Y f$ are not absolutely arbitrary, but they are linked together through the relations (4). So under the assumptions made above, we receive:

$$
\begin{equation*}
\frac{\partial\left(X_{k}^{\prime} f\right)}{\partial t}=\sum_{\nu=1}^{q} g_{k \nu} \cdot X_{\nu}^{\prime} f \quad(k=1 \cdots q) \tag{6}
\end{equation*}
$$

Now, if we form the differential quotient of $\sum e_{k}^{\prime} X_{k}^{\prime} f$ with respect to $t$, we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial t} \sum_{k=1}^{q} e_{k}^{\prime} \cdot X_{k}^{\prime} f & =\sum_{k=1}^{q} \frac{d e_{k}^{\prime}}{d t} X_{k}^{\prime} f+\sum_{k=1}^{q} e_{k}^{\prime} \sum_{\nu=1}^{q} g_{k \nu} \cdot X_{\nu}^{\prime} f \\
& =\sum_{k=1}^{q}\left\{\frac{d e_{k}^{\prime}}{d t}+\sum_{\nu=1}^{q} g_{\nu k} e_{\nu}^{\prime}\right\} X_{k}^{\prime} f .
\end{aligned}
$$

Obviously, this expression vanishes only when the $e_{k}^{\prime}$ satisfy the differential equations:

$$
\begin{equation*}
\frac{d e_{k}^{\prime}}{d t}+\sum_{\nu=1}^{q} g_{\nu k} e_{\nu}^{\prime}=0 \quad(k=1 \cdots q) \tag{7}
\end{equation*}
$$

But from this the $e_{k}^{\prime}$ can be determined as functions of $t$ in such a way that for $t=0$, each $e_{k}^{\prime}$ converts into the corresponding $e_{k}$; in addition, the $e^{\prime}$ are linear homogeneous functions of the $e$.

If one puts the value in question of the $e^{\prime}$ in the expression $\sum e_{k}^{\prime} X_{k}^{\prime} f$ and then returns from the $x^{\prime}$ to the initial variables $x_{1}, \ldots, x_{n}$, then $\sum e_{k}^{\prime} X_{k}^{\prime}$ will be independent of $t$, that is to say, it will be equal to
$\sum e_{k} X_{k} f$. Consequently, the family of the infinitesimal transformations $\sum e_{k} X_{k} f$ effectively remains invariant by the change of variables in question.

As a result, we can state the following theorem:
Theorem 43.*) A family of $\infty^{q-1}$ infinitesimal transformations $e_{1}$. $X_{1} f+\cdots+e_{q} \cdot X_{q} f$ remains invariant, through the introduction of new variables $x^{\prime}$ which are defined by the equations of a one-term group:

$$
x_{i}^{\prime}=x_{i}+t \cdot Y x_{i}+\cdots \quad(i=1 \cdots n),
$$

if and only if between $Y f$ and the $X_{k} f$ there are $q$ relations of the form:

$$
\begin{equation*}
\left[Y, X_{k}\right]=\sum_{\nu=1}^{q} g_{k \nu} \cdot X_{\nu} f \quad(k=1 \cdots q) \tag{4}
\end{equation*}
$$

in which the $g_{k \nu}$ denote constants. If these conditions are fulfilled, then by the concerned change of variables, $\sum e_{k} X_{k} f$ receives the form $\sum e_{k}^{\prime} X_{k}^{\prime} f$, where $e_{1}^{\prime}, \ldots, e_{q}^{\prime}$ determine themselves from the differential equations:

$$
\frac{d e_{k}^{\prime}}{d t}+\sum_{\nu=1}^{q} g_{\nu k} e_{\nu}^{\prime}=0 \quad(k=1 \cdots q)
$$

while taking account of the initial conditions: $e_{k}^{\prime}=e_{k}$ for $t=0$.
*) Lie, Archiv for Mathematik og Naturvidenskab Vol. 3, Christiania 1878.
If one performs the integration of which the preceding theorem speaks, hence determines $e_{1}^{\prime}, \ldots, e_{r}^{\prime}$ from the differential equations:

$$
\frac{d e_{k}^{\prime}}{d t}=-\sum_{\nu=1}^{q} g_{\nu k} e_{\nu}^{\prime} \quad(k=1 \cdots q)
$$

taking as a basis the initial conditions: $e_{k}^{\prime}=e_{k}$ for $t=0$, then one obtains equations of the form:

$$
e_{k}^{\prime}=\sum_{j=1}^{q} d_{k j}(t) \cdot e_{j} \quad(k=1 \cdots q)
$$

It is clear that these equations represent the finite transformations of a certain one-term group, namely the one which is engendered by the infinitesimal transformation:

$$
\sum_{k=1}^{q}\left\{\sum_{\nu=1}^{q} g_{\nu k} e_{\nu}\right\} \frac{\partial f}{\partial e_{k}}
$$

(cf. Chap. 3, pages 47 and 48 [here: Chap. C]).

## Chapter L

## The Adjoint Group

## Fundamental Differential Equations for the Inverse Transformations

According to a fundamental theorem stated on p . ??, a general $r$-term continuous transformation group $x_{i}^{\prime}=f_{i}\left(x ; a_{1}, \ldots, a_{r}\right)$ satisfies partial differential equations: $\frac{\partial f_{i}}{\partial a_{k}}=\sum_{j=1}^{r} \psi_{k j}(a) \cdot \xi_{j i}\left(f_{1}, \ldots, f_{n}\right)$ that are used everywhere in the basic Lie theory. For the study of the adjoint group, we must also know how to write precisely the fundamental differential equations that are satisfied by the group of inverse transformations:

$$
x_{i}=f_{i}\left(x^{\prime} ; \mathbf{i}(a)\right) \quad(i=1 \cdots n),
$$

and this is easy. Following an already known path, we must indeed begin by differentiating these equations with respect to the parameters $a_{k}$ :

$$
\frac{\partial x_{i}}{\partial a_{k}}=\sum_{l=1}^{r} \frac{\partial f_{i}}{\partial a_{l}}\left(x^{\prime} ; \mathbf{i}(a)\right) \frac{\partial \mathbf{i}_{l}}{\partial a_{k}}(a) \quad(i=1 \cdots n ; k=1 \cdots r)
$$

Naturally, we replace here the $\frac{\partial f_{i}}{\partial a_{l}}$ by their values $\sum_{j=1}^{r} \psi_{l j} \cdot \xi_{j i}$ given by the fundamental differential equations, and we obtain a double sum:

$$
\begin{aligned}
\frac{\partial x_{i}}{\partial a_{k}} & =\sum_{l=1}^{r} \sum_{j=1}^{r} \psi_{l j}(\mathbf{i}(a)) \xi_{j i}\left(f\left(x^{\prime} ; \mathbf{i}(a)\right) \frac{\partial \mathbf{i}_{l}}{\partial a_{k}}(a)\right. \\
& =: \sum_{j=1}^{r} \vartheta_{k j}(a) \cdot \xi_{j i}(x) \quad(i=1 \cdots n ; k=1 \cdots r),
\end{aligned}
$$

which we contract to a single sum by simply introducing the following new $r \times r$ auxiliary matrix of parameter functions:

$$
\vartheta_{k j}(a):=\sum_{l=1}^{r} \psi_{l j}(\mathbf{i}(a)) \frac{\partial \mathbf{i}_{l}}{\partial a_{k}}(a) \quad(k, j=1 \cdots r)
$$

whose precise expression will not matter anymore. It now remains to check that this matrix $\left(\vartheta_{k j}(a)\right)_{1 \leqslant k \leqslant r}^{1 \leqslant \leqslant r}$ is invertible for all $a$ in a neighborhood of the identity element $e=\left(e_{1}, \ldots, e_{r}\right)$. We in fact claim that:

$$
\vartheta_{k j}(e)=\delta_{k}^{j},
$$

which will clearly assure the invertibility in question. At first, we remember from Theorem 3 on p. ?? that $\psi_{l j}(e)=-\delta_{l}^{j}$. Thus secondly, it remains now only to check that $\frac{\partial \mathbf{i}_{i}}{\partial a_{k}}(e)=-\delta_{k}^{l}$, a rather known fact.

To check this, we differentiate with respect to $a_{k}$ the trivial identities: $e_{j} \equiv \mathbf{m}_{j}(a, \mathbf{i}(a)), j=1, \ldots, r$, getting:

$$
0 \equiv \frac{\partial \mathbf{m}_{j}}{\partial a_{k}}(e, e)+\sum_{l=1}^{r} \frac{\partial \mathbf{m}_{j}}{\partial b_{l}}(e, e) \frac{\partial \mathbf{i}_{l}}{\partial a_{k}}(e) \quad(j=1 \cdots r)
$$

From another side, by differentiating the two families of $r$ identities $a_{j} \equiv$ $\mathbf{m}_{j}(a, e)$ and $b_{j} \equiv \mathbf{m}_{j}(e, b)$ with respect to $a_{k}$ and with respect to $b_{l}$, we immediately get two expressions:

$$
\frac{\partial \mathbf{m}_{j}}{\partial a_{k}}(e, e)=\delta_{k}^{j} \quad \text { and } \quad \frac{\partial \mathbf{m}_{j}}{\partial b_{l}}(e, e)=\delta_{l}^{j}
$$

which, when inserted just above, yield the announced $\frac{\partial \mathbf{i}_{l}}{\partial a_{k}}(e)=-\delta_{k}^{l}$. Sometimes, we will write $g(x ; a)$ instead of $f(x ; \mathbf{i}(a))$. As a result:
Lemma. The finite continuous transformation group $x_{i}^{\prime}=f_{i}(x ; a)$ and its inverse transformations $x_{i}=g_{i}(x ; a):=f_{i}(x ; \mathbf{i}(a))$ both satisfy fundamental partial differential equations of the form:

$$
\left\{\begin{align*}
\frac{\partial x_{i}^{\prime}}{\partial a_{k}}(x ; a)=\sum_{j=1}^{r} \psi_{k j}(a) \cdot \xi_{j i}\left(x^{\prime}(x ; a)\right) & (i=1 \cdots n ; k=1 \cdots r)  \tag{1}\\
\frac{\partial x_{i}}{\partial a_{k}}\left(x^{\prime} ; a\right)=\sum_{j=1}^{r} \vartheta_{k j}(a) \cdot \xi_{j i}\left(x\left(x^{\prime} ; a\right)\right) & (i=1 \cdots n ; k=1 \cdots r),
\end{align*}\right.
$$

where $\psi$ and $\vartheta$ are some two $r \times r$ matrices of analytic functions with $-\psi_{k j}(e)=\vartheta_{k j}(e)=\delta_{k}^{j}$, and where the functions $\xi_{j i}$ appearing in both systems of equations:

$$
\xi_{j i}(x):=-\frac{\partial f_{i}}{\partial x_{j}}(x ; e) \quad(i=1 \cdots n ; j=1 \cdots r)
$$

are, up to an overall minus sign, just the coefficients of the r infinitesimal transformations

$$
X_{1}^{e}=\frac{\partial f}{\partial a_{1}}(x ; e), \ldots \ldots, X_{r}^{e}=\frac{\partial f}{\partial a_{r}}(x ; e)
$$

obtained by differentiating the finite equations with respect to the parameters at the identity element.

## Transfer of Infinitesimal Transformations by the Group

We now differentiate with respect to $a_{k}$ the identically satisfied equations:

$$
x_{i}^{\prime} \equiv f_{i}\left(g\left(x^{\prime} ; a\right) ; a\right) \quad(i=1 \cdots n),
$$

which just say that an arbitrary transformation of the group followed by its inverse regives the identity transformation, and we immediately get:

$$
0 \equiv \sum_{\nu=1}^{n} \frac{\partial f_{i}}{\partial x_{\nu}} \frac{\partial g_{\nu}}{\partial a_{k}}+\frac{\partial f_{i}}{\partial a_{k}} \quad(i=1 \cdots n ; k=1 \cdots r)
$$

Thanks to the above two systems of partial differential equations, we may then replace $\frac{\partial g_{\nu}}{\partial a_{k}}$ by its value from $(1)_{2}$ and also $\frac{\partial f_{i}}{\partial a_{k}}$ by its value from $(1)_{1}$ :

$$
\begin{gather*}
0 \equiv \sum_{\nu=1}^{n}\left\{\sum_{j=1}^{r} \vartheta_{k j}(a) \xi_{j \nu}(g)\right\} \frac{\partial f_{i}}{\partial x_{\nu}}+\sum_{j=1}^{r} \psi_{k j}(a) \xi_{j i}(f)  \tag{2}\\
(i=1 \cdots n ; k=1 \cdots r)
\end{gather*}
$$

In order to bring these equations to a more symmetric form, following [25] pp. 44-45, we fix $k$ and we multiply, for $i=1$ to $n$, the $i$-th equation by $\frac{\partial}{\partial x_{i}^{\prime}}$, we apply the summation $\sum_{i=1}^{n}$, we use the fact that, through the diffeomorphism $x \mapsto f_{a}(x)=x^{\prime}$, the coordinate vector fields transform as $\frac{\partial}{\partial x_{\nu}}=\sum_{i=1}^{n} \frac{\partial f_{i}}{\partial x_{\nu}} \frac{\partial}{\partial x_{i}^{\prime}}$, which just means in contemporary notation that:

$$
\left(f_{a}\right)_{*}\left(\frac{\partial}{\partial x_{\nu}}\right)=\sum_{i=1}^{n} \frac{\partial f_{i}}{\partial x_{\nu}} \frac{\partial}{\partial x_{i}^{\prime}} \quad(\nu=1 \cdots n)
$$

and we obtain, thanks to this observation, completely symmetric equations:
$0 \equiv \sum_{\nu=1}^{n} \vartheta_{k j}(a) \sum_{\nu=1}^{n} \xi_{j \nu}(x) \frac{\partial}{\partial x_{\nu}}+\sum_{j=1}^{r} \psi_{k j}(a) \sum_{\nu=1}^{r} \xi_{j \nu}\left(x^{\prime}\right) \frac{\partial}{\partial x_{\nu}^{\prime}} \quad(k=1 \ldots r)$.
in which the push-forwards $\left(f_{a}\right)_{*}\left(\frac{\partial}{\partial x_{\nu}}\right)$ are now implicitly understood. It is easy to see that exactly the same equations, but with the opposite pushforwards $\left(g_{a}\right)_{*}\left(\frac{\partial}{\partial x_{\nu}^{\prime}}\right)$, are obtained by subjecting to similar calculations the reverse, identically satisfied equations: $x_{i} \equiv g_{i}(f(x ; a) ; a)$. Consequently, we have obtained two families of equations:

$$
\left\{\begin{align*}
0 & \left.\equiv \sum_{\nu=1}^{n} \vartheta_{k j}(a) \sum_{\nu=1}^{n} \xi_{j \nu}(x) \frac{\partial}{\partial x_{\nu}}\right|_{x \mapsto g_{a}\left(x^{\prime}\right)}+\sum_{j=1}^{r} \psi_{k j}(a) \sum_{\nu=1}^{r} \xi_{j \nu}\left(x^{\prime}\right) \frac{\partial}{\partial x_{\nu}^{\prime}},  \tag{3}\\
0 & \equiv \sum_{\nu=1}^{n} \vartheta_{k j}(a) \sum_{\nu=1}^{n} \xi_{j \nu}(x) \frac{\partial}{\partial x_{\nu}}+\left.\sum_{\substack{j=1 \\
(k=1 \cdots r)}}^{r} \psi_{k j}(a) \sum_{\nu=1}^{r} \xi_{j \nu}\left(x^{\prime}\right) \frac{\partial}{\partial x_{\nu}^{\prime}}\right|_{x^{\prime} \mapsto f_{a}(x)}
\end{align*}\right.
$$

in which we represent push-forwards of vector fields by the sy mbol of variable replacement $x \mapsto g_{a}\left(x^{\prime}\right)$ in the first line, and similarly in the second line, by $x^{\prime} \mapsto f_{a}(x)$.

Synthetic, geometric counterpart of the computations. To formulate the adequate interpretation, we must introduce the two systems of $r$ infinitesimal transformations $(1 \leqslant k \leqslant r)$ :

$$
X_{k}:=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial}{\partial x_{i}} \quad \text { and } \quad X_{k}^{\prime}:=\sum_{i=1}^{n} \xi_{k i}\left(x^{\prime}\right) \frac{\partial}{\partial x_{i}^{\prime}},
$$

where the second ones are defined to be exactly the same vector fields as the first ones, though considered on the $x^{\prime}$-space. This target, auxiliary space $x^{\prime}$ has in fact to be considered to be the same space as the $x$-space, because the considered transformation group acts on a single individual space. So we can also consider that $X_{k}^{\prime}$ coincides with the value of $X_{k}$ at $x^{\prime}$ and we shall sometimes switch to another notation:

$$
\left.X_{k}^{\prime} \equiv X_{k}\right|_{x^{\prime}} .
$$

Letting now $\widetilde{\psi}$ and $\widetilde{\vartheta}$ be the inverse matrices of $\psi$ and of $\vartheta$, namely:

$$
\sum_{k=1}^{r} \widetilde{\psi}_{l k}(a) \psi_{k j}(a)=\delta_{l}^{j}, \quad \sum_{k=1}^{r} \widetilde{\vartheta}_{l k}(a) \vartheta_{k j}(a)=\delta_{l}^{j},
$$

we can multiply the first (resp. the second) line of (3) by $\widetilde{\psi}_{l k}(a)$ (resp. by $\left.\widetilde{\vartheta}_{l k}(a)\right)$ and then make summation over $k=1, \ldots, r$ in order to get resolved equations:

$$
\begin{cases}0 \equiv \sum_{k=1}^{r} \sum_{j=1}^{r} \widetilde{\psi}_{l k}(a) \vartheta_{k j}(a) X_{j}+X_{l}^{\prime} & (k=1 \cdots r) \\ 0 \equiv X_{l}+\sum_{k=1}^{r} \sum_{j=1}^{r} \widetilde{\vartheta}_{l k}(a) \psi_{k j}(a) X_{j}^{\prime} & (k=1 \cdots r)\end{cases}
$$

in which we have suppressed the push-forward symbols. We can readily rewrite such equations under the contracted form:

$$
X_{k}=\sum_{l=1}^{r} \rho_{j k}(a) X_{j}^{\prime} \quad \text { and } \quad X_{k}^{\prime}=\sum_{l=1}^{r} \widetilde{\rho}_{j k}(a) X_{j}
$$

by introducing some two appropriate auxiliary $r \times r$ matrices $\rho_{j k}(a):=$ $-\sum_{l=1}^{r} \widetilde{\vartheta}_{k l}(a) \psi_{l j}(a)$ and $\widetilde{\rho}_{j k}(a):=-\sum_{l=1}^{r} \widetilde{\psi}_{k l}(a) \vartheta_{l j}(a)$ of analytic functions (whose precise expression does not matter here) which depend only
upon $a$ and which, naturally, are inverses of each other. A diagram illustrating what we have gained at that point is welcome and intuitively helpful.


Fig. : Transfer of infinitesimal transformations by the group
Proposition. If, in each one of the r basic infinitesimal transformations of the finite continuous transformation group $x^{\prime}=f(x ; a)=f_{a}(x)$ having the inverse transformations $x=g_{a}\left(x^{\prime}\right)$, namely if in the vector fields:

$$
X_{k}=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial}{\partial x_{i}} \quad(k=1 \cdots r), \quad \xi_{k i}(x):=-\frac{\partial f_{i}}{\partial a_{k}}(x ; e)
$$

one introduces the new variables $x^{\prime}=f_{a}(x)$, that is to say: replaces $x$ by $g_{a}\left(x^{\prime}\right)$ and $\frac{\partial}{\partial x_{i}}$ by $\sum_{\nu=1}^{n} \frac{\partial f_{\nu}}{\partial x_{i}}(x ; a) \frac{\partial}{\partial x_{\nu}^{\prime}}$, then one necessarily obtains a linear combination of the same infinitesimal transformations $X_{l}^{\prime}=\sum_{i=1}^{n} \xi_{k i}\left(x^{\prime}\right) \frac{\partial}{\partial x_{i}^{\prime}}$ at the point $x^{\prime}$ with coefficients depending only upon the parameters $a_{1}, \ldots, a_{r}$ :

$$
\left(f_{a}\right)_{*}\left(\left.X_{k}\right|_{x}\right)=\left(g_{a}\right)^{*}\left(\left.X_{k}\right|_{g_{a}\left(x^{\prime}\right)}\right)=\left.\sum_{l=1}^{r} \rho_{l k}\left(a_{1}, \ldots, a_{r}\right) \cdot X_{l}\right|_{x^{\prime}} \quad(k=1 \cdots r)
$$

Of course, through the inverse change of variable $x^{\prime} \mapsto f_{a}(x)$, the infinitesimal transformations $X_{k}^{\prime}$ are subjected to similar linear substitutions:

$$
\left(g_{a}\right)_{*}\left(\left.X_{k}^{\prime}\right|_{x^{\prime}}\right)=\left(f_{a}\right)^{*}\left(\left.X_{k}^{\prime}\right|_{f_{a}(x)}\right)=\left.\sum_{l=1}^{r} \widetilde{\rho}_{l k}(a) \cdot X_{l}\right|_{x} \quad(k=1 \cdots r)
$$

## Coincidence with the Contemporary Presentation

Afterwards, thanks to the linearity of the tangent map, we deduce that the general transformation of our group:

$$
X:=e_{1} X_{1}+\cdots+e_{r} X_{r}
$$

coordinatized in the basis $\left(X_{k}\right)_{1 \leqslant k \leqslant r}$ by means of some $r$ arbitrary constants $e_{1}, \ldots, e_{r} \in \mathbb{K}$, then transforms as:

$$
\begin{aligned}
\left(g_{a}\right)^{*}\left(e_{1} X_{1}+\cdots+\left.e_{r} X_{r}\right|_{g_{a}\left(x^{\prime}\right)}\right) & =\left.\sum_{k=1}^{r} e_{k} \sum_{l=1}^{r} \rho_{l k}(a) X_{l}\right|_{x^{\prime}} \\
& =:\left.e_{1}^{\prime}(e ; a) X_{1}\right|_{x^{\prime}}+\cdots+\left.e_{r}^{\prime}(e ; a) X_{r}\right|_{x^{\prime}}
\end{aligned}
$$

and hence we obtain that the change of variables $x^{\prime}=f_{a}(x)$ performed by a general transformation of the group then acts linearly on the space $\simeq \mathbb{K}^{r}$ of its infinitesimal transformations:

$$
e_{k}^{\prime}(e ; a):=\sum_{l=1}^{r} \rho_{k l}(a) \cdot e_{l} \quad(k=1 \cdots r),
$$

by just multiplying the coordinates $e_{l}$ by the matrix $\rho_{k l}(a)$.
In contemporary treatises, the action of the group on its infinitesimal transformations coincides in substance with what Lie had devised in the 1870's. Indeed, to bridge the Babelian-like gap, we consider the general infinitesimal transformation $\left.X\right|_{x^{\prime}}=e_{1} X_{1}+\cdots+\left.e_{r} X_{r}\right|_{x^{\prime}}$ of the group as being based at the point $x^{\prime}$, and we compute the adjoint action $\operatorname{Ad} f_{a}\left(\left.X\right|_{x^{\prime}}\right)$ of $f_{a}$ on $\left.X\right|_{x^{\prime}}$; this expression is nowadays defined by just differentiating at $t=0$ the composition $f_{a} \circ \exp (t X) \circ f_{a}^{-1}$ which represents the action of the interior automorphism associated to $f_{a}$ on the one-parameter subgroup $\exp (t X)(\cdot)$ generated by $X$ :

$$
\begin{aligned}
\operatorname{Ad} f_{a}\left(\left.X\right|_{x^{\prime}}\right) & :=\left.\frac{d}{d t}\left(f_{a} \circ \exp (t X)(\cdot) \circ f_{a}^{-1}\left(x^{\prime}\right)\right)\right|_{t=0} \\
& =\left.\left(f_{a}\right)_{*} \frac{d}{d t}\left(\exp (t X)\left(f_{a}^{-1}\left(x^{\prime}\right)\right)\right)\right|_{t=0} \\
& =\left(f_{a}\right)_{*}\left(\left.X\right|_{f_{a}^{-1}\left(x^{\prime}\right)}\right) \\
& =\left(g_{a}\right)^{*}\left(\left.X\right|_{g_{a}\left(x^{\prime}\right)}\right) \\
& =\left(g_{a}\right)^{*}\left(e_{1} X_{1}+\cdots+\left.e_{r} X_{r}\right|_{g_{a}\left(x^{\prime}\right)}\right) \\
& =\left.e_{1}^{\prime}(e ; a) X_{1}\right|_{x^{\prime}}+\cdots+\left.e_{r}^{\prime}(e ; a) X_{r}\right|_{x^{\prime}}
\end{aligned}
$$

We thus recover exactly the linear action $e_{k}^{\prime}=e_{k}^{\prime}\left(e ; a_{1}, \ldots, a_{r}\right)$ boxed above.


Fig. : Differentiating the action of an interior automorphism

## Infinitesimal Generators of the Adjoint Group

After these preliminaries devoted to survey, to modernize and to clarify selected topics of the first chapters of [25], we can now just translate the
very clear presentation of Lie's theory of the adjoint group written out by Engel and Lie.

## Chapter 16 (Vol. I). <br> The adjoint group.

Let $x_{i}^{\prime}=f_{i}\left(x_{1}, \ldots, x_{n}, a_{1}, \ldots, a_{n}\right)$ be an $r$-term group with the $r$ infinitesimal transformations:

$$
X_{k} f=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial f}{\partial x_{i}} \quad(k=1 \cdots r)
$$

If one introduces the $x_{i}^{\prime}$ as new variables in the expression $\sum e_{k} X_{k} f$, then as it has been already shown in Chap. 4, Prop. 4, p. 81 [reconstituted just above], one gets for all values of the $e_{k}$ an equation of the form:

$$
\sum_{k=1}^{r} e_{k} \cdot X_{k} f=\sum_{k=1}^{r} e_{k}^{\prime} \cdot X_{k}^{\prime} f
$$

Here, the $e_{k}^{\prime}$ are certain linear, homogeneous functions of the $e_{k}$ with coefficients that depend upon $a_{1}, \ldots, a_{r}$ :

$$
\begin{equation*}
e_{k}^{\prime}=\sum_{j=1}^{r} \rho_{k j}\left(a_{1}, \ldots, a_{r}\right) \cdot e_{j} \tag{1}
\end{equation*}
$$

If one again introduces in $\sum e_{k}^{\prime} X_{k}^{\prime} f$ the new variables $x_{i}^{\prime \prime}=f_{i}(x, b)$, then one receives:

$$
\sum_{k=1}^{r} e_{k}^{\prime} \cdot X_{k}^{\prime} f=\sum_{k=1}^{r} e_{k}^{\prime \prime} \cdot X_{k}^{\prime \prime} f
$$

where:

$$
\begin{equation*}
e_{k}^{\prime \prime}=\sum_{j=1}^{r} \rho_{k j}\left(b_{1}, \ldots, b_{r}\right) \cdot e_{j}^{\prime} . \tag{1’}
\end{equation*}
$$

But now because the equations $x_{i}^{\prime}=f_{i}(x, a)$ represent a group, the $x^{\prime \prime}$ are consequently linked with the $x$ through relations of the form $x_{i}^{\prime \prime}=f_{i}(x, c)$ in which the $c$ depend only upon $a$ and $b$ :

$$
c_{k}=\varphi_{k}\left(a_{1}, \ldots, a_{r}, b_{1}, \ldots, b_{r}\right)
$$

Hence one passes directly from the $x$ to the $x^{\prime \prime}$ so one finds:

$$
\sum_{k=1}^{r} e_{k} \cdot X_{k} f=\sum_{k=1}^{r} e_{k}^{\prime \prime} \cdot X_{k}^{\prime \prime} f
$$

and to be precise, one has:
(1") $\quad e_{k}^{\prime \prime}=\sum_{j=1}^{r} \rho_{k j}\left(c_{1}, \ldots, c_{r}\right) \cdot e_{j}=\sum_{j=1}^{r} \rho_{k j}\left(\varphi_{1}(a, b), \ldots, \varphi_{r}(a, b)\right) \cdot e_{j}$.
From this, it can be deduced that the totality of all transformations $e_{k}^{\prime}=\sum \rho_{k j}(a) \cdot e_{j}$ forms a group. Indeed, by combination of the equations (1) and ( $1^{\prime}$ ) it comes out:

$$
e_{k}^{\prime \prime}=\sum_{j, \nu=1 \ldots r} \rho_{k j}\left(b_{1}, \ldots, b_{r}\right) \cdot \rho_{j \nu}\left(a_{1}, \ldots, a_{r}\right) \cdot e_{\nu}
$$

what must naturally coincide with the equations (1") and in fact, for all values of the $e$, the $a$ and the $b$. Consequently, there are the $r^{2}$ identities:

$$
\rho_{k \nu}\left(\varphi_{1}(a, b), \ldots, \varphi_{r}(a, b)\right) \equiv \sum_{j=1}^{r} \rho_{j \nu}\left(a_{1}, \ldots, a_{r}\right) \cdot \rho_{k j}\left(b_{1}, \ldots, b_{r}\right)
$$

from which it results that the family of the transformations $e_{k}^{\prime}=\sum \rho_{k j}(a)$. $e_{j}$ effectively forms a group.

To every $r$-term group $x_{i}^{\prime}=f_{i}(x, a)$ therefore belongs a fully determined linear homogeneous group:

$$
\begin{equation*}
e_{k}^{\prime}=\sum_{j=1}^{r} \rho_{k j}\left(a_{1}, \ldots, a_{r}\right) \cdot e_{j} \quad(k=1 \cdots r) \tag{1}
\end{equation*}
$$

which we want to call the adjoint group*) [ADJUNGIRTE GRUPPE] of the group $x_{i}^{\prime}=f_{i}(x, a)$.
*) Lie, Archiv for Math., Vol. 1, Christiania 1876.
We consider for example the two-term group $x^{\prime}=a x+b$ with the two independent infinitesimal transformations: $\frac{d f}{d x}, x \frac{d f}{d x}$. We find:

$$
e_{1} \frac{d f}{d x}+e_{2} x \frac{d f}{d x}=e_{1} a \frac{d f}{d x^{\prime}}+e_{2}\left(x^{\prime}-b\right) \frac{d f}{d x^{\prime}}=e_{1}^{\prime} \frac{d f}{d x^{\prime}}+e_{2}^{\prime} x^{\prime} \frac{d f}{d x^{\prime}},
$$

whence we obtain for the adjoint group of the group $x^{\prime}=a x+b$ the following equations:

$$
e_{1}^{\prime}=a e_{1}-b e_{2}, \quad e_{2}^{\prime}=e_{2},
$$

which visibly really represent a group.
The adjoint group of the group $x_{i}^{\prime}=f_{i}(x, a)$ contains, under the form in which it has been found above, precisely $r$ arbitrary parameters: $a_{1}, \ldots, a_{r}$. But for every individual group $x_{i}^{\prime}=f_{i}(x, a)$, a special research is required to investigate whether the parameters $a_{1}, \ldots, a_{r}$ are all essential
in the adjoint group. Actually, we shall shortly see that there are $r$-term groups whose adjoint group does not contain $r$ essential parameters.

Besides, in all circumstances, one transformation comes in the adjoint group of the group $x_{i}^{\prime}=f_{i}(x, a)$, namely the identity transformation; for if one sets for $a_{1}, \ldots, a_{r}$ in the equations (1) the system of values which produces the identity transformation $x_{i}^{\prime}=x_{i}$ in the group $x_{i}^{\prime}=f_{i}(x, a)$, then one obtains the transformation: $e_{1}^{\prime}=e_{1}, \ldots, e_{r}^{\prime}=e_{r}$, which hence is always present in the adjoint group. However, as we shall see, it can happen that the adjoint group consists only of the identity transformation: $e_{1}^{\prime}=e_{1}, \ldots, e_{r}^{\prime}=e_{r}$.

## § 76.

In order to make accessible the study of the adjoint group, we must above all determine its infinitesimal transformations. We easily reach this end by an application of the Theorem 43, Chap. 15, p. 252 [here: p. 7] ; yet we must in the process replace the equations $x_{i}^{\prime}=f_{i}(x, a)$ of our group by the equivalent canonical equations:

$$
\begin{equation*}
x_{i}^{\prime}=x_{i}+\frac{t}{1} \sum_{k=1}^{r} \lambda_{k} \cdot X_{k} x_{i}+\cdots \quad(i=1 \cdots n) \tag{2}
\end{equation*}
$$

which represent the $\infty^{r-1}$ one-term subgroups of the group $x_{i}^{\prime}=f_{i}(x, a)$. According to Chap. 4, p. 69 [here: p. ??], the $a_{k}$ are defined here as functions of $t$ and $\lambda_{1}, \ldots, \lambda_{r}$ by the simultaneous system:

$$
\begin{equation*}
\frac{d a_{k}}{d t}=\sum_{j=1}^{r} \lambda_{j} \cdot \alpha_{j k}\left(a_{1}, \ldots, a_{r}\right) \quad(k=1 \cdots r) \tag{3}
\end{equation*}
$$

By means of the equations (2), we have therefore to introduce the new variables $x_{i}^{\prime}$ in $\sum e_{k} X_{k} f$ and we must as a result obtain a relation of the form:

$$
\sum_{k=1}^{r} e_{k} \cdot X_{k} f=\sum_{k=1}^{r} e_{k}^{\prime} \cdot X_{k}^{\prime} f .
$$

The infinitesimal transformation denoted by $Y f$ in Theorem 43 on p. 252 [here: p. 7] now writes: $\lambda_{1} X_{1} f+\cdots+\lambda_{r} X_{r} f$; we therefore receive in our case:

$$
\begin{aligned}
Y\left(X_{k}(f)\right)-X_{k}(Y(f)) & =\sum_{\nu=1}^{r} \lambda_{\nu}\left[X_{\nu}, X_{k}\right] \\
& =\sum_{s=1}^{r}\left\{\sum_{\nu=1}^{r} \lambda_{\nu} c_{\nu k s}\right\} X_{s} f .
\end{aligned}
$$

Consequently, we obtain the following differential equations for $e_{1}^{\prime}, \ldots, e_{r}^{\prime}$ :

$$
\begin{equation*}
\frac{d e_{s}^{\prime}}{d t}+\sum_{\nu=1}^{r} \lambda_{\nu} \sum_{k=1}^{r} c_{\nu k s} e_{k}^{\prime}=0 \quad(s=1 \cdots r) \tag{4}
\end{equation*}
$$

We consider the integration of these differential equations as an executable operation, for it is known that it requires only the resolution of an algebraic equation of $r$-th degree. So if we perform the integration on the basis of the initial condition: $e_{k}^{\prime}=e_{k}$ for $t=0$, we obtain $r$ equations of the form:

$$
\begin{equation*}
e_{k}^{\prime}=\sum_{j=1}^{r} \psi_{k j}\left(\lambda_{1} t, \ldots, \lambda_{r} t\right) \cdot e_{j} \quad(k=1 \cdots r), \tag{5}
\end{equation*}
$$

which are equivalent to the equations (1), as soon as the $a_{k}$ are expressed as functions of $\lambda_{1} t, \ldots, \lambda_{r} t$ in the latter.

It follows from this that the equations (5) represent the adjoint group too. But now we have derived the equations (5) in exactly the same way as if we would have wanted to determine all finite transformations which are engendered by the infinitesimal transformations:

$$
\sum_{\nu=1}^{r} \lambda_{\nu} \sum_{k, s=1 \ldots r} c_{k \nu s} e_{k} \frac{\partial f}{\partial e_{s}}=\sum_{\nu=1}^{r} \lambda_{\nu} \cdot E_{\nu} f
$$

(cf. p. 51 above). Consequently we conclude that the adjoint group (1) consists of the totality of all one-term groups of the form $\lambda_{1} E_{1} f+\cdots+$ $\lambda_{r} E_{r} f$.

If amongst the family of all infinitesimal transformations $\lambda_{1} E_{1} f+$ $\cdots+\lambda_{r} E_{r} f$ there are exactly $\rho$ transformations and not more which are indepedent, say $E_{1} f, \ldots, E_{\rho} f$, then all the finite transformations of the one-term groups $\lambda_{1} E_{1} f+\cdots+\lambda_{r} E_{r} f$ are already contained in the totality of all finite transformations of the $\infty^{\rho-1}$ groups $\lambda_{1} E_{1} f+\cdots+\lambda_{\rho} E_{\rho} f$. The totality of these $\infty^{\rho}$ finite transformations forms the adjoint group: $e_{k}^{\prime}=\sum \rho_{k j}(a) \cdot e_{j}$, which therefore contains only $\rho$ essential parameters (Chap. 3, Theorem 8, p. 65 [here: p. ??]).

According to what precedes, it is to be supposed that $E_{1} f, \ldots, E_{\rho} f$ are linked together through relations of the form:

$$
\left[E_{\mu}, E_{\nu}\right]=\sum_{s=1}^{\rho} g_{\mu \nu s} \cdot E_{s} f
$$

|we can also confirm this by a computation. By a direct calculation, it comes:

$$
E_{\mu}\left(E_{\nu}(f)\right)-E_{\nu}\left(E_{\mu}(f)\right)=\sum_{\sigma, k, \pi}^{1 \ldots r}\left(c_{\pi \mu k} c_{k \nu \sigma}-c_{\pi \nu k} c_{k \mu \sigma}\right) e_{\pi} \frac{\partial f}{\partial e_{\sigma}}
$$

But between the $c_{i k s}$, there exist the relations:

$$
\sum_{k=1}^{r}\left(c_{\pi \mu k} c_{k \nu \sigma}+c_{\mu \nu k} c_{k \pi \sigma}+c_{\nu \pi k} c_{k \mu \sigma}\right)=0
$$

which we have deduced from the Jacobi identity some time ago ( $c f$. Chap. 9, Theorem 27, p. 170 [here: ??]). If we yet use for this that $c_{\nu \pi k}=-c_{\pi \nu k}$ and $c_{k \pi \sigma}=-c_{\pi k \sigma}$, we can bring the right hand-side of our equation for $\left[E_{\mu}, E_{\nu}\right]$ to the form:

$$
\sum_{k=1}^{r} c_{\mu \nu k} \sum_{\sigma, \pi}^{1 \ldots r} c_{\pi k \sigma} e_{\pi} \frac{\partial f}{\partial e_{\sigma}},
$$

whence it comes:

$$
\left[E_{\mu}, E_{\nu}\right]=\sum_{k=1}^{r} c_{\mu \nu k} \cdot E_{k} f .
$$

Lastly, under the assumptions made above, the right hand side can be expressed by means of $E_{1} f, \ldots, E_{\rho} f$ alone, so that relations of the form:

$$
\left[E_{\mu}, E_{\nu}\right]=\sum_{s=1}^{\rho} g_{\mu \nu s} \cdot E_{s} f
$$

really hold, in which the $g_{\mu \nu s}$ denote constants.
Before we continue, we want yet to recapitulate in cohesion [IM ZUSAMMENHANGE WIEDERHOLEN] the results of the chapter obtained up till now.

Theorem 48. If one introduces in the general infinitesimal transformation $e_{1} X_{1} f+\cdots+e_{r} X_{r} f$ of the r-term group $x_{i}^{\prime}=f_{i}(x, a)$ the new variable $x^{\prime}$ in place of $x$, then one obtains an expression of the form:

$$
e_{1}^{\prime} \cdot X_{1}^{\prime} f+\cdots+e_{r}^{\prime} \cdot X_{r}^{\prime} f
$$

in the process, the $e^{\prime}$ are linked with the e through equations of the shape:

$$
e_{k}^{\prime}=\sum_{j=1}^{r} \rho_{k j}\left(a_{1}, \ldots, a_{r}\right) \cdot e_{j} \quad(k=1 \cdots r),
$$

which represent a group in the variables e, the so-called adjoint group of $\mid$ the group $x_{i}^{\prime}=f_{i}(x, a)$. This adjoint group contains the identity transformation and is engendered by certain infinitesimal transformations; if, between $X_{1} f, \ldots, X_{r} f$, there exist the Relations:

$$
\left[X_{i}, X_{k}\right]=\sum_{s=1}^{r} c_{i k s} \cdot X_{s} f \quad(i, k=1 \cdots r)
$$

and if one sets:

$$
E_{\mu} f=\sum_{k, j}^{1 \ldots r} c_{j \mu k} e_{j} \frac{\partial f}{\partial e_{k}} \quad(\mu=1 \cdots r),
$$

then $\lambda_{1} E_{1} f+\cdots+\lambda_{r} E_{r} f$ is the general infinitesimal transformation of the adjoint group and between $E_{1} f, \ldots, E_{r} f$, there are at the same time the Relations:

$$
\left[E_{i}, E_{k}\right]=\sum_{s=1}^{r} c_{i k s} \cdot E_{s} f \quad(i, k=1 \cdots r)
$$

If two $r$-term groups $X_{1} f, \ldots, X_{r} f$ and $Y_{1} f, \ldots, Y_{r} f$ are constituted in such a way that one has equally:

$$
\left[X_{i}, X_{k}\right]=\sum_{s=1}^{r} c_{i k s} \cdot X_{s}, \quad\left[Y_{i}, Y_{k}\right]=\sum_{s=1}^{r} c_{i k s} \cdot Y_{s} f,
$$

with the same constants $c_{i k s}$ in the two cases, then both groups obviously have the same adjoint group. Later, we will see that in certain circumstances, also certain groups which do not possess an equal number of terms can nonetheless have the same adjoint group.

## Excellent Infinitesimal Transformations

## § 77.

Now, by what can one recognize how many independent infinitesismal transformation there are amongst $E_{1} f, \ldots, E_{r} f$ ?

If $E_{1} f, \ldots, E_{r} f$ are not all independent of each other, then there is at least one infinitesimal transformation $\sum g_{\mu} E_{\mu} f$ that does vanish identically. From the identity:

$$
\sum_{\mu=1}^{r} g_{\mu} \sum_{k, j}^{1 \ldots r} c_{j \mu k} e_{j} \frac{\partial f}{\partial e_{k}} \equiv 0
$$

it comes out:

$$
\sum_{\mu=1}^{r} g_{\mu} c_{j \mu k}=0
$$

for all values of $j$ and $k$, and consequently the expression:

$$
\left[X_{j}, \sum_{\mu=1}^{r} g_{\mu} \cdot X_{\mu} f\right]=\sum_{k=1}^{r}\left\{\sum_{\mu=1}^{r} g_{\mu} c_{j \mu k}\right\} X_{k} f
$$

vanishes, that is to say: the infinitesimal transformation $\sum g_{\mu} X_{\mu} f$ is exchangeable with all the $r$ infinitesimal transformations $X_{j} f$. Conversely, if the group $X_{1} f, \ldots, X_{r} f$ comprises an infinitesimal transformation $\sum g_{\mu} X_{\mu} f$ which is exchangeable with all the $X_{k} f$, then if follows in the same way that the infinitesimal transformation $\sum g_{\mu} E_{\mu} f$ vanishes identically.

In order to express this relationship in an as brief as possible manner, we introduce the following naming:

An infinitesimal transformation $\sum g_{\mu} X_{\mu} f$ of the r-term group $X_{1} f, \ldots, X_{r} f$ is called and excellent infinitesimal transformation of this group if it is exchangeable with all the $X_{k} f$.

Incidentally, the excellent infinitesimal transformations of the group $X_{1} f, \ldots, X_{r} f$ are also characterized by the fact that they keep their form through the introduction of the new variables $x_{i}^{\prime}=f_{i}(x, a)$, whichever values the parameters $a_{1}, \ldots, a_{r}$ can have. Indeed, if the infinitesimal transformation $\sum g_{\mu} X_{\mu} f$ is excellent, then according to Chap. 15, p. 259, there is a relation of the form:

$$
\sum g_{\mu} \cdot X_{\mu} f=\sum g_{\mu} \cdot X_{\mu}^{\prime} f
$$

In addition, the cited developments show that each finite transformation of the one-term group $\sum g_{\mu} X_{\mu} f$ is exchangeable with every finite transformation of the group $X_{1} f, \ldots, X_{r} f$.
$\triangleright$ The cited developments.

## Chapter M

## The Projective Group

## Projective Space $\mathbb{K} \mathbb{P}^{n}$ and Homogeneous Coordinates

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. The $n$-dimensional (" $n$-fold-extended") projective space (over $\mathbb{K}$ ) is the set $\mathbb{K} \mathbb{P}^{n}$ of (vector) lines in the vector space $\mathbb{K}^{n+1}$. One can see $\mathbb{K} \mathbb{P}^{n}$ as the quotient set $\left(\mathbb{K}^{n+1} \backslash\{0\}\right) / \sim$ of nonzero vectors $e \in \mathbb{K}^{n+1} \backslash\{0\}$ modulo the equivalence relation $e^{\prime} \sim e$ if and only if $e^{\prime}=\lambda e$, for some $\lambda \in \mathbb{K}$ (naturally, $\lambda \neq 0$ ). Thus, we have a canonical projection map $\pi: \mathbb{K}^{n+1} \backslash\{0\} \longrightarrow \mathbb{K} \mathbb{P}^{n}$ that associates to each vector $v$ the vector line $\mathbb{K} v$ it spans. Here, $\pi(\mu e)=\pi(e)$ for every $e \in \mathbb{K}^{n+1} \backslash\{0\}$ and all $\mu \neq 0$.

When considered as a basis of $\mathbb{K}^{n+1}$, any collection of $(n+1)$ linearly independent vectors $\left(e_{0}, e_{1}, \ldots, e_{n}\right)$ in $\mathbb{K}^{n+1} \backslash\{0\}$ is enough to determine uniquely every nonzero vector $e=x_{0} e_{0}+x_{1} e_{1}+\cdots+x_{n} e_{n}$ of $\mathbb{K}^{n+1}$ by its coordinates $x_{0}, x_{1}, \ldots, x_{n}$, and hence also, to determine uniquely the projected point $P=\pi(e)=\pi(\mu e)$ of the projective space $\mathbb{K} \mathbb{P}^{n}$.

Given a basis $\left(e_{1}, e_{1}, \ldots, e_{n}\right)$ of $\mathbb{K}^{n+1}$, we can therefore associate to every projective point $P=\pi(e)$ a certain $(n+1)$-tuple of elements of $\mathbb{K}$, called homogeneous coordinates of $P$ (relative to the basis in question) and denoted $\left[x_{0}: x_{1}: \cdots: x_{n}\right]$, namely the coordinates of any vector $e=x_{0} e_{0}+$ $x_{1} e_{1}+\cdots+x_{n} e_{n}$ with $\pi(e)=P$. By definition, these $(n+1)$-tuples have at least one nonzero component, and for reasons of coherence, they must be left unchanged by any homothety of nonzero ratio $\mu \in \mathbb{K}$ :

$$
\left[x_{0}: x_{1}: \cdots: x_{n}\right] \equiv\left[\mu x_{0}: \mu x_{1}: \cdots: \mu x_{n}\right] .
$$

Thus, each representative of any equivalence class under $[x] \equiv[\mu x]$ provides homogeneous coordinates for a well defined point $P \in \mathbb{K} \mathbb{P}^{n}$. Notice that the $(n+1)$ points $P_{0}=\pi\left(e_{0}\right), P_{1}=\pi\left(e_{1}\right), \ldots, P_{n}=\pi\left(e_{n}\right)$ have homogeneous coordinates $[1: 0: \cdots: 0],[0: 1: \cdots: 0], \ldots,[0: 0: \cdots: 1]$.

However, while thinking intrinsically inside the projective space, no collection of $(n+1)$ projective points in general position $P_{0}, P_{1}, \ldots$, $P_{n}$ can be sufficient to "coordinatize" uniquely all points $P$ of $\mathbb{K}^{1} \mathbb{P}^{n}$ by means of some lifted basis $\left(e_{0}, e_{1}, \ldots, e_{n}\right)$, where $e_{i} \in \pi^{-1}\left(P_{i}\right)$; indeed, any other such basis $\left(e_{1}^{\prime}, e_{2}^{\prime}, \ldots, e_{n}^{\prime}\right)$ with $P_{i}=\pi\left(e_{i}^{\prime}\right)$ is necessarily of the
form $\left(e_{1}^{\prime}, e_{2}^{\prime}, \ldots, e_{n}^{\prime}\right)=\left(\lambda_{0} e_{0}, \lambda_{1} e_{1}, \ldots, \lambda_{n} e_{n}\right)$ with arbitrary $\lambda_{i} \neq 0$, but a variable vector

$$
e=x_{0} e_{0}+x_{1} e_{1}+\cdots+x_{n} e_{n}=x_{0}^{\prime} e_{0}^{\prime}+x_{1}^{\prime} e_{1}^{\prime}+\cdots+x_{n}^{\prime} e_{n}^{\prime}
$$

which is a lift of a variable projective point $P=\pi(e)$ has homogeneous coordinates:

$$
\begin{aligned}
{\left[x_{0}^{\prime}: x_{1}^{\prime}: \cdots: x_{n}^{\prime}\right] } & =\left[x_{0} / \lambda_{0}: x_{1} / \lambda_{1}: \cdots: x_{n} / \lambda_{n}\right] \\
& \neq\left[x_{0}: x_{1}: \cdots: x_{n}\right]
\end{aligned}
$$

which are in general different in the alternative lifted basis $\left(e_{1}^{\prime}, e_{2}^{\prime}, \ldots, e_{n}^{\prime}\right)$. The introduction of an $(n+2)$-th point $P_{n+1} \in \mathbb{K} \mathbb{P}^{n}$ shall insure here that $\lambda_{0}=\lambda_{1}=\cdots=\lambda_{n}=: \mu$ must all be equal, whence homogeneous coordinates will be uniquely defined by $P_{0}, P_{1}, \ldots, P_{n}, P_{n+1}$, as we now explain.

Projective frames. A $(n+2)$-tuple of points $P_{0}, P_{1}, \ldots, P_{n}, P_{n+1}$ of $\mathbb{K} \mathbb{P}^{n}$ is called a projective frame if any $(n+1)$ among the $(n+2)$ lines $\pi^{-1}\left(P_{0}\right)$, $\pi^{-1}\left(P_{1}\right), \ldots, \pi^{-1}\left(P_{n}\right), \pi^{-1}\left(P_{n+1}\right)$ span $\mathbb{K}^{n+1}$. This is a precise sense of being "mutually in general position".
Lemma. For any projective frame $P_{0}, P_{1}, \ldots, P_{n}, P_{n+1}$, there exist $(n+2)$ vectors $e_{0}, e_{1}, \ldots, e_{n}, e_{n+1}$ in $\mathbb{K}^{n+1}$ with the first $(n+1)$ ones $e_{0}, e_{1}, \ldots, e_{n}$ constituting a basis of $\mathbb{K}^{n+1}$, and with:

$$
e_{n+1}=e_{0}+e_{1}+\cdots+e_{n}
$$

such that they provide a lift of the projective frame, namely:
$\pi\left(e_{0}\right)=P_{0}, \quad \pi\left(e_{1}\right)=P_{1}, \cdots \cdots, \pi\left(e_{n}\right)=P_{n}, \quad$ and $\quad \pi\left(e_{n+1}\right)=P_{n+1}$.
Any other such lift $\left(e_{0}^{\prime}, e_{1}^{\prime}, \ldots, e_{n}^{\prime}, e_{n+1}^{\prime}\right)$ differs from $\left(e_{0}, e_{1}, \ldots, e_{n}, e_{n+1}\right)$ just up to a homothety: $e_{i}^{\prime}=\mu e_{i}, i=0,1, \ldots, n, n+1$, for some nonzero $\mu \in \mathbb{K}$.

Proof. Lift the first $(n+1)$ points $P_{0}, P_{1}, \ldots, P_{n}$ to any basis $\left(e_{0}, e_{1}, \ldots, e_{n}\right)$ of $\mathbb{K}^{n+1}$, namely $\pi\left(e_{0}\right)=P_{0}, \pi\left(e_{1}\right)=P_{1}, \ldots, \pi\left(e_{n}\right)=P_{n}$ and consider the coordinates $\left(x_{0}, x_{1}, \ldots, x_{n}\right)$ in this basis of some nonzero vector $e_{n+1}=x_{0} e_{0}+x_{1} e_{1}+\cdots+x_{n} e_{n}$ chosen in the line $\pi^{-1}\left(P_{n+1}\right)$ associated to the last point. Then all the $x_{i}$ here must be nonzero; otherwise, if say $x_{0}=0$, the $(n+1)$ lines $\mathbb{K} e_{1}, \ldots, \mathbb{K} e_{n}, \mathbb{K} e_{n+1}$ would be contained in the hyperplane $\left\{x_{0}=0\right\}$, in contradiction to general position. So we can replace the basis $\left(e_{0}, e_{1}, \ldots, e_{n}\right)$ by just $\left(x_{0} e_{0}, x_{1} e_{1}, \ldots, x_{n} e_{n}\right)$, and we then have $e_{n+1}=e_{0}+e_{1}+\cdots+e_{n}$, as desired.

Next, supposing $\pi\left(e_{i}^{\prime}\right)=\pi\left(e_{i}\right)$ for the $(n+1)$ first indices $i=0,1, \ldots, n$, there must, as already seen, exist nonzero $\lambda_{i} \in \mathbb{K}$ such that $e_{i}^{\prime}=\lambda_{i} e_{i}$, but if in addition, also for the last index, one requires $\pi\left(e_{n+1}^{\prime}\right)=\pi\left(e_{n+1}\right)$, namely
$e_{n+1}^{\prime}=\mu e_{n+1}$, we deduce by inserting $e_{n+1}^{\prime}=e_{0}^{\prime}+e_{1}^{\prime}+\cdots+e_{n}^{\prime}$ and $e_{n+1}=e_{0}+e_{1}+\cdots+e_{n}$ :

$$
\lambda_{0} e_{0}+\lambda_{1} e_{1}+\cdots+\lambda_{n} e_{n}=e_{n+1}^{\prime}=\mu e_{n+1}=\mu e_{0}+\mu e_{1}+\cdots+\mu e_{n},
$$

whence $\lambda_{0}=\mu=\lambda_{1}=\cdots=\lambda_{n}$, as claimed.
Given a basis $\left(e_{0}, e_{1}, \ldots, e_{n}\right)$ for $\mathbb{K}^{n+1}$, the mapping which sends $P=$ $\pi\left(x_{0} e_{0}+x_{1} e_{1}+\cdots+x_{n} e_{n}\right)$ to $\left[x_{0}: x_{1}: \cdots: x_{n}\right]$ is called a projective coordinate system.
Proposition. Projective coordinate systems $\left[x_{0}: x_{1}: \cdots: x_{n}\right]$ are in one-toone correspondence with projective frames $P_{0}, P_{1}, \ldots, P_{n}, P_{n+1}$, namely:

$$
\begin{aligned}
P_{0} \longleftrightarrow[1: 0: \cdots: 0], \quad P_{1} \longleftrightarrow[0: 1: \cdots: 0], \ldots, \\
\ldots, P_{n} \longleftrightarrow[0: 0: \cdots: 1], \quad P_{n+1} \longleftrightarrow[1: 1: \cdots: 1] .
\end{aligned}
$$

Proof.

## Projective Frames

Any linear automorphism $u \in \mathrm{GL}_{n+1}(\mathbb{K})$ of $\mathbb{K}^{n+1}$ sends lines of $\mathbb{K}^{n+1}$ to lines, so passing to the quotient map, $u$ defines $\mathbb{P}(u): \mathbb{K} \mathbb{P}^{n} \rightarrow \mathbb{K} \mathbb{P}^{n}$. The maps obtained this way are called projective transformations.
Lemma. Two linear automorphisms $u_{1}, u_{2} \in \mathrm{GL}_{n+1}(\mathbb{K})$ yield the same projective transformation $\mathbb{P}\left(u_{1}\right)=\mathbb{P}\left(u_{2}\right)$ of $\mathbb{K}^{\mathbb{P}^{n}}$ if and only if there exists a nonzero constant $\lambda \in \mathbb{K}$ such that $u_{2}(e)=\lambda u_{1}(e)$ for all $e \in \mathbb{K}^{n+1}$.

Proof. If $u_{2}=\lambda u_{1}$, obviously $\mathbb{P}\left(u_{2}\right)=\mathbb{P}\left(u_{1}\right)$. Conversely, if $\mathbb{P}\left(u_{2}\right)=$ $\mathbb{P}\left(u_{1}\right)$, then for every $e \in E \backslash\{0\}$, there exists a nonzero constant $\lambda_{e}$, depending a priori on $e$, such that $u_{2}(e)=\lambda_{e} u_{1}(e)$. Here, $\lambda_{e^{\prime}}=\lambda_{e}$ at least when $e^{\prime}=\mu e$ is collinear to $e$. On the other hand, taking $e$ and $e^{\prime}$ linearly independent and expressing $u_{2}\left(e+e^{\prime}\right)$ in two ways:

$$
\begin{aligned}
& \lambda_{e} u_{1}(e)+\lambda_{e^{\prime}} u_{1}\left(e^{\prime}\right)=u_{2}(e)+u_{2}\left(e^{\prime}\right)= \\
& =u_{2}\left(e+e^{\prime}\right)=\lambda_{e+e^{\prime}} u_{1}\left(e+e^{\prime}\right)= \\
& =\lambda_{e+e^{\prime}} u_{1}(e)+\lambda_{e+e^{\prime}} u_{1}\left(e^{\prime}\right),
\end{aligned}
$$

we get $\lambda_{e}=\lambda_{e+e^{\prime}}=\lambda_{e^{\prime}}$ for any $e$ and $e^{\prime}$, so $\lambda_{e} \equiv \lambda$ is constant.
Given another $v \in G \mathrm{~L}_{n+1}(\mathbb{K})$, we can write $\mathbb{P}(v \circ u)=\mathbb{P}(v) \circ \mathbb{P}(u)$ and we also clearly have $\mathbb{P}\left(u \circ u^{-1}\right)=\mathbb{P}\left(u^{-1} \circ u\right)=\mathbb{P}\left(\operatorname{Id}_{\mathbb{K}^{n+1}}\right)=\operatorname{Id}_{\mathbb{K}^{P}}$. It follows that the projective transformations of $\mathbb{K}^{p}$ into itself form a group, called the projective group of $\mathbb{K} \mathbb{P}^{n}$ and denoted $\mathrm{PGL}_{n}(\mathbb{K})$. The lemma shows that

$$
\mathrm{PGL}_{n}(\mathbb{K})=\mathrm{GL}_{n+1}(\mathbb{K}) / \mathbb{K} \cdot I_{n+1} \simeq \mathrm{SL}_{n+1}(\mathbb{K})
$$

Theorem. Let $\left(P_{0}, P_{1}, \ldots, P_{n}, P_{n+1}\right)$ and $\left(P_{0}^{\prime}, P_{1}^{\prime}, \ldots, P_{n}^{\prime}, P_{n+1}^{\prime}\right)$ be two $(n+2)$-tuples of points of $\mathbb{K}^{\mathbb{P}^{n}}$ which both constitute a projective frame. Then there exists a unique projective transformation $h=\mathbb{P}(u) \in \mathrm{PGL}_{n}(\mathbb{K})$,
$u \in \mathrm{GL}_{n+1}(\mathbb{K})$, which maps the first frame to the second one, namely such that $h\left(P_{i}\right)=P_{i}^{\prime}$ for $i=0,1, \ldots, n, n+1$.

Proof. The lemma p. 22 enables us to lift $P_{0}, P_{1}, \ldots, P_{n}$ as a basis $\left(e_{0}, e_{1}, \ldots, e_{n}\right)$ such that $\pi\left(e_{0}+e_{1}+\cdots+e_{n}\right)=P_{n+1}$ and similarly, to lift $P_{0}^{\prime}, P_{1}^{\prime}, \ldots, P_{n}^{\prime}$ as $\left(e_{0}^{\prime}, e_{1}^{\prime}, \ldots, e_{n}^{\prime}\right)$ with $\pi\left(e_{0}^{\prime}+e_{1}^{\prime}+\cdots+e_{n}^{\prime}\right)=P_{n+1}^{\prime}$. The map $u$ defined by $u\left(e_{0}\right)=e_{0}^{\prime}, u\left(e_{1}\right)=e_{1}^{\prime}, \ldots, u\left(e_{n}\right)=e_{n}^{\prime}$ and simply extended by linearity, whence $u\left(e_{0}+e_{1}+\cdots+e_{n}\right)=e_{0}^{\prime}+e_{1}^{\prime}+\cdots+e_{n}^{\prime}$, does the job: its projectivization $h:=\mathbb{P}(u)$ clearly satisfies $h\left(P_{i}\right)=P_{i}^{\prime}$, for $i=0,1, \ldots, n, n+1$.

On the other hand, if $k=\mathbb{P}(v)$ is another such projective transformation performing $k\left(P_{j}\right)=P_{j}^{\prime}$, namely $v\left(e_{i}\right)=\lambda_{i} e_{i}^{\prime}$ for $i=0,1, \ldots, n$ and also $v\left(e_{n+1}\right)=\mu e_{n+1}^{\prime}$, we deduce by inserting $e_{n+1}=e_{0}+e_{1}+\cdots+e_{n}$ and $e_{n+1}^{\prime}=e_{0}^{\prime}+e_{1}^{\prime}+\cdots+e_{n}^{\prime}:$

$$
\lambda_{0} e_{0}^{\prime}+\lambda_{1} e_{1}^{\prime}+\cdots+\lambda_{n} e_{n}^{\prime}=v\left(e_{n+1}\right)=\mu e_{n+1}^{\prime}=\mu e_{0}^{\prime}+\mu e_{1}^{\prime}+\cdots+\mu e_{n}^{\prime}
$$

whence $\lambda_{0}=\mu=\lambda_{1}=\cdots=\lambda_{n}$, so $v=\mu u$ and hence $k=\mathbb{P}(v)=$ $\mathbb{P}(u)=h$ : this shows uniqueness.

## Structural Properties

## Chapter 26 (Voll).

## The General Projective Group.

The equations:

$$
\begin{equation*}
x_{\nu}^{\prime}=\frac{a_{1 \nu} x_{1}+\cdots+a_{n \nu} x_{n}+a_{n+1, \nu}}{a_{1, n+1} x_{1}+\cdots+a_{n, n+1} x_{n}+a_{n+1, n+1}} \quad \quad(\nu=1 \cdots n) \tag{1}
\end{equation*}
$$

determine a group, as one easily convinces oneself, the so-called general projective group of the manifold $x_{1}, \ldots, x_{n}$. In the present chapter, we want to study somehow more closely this important group, which is also called the group of all collineations of the space $x_{1}, \ldots, x_{n}$, by focusing our attention especially on its subgroups.

## § 134.

The $(n+1)^{2}$ parameters $a$ are not all essential: there indeed appears just their ratios; one of the parameters, best $a_{n+1, n+1}$, can hence be set equal to 1 . The values of the parameters are subjected to the restriction that the substitution determinant [SUBSTITUTIONSDETERMINANT] $\sum \pm a_{11} \cdots a_{n+1, n+1}$ should not be equal to zero; because at the same
|time with it the functional determinant [FUNCTIONALDETERMINANT]: $\sum \pm \frac{\partial x_{1}^{\prime}}{\partial x_{1}} \ldots \frac{\partial x_{n}^{\prime}}{\partial x_{n}}$ would also vanish.

The identical transformation is contained in our group, it corresponds to the values of the parameters:

$$
a_{\nu \nu}=1, \quad a_{\mu \nu}=0 \quad(\mu, \nu=1 \cdots n+1, \mu \neq \nu),
$$

for which indeed it comes $x_{i}^{\prime}=x_{i}$. As a consequence of that, one obtains the infinitesimal transformations of the group by giving to the $a_{\mu \nu}$ the values:

$$
a_{\nu \nu}=1+\omega_{\nu \nu}, \quad a_{n+1, n+1}=1, \quad a_{\mu \nu}=\omega_{\mu \nu}
$$

where the $\omega_{\mu \nu}$ mean infinitesimal quantities. Thus one finds:

$$
x_{\nu}^{\prime}=\left(x_{\nu}+\sum_{1 \leqslant \mu \leqslant n} \omega_{\mu \nu} x_{\mu}+\omega_{n+1, \nu}\right)\left(1-\sum_{1 \leqslant \mu \leqslant n} \omega_{\mu, n+1} x_{\mu}+\cdots\right),
$$

or by leaving out the quantities of second or higher order:

$$
x_{\nu}^{\prime}-x_{\nu}=\sum_{1 \leqslant \mu \leqslant n} \omega_{\mu \nu} x_{\mu}+\omega_{n+1, \nu}-x_{\nu} \sum_{1 \leqslant \mu \leqslant n} \omega_{\mu, n+1} x_{\mu} .
$$

If one sets here all the $\omega_{\mu \nu}$ with the exception of a single one equal to zero, then one recognizes bit by bit that our group comprises the $n(n+2)$ independent infinitesimal transformations:
(2) $\frac{\partial f}{\partial x_{i}}, \quad x_{i} \frac{\partial f}{\partial x_{k}}, \quad x_{i} \sum_{j=1}^{n} x_{j} \frac{\partial f}{\partial x_{j}} \quad(i, k=1 \cdots n)$

The general projective group of the $n$-fold extended space $x_{1}, \ldots, x_{n}$ therefore contains $n(n+2)$ essential parameters and is engendered by infinitesimal transformations. The analytic expressions of the latter behave regularly for every point of the space.

From now on, we will as a rule write $p_{i}$ for $\frac{\partial f}{\partial x_{i}}$. In addition, for reasons of convenience, we want to introduce the abbreviations:

$$
x_{i} p_{k}=T_{i k}, \quad x_{i} \sum_{k=1}^{n} x_{k} p_{k}=P_{i}
$$

in this chapter. Lastly, we still want to agree on that $\varepsilon_{i k}$ should mean zero every time $i$ and $k$ are distinct from each other, whereas by contrast $\varepsilon_{i i}$ shall have the value 1 ; a terminology fixing that we have already adopted from time to time. On such a basis, we can write as follows the relations which
come out through Combination [bracketting] of the infinitesimal transformations $p_{i}, T_{i k}, P_{i}$ :

$$
\begin{gathered}
{\left[p_{i}, p_{k}\right]=0, \quad\left[P_{i}, P_{k}\right]=0, \quad\left[p_{i}, P_{k}\right]=T_{k i}+\varepsilon_{i k} \sum_{\nu=1}^{n} T_{\nu \nu}} \\
{\left[p_{i}, T_{k \nu}\right]=\varepsilon_{i k} p_{\nu}, \quad\left[P_{i}, T_{k \nu}\right]=-\varepsilon_{i \nu} P_{k}} \\
{\left[T_{i k}, T_{\mu \nu}\right]=\varepsilon_{k \mu} T_{i \nu}-\varepsilon_{\nu i} T_{\mu k}}
\end{gathered}
$$

One easily convinces oneself that these relations remain unchanged when one substitutes in them the $p_{i}, T_{i k}$ and $P_{i}$ by the respective expressions standing under them in the pattern:

$$
\begin{array}{ccc}
p_{i}, & T_{i k}, & P_{i} \\
P_{i}, & -T_{k i}, & p_{i} . \tag{3}
\end{array}
$$

Thus in this way, the general projective group can be referred to as holoedric Isomorph to itself.

One could presume that there is a transformation: $x_{i}^{\prime}=\Phi_{i}\left(x_{1}, \ldots, x_{n}\right)$ which transfers the infinitesimal transformations:

$$
p_{i}, \quad x_{i} p_{k}, \quad x_{i} \sum_{k=1}^{n} x_{k} p_{k}
$$

respectively to:

$$
x_{i}^{\prime} \sum_{k=1}^{n} x_{k}^{\prime} p_{k}^{\prime}, \quad-x_{k}^{\prime} p_{i}^{\prime}, \quad p_{i}^{\prime} .
$$

But there is no such transformation,
because the $n$ infinitesimal transformations $p_{1}, \ldots, p_{n}$ engender an $n$-term transitive group, while: $x_{1}^{\prime} \sum x_{k}^{\prime} p_{k}^{\prime}, \ldots, x_{n}^{\prime} \sum x_{k}^{\prime} p_{k}^{\prime}$ engender an $n$-term intransitive group.

First in the next chapter we will learn to see the full signification of this important property of the projective group, when the concept of contact transformation [BERÜHRUNGSTRANSFORMATION] and especially the duality will be introduced.
$\triangleright$ Duality and contact transformations.

The general infinitesimal transformation:

$$
\sum_{i=1}^{n} a_{i} p_{i}+\sum_{i, k=1}^{n} b_{i k} T_{i k}+\sum_{i=1}^{n} c_{i} P_{i}
$$

of our group is [already per se] expanded in powers of $x_{1}, \ldots, x_{n}$ and visibly contains only terms of zeroth, first and second order in the $x$. One
easily realizes that the group comprises $n$ independent infinitesimal transformations of zeroth order in $x$, out of which no infinitesimal transformation of first or second order in the $x$ can be deduced linearly. For instance, $p_{1}, \ldots, p_{n}$ are $n$ such infinitesimal transformations. From this it follows that the general projective group is transitive.

Besides, there are $n^{2}$ infinitesimal transformations of first order in the $x_{i}$, for instance all $x_{i} p_{k}=T_{i k}$, out of which no one of second order can be deduced linearly. Finally it yet arises $n$ transformations of second order in the $x$ :

$$
x_{i} \sum_{k=1}^{n} x_{k} p_{k}=P_{i} .
$$

In agreement with the Proposition 9 of the Chap. 15 on p. 264 the $P_{i}$ are exchangeable in pairs and in addition, the $T_{i k}$ together with the $P_{i}$ engender a subgroup in which the group of the $P_{i}$ is contained as invariant subgroup.
$\triangleright \boldsymbol{A}$ check. A (local) Lie subgroup $H$ of a (local) Lie group $G$ is invariant in $G$, meaning that $g H g^{-1}=H$ for every $g \in G$, if and only if, at the level of the two corresponding Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$, on has $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. Here, the concerned bracket relations are: $\left[P_{i}, T_{k \nu}\right]=-\varepsilon_{i \nu} P_{k}$.

As one sees, and also as it follows from our remark above about the relationship between the $p_{i}$ and the $P_{i}$, the $p_{i}$ are also exchangeable in pairs and they engender together with the $T_{i k}$ a subgroup in which the group of the $p_{i}$ is invariant.

## § 135.

For the most important subgroups of the general projective group, it is advisable to employ special names. If, in the general expression (1) of a projective transformation, one lets the denominator reduce to 1 , then one gets a linear transformation:

$$
x_{\nu}^{\prime}=a_{1 \nu} x_{1}+\cdots+a_{n \nu} x_{n}+a_{n+1, \nu} \quad(\nu=1 \cdots n) ;
$$

all transformations of this kind constitute the so-called general linear group. We have already indicated at the end of the previous paragraph the infinitesimal transformations of this group; they are deduced by linear combination from the the following $n(n+1)$ ones:

$$
p_{i}, \quad x_{i} p_{k} \quad(i, k=1 \cdots n)
$$

If one interprets $x_{1}, \ldots, x_{n}$ as coordinates of an $n$-fold extended space $R_{n}$ and if one translates the way of expressing into the ordinary space, then
one can say that the general linear group consists of all projective transformations which leave invariant the infinitely far $(n-1)$-fold extended even manifold, or briefly, the infinitely far plane [UNDENDLICH FERNE EbENE] $M_{n-1}$.

Next, if one remembers that by execution of two finite linear transformations one after the other, the substitution determinants: $\sum \pm a_{11} \cdots a_{n n}$ multiply them, then one realizes without difficulty that the totality of all linear transformations whose determinant equals 1 constitutes a subgroup, and in fact, an invariant subgroup, which we want to call the special linear group. One finds easily that as the $n(n+1)-1$ independent infinitesimal transformations of this group, the following can be chosen:

$$
p_{i}, \quad x_{i} p_{k}, \quad x_{i} p_{i}-x_{k} p_{k} \quad(i \gtrless k) .
$$

If, amongst all linear transformations, one restricts oneself to those homogeneous in $x$, then one obtains the general linear homogeneous group:

$$
x_{\nu}^{\prime}=a_{1 \nu} x_{1}+\cdots+a_{n \nu} x_{n} \quad(\nu=1 \cdots n),
$$

whose infinitesimal transformations all possess the form: $\sum b_{i k} x_{i} p_{k}$ and hence can be linearly deduced from the $n^{2}$ transformations: $x_{i} p_{k}$. Also this group visibly contains an invariant subgroup, the special linear homogeneous group, for which: $\sum \pm a_{11} \cdots a_{n n}$ has the value 1 . The $n^{2}-1$ infinitesimal transformations of this latter are:

$$
x_{i} p_{k}, \quad x_{i} p_{i}-x_{k} p_{k} \quad(i \gtrless k) ;
$$

therefore the general infinitesimal transformation of the group in question has the form: $\sum i, k \alpha_{i k} x_{i} p_{k}$, where the $n^{2}$ arbitrary constants $\alpha_{i k}$ are only subjected to the condition $\sum \alpha_{i i}=0$.

Since the expression: $\left[x_{i} p_{k}, \sum_{j} x_{j} p_{j}\right]$ always vanishes, it is obvious that the last two named groups are systatic and consequently imprimitive. Indeed, if one sets:

$$
\frac{x_{i}}{x_{n}}=y_{i}, \quad \frac{x_{i}^{\prime}}{x_{n}^{\prime}}=y_{i}^{\prime} \quad(i=1 \cdots n-1),
$$

then one receives:

$$
y_{\nu}^{\prime}=\frac{a_{1 \nu} y_{1}+\cdots+a_{n-1, \nu} y_{n-1}+a_{n \nu}}{a_{1, n} y_{1}+\cdots+a_{n-1, n} y_{n-1}+a_{n, n}} \quad \quad(i=1 \cdots n-1) .
$$

It results from this that in both cases the $y$ are transformed by the $\left(n^{2}-\right.$ 1 )-term general projective group of the $(n-1)$-fold extended manifold $y_{1}, \ldots, y_{n-1}$. Consequently, this group is Isomorph with the general linear homogeneous group of an $n$-fold extended manifold and with the special
linear homogeneous group as well, though the Isomorphism is holoedric only for the special linear homogeneous group, since this one contains $n^{2}-1$ parameters.

Theorem 96. The special linear homogeneous group:

$$
x_{i} p_{k}, \quad x_{i} p_{i}-x_{k} p_{k} \quad(i \gtrless k=1 \cdots n)
$$

in the variables $x_{1}, \ldots, x_{n}$ is imprimitive and holoedric Isomorph with the general projective group of an $(n-1)$-times extended manifold.

The formally simplest infinitesimal transformations of the general projective group are $p_{1}, \ldots, p_{n}$; these generate, as already observed, a group actually: the group of all translations:

$$
x_{i}^{\prime}=x_{i}+a_{i} \quad(i=1 \cdots n),
$$

which obviously is simply transitive.
Generally, $m$ arbitrary infinitesimal translations, for instance $p_{1}, \ldots, p_{m}$, always generate an $m$-term group. For all of these groups, the following holds:

Proposition 1. All m-term groups of translations are conjugate to each other inside the general projective group, and even inside the general linear group.

Indeed, the $m$ independent infinitesimal transformations of such a group always have the form:

$$
\sum_{\nu=1}^{n} b_{\mu \nu} p_{\nu} \quad(\mu=1 \cdots m)
$$

where not all $m \times m$ determinants of the $b_{\mu \nu}$ vanish.
But we can very easily show that by means of some linear transformation, new variables $x_{1}^{\prime}, \ldots, x_{n}^{\prime}$ can be introduced for which one has:

$$
p_{\mu}^{\prime}=\sum_{\nu=1}^{n} b_{\mu \nu} p_{\nu} \quad(\mu=1 \cdots m)
$$

In fact, let $p_{\mu}^{\prime}=p_{1} \frac{\partial x_{1}}{\partial x_{\mu}^{\prime}}+\cdots+p_{n} \frac{\partial x_{n}}{\partial x_{\mu}^{\prime}}$, then we only need to set:

$$
\frac{\partial x_{\nu}}{\partial x_{\mu}^{\prime}}=b_{\mu \nu} \quad(\nu=1 \cdots n ; \mu=1 \cdots m),
$$

while the $\frac{\partial x_{\nu}}{\partial x_{m+1}^{\prime}}, \ldots, \frac{\partial x_{\nu}}{\partial x_{n}^{\prime}}$ remain arbitrary. We can give to these last ones some values such that the equations:

$$
x_{\nu}=\sum_{\mu=1}^{n} b_{\mu \nu} x_{\mu}^{\prime}+\sum_{\pi=m+1}^{n} c_{\pi \nu} x_{\pi}^{\prime} \quad(\nu=1 \cdots n)
$$

determine a transformation, and this transformation transfers the given group of translations to the group $p_{1}^{\prime}, \ldots, p_{m}^{\prime}$. From this, our proposition follows immediately.

We want to at least indicate a second proof of the same proposition. As already observed, the general linear group leaves invariant the infinitely far plane $M_{n-1}$, and in fact, it is even the most general projective group of this nature. Now, every infinitesimal translation is directed by an infinitely far point and is completely determined by this point; every m-term group of translations can therefore be represented by an m-fold extended, infinitely far, straight manifold $M_{m}$. But two infinitely far straight $M_{m}$ always can be transferred one to the other by a linear transformation which leaves invariant the infinitely far plane. Consequently, all $m$-term groups of translations are conjugate to each other inside the general linear group, and in the same way, inside the general projective group.

The correspondence indicated earlier which takes place between the $p_{i}$ and the $P_{i}$ yields, as we prove instantly, the

Proposition 2. All m-term groups, whose infinitesimal transformations possess the form $\sum e_{i} P_{i}$, are conjugate to each other inside the general projective group.

For the proof, we start from the fact that two subgroups are conjugate inside a group $G_{r}$ when the one can be, by means of a transformation of the adjoint group of $G_{r}$, transferred to the other; here, we have to imagine the subgroups as an even manifold in the space $e_{1}, \ldots, e_{r}$, which is transformed by the adjoint group (cf. Chap. 16, p. 280; [here: ??p. ??]). If we now write the transformations of the projective group firstly in the sequence $p_{i}, T_{i k}, P_{i}$ and next in the sequence $P_{i},-T_{k i}, p_{i}$, then in the two cases we get the same adjoint group. But since two $m$-term groups of translations can always be transferred one to the other by the adjoint group, this must also always be the case with two $m$-term groups whose infinitesimal transformations can be deduced linearly from the $P_{i}$. Furthermore, it even immediately comes out that two $m$-term groups of this sort are already conjugate to each other inside the group $P_{i}, T_{i k}$. With that, our proposition is proved.
$\triangleright$ Explanation. Structure is invariant by this involution.

## § 136.

We consider now one after the other the general projective group, the general linear group and the linear homogeneous group, and to be precise, we want to examine whether there are invariant subgroups and which one are contained in these three groups.

At first, the general projective group. Let:

$$
S=\sum_{i=1}^{n} \alpha_{i} p_{i}+\sum_{i=1}^{n} \sum_{k=1}^{n} \beta_{i k} x_{i} p_{k}+\sum_{i=1}^{n} \gamma_{i} x_{i} \sum_{k=1}^{n} x_{k} p_{k}
$$

be an infinitesimal transformation of an invariant subgroup; then necessarily $\left[p_{\nu}, S\right]$ and $\left[p_{\mu},\left[p_{\nu}, S\right]\right]$ are also tranformations of the same subgroup. Consequently, in our invariant subgroup, there would certainly appear an infinitesimal translation $\sum \rho_{i} p_{i}$.
$\triangleright$ A check. It is nonzero.
$\triangleleft$
But because all infinitesimal translations are conjugate to each other inside the general projective group, they would all appear. Furthermore, since it is invariant, the subgroup would necessarily contain all transformations: $\left[p_{i}, x_{i} \sum_{j} x_{j} p_{j}\right]$, or after computation:

$$
x_{i} p_{k} \quad(i \gtrless k), \quad x_{i} p_{i}+\sum_{j=1}^{n} x_{j} p_{j} .
$$

Adding the $n$ transformations: $x_{i} p_{i}+\sum_{j} x_{j} p_{j}$, one obtains: $(n+$ 1) $\sum x_{j} p_{j}$, whence $x_{i} p_{i}$ and therefore in general all $x_{i} p_{k}$. Finally, the invariant subgroup would yet contain all transformations: $\left[x_{i} p_{i}, x_{i} \sum_{k} x_{k} p_{k}\right]$, hence all $x_{i} \sum_{k} x_{k} p_{k}$ and thus it would be identical to the general projective group itself. Thus, our first result is:

Theorem 97. The general projective group in $n$ variables is simple. *)
*) Lie, Math. Ann., Vol. XXV, p. 130.
Correspondingly, the special linear homogeneous group:

$$
\begin{equation*}
x_{i} p_{k}, \quad x_{i} p_{i}-x_{k} p_{k} \quad(i \gtrless k) \tag{4}
\end{equation*}
$$

is also simple.
The general linear homogeneous group with the $n^{2}$ infinitesimal transformations $x_{i} p_{k}$ contains, as we have seen above, an invariant subgroup with $n^{2}-1$ parameters, namely the just named group (4).

If there is yet a second invariant subgroup, then this one obviously cannot contain the group (4), and in the same way, it even cannot have
an infinitesimal transformation in common with the same group, since such transformations would constitute an invariant subgroup in the simple group (4) (cf. Prop. 10 of the Chap. 15 on p. 264).
$\triangleright$ The cited proposition. It just says that the infinitesimal transformations that are in common between two invariant subgroups of a group $G$ do likewise form an invariant subgroup of $G$. For abstract or for vector field Lie algebras, it says: $\left[\mathfrak{h}_{1} \cap \mathfrak{h}_{2}, \mathfrak{g}\right] \subset \mathfrak{h}_{1} \cap \mathfrak{h}_{2}$ whenever two Lie subalgebras $\mathfrak{h}_{1}$ and $\mathfrak{h}_{2}$ of a Lie algebra $\mathfrak{g}$ are ideals in it: $\left[\mathfrak{h}_{i}, \mathfrak{g}\right] \subset \mathfrak{h}_{i}$ for $i=1,2$. $\quad \triangleleft$

Taking the Proposition 7 of Chap. 12 on p. 211 into account, it follows that a possible second invariant subgroup can contain only one infinitesimal transformation, and to be precise, one of the form:

$$
\sum_{i=1}^{n} x_{i} p_{i}+\sum_{i, k}^{1 \ldots n} \alpha_{i k} x_{i} p_{k} \quad\left(\sum_{i=1}^{n} \alpha_{i i}=0\right) .
$$

Besides, according to Proposition 11 of Chap. 15 on p. 264, the same transformation must be exchangeable with every transformation of the group (4), from which it follows that the transformation:

$$
\sum_{i, k}^{1 \ldots r} \alpha_{i k} x_{i} p_{k} \quad\left(\sum_{i=1}^{n} \alpha_{i i}=0\right)
$$

must be excellent inside the group (4).
$\triangleright$ The cited propositions. The first one boils down to the dimension formula for intersections of Lie algebras: if an $r$-term group contains two subgroups with $m$ and $\mu$ parameters, then these two have at least $m+\mu-r$ independent infinitesimal transformations in common, and the ones in common do actually form a subgroup.

The second cited proposition states that if two invariant subgroups $Y_{1} f, \ldots, Y_{m} f$ and $Z_{1} f, \ldots, Z_{p} f$ of a group $G$ have no infinitesimal transformations in common, then all the brackets $\left[Y_{i}, Z_{k}\right]$ vanish. It is just because, by the invariancy assumption, each bracket $\left[Y_{i}, Z_{k}\right]$ must be expressible as a linear combination of the $Y_{i}$ and also as a linear combination of the $Z_{k}$ as well, but since no infinitesimal transformation is shared, brackets must hence all be zero.

So applying this observation, $\sum_{i} x_{i} p_{i}+\sum_{i, k} \alpha_{i k} x_{i} p_{k}$ must be excellent inside the group (4), as was $\sum_{i} x_{i} p_{i}$ for free.

But there is no such transformation, whence all the $\alpha_{i k}$ vanish and it shows up that $x_{1} p_{1}+\cdots+x_{n} p_{n}$ and (4) are the only two invariant subgroups of the group $x_{i} p_{k}$.

Theorem 98. The general linear homogeneous group $x_{i} p_{k}$ in $n$ variables contains only two invariant subgroups, namely the special linear homogeneous group and the one-term group: $x_{1} p_{1}+\cdots+x_{n} p_{n}$.

At present, one easily manages to set up all invariant groups of the general linear group. Let:

$$
S=\sum_{i=1}^{n} \alpha_{i} p_{i}+\sum_{i=1}^{n} \sum_{k=1}^{n} \beta_{i k} x_{i} p_{k}
$$

be a transformation of such a subgroup. Then together with $S$, also $\left[p_{j}, S\right]$ belongs to the invariant subgroup; hence the same certainly contains a translation, and because of Proposition 1, p. 29, it contains all of them.

Check. It is nonzero.
The smallest invariant subgroup therefore consists of the translations themselves; every other one must, aside from the translations, still contain a series of infinitesimal transformations of the form: $\sum_{i} \sum_{k} \alpha_{i k} x_{i} p_{k}$. But these latter ones visibly engender an invariant subgroup, the linear homogeneous group $x_{i} p_{k}$. So we find:

Theorem 99. The general linear group: $p_{i}, x_{i} p_{k}$ contains only three invariant subgroups*), namely the three ones:
$p_{i} \quad p_{i}, x_{1} p_{1}+\cdots+x_{n} p_{n} \quad p_{i}, x_{i} p_{k}, x_{i} p_{i}-x_{k} p_{k} \quad(i \gtrless k)$, with respectively $n, n+1$ and $n^{2}+n-1$ parameters.
*) Lie, Math. Ann., Vol. XXV, p. 130.
If, as already done several times, we employ the terminology which is common for the ordinary space, we can say: the three invariant subgroups of the general linear group are firstly the group of all translations, secondly the group of all similitudes [AEHNLICHKEITSTRANSFORMATIONENEN]: $\left(x_{1}-x_{1}^{0}\right) p_{1}+\cdots+\left(x_{n}-x_{n}^{0}\right) p_{n}$, and lastly the most general linear group which leaves all volumes unchanged.

## Prologue for Part II

## Classification of Lie algebras of holomorphic vector fields

Before launching ourselves on the classification theorems joint with Engel and his Master Lie, we provide brief recalls of the basic fundamental theory (Part I).

## Suppressing in Advance Illusory Parameters

As an example, we illustrate how do these three principles work to get rid of redundants $a_{k}$ 's. Developing the $f_{i}$ of $x_{i}^{\prime}=f_{i}(x, a)$ in power series with respect to $x-x_{0}$ in some (unnamed, connected) neighbourhood of a fixed point $x_{0}$ :

$$
f_{i}(x, a)=\sum_{\alpha \in \mathbb{N}^{n}} \mathcal{U}_{\alpha}^{i}(a)\left(x-x_{0}\right)^{\alpha}
$$

we get an infinite number of analytic functions $\mathcal{U}_{\alpha}^{i}=\mathcal{U}_{\alpha}^{i}(a)$ of the parameters that are defined in some uniform domain, say $U$, of $\mathbb{K}^{r}$. Then we claim that superfluous $a_{k}$ 's can be visible by just looking at the rank of the coefficient mapping $\mathrm{U}_{\infty}$, in its wholeness:

$$
\mathrm{U}_{\infty}: \quad \mathbb{K}^{r} \ni a \longmapsto\left(\mathcal{U}_{\alpha}^{i}(a)\right)_{\alpha \in \mathbb{N}^{n}}^{1 \leqslant i \leqslant n} \in \mathbb{K}^{\infty}
$$

If for instance there is one parameter, say $a_{1}$, upon which absolutely no $\mathcal{U}_{\alpha}^{i}$ does depend, then this map $\mathrm{U}_{\infty}$ clearly has rank $\leqslant r-1$ at every point. Specifically, one looks at the generic rank $\rho_{\infty}$ of $\mathrm{U}_{\infty}$, an integer satisfying $0 \leqslant \rho_{\infty} \leqslant r$, namely the maximal possible rank of $U_{\infty}$, which is in fact attained at every point $a \in U \backslash \mathrm{D}$ outside a certain ${ }^{1}$ proper closed analytic set D. So, avoiding ${ }^{2}$ D, if we relocalize to a small neighbourhood of some point $a \in U \backslash \mathrm{D}$, a suitable application of the constant rank theorem, followed by an appropriate local diffeomorphism $a \mapsto \bar{a}=\bar{a}(a)$ of the parameter space, enables to show ([25]; [here: see Chapter A]) that the new coefficients $\overline{\mathcal{U}}_{\alpha}^{i}(\bar{a})$ become absolutely independent of the $r-\rho_{\infty}$ last parameters $\bar{a}_{\rho_{\infty}+1}, \ldots, \bar{a}_{r}$ : they thus have become visibly superfluous.

[^0]Definition. The parameters $\left(a_{1}, \ldots, a_{r}\right)$ of given point transformation equations $\bar{x}_{i}=f_{i}(x, a)$ are called essential if, after developing $f_{i}(x, a)=\sum_{\alpha \in \mathbb{N}^{n} \boldsymbol{u}} \mathcal{U}_{\alpha}^{i}(a)\left(x-x_{0}\right)^{\alpha}$ in power series at some $x_{0}$, the generic rank $\rho_{\infty}$ of the coefficient mapping $U_{\infty}: a \longmapsto\left(\mathcal{U}_{\alpha}^{i}(a)\right)_{\alpha \in \mathbb{N}^{n}}^{1 \leqslant i \leqslant n}$ is maximal, equal to the number $r$ of parameters: $\rho_{\infty}=r$.

In this case, the transformation equations are called $r$-term [ $r$ GLIEDRIG]; we adopt the translation of [1]. From now on, parameters will always be assumed to be essential.

## Concept of local Lie group

We restitute here basic definitions and theorems without emphasizing the formal rigor about (shrunk) domains that Chap. B will fully provide.

In arbitrary dimension $n \geqslant 1$, a finite continuous transformation group on $\mathbb{K}^{n}$ is a finitely parametrized family of analytic point diffeomorphic transformations:

$$
x_{i}^{\prime}=f_{i}\left(x_{1}, \ldots, x_{n} ; a_{1}, \ldots, a_{r}\right) \quad(i=1 \cdots n)
$$

enjoying the following three properties.
Group composition law: Whenever it is well defined, the succession $x^{\prime}=$ $f(x ; a)$ and $x^{\prime \prime}=f\left(x^{\prime} ; b\right)$ of any two such transformations, namely:

$$
x^{\prime \prime}=f(f(x ; a) ; b)=f(x ; c)
$$

always identifies to an element of the same family, for some new parameter:

$$
c=\mathbf{m}(a, b)
$$

uniquely and precisely defined by a certain local analytic map $\mathbf{m}$ : $\mathbb{K}^{r} \times \mathbb{K}^{r} \rightarrow \mathbb{K}^{r}$, which, from its side, inherits automatically the property $\mathbf{m}(\mathbf{m}(a, b), c)=\mathbf{m}(a, \mathbf{m}(b, c))$ from the associativity of diffeomorphism composition.
Existence of an identity element: There exists a special parameter $e=$ $\left(e_{1}, \ldots, e_{r}\right)$ such that $f(x ; e) \equiv x$ is just the identity mapping.
Underlying group multiplication law: The analytic map $(a, b) \longmapsto$ $\mathbf{m}(a, b)$, which can sometimes also be alternatively written shortly $(a, b) \longmapsto a \cdot b$, is a local continuous group law, in the sense that:

- For all $a$, one should have: $a \cdot e=e \cdot a=a$, a property which follows in fact from:

$$
f(x ; a \cdot e)=f(f(x ; a) ; e)=f(x ; a)=f(f(x ; e) ; a)=f(x ; e \cdot a)
$$

thanks to the postulated uniqueness of $c=\mathbf{m}(a, b)$.

- Also, the inherited associativity $(a \cdot b) \cdot c=a \cdot(b \cdot c)$ should hold.
- Inverse-element map: Finally, as a last axiom which is not a consequence of the group composition law, there must exist a local diffeomor$\operatorname{phism} \mathbf{i}: \mathbb{K}^{r} \rightarrow \mathbb{K}^{r}$ defined in a neighbourhood of $e$ with $\mathbf{i}(e)=e$ such that $a \cdot \mathbf{i}(a)=\mathbf{i}(a) \cdot a=e$, namely $\mathbf{i}(a)$ represents the group inverse of $a$, and moreover, $a \mapsto \mathbf{i}(a)$ is an analytic, necessarily diffeomorphic, local map. In particular, writing now that the composition:

$$
f(f(x ; a) ; b)=f(x ; a \cdot b)
$$

is just performed by group multiplication between parameters, one then formally deduces:

$$
f(f(x ; a) ; \mathbf{i}(a)) \equiv f(x ; a \cdot \mathbf{i}(a)) \equiv x \equiv f(x ; \mathbf{i}(a) \cdot a) \equiv f(f(x ; \mathbf{i}(a)) ; a) .
$$

## Notion of $r$-term continuous transformation group

It is useful for further reading to remember that in Lie's terminology, a "finite continous (transformation) group of a space" precisely means a finite-dimensional, local, analytic Lie group action $x^{\prime}=f(x ; a)$ as above; Lie does not emphasizes the everywhere presupposed analyticity, but he uses instead the word continuous to make clear the contrast of his own theory with the discrete Galois theory of algebraic equations that inspired him (thorough, exciting history appears in [17]). What we nowadays call a local Lie group, namely a $\mathbb{K}^{r}$ around some identity element $e$ equipped with a local analytic group multiplication $(a, b) \longmapsto \mathbf{m}(a, b)=a \cdot b$ together with an analytic inverse-element map $a \mapsto \mathbf{i}(a)$, is called by Lie the "parameter group of a transformation group"; pages 401-429 of Vol. I are devoted to its general study. Finally, for Lie, the adjective " $r$-term" means that the $r$ written parameters $\left(a_{1}, \ldots, a_{r}\right)$ are essential, or equivalently, that the dimension of the parameter group is exactly $r$. In summary:
" $r$-term group of $x_{1}, \ldots, x_{n} " \Longleftrightarrow r$-dimensional Lie group acting on $\mathbb{K}^{n}$

## Introduction of Infinitesimal Transformations

Next, letting $\varepsilon$ denote either an infinitesimal quantity in the sense of Leibniz, or a small quantity subjected to Weierstrass' rigorous epsilon-delta formalism, for fixed $k \in\{1,2, \ldots, r\}$, we consider all the points:

$$
\begin{aligned}
x_{i}^{\prime} & =f_{i}\left(x ; e_{1}, \ldots, e_{k}+\varepsilon, \ldots, e_{n}\right) \\
& =x_{i}+\frac{\partial f_{i}}{\partial a_{k}}(x ; e) \cdot \varepsilon+\cdots \quad(i=1 \cdots n)
\end{aligned}
$$

that are infinitesimally pushed from the starting points $x=f(x ; e)$ by adding the tiny increment $\varepsilon$ to only the $k$-th identity parameter $e_{k}$. One may
reinterpret this common spatial move by introducing the vector field (and a new notation for its coefficients):

$$
X_{k}^{e}:=\sum_{i=1}^{n} \frac{\partial f_{i}}{\partial a_{k}}(x ; e) \frac{\partial}{\partial x_{i}}:=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial}{\partial x_{i}},
$$

which is either written as a derivation in modern style, or considered as a column vector:

$$
\left.\tau\left(\frac{\partial f_{1}}{\partial a_{k}}, \cdots, \frac{\partial f_{n}}{\partial a_{k}}\right)\right|_{x}=\left.^{\tau}\left(\xi_{k 1}, \ldots, \xi_{k n}\right)\right|_{x}
$$

based at $x$, where ${ }^{\tau}(\cdot)$ denotes a transposition, yielding column vectors. Then $x^{\prime}=x+\varepsilon X_{k}^{e}+\cdots$, or equivalently:

$$
x_{i}^{\prime}=x_{i}+\varepsilon \xi_{k i}+\cdots \quad \quad(i=1 \cdots n),
$$

where the left out terms " $+\cdots$ " are of course an $\mathrm{O}\left(\varepsilon^{2}\right)$, so that from the geometrical viewpoint, $x^{\prime}$ is infinitesimally pushed along the vector $\left.X_{k}^{e}\right|_{x}$ up to a length $\varepsilon$.


Fig. : Infinitesimal displacement $x^{\prime}=x+\varepsilon X^{e}$ of all points
More generally, still starting from the identity parameter $e$, when we add to $e$ an arbitrary infinitesimal increment:

$$
\left(e_{1}+\varepsilon \lambda_{1}, \ldots, e_{k}+\varepsilon \lambda_{k}, \ldots, e_{r}+\varepsilon \lambda_{r}\right),
$$

where $\left.{ }^{\tau}\left(\lambda_{1}, \ldots, \lambda_{r}\right)\right|_{e}$ is a fixed, constant vector based at $e$ in the parameter space, it follows by linearity of the tangential map, or else just by the chain rule in coordinates, that:

$$
\begin{aligned}
f_{i}(x ; e+\varepsilon \lambda) & =x_{i}+\sum_{k=1}^{n} \varepsilon \lambda_{k} \cdot \frac{\partial f_{i}}{\partial a_{k}}(x ; e)+\cdots \\
& =x_{i}+\varepsilon \sum_{k=1}^{n} \lambda_{k} \cdot \xi_{k i}(x)+\cdots
\end{aligned}
$$

so that all points $x^{\prime}=x+\varepsilon X+\cdots$ are infinitesimally and simultaneously pushed along the vector field:

$$
X:=\lambda_{1} X_{1}^{e}+\cdots+\lambda_{r} X_{r}^{e}
$$

which is the general linear combination of the $r$ previous basic vector fields $X_{k}^{e}, k=1, \ldots, r$.

Occasionally, Lie wrote that such a vector field $X$ belongs to the group $x^{\prime}=f(x ; a)$, to mean that $X$ comes itself with the infinitesimal move $x^{\prime}=$ $x+\varepsilon X$ it is supposed to perform (dots should now be suppressed in intuition), and hence accordingly, Lie systematically called such an $X$ an infinitesimal transformation, viewing indeed $x^{\prime}=x+\varepsilon X$ as just a case of $x^{\prime}=f(x, a)$. Another, fundamental and very deep reason why Lie said that $X$ belongs to the group $x^{\prime}=f(x, a)$ is that he showed that local transformation group actions are in one-to-one correspondence with the purely linear vector spaces:

$$
\operatorname{Vect}_{\mathbb{K}}\left(X_{1}, X_{2}, \ldots, X_{r}\right)
$$

of infinitesimal transformations, which in fact also inherit a crucial additional algebraic structure directly from the group multiplication law.

## Lie's Basic Main Theorem and Its Converse

Indeed, the major discovery that Lie made in the winter ${ }^{3}$ 187374 was that the infinitesimal transformations $X_{1}, \ldots, X_{r}$ are not only closed under Jacobi bracket as in the so-called Frobenius theorem: $\left[X_{k}, X_{j}\right]=\sum_{s=1}^{r} \mathbf{c}_{k j s}(x) X_{s}$, a condition which insures the existence of local foliations (integrability), but also and principally: the concerned coefficient functions $\mathbf{c}_{k j s}(x)$ are in fact constant: $\mathbf{c}_{k j s}(x) \equiv c_{k j s} \in \mathbb{K}$. From Vol. I of the Theorie der Transformationsgruppen, we translate both Lie's bracket statement and its converse.
Theorem I.22. If an r-term continuous transformation group in the variables $x_{1}, \ldots, x_{n}$ contains the $r$ infinitesimal transformations:

$$
X_{k}(f)=\sum_{1 \leqslant i \leqslant n} \xi_{k i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}} \quad(k=1 \cdots r),
$$

then between these infinitesimal transformations, there exist pairwise relations of the form:

$$
X_{k}\left(X_{i}(f)\right)-X_{i}\left(X_{k}(f)\right)=\sum_{1 \leqslant s \leqslant r} c_{k j s} X_{s}(f),
$$

where the $c_{k j s}$ designate numerical constants.
In contemporary mathematics, one calls $X_{1}, \ldots, X_{n}$ a Lie algebra of (local) analytic vector fields. The assumption that the parameters are essential

[^1]is shown to imply that the $r$ infinitesimal transformations $X_{1}, \ldots, X_{r}$ are linearly independent.

Most importantly, Lie showed conversely that the infinitesimal, linear, algebraic datum of any local Lie algebra of vector fields on the space $x_{1}, \ldots, x_{n}$, enables one to reconstitute readily a local transformation group.
Theorem I.24. If $r$ independent infinitesimal transformations stand pairwise in the relationships:

$$
X_{k}\left(X_{j}(f)\right)-X_{j}\left(X_{k}(f)\right)=\left[X_{k}, X_{j}\right]=\sum_{1 \leqslant s \leqslant r} c_{k j s} X_{s} f,
$$

where the $c_{k j s}$ are constants, then the collection of the $\infty^{r-1}$ one-term groups ${ }^{4}$ :

$$
\lambda_{1} X_{1} f+\cdots+\lambda_{r} X_{r} f
$$

constitutes an r-term group which contains the identity transformation and whose transformations organize together as inverses in pairs.

Here, one should think that what we nowadays call the local exponential map, here viewed as the integration of a (parametrized) vector field, is implicitly applied to $\lambda_{1} X_{1} f+\cdots+\lambda_{r} X_{r} f$, namely:

$$
\exp \left(\lambda_{1} X_{1}+\cdots+\lambda_{r} X_{r}\right)(x)=: f(x ; \lambda)
$$

reconstitutes the finite equations $x^{\prime}=f(x ; \lambda)$ of the group. Lie's exponential Theorem I. 11 indeed states that the $r$ linearly independent infinitesimal transformations $X_{1}, \ldots, X_{r}$ engender a transformation group, in the sense that the equations:

$$
x_{i}^{\prime}=x_{i}+\sum_{k=1}^{r} \lambda_{k} \xi_{k i}(x)+\sum_{k, j=1}^{r} \frac{\lambda_{k} \lambda_{j}}{1 \cdot 2} X_{k}\left(\xi_{j i}\right)+\cdots \quad(i=1 \cdots n)
$$

deliver the finite transformations $x_{i}^{\prime}=f_{i}(x ; \lambda)$ of the group, "so that the totality of all these finite transformations is identical with the totality of all transformations of the group $x_{i}^{\prime}=f_{i}(x ; \lambda)$ " ([25], p. 75).

As exemplified by the above statement of Theorem I.24, it is typical of Lie's thought to identify plainly a transformation group with the corresponding Lie algebra. In fact, after a Lie algebra has been classified by means of several normalization procedures, taking the exponential to get some finite equations follows (in principle) by direct, unproblematic computations. Here is a relevant excerpt from Vol. I, p. 55.

[^2]
## Lie's main classification problem

For Lie, the central question of the monumental theory he erected was to classify, up to equivalence all possible finite transformation groups, locally, generically, and principally up to the "physically meaningful" threedimensional space.

## Classification of Lie algebras of local analytic vector fields in dimensions $\mathbf{1 , 2}$ and $\mathbf{3}$ in neighbourhoods of generic points

Naturally, two $r$-term transformation groups $x^{\prime}=f(x ; a)$ and $y^{\prime}=g(y ; b)$ acting on spaces of the same dimension with the same number of essential parameters are equivalent [ÄHNLICH] if there exist both a change of parameters $b=\beta(a)$, and a change of coordinates $y=\varphi(x)$ of the source space which acts simultaneously as $y^{\prime}=\varphi\left(x^{\prime}\right)$ on the target space, such that, after plugging in as one should, one has the last following relation:

$$
x^{\prime}=\varphi^{-1}\left(y^{\prime}\right)=\varphi^{-1}(g(y) ; b)=\varphi^{-1}(g(\varphi(x)) ; \beta(a)) \equiv f(x ; a),
$$

to be identically satisfied for any $x$ and $a$.
Accordingly, at the infinitesimal level, two Lie algebras of local holomorphic vector fields $X_{1}, \ldots, X_{r}$ and $Y_{1}, \ldots, Y_{r}$ of the same dimension $r$ acting on two spaces $x_{1}, \ldots, x_{n}$ and $y_{1}, \ldots, y_{n}$ of the same dimension $n$ are (locally) equivalent if there exists a diffeomorphism $x \mapsto y=y(x)$ which sends ${ }^{5}$ each $X_{k}$ to some linear combination $\lambda_{k 1} Y_{1}+\cdots+\lambda_{k r} Y_{r}$ of the $Y_{l}$ with constant coefficients $\lambda_{k l}$.

Thus, for Lie and for us who will in this work follow his wake, the question amounts to the following main problem that we now describe in length for future comprehension.

1) To find all possible finite-dimensional Lie algebras $X_{1}, \ldots, X_{r}$ of local holomorphic vector fields

$$
X_{k}=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial}{\partial x_{i}} \quad(k=1 \cdots r)
$$

defined in some initial domain $U \subset \mathbb{C}^{n}$, the mathematical rules of the game allowing a finite number of free relocalizations, namely: the rules allow to restrict a finite number of times the considerations to a smaller subdomain in order to perform every appearing mathematical operation which would necessitate that a certain analytical quantity is nondegenerate ${ }^{6}$.

[^3]2) To bring each such system $X_{1}, \ldots, X_{r}$ of vector fields to an as simple as possible normal form, e.g. to achieve that a majority of $\xi_{k i}$ be null, and that the remaining ones to be either monomials of small degree, or very simple polynomials, or exponentials of linear functions, or maybe possibly arbitrary functions, but of a number of variables that should be smaller than $n$.
3) To precisely distinguish the possible systems of vector fields by devising and introducing either geometrical, coordinate-independent, or calculatory, purely algebraical concepts, which enable one to build classification boxes, sub-boxes, and further sub-boxes to which to drive the sought groups.
4) To draw up extensive tables of the found Lie algebras of vector fields that are complete and mutually exclusive.

Finest classification theorems should indeed not only provide lists that capture as a first step all possible objects enjoying some definite mathematical properties, but they should also, as a second step, clean up the obtained tables, namely remove thoroughly the various overlaps which can occur between the found classes. Experience in various mathematical fields indeed shows that quite often, the branches of a given classification tree obtained by one, or by another means, do penetrate into each other, unavoidably.
Working principally over $\mathbb{C}$ Sometimes (but not in the present Chap. 1), the algebraic closedness of the ground field appears to be needed, especially for the classification theorems of primitive Lie algebras in dimensions 2 (Chap. 3 and Chap. I.29) and 3 (Chap. 7). Although the majority of the general statements reminded here in Vol. III by Engel and Lie do hold over $\mathbb{C}$ and over $\mathbb{R}$ as well, it is safer to plainly understand, when nothing is said about the field, that all the considerations are restricted to complex numbers.

Comment principles Finally, apologizing for having interrupted so lengthily the flow of thought just when Engel and Lie do launch the classification achievements of Vol. III, we briefly list our comment principles:

- Reconstitute details of proof that rely upon preliminary knowledge of Vols. I and II;
- Do not translate the contents into abstract mathematical language;
- Insert geometrical illustrations and summarizing tables as well.
unresolved in full generality, even in dimension $n=2$ ), because at a generic point, such a nonzero $X$ may be straightened simply to $\frac{\partial}{\partial x_{1}}$.


## Division I.

## The Finite Continuous Groups of The Straight Line and of the Plane.

The present first Division comprises the determination [BESTIMMUNG] of all finite continuous groups of point transformations on the straight line and on the plane (Chapters 1, 3, 4). Besides, it comprises the determination of all projective groups on the line and on the plane (Chap. 2, § 4 and Chap. 5). Subsequent to these studies, all linear homogeneous groups in two and in three variables will in addition be drawn up [AUFGESTELLT] (Chap. 2, § 5 and Chap. 6). Moreover, it is still to be mentioned that, through the developments of Chaps. 3 and 4 in conjunction with Chap. 23 of the Volume II, the determination of all finite continuous groups of contact transformations of a plane is also accomplished.

In what has been said, the results of the first Division are broadly identified. Notably, one can underline what follows concerning the form of the specific groups, namely: it turns out that the finite continuous groups of the straight line are all equivalent [ÄHNLICH] to projective groups. However, for the plane, this is no longer valid, although in the plane, the infinitesimal transformations of any finite continuous group can be also brought to a very simple form: for the transitive groups, aside from completely rational functions, only exponential functions occur in such a form; for the intransitive groups, arbitrary functions appear. Lie published these important results about the form of the groups on the line and on the plane as early as in the year 1874, in no. 22 of the Gött. Nachr.
$\triangleright$ Translation note. Two continuous transformation groups which transform one into the other by an invertible change of coordinates, and by a simultaneous invertible change of parameters as well, are called "ÄHNLICH" by Lie (vol. I, p. 24); since the adjective "similar" belongs mostly to a nonconceptual lexical field, we translate "ÄHNLICH" by "equivalent", assuming that contemporary readers know well of the problem of equivalence, of the problem of classification and of the problem of providing as simple as possible normal forms, for transformation groups or for various other mathematical objects. Apud Lie notably, the word "Bestimmung" denotes a complete solution of the problem which embraces all its three aspects. Final results are drawn up as extensive tables [TABELLE] of groups.

## Chapter 1.

## Determination of all Finite Continuous Transformation Groups of the Once-Extended [Einfach Ausgedehnten] Manifold.

At first, we develope two different methods which deliver us without difficulty all transformation groups of the once-extended manifold. After that, we show that the determination of these groups also follows already almost immediately from the results of Chap. 29 in Vol. I.
$\triangleright$ Reminding Lie's principles of thought. From the beginning, it will be assumed that:

- mathematical objects are analytic;
- relocalization is freely allowed;
- open sets are small, usually unnamed, and always connected.
§ 1.

An $r$-term group [ $r$-GLIEDRIGE GRUPPE] of the once-extended manifold $x$ is represented by an equation of the form:

$$
x^{\prime}=f\left(x, a_{1}, \ldots, a_{r}\right),
$$

with $r$ parameters $a_{1}, \ldots, a_{r}$. It is engendered by $r$ independent infinitesimal transformations:

$$
X_{1} f=\xi_{1}(x) \frac{d f}{d x}, \ldots, X_{r} f=\xi_{r}(x) \frac{d f}{d x},
$$

which satisfy relations in pairs of the form:

$$
\begin{gathered}
{\left[X_{i}, X_{k}\right]=\left(\xi_{i} \frac{d \xi_{k}}{d x}-\xi_{k} \frac{d \xi_{i}}{d x}\right) \frac{d f}{d x}=\sum_{s=1}^{r} c_{i k s} X_{s} f} \\
(i, k=1 \cdots r)
\end{gathered}
$$

Lie algebras and local Lie groups. At this very beginning of Volume III, Lie and Engel of course take for granted the one-to-one correspondence between finite-dimensional Lie algebras of local holomorphic vector fields and local Lie groups that they already established in great details in Chapter 9 of Volume I. So the first goal here is to classify Lie algebras on the onedimensional $x$-space.

Here, the symbol $f$ in $X f$ should not be confused with the $f$ in $x^{\prime}=$ $f(x, a)$. In fact, Lie always writes a vector field derivation as acting on a test function which he always designates by the symbol $f$.

Every transformation of the one-term group:

$$
x_{i}^{\prime}=x_{i}+\frac{t}{1} \xi_{i}+\frac{t^{2}}{1 \cdot 2} X\left(\xi_{i}\right)+\cdots \quad(i=1 \cdots n)
$$

is obtained by repeating infinitely many times [UNENDLICHMALIGE WIEDERHOLUNG] the infinitesimal transformation:

$$
x_{i}^{\prime}=x_{i}+\xi_{i} \delta t \quad \text { or } \quad X(f)=\xi_{1} \frac{\partial f}{\partial x_{1}}+\cdots+\xi_{n} \frac{\partial f}{\partial x_{n}} .
$$

Or yet more briefly:
The one-term group in question is engendered by its infinitesimal transformations.

In contrast to the infinitesimal transformation $X(f)$, we call the equations:

$$
x_{i}^{\prime}=x_{i}+\frac{t}{1} \xi_{i}+\frac{t^{2}}{1 \cdot 2} X\left(\xi_{i}\right)+\cdots
$$

the finite equations of the one-term group in question.
$\triangleright$ Comment. This is just a brief reminder of the general theory: exponentiating an infinitesimal transformation yields the finite equations $x_{i}^{\prime}=$ $\exp (t X)\left(x_{i}\right), i=1, \ldots, n$ that are written here after expanding them with respect to $t$. Intuitively, they derive from the infinitesimal moves $x_{i}^{\prime}=x_{i}+$ $\xi_{i} \delta t$ by means of infinite iteration, namely integration.

The general infinitesimal transformation of our group reads:

$$
X f=\sum_{k=1}^{r} e_{k} \xi_{k}(x) \cdot \frac{d f}{d x}=\xi(x) \frac{d f}{d x},
$$

where $e_{1}, \ldots, e_{r}$ denote arbitrary constants. Now, since $X_{1} f, \ldots, X_{r} f$ are independent infinitesimal transformations and since as a consequence of that, $\xi_{1}, \ldots, \xi_{r}$ satisfy no linear relation:

$$
a_{1} \xi_{1}+\cdots+a_{r} \xi_{r}=0
$$

with constant coefficients, it follows that the function:

$$
\xi=e_{1} \xi_{1}+\cdots+e_{r} \xi_{r}
$$

with $r$ arbitrary constants $e_{1}, \ldots, e_{r}$ is the general solution of an $r$-th order linear differential equation:

$$
\frac{d^{r} \xi}{d x^{r}}+\alpha_{1}(x) \cdot \frac{d^{r-1} \xi}{d x^{r-1}}+\cdots+\alpha_{r-1}(x) \cdot \frac{d \xi}{d x}+\alpha_{r}(x) \cdot \xi=0
$$

which, on its side, completely determines the general infinitesimal transformation of our group and hence, the group itself. We thus see: The
defining equations (cf. Vol. I, Chap. 11) of an r-term transformation group of the once-extended manifold are made of an $r$-th order linear ordinary differential equation.

Explanation. Let us restate and prove what is considered to be known.
Assertion. If $r$ given arbitrary analytic functions $\xi_{1}(x), \ldots, \xi_{r}(x)$ are linearly independent, then possibly after relocalization, there exists a monic $r$-th order ordinary differential equation:

$$
\xi^{(r)}+\alpha_{1}(x) \cdot \xi^{(r-1)}+\cdots+\alpha_{r}(x) \cdot \xi=0
$$

whose general solution is the general linear combination $\xi=e_{1} \xi_{1}+\cdots+$ $e_{r} \xi_{r}$.

Lemma. Let $\xi_{1}(x), \ldots, \xi_{r}(x)$ be $r$ analytic functions. Then there exist constants $a_{1}, \ldots, a_{r}$ not all zero making a linear dependence relation $a_{1} \xi_{1}(x)+$ $\cdots+a_{r} \xi_{r}(x) \equiv 0$ between the $\xi_{i}$, if and only if their Wronskian:

$$
\mathbf{W}\left(\xi_{1}, \ldots, \xi_{r}\right):=\left|\begin{array}{ccc}
\xi_{1} & \cdots & \xi_{r} \\
\xi_{1}^{\prime} & \cdots & \xi_{r}^{\prime} \\
\vdots & \ddots & \vdots \\
\xi_{1}^{(r-1)} & \cdots & \xi_{r}^{(r-1)}
\end{array}\right| \equiv 0
$$

vanishes identically.

Proof of the lemma. In one direction, the existence of constants $a_{i}$ not all zero such that $0 \equiv a_{1} \xi_{1}+\cdots+a_{r-1} \xi_{r-1}+a_{r} \xi_{r}$ with, say: $a_{r}=-1$ after renumbering and dilation, implies that the Wronskian:
(a)

$$
\mathbf{W}\left(\xi_{1}, \ldots, \xi_{r-1}, \xi_{r}\right)=\left|\begin{array}{cccc}
\xi_{1} & \cdots & \xi_{r-1} & a_{1} \xi_{1}+\cdots+a_{r-1} \xi_{r-1} \\
\xi_{1}^{\prime} & \cdots & \xi_{r-1}^{\prime} & a_{1} \xi_{1}^{\prime}+\cdots+a_{r-1} \xi_{r-1}^{\prime} \\
\vdots & \ddots & \vdots & \vdots \\
\xi_{1}^{(r-1)} & \cdots & \xi_{r-1}^{(r-1)} & a_{1} \xi_{1}^{(r-1)}+\cdots+a_{r-1} \xi_{r-1}^{(r-1)}
\end{array}\right| \equiv 0
$$

obviously vanishes, because of colum linear dependence.
Conversely, suppose that the Wronskian $\mathbf{W}\left(\xi_{1}, \ldots, \xi_{r-1}, \xi_{r}\right) \equiv 0$ vanishes identically and establish linear dependence of the $\xi_{i}$. Reasoning by induction on $r$, we can assume that the subWronskian $\mathrm{W}\left(\xi_{1}, \ldots, \xi_{r-1}\right)$ does not vanish identically, since otherwise $\xi_{1}, \ldots, \xi_{r-1}$ (and hence $\xi_{1}, \ldots, \xi_{r-1}, \xi_{r}$ too) would already be, without any effort, linearly dependent. We then expand the determinant $\mathbf{W}\left(\xi_{1}, \ldots, \xi_{r-1}, \xi_{r}\right)$ along its last
column:
(b)

$$
\begin{aligned}
0 & \equiv\left|\begin{array}{cccc}
\xi_{1} & \cdots & \xi_{r-1} & \xi_{r} \\
\vdots & \ddots & \vdots & \vdots \\
\xi_{1}^{(r-1)} & \cdots & \xi_{r-1}^{(r-1)} & \xi_{r}^{(r-1)}
\end{array}\right| \\
& =\xi_{r}^{(r-1)} \cdot \mathbf{W}\left(\xi_{1}, \ldots, \xi_{r-1}\right)-\cdots+(-1)^{r-1} \xi_{r} \cdot\left|\begin{array}{ccc}
\xi_{1}^{\prime} & \cdots & \xi_{r-1}^{\prime} \\
\vdots & \ddots & \vdots \\
\xi_{1}^{(r-1)} & \cdots & \xi_{r}^{(r-1)}
\end{array}\right|
\end{aligned}
$$

we relocalize to a neighbourhood of a point where the mentioned subWronksian does not vanish and we divide the above equation by this leading coefficient, getting:

$$
\begin{equation*}
0 \equiv \xi_{r}^{(r-1)}+\alpha_{1} \xi_{r}^{(r-2)}+\cdots+\alpha_{r-1} \xi \tag{c}
\end{equation*}
$$

for some analytic functions $\alpha_{1}(x), \ldots, \alpha_{r-1}(x)$ defined in some subdomain. Recall ([35]) that the space of solutions of such an $(r-1)$-th order ordinary differential equation is a vector space of dimension $(r-1)$. But we in fact already know thanks to (a) that the general linear combination $e_{1} \xi_{1}+$ $\cdots+e_{r} \xi_{r-1}$ constitutes trivially a solution of (c), by just replacing in the first, big determinant of (b), and since this combination generates an $(r-$ 1)-dimensional space, we must have $\xi_{r}=a_{1} \xi_{1}+\cdots+a_{r} \xi_{r}$, form some appropriate constants $a_{i}$. Finally, we remark that thanks to the principle of analytic continuation, the relation $0 \equiv a_{1} \xi_{1}(x)+\cdots+a_{r-1} \xi_{r-1}(x)-\xi_{r}(x)$ propagates from the subdomain where we could divide by $\mathrm{W}\left(\xi_{1}, \ldots, \xi_{r-1}\right)$ to the original domain of definition of the $\xi_{i}$.

Proof of the assertion. Again and similarly, the Wronskian of the $(r+1)$ functions linked by the relation $0 \equiv e_{1} \xi_{1}+\cdots+e_{r} \xi_{r}-\xi$, vanishes identically:

$$
\begin{aligned}
0 & \equiv \mathbf{W}\left(\xi_{1}, \ldots, \xi_{r}, \xi\right)=\left|\begin{array}{cccc}
\xi_{1} & \cdots & \xi_{r} & \xi^{\prime} \\
\xi_{1}^{\prime} & \cdots & \xi_{r}^{\prime} & \xi^{\prime} \\
\vdots & \ddots & \vdots & \vdots \\
\xi_{1}^{(r)} & \cdots & \xi_{r}^{(r)} & \xi^{(r)}
\end{array}\right| \\
& =\xi^{(r)} \cdot \mathbf{W}\left(\xi_{1}, \ldots, \xi_{r}\right)-\cdots+(-1)^{r} \xi \cdot\left|\begin{array}{ccc}
\xi_{1}^{\prime} & \cdots & \xi_{r}^{\prime} \\
\vdots & \cdots & \vdots \\
\xi_{1}^{(r)} & \cdots & \xi_{r}^{(r)}
\end{array}\right|
\end{aligned}
$$

and by expanding it along its last column, we get an ordinary $r$-th order differential equation which we may bring to a monic form in the set where the subWronskian $\mathbf{W}\left(\xi_{1}, \ldots, \xi_{r}\right)$ is different from zero.

We now imagine in our mind that, in the neighbourhood of a point in general position which we choose as the origin of coordinates, the infinitesimal transformations of our group are expanded in powers of $x$. The defining equation that is solved with respect to the $r$-th order differential quotient of $\xi$ shows (loc. cit., p. 188 sq.) that in our group no infinitesimal transformation of $r$-th or higher order in $x$ is available, hence no infinitesimal transformation exists whose power series with respect to $x$ begins with terms of $r$-th or higher order. Consequently, we can always imagine $r$ independent infinitesimal transformations of the group chosen in such a way that the one of zeroth, the one of first, ..., the one of $(r-1)$-th order in $x$ are:
(1)

$$
\left\{\begin{array}{c}
X_{0} f=\left(1+a_{0} x+\cdots\right) \frac{d f}{d x} \\
X_{1} f=\left(x+a_{1} x^{2}+\cdots\right) \frac{d f}{d x} \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
X_{r-1} f=\left(x^{r-1}+a_{r-1} x^{r}+\cdots\right) \frac{d f}{d x} .
\end{array}\right.
$$

We can naturally use these $r$ independent infinitesimal transformations in place of the initially chosen: $X_{1} f, \ldots, X_{r} f$.
$\triangleright$ Explanation. If the coefficient $\xi=e_{1} \xi_{1}+\cdots+e_{r} \xi_{r}$ of an arbitrary infinitesimal transformation $X f$ of the group satisfies $\xi=\mathrm{O}\left(x^{r}\right)$, i.e. $\xi(0)=$ $\cdots=\xi^{(r-1)}(0)=0$, then uniqueness of solutions to the above ODE implies $\xi \equiv 0$. One thus gets the invertibility of the $r \times r$ matrix $\left(a_{i}^{j}\right)$ associated to the truncated expansions $\xi_{i}(x)=a_{i}^{0}+a_{i}^{1} x+\cdots+a_{i}^{r-1} x^{r-1}+\mathrm{O}\left(x^{r}\right)$ of the $r$ linearly independent coefficients $\xi_{1}, \ldots, \xi_{r}$, whence lastly, a triangulation may be performed.

At present, we remember that two infinitesimal transformations of $i$ th and $k$-th order respectively produce by combination a transformation of $(i+k-1)$-th or higher order (loc. cit. p. 193, Theor. 30); in our case, we find:

$$
\begin{equation*}
\left[\left(x^{i}+\cdots\right) \frac{d f}{d x}, \quad\left(x^{k}+\cdots\right) \frac{d f}{d x}\right]=\left((k-i) x^{i+k-1}+\cdots\right) \frac{d f}{d x}, \tag{2}
\end{equation*}
$$

where on the right-hand side the term of $(i+k-1)$-th order visibly never can vanish, when $i$ and $k$ are different from each other. A short while ago, we have seen that our $r$-term group contains no infinitesimal transformation of $r$-th or higher order, so we can conclude that in our group the numbers $i, k$ and $i+k-1$ must always be smaller than $r$. Hence if we
choose for $i$ the largest possible value: $i=r-1$ and likewise for $k \neq i \mid$ the largest possible value: $k=r-2$, then we obtain for the number $r$ of terms of our group the condition:

$$
r-1+r-2-1<r
$$

that is to say: $r<4$.
Thus in the once-extended manifold there is no finite continuous group with more than three parameters.
$\triangleright$ Vanishing order of infinitesimal transformations. In $n$ variables $\left(x_{1}, \ldots, x_{n}\right)$, an analytic function $\xi\left(x_{1}, \ldots, x_{n}\right)$ is said to be of order $\geqslant \mu$ with respect to $x_{1}-x_{1}^{0}, \ldots, x_{n}-x_{n}^{0}$ if, in its power series expansion $\xi=\sum_{\alpha \in \mathbb{N}^{n}} c_{\alpha_{1} \ldots \alpha_{n}}\left(x_{1}-x_{1}^{0}\right)^{\alpha_{1}} \cdots\left(x_{n}-x_{n}^{0}\right)^{\alpha_{n}}$ at $x^{0}$, the coefficients $c_{\alpha}$ vanish for all multiindices $\alpha$ with $\alpha_{1}+\cdots+\alpha_{n} \leqslant \mu-1$. An infinitesimal transformation $X f=\sum_{i=1}^{n} \xi_{i}(x) \frac{\partial}{\partial x_{i}}$ is said to be of order $\mu$ at $x^{0}$ if its $n$ coefficients $\xi_{i}$ are of order $\geqslant \mu$ at $x^{0}$, and one of them at least is not of order $\geqslant \mu+1$, namely it is of order $=\mu$.
$\triangleright$ The cited theorem. Its precise statement, valuable for an arbitrary number $n$ of variables, implicitly offers a direct proof.
Theorem I.30. If $X f$ and $Y f$ are two infinitesimal tranformations:
$X f=\sum_{1 \leqslant k \leqslant n}\left(\xi_{k}^{(\mu)}+\cdots\right) \frac{\partial f}{\partial x_{k}}, \quad Y f=\sum_{1 \leqslant j \leqslant n}\left(\eta_{j}^{(\nu)}+\cdots\right) \frac{\partial f}{\partial x_{j}}$,
whose power series expansions with respect to powers of $x_{1}-x_{1}^{0}, \ldots, x_{n}-x_{n}^{0}$ begin respectively with terms of $\mu$-th order and with terms of $\nu$-th order, namely each $\xi_{k}^{(\mu)}$ (resp. each $\eta_{j}^{(\mu)}$ ) is a homogeneous polynomial of degree $\mu(r e s p . \nu)$ in $x-x_{0}$, then the power series expansion of the infinitesimal transformation
$X Y f-Y X f=[X, Y] f=\sum_{j=1}^{n}\left\{\sum_{k=1}^{n}\left(\xi_{k}^{(\mu)} \frac{\partial \eta_{j}^{(\nu)}}{\partial x_{k}}-\eta_{k}^{(\nu)} \frac{\partial \xi_{j}^{(\mu)}}{\partial x_{k}}\right)+\cdots\right\} \frac{\partial f}{\partial x_{j}}$
begins with terms of $(\mu+\nu-1)$-th order which are perfectly determined by the terms of $\mu$-th order of $X f$ and by the terms of $\nu$-th order of $Y f$. If these terms of $(\mu+\nu-1)$-th order vanish, then one can only say about the power series expansion of $[X, Y]$ that it starts with terms of $(\mu+\nu)$-th or of higher order.
$\triangleleft$
We now treat the three possible cases one after the other: $r=1,2,3$.
If $r=1$, the group contains only one infinitesimal transformation of the form:

$$
X_{0} f=(1+\cdots) \frac{d f}{d x}
$$

We now introduce:

$$
x_{1}=\int_{0}^{x} \frac{d x}{1+\cdots}
$$

as a new variable in place of $x$. This is allowed, since $x_{1}$ is an ordinary power series in $x$ which vanishes for $x=0$ and in the same way, $x$ is an ordinary power series in $x_{1}$ which vanishes for $x_{1}=0$. In the new variable $x_{1}, X_{0} f$ becomes of the form:

$$
X_{0} f=\frac{d f}{d x_{1}}
$$

This infinitesimal transformation engenders a once-term group whose finite transformations read: $x_{1}^{\prime}=x_{1}+a$; this is the group of all translations of the once-extended manifold.

In case $r=2$, we have two infinitesimal transformations:

$$
X_{0} f=(1+\cdots) \frac{d f}{d x}, \quad X_{1} f=(x+\cdots) \frac{d f}{d x}
$$

which give by combination [COMBINATION]:

$$
\left[X_{0}, X_{1}\right]=(1+\cdots) \frac{d f}{d x}
$$

and consequently there is a relation of the form:

$$
\left[X_{0}, X_{1}\right]=X_{0} f+\lambda \cdot X_{1} f
$$

or, if we introduce $X_{0} f+\lambda X_{1} f$ as a new $X_{0} f$ :

$$
\begin{equation*}
\left[X_{0}, X_{1}\right]=X_{0} f \tag{3}
\end{equation*}
$$

If we now choose as in the first case the variable $x_{1}$ so that $X_{0} f$ takes the form: $\frac{d f}{d x_{1}}$, then it becomes: $X_{1} f=\xi_{1} \frac{d f}{d x_{1}}$ and because of the relation (3):

$$
\frac{d \xi_{1}}{d x_{1}}=1, \quad \xi_{1}=x_{1}+\text { Const. }
$$

where, incidentally, the constant of integration vanishes, since the infinitesimal transformation $X_{1} f$ must also be of the first order in the new variable $x_{1}$ (Vol. I, p. 197, Prop. 1).

The cited general proposition. Even in an arbitary number $n$ of variables, its proof is straightforward.
Proposition. If one introduces, into an infinitesimal transformation $X=$ $\sum_{i=1}^{n} \xi_{i}(x) \frac{\partial}{\partial x_{i}}$ supposed to be of $\mu$-th order with respect to $x_{1}-x_{1}^{0}, \ldots, x_{n}-$ $x_{n}^{0}$, new variables $y_{1}, \ldots, y_{n}$ :

$$
y_{k}=y_{k}^{0}+\sum_{1 \leqslant i \leqslant n} a_{k i}\left(x_{i}-x_{i}^{0}\right)+\sum_{1 \leqslant i, j \leqslant n} a_{k i j}\left(x_{i}-x_{i}^{0}\right)\left(x_{j}-x_{j}^{0}\right)+\cdots
$$

where the determinant $\left|a_{i j}\right|$ of the first order part is nonzero, then $X$ transforms into an infinitesimal transformation of order $\mu$ with respect to $y_{1}-$ $y_{1}^{0}, \ldots, y_{n}-y_{n}^{0}$.

Both infinitesimal transformations:

$$
X_{0} f=\frac{d f}{d x_{1}}, \quad X_{1} f=x_{1} \frac{d f}{d x_{1}}
$$

engender a two-term group with the finite transformations: $x_{1}^{\prime}=a_{1} x_{1}+a_{2}$; this is the general linear group of the once-extended manifold.

Lastly, if $r=3$, then the group comprises three infinitesimal transformations of the form:

$$
X_{0} f=(1+\cdots) \frac{d f}{d x}, \quad X_{1} f=(x+\cdots) \frac{d f}{d x}, \quad X_{2} f=\left(x^{2}+\cdots\right) \frac{d f}{d x},
$$

whence there exist relations of the following shape:

$$
\begin{aligned}
& {\left[X_{0}, X_{1}\right]=X_{0} f+\lambda_{1} X_{1} f+\lambda_{2} X_{2} f} \\
& {\left[X_{0}, X_{2}\right]=2 X_{1} f+\mu X_{2} f} \\
& {\left[X_{1}, X_{2}\right]=X_{2} f}
\end{aligned}
$$

It we set:

$$
\bar{X}_{0} f=X_{0} f+\alpha_{1} X_{1} f+\alpha_{2} X_{2} f
$$

then it follows:

$$
\begin{aligned}
{\left[\bar{X}_{0}, X_{1}\right] } & =\bar{X}_{0} f+\left(\lambda_{1}-\alpha_{1}\right) X_{1} f+\left(\lambda_{2}-2 \alpha_{2}\right) X_{2} f \\
{\left[\bar{X}_{0}, X_{2}\right] } & =2 X_{1} f+\left(\alpha_{1}+\mu\right) X_{2} f
\end{aligned}
$$

or, when we choose $\alpha_{1}=\lambda_{1}$ and $2 \alpha_{2}=\lambda_{2}$ :

$$
\left[\bar{X}_{0}, X_{1}\right]=\bar{X}_{0} f, \quad\left[\bar{X}_{0}, X_{2}\right]=2 X_{1} f+\left(\lambda_{1}+\mu\right) X_{2} f
$$

From the Jacobian identity:

$$
\left.\left[\left[\bar{X}_{0}, X_{1}\right], X_{2}\right]+\left[\left[X_{1}, X_{2}\right], \bar{X}_{0}\right]+\left[X_{2}, \bar{X}_{0}\right], X_{1}\right]=0
$$

it ensues finally:

$$
\left[\bar{X}_{0}, X_{2}\right]+\left[X_{2}, \bar{X}_{0}\right]+\left(\lambda_{1}+\mu\right) X_{2} f=0
$$

whence: $\lambda_{1}+\mu=0$, and we have:

$$
\begin{equation*}
\left[\bar{X}_{0}, X_{1}\right]=\bar{X}_{0}, \quad\left[\bar{X}_{0}, X_{2}\right]=2 X_{1}, \quad\left[X_{1}, X_{2}\right]=X_{2} \tag{4}
\end{equation*}
$$

$\triangleright$ Comment about notation. In the original German text, a Lie "bracket" is called a "Combination" (between two infinitesimal transformations), or sometimes named as just an "equation" [Gleichung]. It is denoted $\left(X_{1} X_{2}\right)$, always with parentheses, usually without comma, but with a
comma when $X_{1}$ and $X_{2}$ are explicit vector fields written in coordinates (see below, throughout), and without the symbol of function $f$, which is traditionally almost always present to denote an individual infinitesimal transformation $X f$. But remarkably in the eq. (4) just above, $f$ has been removed in the three right-hand sides.

As the only update of notation we allow in comparison to the German text, we systematically translate brackets as $[\cdot, \cdot]$.

The infinitesimal transformations $\bar{X}_{0} f$ and $X_{1} f$ obviously engender for themselves a two-term group which falls under the previous case, and which hence can, through an appropriate change of the variable $x$, be brought to the form:

$$
\bar{X}_{0} f=\frac{d f}{d x}, \quad X_{1} f=x \frac{d f}{d x} .
$$

At the same time, $X_{2} f$ receives a certain new form: $\xi_{2} \frac{d f}{d x}$, where $\xi_{2}$, on account of the relation (4), must satisfy the equations:

$$
\frac{d \xi_{2}}{d x}=2 x, \quad x \frac{d \xi_{2}}{d x}-\xi_{2}=\xi_{2}
$$

in consequence of what it is identically equal to $x^{2}$. Thus we have:

$$
\bar{X}_{0} f=\frac{d f}{d x}, \quad X_{1} f=x \frac{d f}{d x}, \quad X_{2} f=x^{2} \frac{d f}{d x} .
$$

The finite equations of the three-term group engendered by these infinitesimal transformations write:

$$
x^{\prime}=\frac{a_{1}+a_{2} x}{1+a_{3} x} ;
$$

this is the general projective group of the once-extended manifold.
So we have gained the important theorem:
Theorem 1.*) Every finite continuous group of the once-extended manifold has at most three parameters; such a group is equivalent either to the one-term group:

$$
x^{\prime}=x+a
$$

of all translations, or to the two-term general linear group:

$$
x^{\prime}=a_{1}+a_{2} x
$$

or finally to the three-term general projective group:

$$
x^{\prime}=\frac{a_{1}+a_{2} x}{1+a_{3} x} .
$$

[^4]coming from the second reference.
$\triangleright$ Lie's discovery of Theorem 1. At the end of the his first synthetical memoir ([24], p. 93; [1], pp. 380-381), Lie explained how the discovery of this first classification Theorem 1 appeared spectacular and motivating to him: "In the course of investigations on first-order partial differential equations, I observed that the formulas that occur in this discipline become amenable to a remarkable interpretation by means of the concept of an infinitesimal transformation. In particular, the so-called Poisson-Jacobi theorem is closely connected with the composition of infinitesimal transformations. By following up on this observation I arrived at the surprising result that all transformation groups of a simply extended manifold can be reduced to the linear form by a suitable choice of variables, and also that the determination of all groups of an $n$-fold extended manifold can be achieved by the integration of ordinary differential equations. This discovery, whose first traces go back to Abel and Helmholtz, became the starting point of my many years of research on transformation groups."
$\triangleright$ Historical note. Thomas Hawkins summarizes as follows the development of Lie's classification problem: ([17], p. 76): "Along with his efforts to polish up his theoretical treatment of the general classification problem, Lie expended considerable effort on the actual determination (up to equivalence) of all groups for small values of $n$. Judging by his brief note in the Göttinger Nachrichten [22], by the end of 1874 he had resolved the problem for $n=2$ to his own satisfaction, using in part geometrical means. But it was not until 1878 that he managed to translate his results into publishable, analytical terms ([23], p. 78). At that time, he also announced that he had solved the problem for $n=3$ but restricted to groups of point transformations. However, the calculations needed to do this remained too extensive to make publication feasible ([27], p. 122), and Lie contented himself with partial results ([27], pp. 122-262). As for the problem for $n$ arbitrary, in [25], p. 598 , he expressed the view that it would probably never be resolved." $\triangleleft$

We have seen that every two-term group of the once-extended manifold can be brought to the form:

$$
\frac{d f}{d x}, \quad x \frac{d f}{d x} .
$$

From this, one can conclude that two independent infinitesimal transformations:

$$
X_{1} f=\xi_{1} \frac{d f}{d x}, \quad X_{2} f=\xi_{2} \frac{d f}{d x}
$$

of the once-extended manifold can never be interchangeable [VERTAUSCHBAR]; indeed, if they were so, then they would engender a twoterm group which would not be of the same composition as the group (5), hence would also not be equivalent to it. Besides, from the condition of interchangeability:

$$
\left[X_{1}, X_{2}\right]=\left(\xi_{1} \frac{d \xi_{2}}{d x}-\xi_{2} \frac{d \xi_{1}}{d x}\right) \frac{d f}{d x}=0
$$

one also finds immediately that $\xi_{2}$ and $\xi_{1}$ differ from each other only by a constant factor, hence that $X_{1} f$ and $X_{2} f$, if they are supposed to be interchangeable, cannot be independent from each other. Consequently, we get the

Proposition 1. Two independent infinitesimal transformations of the once-extended manifold are never interchangeable.
$\triangleright$ Translation note. Present-day commutativity is called by Engel-Lie "VERTAUSCHBARKEIT", a concept that we translate by interchangeability, so as to be faithful to the text. It just means vanishing of Lie brackets. $\triangleleft$
$\triangleright$ Transformation groups having the same composition. Two (local) Lie algebras of linearly independent vector fields:

$$
X_{k}=\sum_{i=1}^{n} \xi_{k i}\left(x_{1}, \ldots, x_{n}\right) \frac{\partial f}{\partial x_{i}} \quad(k=1 \cdots r)
$$

of the same dimension, but not necessarily acting on a space of the same dimension:

$$
Y_{k}=\sum_{\mu=1}^{m} \eta_{k \mu}\left(y_{1}, \ldots, y_{m}\right) \frac{\partial f}{\partial y_{\mu}} \quad(k=1 \cdots r)
$$

are said to have identical composition [GLEICHZUSAMMENGESETZ SEIN] by Lie if they are isomorphic [HOLOEDRISCH ISOMORPH] as plain Lie algebras, i.e. if among all the infinitesimal transformations $e_{1} Y_{1}+\cdots+e_{r} Y_{r}$ of the second family, there are $r$ linearly independent linear combinations $\mathfrak{Y}_{k}=\sum_{l=1}^{r} \lambda_{k l} Y_{l}, k=1, \ldots, r$, having the same structure constants as the $X_{k}$, namely $\left[X_{k}, X_{l}\right]=\sum_{s=1}^{r} c_{k l s} X_{s}$ and $\left[\mathfrak{Y}_{k}, \mathfrak{Y}_{l}\right]=\sum_{s=1}^{r} c_{k l s} \mathfrak{Y}_{s}$, with identical $c_{k l s}$.
$\triangleright$ Vector fields and Lie brackets under change of coordinates. The text uses the general fact that equivalence of two local Lie algebras under a change of coordinates $x \mapsto \bar{x}=\bar{x}(x)$ implies that they have the same composition. Let us explain this (of course obvious) claim, and, on this occasion, recall some basics about variable changes.

Under such a (local) diffeomorphism $x \mapsto \bar{x}=\bar{x}(x)$, a function $f(x)$ transforms to the function $\bar{f}(\bar{x})$ defined by the identity $f(x) \equiv \bar{f}(\bar{x}(x))$, or
by the equivalent identity $\bar{f}(\bar{x}) \equiv f(x(\bar{x}))$, with $\bar{x} \mapsto x=x(\bar{x})$ simply denoting the inverse (local) diffeomorphism. Differentiating these two identities with respect to $x_{i}$ and with respect to $\bar{x}_{i}$, we get the classical (tensoriallike) transformation rule for coordinate vector fields:

$$
\frac{\partial}{\partial x_{i}}=\sum_{j=1}^{n} \frac{\partial \bar{x}_{j}}{\partial x_{i}} \frac{\partial}{\partial \bar{x}_{j}} \quad \text { and } \quad \frac{\partial}{\partial \bar{x}_{i}}=\sum_{j=1}^{n} \frac{\partial x_{j}}{\partial \bar{x}_{i}} \frac{\partial}{\partial x_{j}} .
$$

Consequently, for $k=1, \ldots, r$, each $X_{k}=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial}{\partial x_{i}}$ transforms by linearity to the infinitesimal transformation defined by the formula:
(a)

$$
\begin{aligned}
\bar{X}_{k} & =\sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{k i}(x(\bar{x})) \frac{\partial \bar{x}_{j}}{\partial x_{i}}(x(\bar{x})) \frac{\partial}{\partial \bar{x}_{j}} \\
& =\sum_{j=1}^{n} X_{k}\left(\bar{x}_{j}(x)\right) \frac{\partial}{\partial \bar{x}_{j}} \\
& =: \sum_{j=1}^{n} \bar{\xi}_{k j}(\bar{x}) \frac{\partial}{\partial \bar{x}_{j}}
\end{aligned}
$$

and having new coefficients: $\bar{\xi}_{k j}(\bar{x}):=\sum_{i=1}^{n} \xi_{k i}(x(\bar{x})) \frac{\partial \bar{x}_{j}}{\partial x_{i}}(x(\bar{x}))$ naturally defined on the target space $\bar{x}$. Throughout, we shall simply write $X_{k}=\bar{X}_{k}$ and $f(x)=\bar{f}(\bar{x})$, without any function symbol for the diffeomorphism, as Lie usually did; contemporary formalism would write instead $\bar{x}=\varphi(x)$ and $\bar{X}_{k}=\varphi_{*}\left(X_{k}\right)$. Then the canonical invariance property of the Lie bracket says: $\left[X_{k}, X_{l}\right]=\left[\bar{X}_{k}, \bar{X}_{l}\right]$, a property that may be checked calculatorily from the coordinatewise definition:

$$
\begin{aligned}
{\left[X_{k}, X_{l}\right] } & =\left[\sum_{j=1}^{n} \xi_{k j} \frac{\partial}{\partial x_{j}}, \sum_{i=1}^{n} \xi_{l i} \frac{\partial}{\partial x_{i}}\right] \\
& =\sum_{i=1}^{n}\left(\sum_{j} \xi_{k j} \frac{\partial \xi_{l i}}{\partial x_{j}}-\sum_{j} \xi_{l j} \frac{\partial \xi_{k i}}{\partial x_{j}}\right) \frac{\partial}{\partial x_{i}}
\end{aligned}
$$

by just inserting the transformation rule (a) in the developments of brackets. Coming back to the claim, it is now obvious that

$$
\left[\bar{X}_{k}, \bar{X}_{l}\right]=\left[X_{k}, X_{l}\right]=\sum_{s=1}^{r} c_{k l s} X_{s}=\sum_{s=1}^{r} c_{k l s} \bar{X}_{s} .
$$

Thus, the two Lie algebras $\left(X_{k}\right)_{1 \leqslant k \leqslant r}$ and $\left(\bar{X}_{k}\right)_{1 \leqslant k \leqslant r}$ have the same structure constants, hence are trivially isomorphic.

One may argue that a conceptual, abstract and coordinate-independent presentation of the transformation rules $\bar{X}=\varphi_{*}(X)$ and $[\bar{X}, \bar{Y}]=$ $\varphi_{*}([X, Y])$ would be less instructive here; indeed, because the main
objective of Lie is to perform several normalization procedures in order to bring systems of infinitesimal transformations to an as simple as possible normal form, explicit computations in coordinates are, and must be, of central importance.

One observes that the groups of the once-extended manifold are all transitive. If one wants, one can even say that they are primitive.
$\triangleright$ Note. Transitivity and primitivity will be dealt with and redefined in a while, see Chap. 3.

## § 2.

In the beginning of the § 1 , we saw that to each $r$-term group of the once-extended manifold belongs a linear ordinary $r$-th order differential equation:

$$
\frac{d^{r} \xi}{d x^{r}}+\alpha_{1}(x) \frac{d^{r-1} \xi}{d x^{r-1}}+\cdots+\alpha_{r}(x) \xi=0
$$

by which it is completely defined: the defining equation of the group in question. One can now also determine the groups of the once-extended manifold in the way that one seeks every differential equation which is the defining equation of a group. This is what we now want to carry out*).
*) Already in the years 1870, Lie has determined the groups on the straight line in this manner, or anyway in 1882. At that time, he occupied himself with the reduction of the differential equation $\xi^{\prime \prime \prime}+2 \alpha \xi^{\prime}+\alpha^{\prime} \xi=0$ to the form $\xi^{\prime \prime \prime}=0$. From his general theory of integration it follows immediately that for this to hold, a Riccati equation of order one must be satisfied.

If the linear $r$-th order differential equation:

$$
\begin{equation*}
\xi^{(r)}+\alpha_{1}(x) \xi^{(r-1)}+\cdots+\alpha_{r-1}(x) \xi^{\prime}+\alpha_{r}(x) \xi=0 \tag{6}
\end{equation*}
$$

is supposed to be the defining equation of a $r$-term group, then according to Vol. I, Theor. 28, p. 187, the following is necessary and sufficient: whenever $\xi(x)$ and $\eta(x)$ are any two solutions of the differential equation (6), then $\xi \eta^{\prime}-\xi^{\prime} \eta$ must also always be a solution of this equation.
$\triangleright$ Explanation, and the cited theorem. The general solution $\xi(x)=$ $e_{1} \xi_{1}(x)+\cdots+e_{r} \xi_{r}(x)$ is a linear combination of $r$ fundamental, linearly independent solutions. Thus, for the $r$ infinitesimal transformations $X_{i}:=\xi_{i}(x) \frac{d f}{d x}, i=1, \ldots, r$, to be a Lie algebra of solutions of (6), it is necessary and sufficient that the coefficient $\xi_{i} \xi_{j}^{\prime}-\xi_{i}^{\prime} \xi_{j}$ of each bracket $\left[X_{i}, X_{j}\right]$ be also a solution of (6).

Theorem I.28. If $\xi_{1}, \ldots, \xi_{n}$ are functions of $x_{1}, \ldots, x_{n}$ be determined by a certain linear homogeneous partial differential equation:

$$
\sum_{\nu=1}^{n} A_{\mu \nu}(x) \cdot \xi_{\nu}+\sum_{\nu, \pi}^{1 \cdots n} B_{\mu \nu \pi}(x) \frac{\partial \xi_{\nu}}{\partial x_{\pi}}+\cdots=0 \quad(\mu=1,2 \cdots),
$$

then the expression $\xi_{1} \frac{\partial f}{\partial x_{1}}+\cdots+\xi_{n} \frac{\partial f}{\partial x_{n}}$ represents the general infinitesimal transformation of a finite continuous group if and only if, firstly the most general system of solutions of these differential equations depends only on a finite number of arbitrary constants, and secondly from two particular systems of solutions $\xi_{k 1}, \ldots, \xi_{k n}$ and $\xi_{j 1}, \ldots, \xi_{j n}$, by formation of the $n$ expressions:

$$
\sum_{\nu=1}^{n}\left(\xi_{k \nu} \frac{\partial \xi_{j i}}{x_{\nu}}-\xi_{j \nu} \frac{\partial \xi_{k i}}{x_{\nu}}\right) \quad(i=1 \cdots n)
$$

one always obtains a new system of solutions.
$\triangleleft$
In order to find the condition which comes out from this for the functions $\alpha_{1}(x), \ldots, \alpha_{r}(x)$, we set up the equation:

$$
\begin{equation*}
\frac{d^{r}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)}{d x^{r}}+\alpha_{1} \cdot \frac{d^{r-1}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)}{d x^{r-1}}+\cdots+\alpha_{r}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)=0 \tag{7}
\end{equation*}
$$

and we express in it the $r$-th and the $(r+1)$-th derivatives of $\xi$ and of $\eta$ by means of:

$$
\begin{equation*}
\xi, \xi^{\prime}, \ldots, \xi^{(r-1)}, \eta, \eta^{\prime}, \ldots, \eta^{(r-1)} \tag{8}
\end{equation*}
$$

thanks to (6) and to:

$$
\begin{equation*}
\eta^{(r)}+\alpha_{1}(x) \eta^{(r-1)}+\cdots+\alpha_{r-1}(x) \eta^{\prime}+\alpha_{r}(x) \eta=0 . \tag{6'}
\end{equation*}
$$

In this way, between the quantities (8), we obtain an equation of the form:

$$
\begin{equation*}
\sum_{0 \leqslant i, k \leqslant r-1} \lambda_{i k}\left(\xi^{(i)} \eta^{(k)}-\xi^{(k)} \eta^{(i)}\right)=0, \tag{9}
\end{equation*}
$$

and this equation must be identically satisfied, whichever $\xi$ and $\eta$ can be, as solutions of the $r$-th order differential equation (6). From this, it follows immediately that (9) must actually hold identically for all values of the quantities (8), and hence that the coefficient of every individual expression: $\xi^{(i)} \eta^{(k)}-\xi^{(k)} \eta^{(i)}$ must be identically zero.

At first, we consider the two cases: $r=1$ and $r=2$.
If $r=1$, the differential equation (6) has the form:

$$
\begin{equation*}
\xi^{\prime}+\alpha(x) \xi=0 . \tag{10}
\end{equation*}
$$

If $\xi(x)$ and $\eta(x)$ are any two solutions of this equation, then the expression $\xi \eta^{\prime}-\xi^{\prime} \eta$ vanishes evidently and hence is again a solution of (10). As
a consequence of that, the function $\alpha(x)$ is submitted to absolutely no restriction.

In case $r=2$, eq. (6) has the form:

$$
\begin{equation*}
\xi^{\prime \prime}+\alpha_{1}(x) \xi^{\prime}+\alpha_{2}(x) \xi=0 \tag{11}
\end{equation*}
$$

Now, one has:

$$
\frac{d\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)}{d x}=\xi \eta^{\prime \prime}-\xi^{\prime \prime} \eta, \quad \frac{d^{2}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)}{d x^{2}}=\xi \eta^{\prime \prime \prime}-\xi^{\prime \prime \prime} \eta+\xi^{\prime} \eta^{\prime \prime}-\xi^{\prime \prime} \eta^{\prime},
$$

so (7) receives the form:

$$
\xi\left(\eta^{\prime \prime \prime}+\alpha_{1} \eta^{\prime \prime}+\alpha_{2} \eta^{\prime}\right)-\eta\left(\xi^{\prime \prime \prime}+\alpha_{1} \xi^{\prime \prime}+\alpha_{2} \xi^{\prime}\right)+\xi^{\prime} \eta^{\prime \prime}-\xi^{\prime \prime} \eta^{\prime}=0,
$$

hence if one expresses $\xi^{\prime \prime}, \eta^{\prime \prime}, \xi^{\prime \prime \prime}, \eta^{\prime \prime \prime}$ by means of $\xi, \eta, \xi^{\prime}, \eta^{\prime}$, then it comes:

$$
\left(-\alpha_{1}^{\prime}+\alpha_{2}\right)\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)=0
$$

and this is for our case the equation (9) discussed above. Here, the factor of $\xi \eta^{\prime}-\xi^{\prime} \eta$ must vanish, whence we find: $\alpha_{1}^{\prime}=\alpha_{2}$, and we realize that the equation (11) is always the defining equation of a two-term group if and only if it possesses the form:

$$
\xi^{\prime \prime}+\alpha(x) \xi^{\prime}+\alpha^{\prime}(x) \xi=0
$$

On its own side, the function $\alpha(x)$ is subjected to no restriction.
We come to the case $r>2$.
As one easily sees, one has:

$$
\begin{aligned}
\frac{d^{m}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)}{d x^{m}} & =\xi \eta^{(m+1)}-\xi^{(m+1)} \eta+(m-1)\left(\xi^{\prime} \eta^{(m)}-\xi^{(m)} \eta^{\prime}\right)+ \\
& +\frac{m(m-3)}{1 \cdot 2}\left(\xi^{\prime \prime} \eta^{(m-1)}-\xi^{(m-1)} \eta^{\prime \prime}\right)+\cdots
\end{aligned}
$$

a series which ends up with the $\frac{1}{2}(m+1)$-th or with the $\frac{1}{2}(m+2)$-th term, according to the entire number $m$ being odd or even. Consequently, if we now only take into consideration all the terms in which appear derivatives of at least $(r-1)$-th order, then we can write the equation (7) as follows:

$$
\begin{aligned}
\xi \eta^{(r+1)} & -\xi^{(r+1)} \eta+(r-1)\left(\xi^{\prime} \eta^{(r)}-\xi^{(r)} \eta^{\prime}\right)+ \\
& +\frac{r(r-3)}{1 \cdot 2}\left(\xi^{\prime \prime} \eta^{(r-1)}-\xi^{(r-1)} \eta^{\prime \prime}\right)+ \\
& +\alpha_{1}\left\{\xi \eta^{(r)}-\xi^{(r)} \eta+(r-2)\left(\xi^{\prime} \eta^{(r-1)}-\xi^{(r-1)} \eta^{\prime}\right)\right\}+ \\
& +\alpha_{2}\left(\xi \eta^{(r-1)}-\xi^{(r-1)} \eta\right)+\cdots=0 .
\end{aligned}
$$

We reshape this equation by using the relations:

$$
\begin{aligned}
& \xi^{(r)}+\alpha_{1} \xi^{(r-1)}+\cdots=0 \\
& \xi^{(r+1)}+\alpha_{1} \xi^{(r)}+\left(\alpha_{1}^{\prime}+\alpha_{2}\right) \xi^{(r-1)}+\cdots=0
\end{aligned}
$$

and the corresponding relations for $\eta$ as well, and we find that the equation (9) has the form:

$$
\begin{aligned}
& -\alpha_{1}^{\prime}\left(\xi \eta^{(r-1)}-\xi^{(r-1)} \eta\right)-\alpha_{1}\left(\xi^{\prime} \eta^{(r-1)}-\xi^{(r-1)} \eta^{\prime}\right)+ \\
& +\frac{r(r-3)}{1 \cdot 2}\left(\xi^{\prime \prime} \eta^{(r-1)}-\xi^{(r-1)} \eta^{\prime \prime}\right)+\cdots=0
\end{aligned}
$$

where the left out terms only contain derivatives of order lower than the $(r-1)$-th order. At present, by setting identically to zero the coefficients of the indivivual expressions $\xi^{(i)} \eta^{(k)}-\xi^{(k)} \eta^{(i)}$, we receive: $r(r-3)=0$ and: $\alpha_{1}=\alpha_{1}^{\prime}=0$. It thus turns out that $r$ can only be larger than 2 when it is at the same time equal to 3 , and that for $r=3$, our differential equation (6) must have the form:

$$
\begin{equation*}
\xi^{\prime \prime \prime}+\alpha_{2}(x) \xi^{\prime}+\alpha_{3}(x) \xi=0 . \tag{12}
\end{equation*}
$$

In order to determine the functions $\alpha_{2}$ and $\alpha_{3}$ more precisely, we form the equation:

$$
\frac{d^{3}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)}{d x^{3}}+\alpha_{2} \cdot \frac{d\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)}{d x}+\alpha_{3}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)=0
$$

and we get:

$$
\xi \eta^{(4)}-\xi^{(4)} \eta+2\left(\xi^{\prime} \eta^{\prime \prime \prime}-\xi^{\prime \prime \prime} \eta^{\prime}\right)+\alpha_{2}\left(\xi \eta^{\prime \prime}-\xi^{\prime \prime} \eta\right)+\alpha_{3}\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)=0 ;
$$

therefore, by taking account of the equation (12), of:

$$
\xi^{(4)}+\alpha_{2} \xi^{\prime \prime}+\left(\alpha_{2}^{\prime}+\alpha_{3}\right) \xi^{\prime}+\alpha_{3}^{\prime} \xi=0
$$

and of the corresponding equations for $\eta$, we find:

$$
\left(-\alpha_{2}^{\prime}+2 \alpha_{3}\right)\left(\xi \eta^{\prime}-\xi^{\prime} \eta\right)=0 .
$$

Consequently $\alpha_{2}^{\prime}$ must be equal to $2 \alpha_{3}$, but this the unique condition that $\alpha_{2}$ and $\alpha_{3}$ have to satisfy.

If we set $\alpha_{2}(x)=2 \alpha(x)$, then the defining equation of the most general three-term group of the once-extended manifold becomes visible in the form:

$$
\xi^{\prime \prime \prime}+2 \alpha(x) \xi^{\prime}+\alpha^{\prime}(x) \xi=0 .
$$

As a result, all the groups of the once-extended manifold are determined and we have the

Proposition 2. The defining equation of a finite continuous group of the once-extended manifold always has one of the three forms:

$$
\left\{\begin{array}{l}
\xi^{\prime}+\alpha(x) \xi \quad=0  \tag{13}\\
\xi^{\prime \prime}+\alpha(x) \xi^{\prime}+\alpha^{\prime}(x) \xi=0 \\
\xi^{\prime \prime \prime}+2 \alpha(x) \xi^{\prime}+\alpha^{\prime}(x) \xi=0
\end{array}\right.
$$

Here, the function $\alpha(x)$ is submitted to no restriction.
Above, we have generally found all the groups of the once-extended manifold, and among the found groups are obviously contained also the three types of groups that we have listed in the preceding paragraph; indeed, if we set $\alpha=0$, we obtain the three defining equations:

$$
\begin{equation*}
\xi^{\prime}=0, \quad \xi^{\prime \prime}=0, \quad \xi^{\prime \prime \prime}=0, \tag{14}
\end{equation*}
$$

which produce the three groups

$$
\frac{d f}{d x} ; \quad \frac{d f}{d x}, \quad x \frac{d f}{d x} ; \quad \frac{d f}{d x}, \quad x \frac{d f}{d x}, \quad x^{2} \frac{d f}{d x},
$$

one after the other. Consequently, it still remains to prove that it is possible, by introduction of a new variable, to reshape the defining equations (13) so that they receive the simple form (14).

In order to produce this proof, we imagine that: $x_{1}=F(x)$ is introduced as a new variable in place of $x$. By this, the infinitesimal transformation:

$$
X f=\xi(x) \frac{d f}{d x}
$$

takes the form:

$$
X f=\xi(x) \frac{d f}{d x}=\xi(x) \cdot F^{\prime}(x) \cdot \frac{d f}{d x_{1}}=\xi_{1}\left(x_{1}\right) \frac{d f}{d x_{1}},
$$

whence one has:

$$
\xi=\frac{\xi_{1}}{F^{\prime}} .
$$

Furthermore:

$$
\begin{aligned}
\xi^{\prime} & =\xi_{1}^{\prime}-\frac{\xi_{1} F^{\prime \prime}}{F^{\prime 2}} \\
\xi^{\prime \prime} & =\xi_{1}^{\prime \prime} F^{\prime}-\xi_{1}^{\prime} \frac{F^{\prime \prime}}{F^{\prime}}-\xi_{1} \frac{F^{\prime} F^{\prime \prime \prime}-2 F^{\prime \prime 2}}{F^{\prime 3}} \\
\xi^{\prime \prime \prime} & =\xi_{1}^{\prime \prime \prime} F^{2}-\xi_{1}^{\prime} \frac{2 F^{\prime} F^{\prime \prime \prime}-3 F^{\prime \prime 2}}{F^{2}}-\xi_{1} \frac{d}{d x} \frac{F^{\prime} F^{\prime \prime \prime}-2 F^{\prime \prime 2}}{F^{3}},
\end{aligned}
$$

where, for reasons of abbreviation, $\xi_{1}^{(\nu)}$ is written in place of $\frac{d^{\nu} \xi_{1}}{d x_{1}^{\nu}}$.

If we now introduce the new variable $x_{1}$ in the equations (13), then these receive the following form:

$$
\begin{aligned}
& \xi_{1}^{\prime}+\left(\frac{\alpha}{F^{\prime}}-\frac{F^{\prime \prime}}{F^{\prime 2}}\right) \xi_{1}=0 \\
& \xi_{1}^{\prime \prime}+\left(\frac{\alpha}{F^{\prime}}-\frac{F^{\prime \prime}}{F^{\prime 2}}\right) \xi_{1}^{\prime}+\frac{1}{F^{\prime}} \frac{d}{d x}\left(\frac{\alpha}{F^{\prime}}-\frac{F^{\prime \prime}}{F^{\prime 2}}\right) \xi_{1}=0, \\
& \xi_{1}^{\prime \prime \prime}+2\left(\frac{\alpha}{F^{\prime 2}}-\frac{F^{\prime} F^{\prime \prime \prime}-\frac{3}{2} F^{\prime \prime 2}}{F^{\prime 4}}\right) \xi_{1}^{\prime}+ \\
& \quad+\frac{1}{F^{\prime}} \frac{d}{d x}\left(\frac{\alpha}{F^{\prime 2}}-\frac{F^{\prime} F^{\prime \prime \prime}-\frac{3}{2} F^{\prime \prime 2}}{F^{\prime 4}}\right) \xi_{1}=0,
\end{aligned}
$$

but if these equations are supposed to take the simple form (14), then in the first two cases we just need to employ for $F$ a solution of the differential equation:

$$
F^{\prime \prime}=\alpha F^{\prime},
$$

whilst in the last case, a solution of the equation:

$$
F^{\prime} F^{\prime \prime \prime}-\frac{3}{2} F^{\prime \prime 2}=\alpha F^{\prime 2}
$$

With that, the required proof is supplied.

$$
\text { § } 3 .
$$

Up to now, we have directly determined the groups of the onceextended manifold, without using more from the theory of the first volume than a few general propositions of the first chapters. But it should not be passed over in silence that the determination of all groups of the once-extended manifold already follows immediately from the result of Chap. 29 in Volume I.
$\triangleright$ Note. This important rigidity Theorem I. 112 (Vol. I, p. 631 [here: see p. ??]), located at the very end of Vol. I, states that the three well known transitive groups: projective $\mathrm{PGL}_{n}(\mathbb{C})$, affine $\mathrm{A}_{n}(\mathbb{C})=\mathrm{GL}_{n}(\mathbb{C}) \ltimes \mathbb{C}^{n}$ and special affine $\mathrm{SA}_{n}(\mathbb{C})=\mathrm{SL}_{n}(\mathbb{C}) \ltimes \mathbb{C}^{n}$ are the only ones which can enjoy maximal free mobility at the infinitesimal, first order level, namely the linearized isotropy group of any point be equal to $G L_{n}(\mathbb{C})$ or to $S L_{n}(\mathbb{C})$. The result is heavily used below for the classification of primitive local Lie group actions on $\mathbb{C}^{2}$ (Chap. 3, § 7, p. 90 sq.). Chap. N?? translates and comments its complete proof.

As we have seen in the beginning of § 1, each $r$-term group of the once-extended manifold comprises, in the neighbourhood of a point $x=0$ in general position, one infinitesimal transformation of zeroth order in $x$,
further, one of first, one of second, $\ldots$, one of $(r-1)$-th order, but not any of $r$-th or higher order.

If we now consider the cases $r=1, r=2, r>2$ somehow into more detail, then we realize at once that they correspond exactly to the three cases distinguished on p. 625 (loc. cit. [here: see p. ??]), if we set $n=1$ there. Indeed, if $r=1$, our $r$-term group comprises in the neighbourhood of $x=0$ exactly $n=1$ infinitesimal transformation of zeroth order, $n^{2}-$ $1=0$ of first order, and not any of higher order. If secondly $r=2$, then the group comprises exactly $n$ transformations of zeroth order, $n^{2}=1$ of first order, but not any of higher order. If finally $r>2$, then the group comprises $n=1$ transformations of zeroth order, $n^{2}=1$ of first order and in addition, still some of higher order. From this, it follows that we can apply immediately to our case the result which is obtained in the Chap. 29 of Vol. I. If we do that, then we receive immediately the result stated in Theorem 1 on p. 6. While doing so, it turns out in particular that the special linear group of a $n$-fold extended space transforms, for $n=1$, to the group of all translations of the once-extended manifold.

## Chapter 2. <br> Determination of all Subgroups <br> of the General Projective Group on the Line and of the General Linear Homogeneous Group in the Plane.

The knowledge of all subgroups of the two groups named in the title is indispensable for later general studies about the groups of the plane*). While determining these subgroups, we naturally get at the same time a determination of all subgroups of every group that has the same composition as one of the two groups referred to in the title (see Vol. I, Theor. 33, p. 210).
> *) The continuous subgroups of the general linear homogeneous group: $x p, y q, x q, y q$ in two variables $x, y$ have been determined for the first time in the 1878 Norvegian Archiv cf. also Math. Ann., vol. 16; on this occasion, the variables $x, y$ are interpreted as Cartesian coordinates in the plane, and also as homogeneous coordinates in a bundle of rays [STRAHLBÜSCHEL]. Later, Stephanos has conducted interesting researchs about the mentioned groups.

$\triangleright$ The cited theorem. Placed at the end of Chap. 12, Theorem I. 33 states that, in principle, the determination of all subgroups of a given finite continuous group $X_{1} f, \ldots, X_{r} f$ involves only algebraic operations on the structure constants appearing in the bracket relations $\left[X_{\rho}, X_{\sigma}\right]=\sum_{\tau=1}^{n} C_{\rho \sigma \tau}$. $X_{\tau}$, so that two isomorphic local groups (Lie algebras) obviously have isomorphic collections of local subgroups (Lie subalgebras). More precisely, Lie describes the following general recipe, which dates back to 1878 .

One wants to determine all possible $m$-dimensional $(1 \leqslant m \leqslant r)$ Lie subalgebras of the form:

$$
Y_{\mu}=\sum_{1 \leqslant \rho \leqslant r} h_{\mu \rho} \cdot X_{\rho}
$$

which are concretely represented by some $m \times r$ unknown constant matrix $\left(h_{\mu \rho}\right)$ supposed to be of rank $m$. Then here the $Y_{\mu}$ generate a Lie subalgebra if and only if their brackets in pairs:

$$
\left[Y_{\mu}, Y_{\nu}\right]=\sum_{1 \leqslant \rho, \sigma \leqslant r} h_{\mu \rho} h_{\nu \sigma} \cdot\left[X_{\rho}, X_{\sigma}\right] \quad(1 \leqslant \mu<\nu \leqslant n)
$$

are linear combinations of themselves alone, namely are of the form $\sum_{\pi=1}^{n} l_{\mu \nu \pi} \cdot Y_{\pi}$ for some $l_{\mu \nu \pi} \in \mathbb{K}$. But since by assumption we have $\left[X_{\rho}, X_{\sigma}\right]=\sum_{\tau=1}^{n} C_{\rho \sigma \tau} \cdot X_{\tau}$ for some structure constants $C_{\rho \sigma \tau} \in \mathbb{K}$,
we can hence plug in these brackets in order to read more precisely the requirement:

$$
\begin{aligned}
\sum_{1 \leqslant \rho, \sigma, \tau \leqslant r} h_{\mu \rho} h_{\nu \sigma} C_{\rho \sigma \tau} \cdot X_{\tau} & =\left[Y_{\mu}, Y_{\nu}\right] \\
& =\sum_{1 \leqslant \pi \leqslant m} l_{\mu \nu \pi} \cdot Y_{\pi} \\
& =\sum_{1 \leqslant \pi \leqslant m} \sum_{1 \leqslant \tau \leqslant r} l_{\mu \nu \pi} h_{\pi \tau} \cdot X_{\tau}
\end{aligned}
$$

Then by identifying the coefficients of $X_{1}$, of $X_{2}, \ldots$, and of $X_{r}$, the sought matrices $\left(h_{\mu \rho}\right)$ should therefore be such that for every pair of indices $(\mu, \nu)$ with $1 \leqslant \mu<\nu \leqslant r$, there exist $m$ solutions $l_{\mu \nu 1}, \ldots, l_{\mu \nu m}$ to the linear nonhomogeneous system of $r$ equations:

$$
\left\{\begin{array}{c}
l_{\mu \nu 1} h_{11}+\cdots+l_{\mu \nu m} h_{m 1}=\sum_{\rho, \sigma} h_{\mu \rho} h_{\nu \sigma} C_{\rho \sigma 1} \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
l_{\mu \nu 1} h_{1 r}+\cdots \cdots l_{\mu \nu m} h_{m r}=\sum_{\rho, \sigma} h_{\mu \rho} h_{\nu \sigma} C_{\rho \sigma r},
\end{array}\right.
$$

whose both sides depend upon the unknowns $h_{\mu \rho}$. But for every fixed pair $(\mu, \nu)$, the existence of such $l_{\mu \nu \pi}$ just amounts to require that the $(m+1) \times r$ matrix:

$$
\left(\begin{array}{cccc}
h_{11} & \cdots & h_{m 1} & \sum_{\rho, \sigma} h_{\mu \rho} h_{\nu \sigma} C_{\rho \sigma 1} \\
\cdots & \cdots & \cdots & \cdots \cdots \cdots \cdots \cdots \cdots \\
h_{1 r} & \cdots & h_{m r} & \sum_{\rho, \sigma} h_{\mu \rho} h_{\nu \sigma} C_{\rho \sigma r}
\end{array}\right)
$$

whose first $m \leqslant r$ columns are already supposed to be of rank $m$, should be of rank $m$ also. Equivalently, all of its $(m+1) \times(m+1)$ minors should vanish. Equating to zero all these minors then furnishes a finite number of algebraic equations for the $h_{\mu \rho}$, which clearly depend only on the structure constants; furthermore, by reasoning backwards, one easily sees that every system of solutions $h_{\mu \rho}$ to these algebraic equations yields an $m$-dimensional Lie subalgebra of $X_{1}, \ldots, X_{r}$, provided of course that one only keeps solution matrices $\left(h_{\mu \rho}\right)$ whose rank equals $m$.

Nonetheless, this brute process rapidly becomes unwieldy as soon as $r \geqslant 3$, and it does not take account of the natural fact that two subgroups $H_{1}$ and $H_{2}$ of a (dis)continuous group $G$ should have equal rights [GLEICHBERECHTIGT SEIN] when they are conjugate to each other by an inner automorphism, namely when $H_{2}=g^{-1} H_{1} g$ for some $g \in G$. Much finer reasonings will be developed by Lie.

For reasons of convenience, we shall from now on shortly write:

$$
\frac{d f}{d x}=p
$$

the infinitesimal transformation in one variable $x$, and in the same way, we shall make use of the abbreviations:

$$
\frac{\partial f}{\partial x}=p, \quad \frac{\partial f}{\partial y}=q
$$

for the infinitesimal transformations in two variables $x, y$. We have in fact used similar designations earlier on (see for instance Vol. I, p. 555).

## § 4.

The general projective group of the once-extended manifold $x$ is 3term, so it shall be shortly named "the $G_{3}$ " in the present paragraph.
$\triangleright$ Translation note. Today, one would write instead: "so it shall be shortly named $G_{3}$ ", without the determinate article "the"; but to be faithful to the text, we maintain it, throughout.

Our $G_{3}$ comprises the $\infty^{3}$ finite transformations:

$$
x^{\prime}=\frac{a_{1} x+a_{2}}{a_{3} x+1}
$$

and is engendered by the three independent infinitesimal transformations:

$$
X_{1} f=p, \quad X_{2} f=x p, \quad X_{3} f=x^{2} p
$$

(see Chap. 1 or Vol. I, p. 554 sq.). Its general infinitesimal transformation therefore possesses the form:

$$
X f=\left(e_{1}+e_{2} x+e_{3} x^{2}\right) p,
$$

where $e_{1}, e_{2}, e_{3}$ indicate arbitrary constants.
The $G_{3}$ is transitive, and even threefold transitive (Vol I, p. 631 sq. [here: see Chap. $N$ ??]), that is to say, it always comprises a transformation by virtue of which any three distinct points of the manifold $x$ can be transferred to any three other points; here, the point at infinity makes absolutely no difference.
$\triangleright$ The concept of composition (structure). In the next paragraph, the word composition [ZuSAMMENSETZUNG] appears. Quoting [17], p. 168, it was Killing in [20] (p. 163) who suggested that "Lie's designation "composition of groups" [ZUSAMMENSETZUNG DER GRUPPEN] was not the best choice to describe the theory he had now so greatly, albeit tentatively advanced. He pointed out that according to Lie, a group was either simple or composite, and yet one also spoke of the composition of simple groups. On the
basis of this inconsistency, he advocated speaking instead of the "shape" [Gestaltung] of a group rather than its composition. Unbeknownst to Killing, E. Vessiot and W. de Tannenberg, the first two graduates of the École Normale Supérieure to study with Lie in Leipzig, had already in a sense met Killing's objection. In their lengthy review of the first volume of Lie's Theorie der Transformationsgruppen they expressed Lie's idea of the Zusammesetzung by the French word structure ([39], p. 137). In his own publications, É. Cartan always referred to la structure des groupes, thereby establishing this expression in the vocabulary of twentieth-century mathematics."

The composition [ZUSAMMENSETZUNG] (Vol. I, Chap. 17) of the $G_{3}$ is determined by the equations:

$$
\begin{equation*}
\left[X_{1}, X_{2}\right]=X_{1} f, \quad\left[X_{1}, X_{3}\right]=2 X_{2} f, \quad\left[X_{2}, X_{3}\right]=X_{3} f \tag{1}
\end{equation*}
$$

and its adjoint group (loc. cit., Chap. 16) therefore reads as follows:

$$
\begin{aligned}
& E_{1} f=-e_{2} \frac{\partial f}{\partial e_{1}}-2 e_{3} \frac{\partial f}{\partial e_{2}} \\
& E_{2} f=e_{1} \frac{\partial f}{\partial e_{1}} \\
& E_{3} f=r e e_{3} \frac{\partial f}{\partial e_{3}}, \\
& 2 e_{1} \frac{\partial f}{\partial e_{2}}+e_{2} \frac{\partial f}{\partial e_{3}} .
\end{aligned}
$$

Since the $G_{3}$ contains no excellent [AUSGEZEICHNETE] infinitesimal transformation, this adjoint group is three-term.
$\triangleright$ The adjoint group. From Vol. I, Chap. 16, we summarize the needed prerequisites. Let $\bar{x}_{i}=f_{i}(x ; a)$ be an arbitrary finite continuous $r$-term group with the $r$ independent infinitesimal transformations $X_{k}=\sum_{i=1}^{n} \xi_{k i}(x) \frac{\partial}{\partial x_{i}}, i=1, \ldots, r$ whose coefficients are defined by $\xi_{k i}(x)=\frac{\partial f_{i}}{\partial a_{k}}(x ; e)$.

Theorem I.48. If one introduces the general transformation of the group $\bar{x}=f(x ; a)$ itself as a change of variable $x$, if $\bar{X}_{k}$ denotes the transformed $X_{k}$, and if one defines on the target space $\bar{x}$ the infinitesimal transformations $\widetilde{X}_{k}:=\sum_{i=1}^{n} \xi_{k i}(\bar{x}) \frac{\partial}{\partial \bar{x}_{i}}$ with the same coefficient functions $\xi_{k i}$ of $\bar{x}$, then the general infinitesimal transformation $e_{1} X_{1}+\cdots+e_{r} X_{r}$ of the r-term group $\bar{x}_{i}=f_{i}(x ; a)$ transforms to:

$$
\begin{aligned}
e_{1} X_{1}+\cdots+e_{r} X_{r} & =e_{1} \bar{X}_{1}+\cdots+e_{r} \bar{X}_{r} \\
& =\bar{e}_{1}(e ; a) \widetilde{X}_{1}+\cdots+\bar{e}_{r}(e ; a) \widetilde{X}_{r},
\end{aligned}
$$

and after reexpressing it in terms of the $\widetilde{X}_{k}$, one obtains that the $\bar{e}_{k}$ are related to the $e_{l}$ by linear equations of the form:

$$
\bar{e}_{k}=\sum_{l=1}^{r} \rho_{k l}\left(a_{1}, \ldots, a_{r}\right) \cdot e_{l} \quad(k=1 \cdots r)
$$

which represent the so-called adjoint group of the group $\bar{x}=f(x ; a), a$ transformation group subjected to the composition law:

$$
\rho_{k \nu}\left(\varphi_{1}(a, b), \ldots, \varphi_{r}(a, b)\right) \equiv \sum_{j=1}^{r} \rho_{j \nu}\left(a_{1}, \ldots, a_{r}\right) \cdot \rho_{k j}\left(b_{1}, \ldots, b_{r}\right)
$$

if one denotes $f(f(x ; a) ; b)=f(x ; \varphi(a, b))$. This adjoint group contains the identity transformation and it is engendered by certain infinitesimal transformations as follows: letting

$$
\left[X_{i}, X_{k}\right]=\sum_{s=1}^{r} c_{i k s} \cdot X_{s} f \quad(i, k=1 \cdots r)
$$

denote the structure of the group, if one introduces the linear homogeneous infinitesimal transformations defined on the linear space equipped with the coordinates $\left(e_{1}, \ldots, e_{r}\right)$ by:

$$
E_{\mu}:=\sum_{k, j=1}^{r} c_{j \mu k} e_{j} \frac{\partial}{\partial e_{k}} \quad(\mu=1 \cdots r),
$$

then $\lambda_{1} E_{1}+\cdots+\lambda_{r} E_{r}$ is the general infinitesimal transformation of the adjoint group and the $E_{\mu}$ have the same structure as the $X_{k}$ :

$$
\left[E_{i}, E_{k}\right]=\sum_{s=1}^{r} c_{i k s} \cdot E_{s} f \quad(i, k=1 \cdots r) .
$$

Although the $E_{\mu} f$ have the same structure as the $X_{k} f$ which are independent by essentiality of the parameters $a$, they need not be likewise linearly independent. In fact, if a certain infinitesimal transformation $X^{e x c}=$ $\sum g_{\mu} X_{\mu} f$ commutes with all the $X_{k} f$, then

For instance, the four-term linear homogeneous group:

$$
x \frac{\partial}{\partial x}, \quad y \frac{\partial}{\partial x}, \quad x \frac{\partial}{\partial y}, \quad y \frac{\partial}{\partial y}
$$

under study in the present chapter ...
Excellent infinitesimal transformations. An infinitesimal transformation $\sum_{\mu=1}^{r} c_{\mu} X_{\mu} f$ of a finite continuous group $X_{1} f, \ldots, X_{r} f$ is called excellent when it commutes with all infinitesimal transformations of the adjoint group.

Theorem I.49. The adjoint group $\bar{e}_{k}=\sum \rho_{k l}(a) \cdot e_{l}$ of an r-term group $X_{1} f, \ldots, X_{r} f$ contains $r$ essential parameters if and only if no one amongst the $\infty^{r-1}$ infinitesimal transformations $\sum g_{\mu} X_{\mu} f$ is excellent; by contrast, the adjoint group has less than $r$, say exactly $r-m$ essential parameters when the group $X_{1} f, \ldots, X_{r} f$ comprises exactly $m$ and not more independent excellent infinitesimal transformations.

At first, we want to study which types of one-term subgroups are contained in our $G_{3}$, or, what amounts to the same, we want to determine all types of infinitesimal transformations existing in it. For that, we make use of the ideas and methods developed in Vol I, pp. 278-287.
$\triangleright$ Summary.

In the general infinitesimal transformation $X f$ of our $G_{3}$, if we interpret the quantities $e_{1}, e_{2}, e_{3}$ as homogeneous point-coordinates [PUNKTCOORDINATEN] of a plane, then every infinitesimal transformation and hence also, every one-term subgroup of the $G_{3}$ will be represented by a point of this plane, and conversely each point of the plane is the image of an infinitesimal transformation and with that at the same time, of a oneterm subgroup of the $G_{3}$.

At present, we imagine that the points of the plane $e_{1}, e_{2}, e_{3}$ are transformed by the adjoint group $E_{1} f, E_{2} f, E_{3} f$ of our $G_{3}$ and we seek all smallest invariant manifolds which appear in the plane (see Vol. I, p. 225), that is to say, all invariant manifolds whose points are transformed by the adjoint group in such a way that every point in general position on such a manifold transfers to all other points of that kind. Every such smallest invariant manifold then represents a type of infinitesimal transformation and hence also, a type of one-term subgroup of the $G_{3}$. In the indicated way, we obtain all such types, because two one-term subgroups belong to the same type, when they are conjugate [GLEICHBERECHTIGT] to each other inside the $G_{3}$, but this happens if and only if the image-point [BILDPUNKT] of the one can be transferred to the image-point of the other by a transformation of the adjoint group, that is to say, when the image-points of the two lie on the same smallest invariant manifold.

Consequently, one now searches for all manifolds of the plane $e_{1}, e_{2}, e_{3}$ which remain invariant by the adjoint group. Next, since $e_{1}, e_{2}, e_{3}$ are homogeneous point-coordinates, the manifolds in question will be represented by systems of equations homogeneous in $e_{1}, e_{2}, e_{3}$, that is to say,
by systems of equations which admit the infinitesimal transformation:

$$
E f=e_{1} \frac{\partial f}{\partial e_{1}}+e_{2} \frac{\partial f}{\partial e_{2}}+e_{3} \frac{\partial f}{\partial e_{3}} .
$$

Our problem therefore amounts to determine all systems of equations in $e_{1}, e_{2}, e_{3}$ that remain invariant by the four-term [VIERGLIEDRIGE] group: $E_{1} f, E_{2} f, E_{3} f, E f$. We undertake this determination under the guidance of Theorem 42, Vol. I, p. 237. We thus form the matrix:

$$
\left(\begin{array}{ccc}
-e_{2} & -2 e_{3} & 0  \tag{2}\\
e_{1} & 0 & -e_{3} \\
0 & 2 e_{1} & e_{2} \\
e_{1} & e_{2} & e_{3}
\end{array}\right)
$$

Since the determinants in three rows [DREIREIHIG] of this matrix do not all vanish identically, if we then set all three-row determinants equal to zero, we receive, disregarding the meaningless system of equations: $e_{1}=$ $e_{2}=e_{3}=0$, the equation:

$$
\begin{equation*}
e_{2}^{2}-4 e_{1} e_{3}=0 \tag{3}
\end{equation*}
$$

which surely represents a manifold of the desired constitution. Furthermore, if we observe that by virtue of (3), not all two-by-two [ZWEIREIHIGEN] determinants of the matrix (2) vanish, and that we only receive, by setting equal to zero all two-by-two determinants, the useless system of equations: $e_{1}=e_{2}=e_{3}=0$, then we recognize that except the conic section (3), the adjoint group leaves invariant no point-figure [PUNKTFIGUR] of the plane $e_{1}, e_{2}, e_{3}$.

With that are found all types of one-term subgroups of the $G_{3}$, they are two: the subgroups of the first type are represented by all the points of the plane which do not lie on the conic section (3), the subgroups of the second type by the points of this conic section. Therefore, two one-term subgroups of the $G_{3}$ are conjugate to each other inside the $G_{3}$ if and only if their point-images lie either both outside the conic section (3), or both on this conic-section.

If we want to have one representative for the two types of one-term subgroups, we need only to select any two points of the plane, of which the first does not lie on the conic section, while the other lies on it. Two such points are: $e_{1}=e_{3}=0$ and $e_{2}=e_{3}=0$, whence the one-term subgroup $x p$ is a representative of the first type, and the one-term subgroup $p$ is a representative of the second type.

The two found types can be characterized in a very simple manner.

Indeed, if one looks for all points of the once-extended manifold $x$ that remain invariant by the one-term group:

$$
X f=\left(e_{1}+e_{2} x+e_{3} x^{2}\right) p
$$

then one only has to solve the second order equation:

$$
\begin{equation*}
e_{3} x^{2}+e_{2} x+e_{1}=0 \tag{4}
\end{equation*}
$$

the roots of this equation are the abscissas of the sought invariant points. Now, if: $e_{2}^{2}-4 e_{1} e_{3} \neq 0$, then $X f$ belongs to the first type, so the equation (4) has two different roots, out of which however one can be infinitely large, and consequently in this case $X f$ leaves invariant two separate points out of which one can also lie at infinity. If on the other hand $e_{2}^{2}-4 e_{1} e_{3}=0$, then the equation (4) has two collapsing roots, which can also be infinitely large, and consequently in this case $X f$ leaves invariant a doubly counting [DOPPELT ZÄHLENDEN] point, which can lie either in the Finite or in the Infinite. One sees easily that in each one of the two discussed cases, the one-term subgroup $X f$ is fully determined by the points that it leaves invariant.
$\triangleright$ Projective line $\mathbb{C P}^{1}$.
Let:

$$
X f=e_{1} X_{1} f+e_{2} X_{2} f+e_{3} X_{3} f, \quad Y f=\varepsilon_{1} X_{1} f+\varepsilon_{2} X_{2} f+\varepsilon_{3} X_{3} f
$$

be any two independent infinitesimal transformations of our $G_{3}$, so that all two-column determinants of the matrix:

$$
\left|\begin{array}{lll}
e_{1} & e_{2} & e_{3} \\
\varepsilon_{1} & \varepsilon_{2} & \varepsilon_{3}
\end{array}\right|
$$

should not vanish. By Combination of $X f$ with $Y f$, we obtain the infinitesimal transformation:

$$
\begin{align*}
{[X, Y]=\left(e_{1} \varepsilon_{2}-e_{2} \varepsilon_{1}\right) X_{1} f } & +2\left(e_{1} \varepsilon_{3}-e_{3} \varepsilon_{1}\right) X_{2} f+ \\
& +\left(e_{2} \varepsilon_{3}-e_{3} \varepsilon_{2}\right) X_{3} f \tag{5}
\end{align*}
$$

Under the assumptions made, $X f$ and $Y f$ are represented by two different points in the plane $e_{1}, e_{2}, e_{3}$, and in the same way $[X, Y]$ by a point with the homogeneous coordinates:

$$
\eta_{1}=e_{1} \varepsilon_{2}-e_{2} \varepsilon_{1}, \quad \eta_{2}=2\left(e_{1} \varepsilon_{3}-e_{3} \varepsilon_{1}\right), \quad \eta_{3}=e_{2} \varepsilon_{3}-e_{3} \varepsilon_{2}
$$

This point can be geometrically defined in a very simple way; indeed, it satisfies obviously the two equations:

$$
e_{2} \eta_{2}-2 e_{1} \eta_{3}-2 e_{3} \eta_{1}=0, \quad \varepsilon_{2} \eta_{2}-2 \varepsilon_{1} \eta_{3}-2 \varepsilon_{3} \eta_{1}=0
$$

therefore it lies both on the polar of $e_{1}, e_{2}, e_{3}$ with respect to the conic section (3) and on the polar of $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}$. In other words:

If $X f$ and $Y f$ are any two independent infinitesimal transformations of our $G_{3}$, one finds the image-point of the infinitesimal transformation $[X, Y]$ by connecting the image-points of $X f$ and of $Y f$ through a straight line and by looking for the polar of this line with respect to the conic section: $e_{2}^{2}-4 e_{1} e_{3}=0$.
$\triangleright$ Polars and the proposed geometric construction.
This geometric construction of the image-point of $[X, Y]$ from the the image-points of $X f$ and of $Y f$ shows clearly that two independent infinitesimal transformations of our $G_{3}$ are never interchangeable (cf. p. 53). Namely, if $X f$ and $Y f$ are independent of each other, then their imagepoints are distinct and the connection line [VERBINDUNGSLINIE] of their image-points always possesses a completely determined polar, whence the expression $[X, Y]$ can never vanish identically.

At present it is very easy to indicate all two-term subgroups of the $G_{3}$.
If $X f$ and $Y f$ are two independent infinitesimal transformations of such a subgroup, then its general infinitesimal transformation writes: $\lambda X f+\mu Y f$, so the subgroup is represented by a line in the plane: $e_{1}, e_{2}, e_{3}$. But now, in order that $X f$ and $Y f$ really engender a two-term subgroup, it is yet necessary and sufficient that an equation of the form:

$$
[X, Y]=c_{1} X f+c_{2} Y f
$$

holds - in other words: a straight line of the plane $e_{1}, e_{2}, e_{3}$ represents a two-term subgroup of the $G_{3}$ if it contains its polar with respect to the conic section (3), that is to say, if it is a tangent to this conic section. Therefore:

The two-term subgroups of the $G_{3}$ in the plane $e_{1}, e_{2}, e_{3}$ are represented by the tangents to the conic-section: $e_{2}^{2}-4 e_{1} e_{3}=0$.
$\triangleright$ Brief explanation.
$\triangleleft$
It is clear that by means of the adjoint group $E_{1} f, E_{2} f, E_{3} f$, every tangent of the conic section can be transferred to any other tangent, and consequently all the two-term subgroups of our $G_{3}$ are conjugate to each other inside the $G_{3}$ : there is one single type of two-term subgroup in the $G_{3}$. As representative of this type we can select the tangent to the point $e_{2}=e_{3}=0$. The equation of this tangent writes: $e_{3}=0$, from which we find as subgroup representative: $p, x p$. This is the largest subgroup contained in the $G_{3}$ which leaves invariant the point at infinity. Next, since
every point $x$, and the point at infinity too, can be transferred by means of our $G_{3}$ to every other point, it turns out that each two-term subgroup of our $G_{3}$ leaves untouched one point in the Finite or in the Infinite, and that it is fully determined by the indication of this invariant point.

We now sum up the gained result.
Theorem 2. Every subgroup of the general projective group: $p, x p$, $x^{2} p$ of the once-extended manifold $x$ is, within this group, conjugate to one of the three subgroups:

$$
\begin{array}{|l|l|}
\hline p, x p & x p \\
\hline
\end{array}
$$

in the first of these three possible cases, it leaves one point invariant, in the second, two separate points, in the third two coinciding points, and in fact, it is completely determined by the indication of the points that are invariant by it.
$\triangleright$ A diagram to summarize the theorem. We draw $\mathbb{K}^{1}{ }^{1}$ as an infinite line whose two extreme points should be identified to the single point at infinity $\infty$ (it is the Riemann sphere in case $\mathbb{K}=\mathbb{C}$ ).


Fig. : Subgroups of $P G L_{1}(\mathbb{K})$ are recognized from their fixed points

The infinitesimal transformation $x^{2} p$ fixes 0 twice. In the left diagram, we therefore encircle 0 twice. Similarly, $p$ fixes $\infty$ twice, because through $x \mapsto$ $\bar{x}=\frac{1}{x}$, it transfers to $-\bar{x}^{2} \bar{p}$. Then the right diagram shows the fixed points of the three groups of the theorem.

It goes without saying that the preceding developments, as far as they only depend upon the composition of the group $p, x p, x^{2} p$, find application to every three-term group $X_{1} f, X_{2} f, X_{3} f$ of the composition:
(1) $\left[X_{1}, X_{2}\right]=X_{1} f, \quad\left[X_{1}, X_{3}\right]=2 X_{2} f, \quad\left[X_{2}, X_{3}\right]=X_{3} f$.

Especially, it comes out immediately that every subgroup of such a group is conjugate, inside the group, to one of the three subgroups:

$$
\begin{array}{|l|}
\hline X_{1} f, X_{2} f \\
X_{2} f \\
X_{1} f . \\
\hline
\end{array}
$$

We want now yet derive one noteworthy proposition that will be useful later on.

Let:

$$
Y_{i} f=\alpha_{i 1} X_{1} f+\alpha_{i 2} X_{2} f+\alpha_{i 3} X_{3} f \quad(i=1,2,3)
$$

be any three infinitesimal transformations of a three-term group $X_{1} f, X_{2} f, X_{3} f$, of which we suppose that it has the composition (1). If we now interpret the above $\infty^{2}$ infinitesimal transformations $e_{1} X_{1} f+$ $e_{2} X_{2} f+e_{3} X_{3} f$ as points of a plane, by understanding $e_{1}, e_{2}, e_{3}$ as homogeneous point-coordinates, then $Y_{1} f, Y_{2} f, Y_{3} f$ are represented by three points and moreover, $\left[Y_{1}, Y_{2}\right]$ is the polar of the straight line between the points $Y_{1} f$ and $Y_{2} f$ with respect to the conic section: $e_{2}^{2}-4 e_{1} e_{3}=0$ and $\left[Y_{1}, Y_{3}\right]$ is the polar of the straight line between $Y_{1} f$ and $Y_{3} f$. From this, it follows that $Y_{1} f$ is the polar of the straight line which connects with each other the two points $\left[Y_{1}, Y_{2}\right]$ and $\left[Y_{1}, Y_{3}\right]$, whence there must exist a relation of the form:

$$
\left[\left[Y_{1}, Y_{2}\right],\left[Y_{1}, Y_{3}\right]\right]=\rho \cdot Y_{1} f
$$

where $\rho$ denotes a constant. By calculation, one finds $\rho$ very easily and as a result one finds the

Proposition 1. If $X_{1} f, X_{2} f, X_{3} f$ is a three-term group of the composition:
(1) $\left[X_{1}, X_{2}\right]=X_{1} f, \quad\left[X_{1}, X_{3}\right]=2 X_{2} f, \quad\left[X_{2}, X_{3}\right]=X_{3} f$, and if:

$$
Y_{i} f=\alpha_{i 1} X_{1} f+\alpha_{i 2} X_{2} f+\alpha_{i 3} X_{3} f \quad(i=1,2,3)
$$

are any three infinitesimal transformations of this group, then there exists a relation of the form:

$$
\left[\left[Y_{1}, Y_{2}\right],\left[Y_{1}, Y_{3}\right]\right]=2\left|\begin{array}{ccc}
\alpha_{11} & \alpha_{12} & \alpha_{13}  \tag{6}\\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{array}\right| \cdot Y_{1} f
$$

§ 5.
We now turn to the general linear homogeneous group:

$$
\left\{\begin{array}{l}
x^{\prime}=a_{1} x+a_{2} y  \tag{7}\\
y^{\prime}=a_{3} x+a_{4} y
\end{array}\right.
$$

of the twice-extended [ZWEIFACH AUSGEDEHNTEN] manifold $x, y$.

This group is four-term, so it shall be shortly named as "the $G_{4}$ " in the present paragraph; it is engendered by the four independent infinitesimal transformations:
(8)

$$
x p, \quad y p, \quad x q, \quad y q .
$$

The theorem cited right below.
It results from Vol. I, Theorem 98, p. 561 that our $G_{4}$ only contains two invariant subgroups, namely the special linear homogeneous:

$$
x q, \quad x p-y q, \quad y p
$$

which is three-term, and in addition also one which is one-term, engendered by the excellent infinitesimal transformation:

$$
x p+y q .
$$

The three-term group (9) is particularly important, because it is (Vol. I, Theorem 96, p. 558) holoedrically isomorphic [HOLOEDRISCH ISOMORPH] to the general projective group of the once-extended manifold; if one chooses as point-coordinates of the once-extended manifold the variable $\mathfrak{x}$ and associates the infinitesimal transformations (9), the one after the other, to the transformations:
(10) $\quad+\mathfrak{p}, \quad-2 \mathfrak{x p}, \quad-\mathfrak{x}^{2} \mathfrak{p}$,
then the holoedric isomorphism between the two groups (9) and (10) immediately comes to light.
$\triangleright \boldsymbol{A}$ check. Consider $[x: y]$ as homogeneous coordinates on $\mathbb{K} \mathbb{P}^{1}$. On the chart $\{x \neq 0\}$, set $\mathfrak{x}:=\frac{y}{x}$. Differentiating the typical functional identity $f(x, y)=\mathfrak{f}(\mathfrak{x})=\mathfrak{f}\left(\frac{y}{x}\right)$ with respect to $x$ and to $y$, one gets as usual the transformation rules for coordinates vector fields:

$$
\partial_{x}=-\frac{y}{x^{2}} \partial_{\mathfrak{x}} \quad \text { and } \quad \partial_{y}=\frac{1}{x} \partial_{\mathfrak{x}}
$$

so that, setting $\mathfrak{p}:=\partial_{\mathfrak{x}}$, we obtain, one after the other, the desired projectivizations:

$$
x q=\mathfrak{p}, \quad x p-y q=-2 \mathfrak{x p}, \quad y p=-\mathfrak{x}^{2} \mathfrak{p}, \quad x p+y q=0,
$$

having indeed the expressions claimed in the text.
Since it is advisable to make visible the existence of the two invariant subgroups of our $G_{4}$, we substitute from here on the four independent infinitesimal transformations (8) of our $G_{4}$ for the following four:

$$
\begin{aligned}
X_{1} f=x q, \quad & X_{2} f=x p-y q, \quad X_{3} f=y p \\
& X_{4} f=x p+y q .
\end{aligned}
$$

The composition of our $G_{4}$ is now represented by the equations:

$$
\begin{gathered}
{\left[X_{1}, X_{2}\right]=-2 X_{1} f, \quad\left[X_{1}, X_{3}\right]=X_{2} f, \quad\left[X_{2}, X_{3}\right]=-2 X_{3} f} \\
{\left[X_{1}, X_{4}\right]=\left[X_{2}, X_{4}\right]=\left[X_{3}, X_{4}\right]=0}
\end{gathered}
$$

To begin with, we again determine all types of one-term subgroups existing in the $G_{4}$. To this purpose, in the general infinitesimal transformation:

$$
e_{1} x q+e_{2}(x p-y q)+e_{3} y p+e_{4}(x p+y q)
$$

of our group, we interpret the quantities $e_{1}, \ldots, e_{4}$ as homogeneous pointcoordinates of a thrice-extended [DREIFACH AUSGEDEHNTEN] space. Then we imagine in our mind that the points of this space are transformed by the adjoint group:

$$
\begin{aligned}
E_{1} f & =2 e_{2} \frac{\partial f}{\partial e_{1}}-e_{3} \frac{\partial f}{\partial e_{2}} \\
E_{2} f & =-2 e_{1} \frac{\partial f}{\partial e_{1}} \\
E_{3} f & = \\
E_{4} f & =0
\end{aligned}
$$

of our $G_{3}$, and we seek all smallest manifolds invariant by the adjoint group - in other words: we look for all systems of equations in the variables $e_{1}, \ldots, e_{4}$ which admit the infinitesimal transformations $E_{1} f, \ldots, E_{4} f$ and in addition yet the transformation:

$$
E f=e_{1} \frac{\partial f}{\partial e_{1}}+e_{2} \frac{\partial f}{\partial e_{2}}+e_{3} \frac{\partial f}{\partial e_{3}}+e_{4} \frac{\partial f}{\partial e_{4}} .
$$

From the beginning, we can for all that leave out the identically vanishing transformation $E_{4} f$.

The transformations $E_{1} f, E_{2} f, E_{3} f, E f$ engender a four-term group whose determinant:

$$
\left|\begin{array}{cccc}
2 e_{2} & -e_{3} & 0 & 0  \tag{11}\\
-2 e_{1} & 0 & 2 e_{3} & 0 \\
0 & e_{1} & -2 e_{2} & 0 \\
e_{1} & e_{2} & e_{3} & e_{4}
\end{array}\right|
$$

vanishes identically, whereas its three-by-three subdeterminants are not all identically null. The four equations:

$$
E_{1} f=0, \quad E_{2} f=0, \quad E_{3} f=0, \quad E f=0
$$

therefore have one solution in common, which, set equal to an arbitrary constant, provides a family of $\infty^{1}$ invariant surfaces, namely the family of the surfaces of degree two:

$$
\begin{equation*}
\frac{e_{2}^{2}+e_{1} e_{3}}{e_{4}^{2}}=\text { const. } \tag{12}
\end{equation*}
$$

among which, as limiting cases, are comprised a cone: $e_{2}^{2}+e_{1} e_{3}=0$ and a doubly counting plane: $e_{4}^{2}=0$.

In order to find the remaining invariant manifolds, we must set equal to zero the three-by-three and the two-by-two subdeterminants of (11). By setting to zero [NULLSETZEN] the three-by-three subdeterminants, we get firstly the system of equations:

$$
\begin{equation*}
e_{2}^{2}+e_{1} e_{3}=0, \quad e_{4}=0, \tag{13}
\end{equation*}
$$

hence an invariant conic section: the cutting-curve [SCHNITTCURVE] of the invariant cone: $e_{2}^{2}+e_{1} e_{3}=0$ with the invariant plane: $e_{4}=0$. Secondly, we obtain:

$$
\begin{equation*}
e_{1}=e_{2}=e_{3}=0 \tag{14}
\end{equation*}
$$

hence an invariant point: the peak point [SPITZE] of the invariant cone: $e_{2}^{2}+e_{1} e_{3}=0$; this is the image-point of the excellent infinitesimal transformation of our $G_{4}$, the transformation $x p+y q$. By setting equal to zero all two-by-two subdeterminants, we obtain only the system of equations (14), hence nothing new.

With this, all manifolds of the space $e_{1}, \ldots, e_{4}$ invariant by the group $E_{1} f, \ldots, E_{4} f$ are found ${ }^{1}$, because the conic section (13) visibly contains no smaller invariant manifold. Besides, we could have predicted the occurence of the plane $e_{4}=0$ and of the conic section (13) lying on it, since the three-term invariant subgroup (9) of our $G_{4}$ is precisely represented by the plane $e_{4}=0$ in the space $e_{1}, \ldots, e_{4}$; on the other hand, the points of the plane $e_{4}=0$ are obviously transformed by the adjoint group $E_{1} f, \ldots, E_{4} f$ exactly as they are by the adjoint group $E_{1} f, E_{2} f, E_{3} f$ of the three-term group (9), since $E_{4} f$ leaves untouched all points of the space and hence also all points of the plane $e_{4}=0$. From the developments of the previous paragraph we now obtain immediately that in the plane $e_{4}=0$, there is no other invariant manifold as a certain conic section.
$\triangleright$ Equivalent reformulation.

[^5]One can still mention that the group $E_{1} f, E_{2} f, E_{3} f$ is only another form of a very well known group. Indeed, in the space $e_{1}, \ldots, e_{4}$, if one imagines the homogeneous system of coordinates chosen in such a way that the plane $e_{4}=0$ is transferred to the plane at infinity, and that the conic section (13) is transferred to the imaginary circle, then our group is nothing else than the group of all rotations around the point $e_{1}=e_{2}=e_{3}=0$; the $\infty^{1}$ second degree surfaces (12) simply are the $\infty^{1}$ spheres with the center $e_{1}=e_{2}=e_{3}=0$.
$\triangleright$ Imaginary circle. $e_{1}^{2}+e_{2}^{2}+e_{3}^{2}=0$ in $\mathbb{C}^{3}$.

After we have found all manifolds invariant by the group $E_{1} f, \ldots, E_{4} f$, we can immediately indicate all types of infinitesimal transformations, or, what amounts to the same, all types of one-term subgroups, of our $G_{4}$. Every such type is indeed represented in the space $e_{1}, \ldots, e_{4}$ by a manifold invariant by the adjoint group $E_{1} f, \ldots, E_{4} f$ and in fact, by a so-called smallest invariant manifold (see p. 67). In this way, one obtains the following:

1) Every nondegenerate surface of second degree among the $\infty^{1}$ ones (12) represents a type, but only at each time, all points of the conic section (13) must be excluded. So these are $\infty^{1}$ different types.
2) The conic $e_{2}^{2}+e_{1} e_{3}=0$ represents one type, when one leaves out the peak point and the conic section (13).

The remaining types are:
3) The plane $e_{4}=0$, to the exclusion of the conic section (13).
4) The conic section (13).
5) The peak point $e_{1}=e_{2}=e_{3}=0$ of the conic $e_{2}^{2}+e_{1} e_{3}=0$.

If we want to have one representative for each of the found types, we must select each time one point on the concerned smallest invariant manifold. In the first case for instance the invariant manifold is represented by an equation of the form:

$$
e_{4}^{2}=c^{2}\left(e_{2}^{2}+e_{1} e_{3}\right),
$$

where $c$ means a finite constant distinct from zero, and where $e_{1}, \ldots, e_{4}$ can take all values which do not satisfy the equation (13). So we can choose:

$$
e_{1}=e_{3}=0, \quad e_{2}=1
$$

and say: $e_{4}=c$, so that for the $\infty^{1}$ types of the first species, we obtain the $\infty^{1}$ representatives:

$$
\text { 1) } \quad x p-y q+c(x p+y q) \quad(c \neq 0) \text {. }
$$

Here, one has to become aware of the fact that two equally opposed values of $c$ [i.e. the two values $+c$ and $-c$ ] always produce two infinitesimal transformations which lie on the same second degree surface, hence which are conjugate to each other inside the $G_{4}$. It's because the equation $e_{4}^{2}=c^{2}$ is satisfied both by $e_{4}=c$ and by $e_{4}=-c$ as well.

In a similar way, we can choose as a representative of the remaining types the following one-term groups:
2) $x q+x p+y q$,
3) $x p-y q$,
4) $x q$,
5) $x p+y q$.

Now it yet remains to determine all two-term and all three-term subgroups of our $G_{4}$.

Let:

$$
Y_{i} f=\alpha_{i} x q+\beta_{i}(x p-y q)+\gamma_{i} y p+\delta_{i}(x p+y q)
$$

be two independent infinitesimal transformations of our $G_{4}$. If we leave out from them the term with $x p+y q$, we then obtain the two reduced [VERKÜRZTEN] infinitesimal transformations:

$$
\begin{gathered}
\bar{Y}_{i} f=\alpha_{i} x q+\beta_{i}(x p-y q)+\gamma_{i} y p \\
(i=1,2) .
\end{gathered}
$$

However, these reduced transformations need not anymore be independent from each other, but one has in any case:

$$
\begin{equation*}
\left[Y_{1}, Y_{2}\right]=\left[\bar{Y}_{1}, \bar{Y}_{2}\right] \tag{15}
\end{equation*}
$$

since indeed $x p+y q$ is exchangeable with all infinitesimal transformations of the $G_{4}$.

The equation (15) leads us to a very simple construction of the imagepoint of $\left[Y_{1}, Y_{2}\right]$ from the image-points of $Y_{1} f$ and of $Y_{2} f$.

Indeed, the reduced infinitesimal transformations $\bar{Y}_{1} f$ and $\bar{Y}_{2} f$ obviously belong to the special linear homogeneous group: $x q, x p-y q$, $y p$, so they are represented by points of the plane $e_{4}=0$. One obtains the image-points of $\bar{Y}_{1} f$ and of $\bar{Y}_{2} f$ when one connects, by means of a straight line, the image-points of $Y_{1} f$ and of $Y_{2} f$ with the point $x p+y q$, i.e. with the peak point of the cone $e_{2}^{2}+e_{1} e_{3}=0$, and when one looks at the intersection point [SCHNITTPUNKTE] of these straight lines with the plane $e_{4}=0$, or more briefly: when one projects the points $Y_{1} f$ and $Y_{2} f \mid$
from the point $x p+y q$ onto the plane $e_{4}=0$. Thus one finds the point $\left[Y_{1}, Y_{2}\right]=\left[\bar{Y}_{1}, \bar{Y}_{2}\right]$ by looking for the polar, with respect to the conic section (13), of the connection line between the two points $\bar{Y}_{1} f, \bar{Y}_{2} f$ in the plane $e_{4}=0$.

In general, the indicated construction for the point $\left[Y_{1}, Y_{2}\right]$ always produces a completely determined point of the plane: $e_{4}=0$, but it does not produce any determined point, only when the connection line of the two points $Y_{1} f$ and $Y_{2} f$ passes through the point $x p+y q$, namely either when the two points $\bar{Y}_{1} f$ and $\bar{Y}_{2} f$ coincide and one has $\left[Y_{1}, Y_{2}\right]=$ $\left[\bar{Y}_{1}, \bar{Y}_{2}\right]=0$, or when one of the points $Y_{1} f$ and $Y_{2} f$ coincides with the point $x p+y q$ and one has $\left[Y_{1}, Y_{2}\right]=0$. It follows from this that two independent infinitesimal transformations of our $G_{4}$ are exchangeable if and only if the connection line of their image-points passes through the point $x p+y q$.

As a result, all two-term subgroups of our $G_{4}$ whose infinitesimal transformations are exchangeable are found; every such subgroup is represented by a straight line which contains the point $x p+y q$. But now, each straight line through the point $x p+y q$ which is not a generator of the conic $e_{2}^{2}+e_{1} e_{3}=0$ can visibly be transferred to every other straight line of the same nature by means of the adjoint group of our $G_{4}$, and in the same way, every generator of this conic can be transferred to any other. In other words:

In our $G_{4}$, there are two types of two-term subgroups with exchangeable infinitesimal transformations. The subgroups of the first type are represented by the straight lines of the space $e_{1}, \ldots, e_{4}$ which contain the point $x p+y q$, but are not generators of the conic $e_{2}^{2}+e_{1} e_{3}=0$ and those of the second type are the generators of this conic. As a representative of the two types we can choose the two-term subgroups:

$$
x p-y q, \quad x p+y q \quad \text { and } \quad x q, x p+y q .
$$

Every still remaining two-term subgroup of the $G_{4}$ is represented by a straight line which does not touch the point $x p+y q$. This straight line lies either completely in the plane $e_{4}=0$ or it has just one point in common with that plane. In the first case, we obviously have to deal with a subgroup of the special linear homogeneous group: $x q, x p-y q, y p$, so the subgroup in question is necessarily represented by a tangent to the conic section (13). In the second case, if $Y_{1} f$ and $Y_{2} f$ are two independent infinitesimal transformations of the subgroup, then the point $\left[Y_{1}, Y_{2}\right]$, which is a fully determined point of the plane $e_{4}=0$, must lie on the straight line
between $Y_{1} f$ and $Y_{2} f$, and consequently $\left[Y_{1}, Y_{2}\right]$ must just be the point that the connection line between $Y_{1} f$ and $Y_{2} f$ has in common with the plane $e_{4}=0$. One then easily recognizes that the point $\left[Y_{1}, Y_{2}\right]$ lies on the conic section (13) and that the straight line between $Y_{1} f$ and $Y_{2} f$ must be contained in a tangential plane to the conic $e_{2}^{2}+e_{1} e_{3}=0$. Conversely, every straight line which hits the conic section (13) and which lies in a tangential plane to the conic $e_{2}^{2}+e_{1} e_{3}=0$ really represents a two-term subgroup.
$\triangleright$ Explanation.
As a result, all two-term subgroups of the $G_{4}$ whose infinitesimal transformations are not exchangeable are found. They are firstly all the tangents to the conic section (13) in the plane $e_{4}=0$, and secondly all tangents to the conic $e_{2}^{2}+e_{1} e_{3}=0$ which hit the conic section (13), but which neither pass through the point $x p+y q$, nor lie in the plane $e_{4}=0$.

The subgroups of the first category are all conjugate with each other inside the $G_{4}$ and they form a type for itself, and as a representative of it, we can choose the group:

$$
x q, \quad x p-y q .
$$

As far as the subgroups of the second category are concerned, one thinks over that the nondegenerate amongst the surfaces (12) of second degree all come into contact with the conic $e_{2}^{2}+e_{1} e_{3}=0$ alongside the conic section (13), hence that the generators of these surfaces of second degree all lie on the tangential planes to the conic $e_{2}^{2}+e_{1} e_{3}=0$. The subgroups of the second category are therefore represented in the space $e_{1}, \ldots, e_{4}$ by the generators of the surfaces of second degree (12) that are nondegenerate. Now, every point of such a surface of second degree which does not lie on the conic (13) is transferred by the adjoint group to every other point, whence each generator of the surface can also be transferred to every other generator of the same family on the surface, whereas such a generator can never be transferred by the adjoint group to a generator of another surface of second degree, and also never to the other family on the same surface; the latter follows from the fact that the adjoint group is continuous. According to that, the two-term subgroups of the second category decompose in infinitely many types. Each nondegenerate one amongst the surfaces (12) of second degree produces two such types, of which one is represented by the first family of generators and the other by the second family. In order to have a representative for each such type, we only need to indicate in an arbitrary tangential plane of the conic $e_{2}^{2}+$
$e_{1} e_{3}=0$ all the tangents of this conic which come into contact with the conic section (13), but which neither lie in the plane $e_{4}=0$, nor pass through the point $x p+y q$, since each tangential plane of the conic is indeed transferred to any other one by the the adjoint group. If for example we choose the tangential plane to the conic which contains the point $x q$, then for our infinitely many types, we receive the representatives:

$$
x q, \quad x p-y q+c(x p+y q),
$$

where $c$ means a finite constant disctinct from 0 .
Here, two equally opposed values of $c$ always furnish subgroups which are represented by two generators of the same surface of second degree, but these generators belong to different families, and consequently, the two subgroups in question are not conjugate to each other inside the $G_{4}$.

Finally, the three-term subgroups of our $G_{4}$ are still to be determined.
A three-term subgroup $g_{3}$ of the $G_{4}$ is represented by a plane in the space $e_{1}, \ldots, e_{4}$. If this plane coincides with the plane $e_{4}=0$, then the $g_{3}$ is nothing but the three-term invariant subgroup:

$$
\begin{equation*}
x q, \quad x p-y q, \quad y p \tag{9}
\end{equation*}
$$

of the $G_{4}$. In every other case, the image-plane [BiLDEBENE] of $g_{3}$ cuts the plane $e_{4}=0$ in a straight line which necessarily represents a two-term subgroup of the $G_{4}$ and at the same time, a subgroup of the group (9). From this, it follows that the straight line in question is a tangent to the conic section (13) and that the image-plane of the subgroup $g_{3}$ comes into contact with this conic section. Next, if $Y_{1} f$ and $Y_{2} f$ are any two independent infinitesimal transformations of $g_{3}$ and if $Y_{1} f$ has its image-point on the mentioned tangent to the conic section, while the image-point of $Y_{2} f$ does not lie on this tangent, then the infinitesimal transformation $\left[Y_{1}, Y_{2}\right]$ must either vanish identically, or have its image-point on this tangent; the first case occurs only when the three points $Y_{1} f, Y_{2} f$ and $x p+y q$ lie in a straight line, the second only when the plane, which is determined by $Y_{2} f$ and by the tangent, also contains the point $x p+y q$. Consequently, the image-plane of $g_{3}$ must pass through the point $x p+y q$, when it does not coincide with the plane $e_{4}=0$, and it must be a tangent plane to the conic $e_{2}^{2}+e_{1} e_{3}=0$. Also, as one easily convinces oneself, every tangential plane of this conic really represents a three-term subgroup of the $G_{4}$. All these subgroups are conjugate to each other inside the $G_{4}$, because every tangential plane to the conic is transferred, by the adjoint group, to every other.

Thus, there are two types of three-term subgroups of our $G_{4}$. The first type is made up of the invariant subgroup:

$$
x q, \quad x p-y q, \quad y p,
$$

alone; the groups of the second type are represented in the space $e_{1}, \ldots, e_{4}$ by the tangential planes to the conic: $e_{2}^{2}+e_{1} e_{3}=0$; a representative of this type is the group:

$$
x q, \quad x p-y q, \quad x p+y q
$$

We now sum up the gained results:
Theorem 3. If a subgroup of the general linear homogeneous group $G_{4}$ :

$$
x q, \quad x p-y q, \quad y p, \quad x p+y q
$$

of the plane $x, y$ is three-term, then it either is the invariant subgroup:

$$
x q, \quad x p-y q, \quad y p,
$$

or it is conjugate, inside the general linear homogeneous group, to the subgroup:
2

$$
x q, \quad x p-y q, \quad x p+y q ;
$$

every two-term subgroup of the $G_{4}$ is, inside the $G_{4}$, conjugate to one of the subgroups:

$$
x q, \quad x p-y q+c(x p+y q) \quad c \neq 0
$$

$x p-y q, \quad x p+y q$
Finally, every one-term subgroup of the $G_{4}$ is conjugate either to one of the subgroups:
7

$$
x p-y q+c(x p+y q) \quad c \neq 0
$$

$8 \quad x p-y q$
10

9

$$
x q+x p+y q
$$

$x q+x p+y q$
or it is engendered by the excellent infinitesimal transformation:
11
$x p+y q$
of the $G_{4}$. The arbitrary constant c appearing in the two cases is an essenial parameter, that is to say, to different values of c correspond subgroups that are not conjugate inside the $G_{4}$.

It goes without saying on the basis of the preceding developments, that for every four-term group which has the same composition as the general linear homogeneous group of the plane $x, y$, we can immediately indicate in general all types of subgroups, and also all subgroups.

While, up to now, we only occupied ourselves with the composition of the $G_{4}$ :

$$
x q, \quad x p-y q, \quad y p, \quad x p+y q,
$$

we yet want, as a conclusion, to make a few observations about the $G_{4}$ in its quality of group of the plane $x, y$.

The $G_{4}$ leaves the point $x=y=0$ invariant and substitutes with each other the $\infty^{1}$ straight lines passing through this point. But also, amongst the infinitesimal transformations of the $G_{4}$, there is only one which leaves untouched every straight line through the point $x=y=0$, it is the excellent infinitesimal transformation $x p+y q$ of the group. Consequently, the $\infty^{1}$ straight lines through the point $x=y=0$ are transformed by our $G_{4}$ by means of a three-term group, which is projective and is meroedrically isomorph [MEROEDRISCH ISOMORPH] to the $G_{4}$. Obviously, this three-term group is nothing but the general projective group of the onceextended manifold; one can easily convince oneself directly of that. Indeed, the variables $x, y$ can be interpreted as homogeneous coordinates of the $\infty^{1}$ straight lines through the point $x=y=0$; if one now replaces these two homogeneous coordinates by the non-homogeneous:

$$
x_{1}=\frac{x}{y},
$$

and if one determines how $x_{1}$ is transformed by the infinitesimal transformations of the $G_{4}$, then one finds that it will be transformed precisely by means of the general projective group $p_{1}, x_{1} p_{1}, x_{1}^{2} p_{1}$ of the once-extended manifold. For the execution of this computation, see Vol. I, p. 579, cf. also ibidem, p. 558, Theorem 96.

## $\triangleright$ The computation.

## $\triangleright$ The cited theorem.

Theorem I.96. The special linear homogeneous group:

$$
x_{i} p_{k}, \quad x_{i} p_{i}-x_{k} p_{k} \quad(i \gtrless k=1 \cdots n)
$$

in the variables $x_{1}, \ldots, x_{n}$ is imprimitive and holoedrically isomorphic to the general projective group of an $(n-1)$-fold extended manifold.

> It results from what has been said that every $r$-term subgroup of our $G_{4}$ transforms the $\infty^{1}$ straight lines through the point $x=y=0$ either as an $r$-term group or as an $(r-1)$-term group and in fact, the first case happens when the infinitesimal transformation $x p+y q$ is lacking in the subgroup, the second one when $x p+y q$ is comprised in the subgroup. The only subgroup of the $G_{4}$ which transforms those straight line in the same
way as the $G_{4}$ is the invariant subgroup $x q, x p-y q, y p$. If we combine with this the results of the preceding paragraph, according to which every subgroup of the general projective group of the once-extended manifold leaves invariant either one point, or two separate points, or two collapsing points, we obtain the

Proposition 2. The general linear homogeneous group $G_{4}$ :

$$
x q, \quad x p-y q, \quad y p, \quad x p+y q
$$

of the plane $x, y$ comprises only one subgroup, which, just as $G_{4}$, leaves untouched no straight line through the invariant point $x=y=0$, it is the three-term invariant subgroup:

$$
x q, \quad x p-y q, \quad y p
$$

of the $G_{4}$. Every other subgroup of the $G_{4}$ leaves at rest at least one straight line through the point $x=y=0$.

If one subgroup of the $G_{4}$ leaves invariant only one straight line through the point $x=y=0$, then it is either three-term and it belongs to the type 2 of the theorem, or it is two-term and it belongs to one of the types 3 and 4. If it leaves untouched two separate straight lines through the point, then it is either two-term and is of type 5 , or it is one-term and is of one of the types 7 and 8 ; if it leaves untouched two collapsing straight lines, then it is two-term of type 6 , or it is one-term of one of the types 9 and 10. It only remains the one-term subgroup $x p+y q$, by which every straight line through the point $x=y=0$ keeps its position.

## Chapter 3.

## Determination of all Finite Continuous Point Transformation Groups of the Plane.

When it is said in the title that all*) finite continuous groups of point transformations of the plane shall be determined, this is not to be understood as actually writing down all these groups. It is not at all our intention to do this; rather, we shall proceed as in Vol. I, Chap. 22, p. 434 sq. for the determination of all $r$-term transitive groups of a given composition. We distribute the finite continuous groups of point transformations of the plane into types, by each time recognizing two of these groups to be of the same type if and only if one is equivalent to the other through a point transformation of the plane. In this way, each of the sought groups belongs to one and only one type; conversely, all groups belonging to one determinate type can be identified without difficulty as soon as one knows one amongst them, and this single group can be regarded as a representative of the entire type. As a result, we can replace the problem referred to in the title by the following:
*) As far back as 1874, Lie has sketched the determination of all groups of the plane in the Göttinger Nachrichten. He gave a justification in great detail in 1878 in the Norwegischer Archiv and later in the Math. Ann., vol 16. Lie has indicated the simple method used in the text for the determination of all imprimitive groups of the plane, firstly in 1884 in his Archiv, and since 1886 in his lectures at the university of Leipzig.

To exhibit, for each type of finite continuous group of point transformations of the plane one, but also only one, representative.

If this problem is solved, then we basically know all finite continuous groups of point transformations of the plane, since each one of these groups is equivalent, through a point transformation of the plane, to a single of the found representatives.

According to Vol. I, p. 220 sq., the groups of the plane are divided in two separate categories, of which the first embraces all primitive groups, and the second all imprimitive groups; besides, it is clear that two groups of the plane belonging to the same type are always either both primitive, or both imprimitive. As a consequence, we can solve the problem, to which treatment we have reduced the problem stated in the title of the chapter,
firstly for the primitive groups, and afterwards for the imprimitive groups, one case after the other. However, before we pass to that, we must yet make clear whether it can be recognized that a given finite continuous group of point transformations of the plane is primitive, or imprimitive. That is why we place in the beginning a paragraph in which we convey the general developments of the Vol. I about primitivity and imprimitivity to the groups of the plane, and in which, as far as it is necessary for this particular case, we complete these developments.

$$
\text { § } 6 .
$$

According to Vol I, p. 220 sq., an $r$-term group:

$$
X_{k} f=\xi_{k}(x, y) p+\eta_{k}(x, y) q, \quad(k=1 \ldots r)
$$

of the plane $x, y$ is imprimitive if and only if it leaves invariant a family of $\infty^{1}$ curves $\varphi(x, y)=$ const. Hence, if the group $X_{1} f, \ldots, X_{r} f$ is intransitive, it is at the same time imprimitive, because an intransitive group of the plane divides the plane in $\infty^{1}$ curves: $\psi(x, y)=$ const. that all remain invariant, whence the group leaves invariant at the same time the totality of the family of curves: $\psi(x, y)=$ const.

The indicated necessary and sufficient condition for the imprimitivity of the group $X_{1} f, \ldots, X_{r} f$ can now be given a different form. At first, it amounts to the fact that by the concerned group, a certain linear partial differential equation:

$$
A f=\alpha(x, y) \frac{\partial f}{\partial x}+\beta(x, y) \frac{\partial f}{\partial y}
$$

remains invariant (loc. cit., p. 221). But if we bear in mind that with the linear partial differential equation $A f=0$ is associated the invariant ordinary differential equation:

$$
\begin{equation*}
\alpha(x, y) \cdot d x-\beta(x, y) \cdot d y=0 \tag{1}
\end{equation*}
$$

then we recognize immediately that we may also say:
The r-term group $X_{1} f, \ldots, X_{r} f$ of the plane $x, y$ is imprimitive if and only if it leaves invariant a first order ordinary differential equation of the form (1).

Next, we remember that by $d x: d y$, a certain direction of progress [FORTSCHREITUNGSRICHTUNG] is determined at every point $x, y$ of the plane, and that consequently, $x, y, d x: d y$ can be interpreted as the $\infty^{3}$ line-elements of the plane. Thus, if we want to know whether our group $X_{1} f, \ldots, X_{r} f$ leaves invariant an ordinary differential equation of the
form (1), we must at first examine in which way it transforms the lineelements of the plane $x, y$, and especially find out whether it leaves invariant a family of $\infty^{2}$ line-elements which is represented by an equation of the special form (1).

To this end, as in Vol.I, p. 524 sq., we consider the variables $x, y$ as functions of an auxiliary variable [HÜLFSVERÄNDERLICHEN] $t$ and we prolong [ERWEITERN] the infinitesimal transformations $X_{k} f$ using the notation of the differential quotients:

$$
\frac{d x}{d t}=x^{\prime}, \quad \frac{d y}{d t}=y^{\prime} .
$$

We therefore obtain the prolonged group:

$$
X_{k}^{\prime} f=\xi_{k} p+\eta_{k} q+\xi_{k}^{\prime} p^{\prime}+\eta_{k}^{\prime} q^{\prime} \quad(k=1 \cdots r)
$$

in the variables $x, y, x^{\prime}, y^{\prime}$, where the abbreviations:

$$
\begin{array}{rlrl}
x^{\prime} \frac{\partial \xi_{k}}{\partial x}+y^{\prime} \frac{\partial \xi_{k}}{\partial y} & =\xi_{k}^{\prime}, & \frac{\partial f}{\partial x^{\prime}}=p^{\prime} \\
x^{\prime} \frac{\partial \eta_{k}}{\partial x}+y^{\prime} \frac{\partial \eta_{k}}{\partial y}=\eta_{k}^{\prime}, & \frac{\partial f}{\partial y^{\prime}}=q^{\prime}
\end{array}
$$

have been employed. Now, since $x, y, x^{\prime}: y^{\prime}$ can obviously be used as coordinates for the $\infty^{3}$ line-elements of the plane $x, y$ as well as $x, y, d x$ : $d y$, the prolonged group $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$ indicates how the line-elements are transformed by the group $X_{1} f, \ldots, X_{r} f$; so now the question is yet to decide whether or not the group $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$ leaves invariant an equation of the form:

$$
\begin{equation*}
\alpha(x, y) \cdot y^{\prime}-\beta(x, y) \cdot x^{\prime}=0 . \tag{1’}
\end{equation*}
$$

In the group $X_{1} f, \ldots, X_{r} f$, there is a certain number, say precisely $r-m$, of indepedent infinitesimal transformations which leave untouched an arbitrarily chosen point $x_{0}, y_{0}$ in general position; here, the number $m$ has the value 2 or the value 1 , according to the group $X_{1} f, \ldots, X_{r} f$ being transitive or intransitive. Naturally, these $r-m$ infinitesimal transformations engender an $(r-m)$-term subgroup of the group $X_{1} f, \ldots, X_{r} f$ (see Vol. I, p. 205, Prop. 2), and their power series developments with respect to powers of $x-x_{0}, y-y_{0}$ are free of terms of zeroth order; hence they have the form:

$$
\begin{aligned}
Y_{k} f=\left\{\lambda_{k}\right. & \left.\left(x-x_{0}\right)+\mu_{k}\left(y-y_{0}\right)+\cdots\right\} p+ \\
& +\left\{\nu_{k}\left(x-x_{0}\right)+\rho_{k}\left(y-y_{0}\right)+\cdots\right\} q \quad(k=1 \cdots r-m)
\end{aligned}
$$

and at the same time, under the assumptions made, the infinitesimal transformation:

$$
e_{1} Y_{1} f+\cdots+e_{r-m} Y_{r-m} f
$$

is the most general transformation of the group $X_{1} f, \ldots, X_{r} f$ which contains no term of zeroth order in $x-x_{0}, y-y_{0}$. If we prolong the $Y_{k} f$ in just the same way as the $X_{k} f$, we then obtain $r-m$ independent infinitesimal transformations in the variables $x, y, x^{\prime}, y^{\prime}$ of the form:

$$
\begin{aligned}
Y_{k}^{\prime} f=Y_{k} f & +\left\{\left(\lambda_{k}+\cdots\right) x^{\prime}+\left(\mu_{k}+\cdots\right) y^{\prime}\right\} p^{\prime}+ \\
& +\left\{\left(\nu_{k}+\cdots\right) x^{\prime}+\left(\rho_{k}+\cdots\right) y^{\prime}\right\} q^{\prime} \quad(k=1 \cdots r-m)
\end{aligned}
$$

which in turn engender an $(r-m)$-term subgroup of the group $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$, namely the largest group contained in this group which leaves invariant the system of equations: $x=x_{0}, y=y_{0}$.

Now, if an equation of the form ( $1^{\prime}$ ) remains invariant under the group $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$, then obviously, the system of equations:

$$
x=x_{0}, \quad y=y_{0}, \quad \alpha(x, y) y^{\prime}-\beta(x, y) x^{\prime}=0
$$

remains invariant under the group $Y_{1}^{\prime} f, \ldots, Y_{r-m}^{\prime} f$, or what amounts to the same, the system of equations:

$$
\begin{equation*}
x=x_{0}, \quad y=y_{0}, \quad \alpha\left(x_{0}, y_{0}\right) y^{\prime}-\beta\left(x_{0}, y_{0}\right) x^{\prime}=0 \tag{2}
\end{equation*}
$$

where, in any case, the coefficients of $x^{\prime}$ and of $y^{\prime}$ do not both vanish, because $x_{0}, y_{0}$ is, indeed, a point in general position. Hence if we remember that a system of equations of the form (2) represents a line-element passing through the point $x_{0}, y_{0}$, we can therefore say: if the group $X_{1} f, \ldots, X_{r} f$ is imprimitive, then aside from the point $x_{0}, y_{0}$, the group $Y_{1}^{\prime} f, \ldots, Y_{r-m}^{\prime} f$ also leaves invariant yet a line-element passing through it.

But the converse also holds true: when the group $Y_{1}^{\prime} f, \ldots, Y_{r-m}^{\prime} f$, together with the point $x_{0}, y_{0}$ in general position, also leaves invariant at the same time a line element $x_{0}, y_{0}, \alpha_{0} y^{\prime}-\beta_{0} x^{\prime}=0$ passing through it, then the group $X_{1} f, \ldots, X_{r} f$ is imprimitive. Actually, to begin with, let the group $X_{1} f, \ldots, X_{r} f$ be transitive, so that the number $m$ has the value 2. In that case, the line-element in question through the point $x_{0}, y_{0}$ admits exactly $r-2$ independent infinitesimal transformations of the $r$ term group $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$, so it takes exactly $\infty^{2}$ different positions by this group, the totality of which remains invariant under this group (see Vol. I, p. 483, Theorem 85). Moreover, if we take into account the fact that the point $x_{0}, y_{0}$ also takes precisely $\infty^{2}$ different positions by the
group $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$, then we realize that the family of $\infty^{2}$ line-elements that are invariant by the group $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$ is represented by an equation of the form ( $1^{\prime}$ ), and hence, under the assumptions made, the group $X_{1} f, \ldots, X_{r} f$ is effectively imprimitive.

## $\triangleright$ Explanation.

There remains the case where the group $X_{1} f, \ldots, X_{r} f$ is intransitive; but in this case, its imprimitivity is sure from the beginning, and thus, the assertion stated above is completely demonstrated.

If one wants to find out whether a given r-term continuous group $X_{1} f, \ldots, X_{r} f$ of the plane $x, y$ is primitive or imprimitive, then one has to proceed as follows: one sets up the largest subgroup contained in the group $X_{1} f, \ldots, X_{r} f$ which leaves invariant an arbitrarily chosen point in general position, and one examines how this subgroup transforms the $\infty^{1}$ line-elements of the plane $x, y$ passing through the point $x_{0}, y_{0}$; if the subgroup leaves untouched one of the $\infty^{1}$ line-elements in question, then the group $X_{1} f, \ldots, X_{r} f$ is imprimitive; if it leaves untouched no line-element, then the group $X_{1} f, \ldots, X_{r} f$ is primitive.

Now it still remains for us to express the criterion found in a convenient analytic form.

Everything comes down to whether or not the group $Y_{1}^{\prime} f, \ldots, Y_{r-m}^{\prime} f$ defined on p. 87 leaves untouched a line-element through the invariant point $x_{0}, y_{0}$. In order to make this clear, we nevertheless do not at all need to determine the entire group $Y_{1}^{\prime} f, \ldots, Y_{r-m}^{\prime} f$, but we only need to determine how this group transforms the $\infty^{1}$ line-elements through the point $x_{0}, y_{0}$, and we can achieve this very easily thanks to Vol. I, pp. 232234: namely, from the $Y_{k}^{\prime} f$, we leave out the terms with $p$ and with $q$, and in the remaining terms, we set $x=x_{0}, y=y_{0}$, so that we obtain a linear homogeneous group:

$$
\begin{gathered}
\mathfrak{Y}_{k}=\left(\lambda_{k} x^{\prime}+\mu_{k} y^{\prime}\right) p^{\prime}+\left(\nu_{k} x^{\prime}+\rho_{k} y^{\prime}\right) q^{\prime} \\
(k=1 \cdots r-m)
\end{gathered}
$$

in the variables $x^{\prime}, y^{\prime}$ which is an Isomorph?? to the group $Y_{1}^{\prime} f, \ldots, Y_{r-m}^{\prime} f$ and which transforms the line-elements through the point $x_{0}, y_{0}$ exactly as this group does.

Now, it still remains to examine whether the linear homogenous group $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ leaves untouched one line-element: $\alpha_{0} y^{\prime}-\beta_{0} x^{\prime}=0$, or, what amounts to the same, whether it leaves untouched, when interpreted as a group of the plane $x^{\prime}, y^{\prime}$, a straight line through the point $x^{\prime}=y^{\prime}=0$.

But according to p. 83, we can answer this question. We therefore obtain the

Theorem 4. Whether a given r-term group of the plane $x, y$ :

$$
X_{k} f=\xi_{k}(x, y) p+\eta_{k}(x, y) q \quad(k=1 \cdots r)
$$

is primitive or not can be decided in the following way: One determines at first the number $r-m$ of the mutually independent infinitesimal transformations $e_{1} X_{1} f+\cdots+e_{r} X_{r} f$ which leave invariant an arbirarily chosen point $x_{0}, y_{0}$ in general position, then one selects amongst the infinitesimal transformations of this nature any $r-m$ independent ones, say $Y_{1} f, \ldots, Y_{r-m} f$, and one writes their power series expansions with respect to powers of $x-x_{0}, y-y_{0}$, though leaving out all terms of second or higher order. If these power series expansions are written as follows:

$$
\begin{aligned}
Y_{k} f=\{ & \left.\lambda_{k}\left(x-x_{0}\right)+\mu_{k}\left(y-y_{0}\right)+\cdots\right\} p+ \\
& +\left\{\nu_{k}\left(x-x_{0}\right)+\rho_{k}\left(y-y_{0}\right)+\cdots\right\} q \quad(k=1 \cdots r-m),
\end{aligned}
$$

then one finally forms the infinitesimal transformations in the variables $x^{\prime}, y^{\prime}$ :

$$
\begin{gathered}
\mathfrak{Y}_{k} f=\left(\lambda_{k} x^{\prime}+\mu_{k} y^{\prime}\right) p^{\prime}+\left(\nu_{k} x^{\prime}+\rho_{k} y^{\prime}\right) q^{\prime} \\
(k=1 \cdots r-m)
\end{gathered}
$$

which engender a linear homogeneous group. After that, if the group $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ has one of the two forms:

$$
\left\{\begin{array}{cl}
x^{\prime} q^{\prime}, & x^{\prime} p^{\prime}-y^{\prime} q^{\prime}, \quad y^{\prime} p^{\prime},  \tag{3}\\
x^{\prime} p^{\prime}+y^{\prime} q^{\prime} ; \\
x^{\prime} q^{\prime}, & x^{\prime} p^{\prime}-y^{\prime} q^{\prime}, \\
y^{\prime} p^{\prime},
\end{array}\right.
$$

then the given group $X_{1} f, \ldots, X_{r} f$ is primitive, while in every other case it is imprimitive.

It follow from this theorem that in order to be able to settle the primitivity or the imprimitivity of the group $X_{1} f, \ldots, X_{r} f$, one even does not at all need to know the infinitesimal transformations $X_{1} f, \ldots, X_{r} f$ themselves, but for this, the defining equations (Vol. I, Chap. 11) of the group are already sufficient. This is because if one knows these defining equations, then one can determine the terms of first order in $x-x_{0}$ and $y-y_{0}$ in the power series developments of the $Y_{k} f$ and hence one can also set up the linear homogeneous group $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$.
$\triangleright$ Explanation.

On the other hand, it follows that an $r$-term group $X_{1} f, \ldots, X_{r} f$ of the plane $x, y$ is always imprimitive when its number $r$ of terms is smaller than five. Indeed, if the group in question is transitive - we obviously need only prove our assertion for this case -, then $m=2$ and so $r-m<3$, whence the associated linear homogeneous group $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ certainly does not have any of the two forms (3).

At present, we can tackle the problem posed on p. 84. As already announced, we carry it out at first for the primitive groups, and then for the imprimitive groups.

## I. The Primitive Groups of the Plane.

## § 7.

If an $r$-term group $X_{1} f, \ldots, X_{r} f$ of the plane $x, y$, or briefly $G_{r}$, is supposed to be primitive, then above all, it must be transitive, and moreover, the linear homogeneous group $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ that we have defined in the preceding paragraph must possess one of the two forms (3). Conversely, according to the developments of the preceding paragraph, every group $G_{r}$ for which these two conditions are fulfilled, must be primitive.

If we imagine in our mind that the infinitesimal transformations of the $G_{r}$ are expanded, in the neighbourhood of a point $x_{0}, y_{0}$ in general position, with respect to powers of $x-x_{0}, y-y_{0}$, then the following comes out:

An $r$-term group $G_{r}$ of the plane $x, y$ is primitive if and only if, in the neighbourhood of a point $x_{0}, y_{0}$ in general position, it comprises the following infinitesimal transformations:

Firstly, two infinitesimal transformations of zeroth order in $x-x_{0}$, $y-y_{0}$ out of which no transformation of first or higher order can be deduced by linear combination, hence in other words, two infinitesimal transformations of the form:

$$
\begin{equation*}
p+\cdots, \quad q+\cdots, \tag{4}
\end{equation*}
$$

where the left out terms are of first or higher order in $x-x_{0}, y-y_{0}$. This demand is synonymous to the one that the $G_{r}$ should be transitive.

Secondly, either four or three infinitesimal transformations of first order, out of which no transformation of second or higher order in $x-x_{0}$,
$y-y_{0}$ can be deduced by linear combination, and actually, these infinitesimal transformations of first order must in fact possess either the form:

$$
\begin{aligned}
\left(x-x_{0}\right) q+\cdots, & \left(x-x_{0}\right) p-\left(y-y_{0}\right) q+\cdots, \quad\left(y-y_{0}\right) p+\cdots \\
& \left(x-x_{0}\right) p+\left(y-y_{0}\right) q+\cdots,
\end{aligned}
$$

or the form:

$$
\left(x-x_{0}\right) q+\cdots, \quad\left(x-x_{0}\right) p-\left(y-y_{0}\right) q+\cdots, \quad\left(y-y_{0}\right) p+\cdots,
$$

where in the two times, the left out terms must be of second or higher order in $x-x_{0}, y-y_{0}$. This form of the infinitesimal transformations of first order follows immediately from the circumstance that the linear homogeneous group $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ discussed earlier must possess one of the two forms (3).

But now, already in Volume I, namely in the $29^{\text {th }}$ Chapter, we have determined all finite continuous groups whose zeroth and first order infinitesimal transformations possess just the indicated form: there, we must only give to $n$ the value 2 . Consequently, we can say:

Theorem 5. If a finite continuous group of point transformations of the plane $x, y$ is primitive, so that it leaves invariant no family of curves: $\varphi(x, y)=$ const., then it has either five, or six, or eight parameters and correspondingly, it is equivalent either to the special linear group:

$$
p, \quad q, \quad x q, \quad x p-y q, \quad y p
$$

or to the general linear group:

$$
p, \quad q, \quad x q, \quad x p-y q, \quad y p, \quad x p+y q,
$$

or finally to the general projective group:

$$
p, \quad q, \quad x q, \quad x p-y q, \quad y p, \quad x p+y q, \quad x^{2} p+x y q, \quad x y p+y^{2} q
$$

As a result, we have found all types of primitive groups on the plane, and at the same time, we have found a representative for each one of these types. As one sees, there are only three types of primitive groups in the plane. Besides, one notices that the developments of the Chap. 29 in Vol. I, when applied to the twice-extended manifold, yet deliver all types of primitive groups of this manifold, while on the other hand, they produce all types of groups of the once-extended manifold (see Chap. 1, p. 60).
$\triangleright$ Comment. Belle remarque structurale.

## II. The Imprimitive Groups of the Plane.

## § 8.

If $X_{1} f, \ldots, X_{r} f$ is an imprimitive $r$-term group of the plane and if: $\varphi(x, y)=$ const. is any family of curves which is invariant by this group, then there exist (cf. Volume I, p. 139, Prop. 1) relations of the form:

$$
X_{k} \varphi=\omega_{k}(\varphi) \quad(k=1 \cdots r) .
$$

Then by introducing $\varphi$ as new $x$, our group receives the form:

$$
X_{k} f=\xi_{k}(x) p+\eta_{k}(x, y) q \quad(k=1 \cdots r)
$$

where we have again employed the customary letters $\xi$ and $\eta$. As one sees, the variable $x$ is transformed for itself [FÜR SICH] by the group: $X_{1} f, \ldots, X_{r} f$, and to be precise (Vol. I, p. 222), it is transformed by means of a group which is engendered by the reduced infinitesimal transformations:

$$
\bar{X}_{k} f=\xi_{k}(x) p \quad(k=1 \cdots r)
$$

This new group is Isomorph with the group: $X_{1} f, \ldots, X_{r} f$, since from the Relations:

$$
\left[X_{i}, X_{k}\right]=\sum_{s=1}^{r} c_{i k s} X_{s} f \quad(i, k=1 \cdots r)
$$

it visibly follows:

$$
\left[\bar{X}_{i}, \bar{X}_{k}\right]=\sum_{s=1}^{r} c_{i k s} \bar{X}_{s} f \quad(i, k=1 \cdots r)
$$

(cf. also Vol. I, p. 307, Prop. 4), but in general, the Isomorphism will be meroedric, because from the independence of the infinitesimal transformations: $X_{1} f, \ldots, X_{r} f$, the independence of the transformations: $\bar{X}_{1} f, \ldots, \bar{X}_{r} f$ does not at all follow.

According to Theorem 1 on p. 51, as a group of the once-extended manifold, the group: $\bar{X}_{1} f, \ldots, \bar{X}_{r} f$ cannot have more than three parameters, hence it is either three-, or two-, or one-term, or finally null-term, that is to say, it reduces to the identity transformation; this last case occurs when all the $\bar{X}_{k} f$ vanish identically. Correspondingly, the group: $X_{1} f, \ldots, X_{r} f$ transforms the curves: $x=$ const. either in three, or in two, or in one, or finally in null terms, that it to say, not at all. We therefore have to distinguish four cases, that we now want to describe one after the other, beginning with the last one.

Firstly. If the curves: $x=$ const. are transformed in null terms, then all the $\bar{X}_{k} f$ vanish. In this case, the group: $X_{1} f, \ldots, X_{r} f$ comprises $r$ independent infinitesimal transformations of the form:

$$
\Phi_{1}(x, y) q, \ldots, \Phi_{r}(x, y) q .
$$

Secondly. The curves: $x=$ const. are transformed in one term. Since the group: $\bar{X}_{1} f, \ldots, \bar{X}_{r} f$ is one-term in this case, then according to the theorem stated above, one can choose the variable $x$ in such a way that every $\bar{X}_{k} f$ receives the form: $a_{k} p$, and consequently every $X_{k} f$ receives the form: $a_{k} p+\eta_{k}(x, y) q$, where naturally, the $r$ constants $a_{1}, \ldots, a_{r}$ should not all vanish. From this, it comes out that, through the choice of $x$ in question, the group: $X_{1} f, \ldots, X_{r} f$ comprises $r$ infinitesimal transformations of the form:

$$
\Phi_{1}(x, y) q, \ldots, \Phi_{r}(x, y) q, \quad p+\eta(x, y) q .
$$

Thirdly. The curves: $x=$ const. are transformed in two terms. In this case, according to the mentioned theorem we are aware of, one can choose the variable $x$ in such a way that every $\bar{X}_{k} f$ receives the form: $\left(a_{k}+b_{k} x\right) p$, and to be precise, not all expressions: $a_{k} b_{j}-a_{j} b_{k}$ vanish here, because the group: $\bar{X}_{1} f, \ldots, \bar{X}_{k} f$ is actually two-term, and hence it must comprise two independent infinitesimal transformations. Now, one sees immediately that, through the choice of $x$ in question, the group: $X_{1} f, \ldots, X_{r} f$ comprises $r$ independent infinitesimal transformations of the form:

$$
\Phi_{1}(x, y) q, \ldots, \Phi_{r-2} q, \quad p+\eta_{0}(x, y) q, \quad x p+\eta_{1}(x, y) q .
$$

Fourthly. The curves: $x=$ const. are transformed in three terms. In this case, one can always choose the variable $x$ so that, in the group: $X_{1} f, \ldots, X_{r} f$, there are $r$ independent infinitesimal transformations of the form:

$$
\begin{gathered}
\Phi_{1}(x, y) q, \ldots, \Phi_{r-3}(x, y) q, \quad p+\eta_{0}(x, y) q, \quad x p+\eta_{1}(x, y) q, \\
x^{2} p+\eta_{2}(x, y) q .
\end{gathered}
$$

As a result, four categories of imprimitive groups of the plane are found. Through an appropriate choice of the variables $x, y$, every imprimitive group of the plane belongs to one of these categories. Thus, in order to find all imprimitive groups of the plane, we only need to determine, for every individual category amongst the four categories, all groups that are comprised in it.

The solution of the problem to which we have thus been led, will be substantially lightened if we bear in mind the following facts: firstly, that
through every transformation of the form:

$$
\begin{equation*}
x_{1}=\text { const. }, \quad y_{1}=\Omega(x, y), \tag{5}
\end{equation*}
$$

every group which belongs to one of our four categories, is transferred to a group belonging to the same category, and secondly, that every $r$-term group amongst one of the last three categories does contain an $(r-1)$ term subgroup which belongs to the preceding category. We can therefore proceed as follows:

To begin with, we determine all groups of the form:

$$
\Phi_{1}(x, y) q, \ldots, \Phi_{r}(x, y) q
$$

and we reduce them, through a transformation of the form (5), to a series of normal forms. After that, to each of the gained normal forms, we add in the most general way an infinitesimal transformation of the form: $p+\eta(x, y) q$, in order that again a group comes out; thus we find all groups of the second category and we bring them, through transformations of the form (5), to simple normal forms. To each one of these normal forms, we again add in the most general way one transformation of the form: $x p+\eta_{1}(x, y) q$ and we thefore obtain the groups of the third category; and finally, we find the groups of the fourth category by adding to the latter one transformation: $x^{2} p+\eta_{2}(x, y) q$.

We now want to realize in details the program set up here.

$$
\text { § } 9 .
$$

The curves: $x=$ const. are transformed in null terms.
The question here is to determine all $r$-term groups of the form:

$$
\begin{equation*}
X_{k} f=\Phi_{k}(x, y) q \quad(k=1 \cdots r) \tag{6}
\end{equation*}
$$

As one sees, the group: $X_{1} f, \ldots, X_{r} f$ transforms only the variable $y$, while it does not transform $x$ at all. Thus, if we confer to $x$ one arbitrary constant value $a$, we then get, in place of the $X_{k} f$, certain infinitesimal transformations:

$$
\begin{equation*}
\mathcal{X}_{k} f=\Phi_{k}(a, y) q \quad(k=1 \cdots r), \tag{6'}
\end{equation*}
$$

engendering a group in the variable $y$ alone, which is Isomorph to the group (6). Now, as long as the constant $a$ does not take any special value, it is obvious that none of the infinitesimal transformations $\mathcal{X}_{k} f$ can vanish identically, so that the group (6') does surely not reduce to the identity transformation. On the other hand, according to Theorem 1 on p. 51, this group cannot contain more than three parameters; consequently, all the
groups (6) are distributed in three classes, and to be precise, one group of the form (6) belongs to the first, to the second, or to the third class, according to whether the group ( 6 ') is one-, two-, or three-term, for a general value of $a$.

We now determine one after the other the groups of each one of these three classes.

If the group (6') is one-term, then according to Theorem 1 on p. 51, it can, through an appropriate choice of $y$, be given the form $q$; in other words, if as a new $y$, one introduces in the $\mathcal{X}_{k} f$, an appropriate function $\Omega(a, y)$ of $y$ and of the constant $a$, then all the $\mathcal{X}_{k} f$ receive the form:

$$
\mathcal{X}_{k} f=F_{k}(a) \cdot q \quad(k=1 \cdots r) .
$$

If one therefore introduces the function: $\Omega(x, y)$ as new $y$ in the group (6) - this is a transformation of the form (5) - , then this group receives the form:

$$
X_{k} f=F_{x}(x) \cdot q \quad(k=1 \cdots r) .
$$

On the other hand, it is clear that $r$ infinitesimal transformations of the form:
[1]

$$
F_{1}(x) q, \quad F_{2}(x) q, \ldots, F_{r}(x) q
$$

do always engender an $r$-term group, whichever also the $F$ can be as functions of $x$, provided only that there exists no relation:

$$
c_{1} F_{1}(x)+\cdots+c_{r} F_{r}(x)=0
$$

with constant coefficients. With this, through a transformation of the form (5), we have brought to a right normal form all groups which belong to our first class.

Secondly, let the group ( $6^{\prime}$ ) be two-term and hence in any case, $r$ is $>1$. Then according to the theorem stated several times, the group (6') can, by an appropriate choice of $y$, be given the form: $q, y q$. Translated into (6), this means: when a suitably chosen function $\Omega(x, y)$ is introduced as new $y$, the group (6) receives the form:

$$
X_{k} f=\left\{F_{k}(x)+G_{k}(x) \cdot y\right\} q \quad(k=1 \cdots r) .
$$

But now, we find by Combination:

$$
\left[X_{i}, X_{k}\right]=\left(F_{i} G_{k}-F_{k} G_{i}\right) q=\Omega_{i k}(x) \cdot q
$$

where in any case $\Omega_{i k}$ do not vanish all the time, since otherwise, $X_{i} f$ and $X_{k} f$ would be linked by a relation of the form: $\alpha_{i k}(x) X_{i} f+\beta_{i k}(x) X_{k} f=$ 0 for all values of $i$ and $k$, so the group ( $6^{\prime}$ ) would be one-term only, against
our assumption. Next, it comes for arbitrary $j$ :

$$
\left[\Omega_{i k} q, X_{j} f\right]=G_{j} \Omega_{i k} q
$$

and when we put $G_{j} \Omega_{i k} q$ in place of $\Omega_{i k} q$ into the left-hand side of this equation, it comes: $G_{j}^{2} \Omega_{i k} q$, and so on, briefly in general: $G_{j}^{m} \cdot \Omega_{i k} q$, where the entire number $m$ can be made arbitrarily large. Form this, it follows that the $G_{j}$ must all be free of $x$, since otherwise, the infinitely many infinitesimal transformations:

$$
\Omega_{i k} q, \quad G_{j} \Omega_{i k} q, \quad G_{j}^{2} \Omega_{i k} q, \cdots
$$

which are all mutually independent, should all belong to the group: $X_{1} f, \ldots, X_{r} f$, but this is impossible, because that group is finite. Consequently, the $G_{j}$ are constant and our group has the form:

$$
X_{k} f=\left\{c_{k} y+F_{k}(x)\right\} q \quad(k=1 \cdots r)
$$

where naturally, the $c_{k}$ do not all vanish. Thus if for instance $c_{r}$ is not equal to zero, then we introduce: $c_{r} y+F_{r}(x)$ as new $y$, and it comes: $X_{r} f=c_{r} y q$, hence our group contains $r$ independent infinitesimal transformations of the form:

$$
\begin{equation*}
F_{1}(x) q, \ldots, F_{r-1}(x) q, y q . \tag{2}
\end{equation*}
$$

Thirdly and lastly, let the group (6') be three-term, so that, according to Theorem 1 on p .51 , it can be given the form: $q, y q, y^{2} q$ through an appropriate choice of $y$. Then it is always possible to introduce as new $y$ a function: $\Omega(x, y)$ such that the group (6) becomes visible under the form:

$$
X_{k} f=\left\{\varphi_{k}(x)+y \cdot \chi_{k}(x)+y^{2} \cdot \psi_{k}(x)\right\} q \quad(k=1 \cdots r) .
$$

If one now thinks that the variable $x$ just plays the role of a constant by the Combination of the two infinitesimal transformations: $X_{i} f$ and $X_{k} f$, and that $q, y q, y^{2} q$ stand in the relationships:

$$
[q, y q]=q, \quad\left[q, y^{2} q\right]=2 y q, \quad\left[y q, y^{2} q\right]=y^{2} q
$$

then one realizes that the Proposition 1 on p. 72 can be applied to any three amongst the infinitesimal transformations $X_{k} f$, hence that between $X_{i} f$, $X_{k} f$ and $X_{j} f$, the identity:

$$
\left[\left[X_{i}, X_{k}\right],\left[X_{i}, X_{j}\right]\right]=2\left|\begin{array}{ccc}
\varphi_{i} & \chi_{i} & \psi_{i} \\
\varphi_{k} & \chi_{k} & \psi_{k} \\
\varphi_{j} & \chi_{j} & \psi_{j}
\end{array}\right|=2 \Delta_{i k j} X_{i} f
$$

holds. If one sets in this identity $\Delta_{i k j} X_{i} f$ in place of $X_{i} f$, then one gets $\Delta_{i k j}^{2} X_{i} f$, and in an analogous way, one gets $\Delta_{i k j}^{3} X_{i} f$, and so on, all of which are infinitesimal transformations belonging to the group: $X_{1} f, \ldots, X_{r} f$. Consequently, the $\Delta_{i k j}$ must be constant: $\Delta_{i k j}=C_{i k j}$, and from the identity:

$$
\left|\begin{array}{cccc}
X_{i} f & \varphi_{i} & \chi_{i} & \psi_{i} \\
X_{k} f & \varphi_{k} & \chi_{k} & \psi_{k} \\
X_{j} f & \varphi_{j} & \chi_{j} & \psi_{j} \\
X_{s} f & \varphi_{s} & \chi_{s} & \psi_{s}
\end{array}\right|=0,
$$

it furthermore comes out that between any four amongst the infinitesimal transformations $X_{1} f, \ldots, X_{r} f$, a relation of the form:

$$
C_{k j s} X_{i} f-C_{i j s} X_{k} f+C_{i k s} X_{j} f-C_{i k j} X_{s} f=0
$$

holds. Now, the $\Delta_{i k j}$ cannot all vanish, because otherwise the group (6') would not be three-term, so we can assume that, say, $\Delta_{123}=C_{123}$ is nonzero. But on admitting this, the latter equation shows immediately that $X_{1} f, \ldots, X_{r} f$ can be deduced linearly with constant coefficients from $X_{1} f, X_{2} f, X_{3} f$, hence that the group $X_{1} f, \ldots, X_{r} f$, just as the associated group ( $6^{\prime}$ ), contains only three infinitesimal transformations. The possibility $r>3$ is therefore excluded and it remains only the possibility: $r=3$.

Thus we now have to bring to an as simple as possible normal form the three-term group:

$$
X_{k} f=\left(\varphi_{k}(x)+y \chi_{k}(x)+y^{2} \psi_{k}(x)\right) q \quad(k=1,2,3)
$$

by means of a transformation of the form (5). To this aim, we remember that according to Vol. I, p. 591, Prop. 5, our group surely contains twoterm subgroups; for reasons of simplicity, we want to admit that $X_{1} f$, $X_{2} f$ engender such a subgroup.

Now, because, as we have seen above, the determinant $\sum \pm \varphi_{1} \chi_{2} \psi_{3}$ does not vanish, then obviously the two infinitesimal transformations:

$$
X_{k} f=\left\{\varphi_{k}(a)+y \chi_{k}(a)+y^{2} \psi_{k}(a)\right\} \quad(k=1,2)
$$

will be independent of each other and will engender a two-term group. But this group $X_{1} f, X_{2} f$ is projective, and hence conjugate to the group $q$, $y q$ inside the general projective group of the once-extended manifold $y$ (see Theorem 2 on p. 71). From this, it follows that, when one introduces as new $y$ an appropriate function of the form:

$$
\frac{\alpha(x) y+\beta(x)}{\gamma(x) y+\delta(x)}
$$

the group $X_{1} f, X_{2} f$ takes the form:

$$
X_{1} f=\left\{F_{1}(x)+y G_{1}(x)\right\} q, \quad X_{2} f=\left\{F_{2}(x)+y G_{2}(x)\right\} q
$$

Here as above (p.96), one realizes that $G_{1}$ and $G_{2}$ are constant, and that the group, by means of an appropriate choice of $y$, can be given the form $F_{1}(x) q$, $y q$; finally, by yet introducing $\frac{1}{F_{1}(x)} y$ as new $y$, one obtains that $F_{1}(x)$ equals 1. In the new variables $x, y$, the group $X_{1} f, X_{2} f, X_{3} f$ now has the form:

$$
q, \quad y q, \quad X_{3} f=\left\{\varphi(x)+y \chi(x)+y^{2} \psi(x)\right\} q,
$$

since all the transformations employed by us possess the form:

$$
x_{1}=x, \quad y_{1}=\frac{\lambda(x) y+\mu(x)}{\nu(x) y+\rho(x)}
$$

so that only the form of the functions $\varphi_{3}, \psi_{3}, \chi_{3}$ change in the initial expression for $X_{3} f$. Thus, one has:

$$
\begin{aligned}
{\left[q, X_{3} f\right] } & =[\chi(x)+2 y \psi(x)] q \\
{\left[q,\left[q, X_{3} f\right]\right] } & =2 \psi(x) q,
\end{aligned}
$$

whence $\psi(x)$ and $\chi(x)$ must be constant. Lastly, from the equation:

$$
\left[y q, X_{3} f\right]=-\varphi(x) q+y^{2} \psi(q) q,
$$

it yet follows that $\varphi(x)$ is a constant too.
As a result, it is proved that every group which belongs to our third class can be brought to the form:
[3] $\quad q, \quad y q, \quad y^{2} q$
by means of a transformation (5).

$$
\S 10 .
$$

The curves: $x=$ const. are transformed in one term.
According to the program set up on p. 94, we now have to add the transformation $p+\eta(x, y) q$ to each one of the groups found in the preceding paragraph and to determine $\eta$ in the most general way in order that a group arises.

At first, we seek to bring all $r$-term groups of the shape:

$$
F_{1}(x) q, \ldots, F_{r-1}(x) q, \quad p+\eta(x, y) q
$$

to a simple normal form.

If $r=1$, we introduce a solution $\omega(x, y)$ of the differential equation:

$$
\frac{\partial \omega}{\partial x}+\eta \frac{\partial \omega}{\partial y}=0
$$

as new $y$, and we get the group:
[4] $\quad p$.
If $r>1$, there must exist an equation of the form:

$$
\left[p+\eta q, F_{i} q\right]=\sum_{k=1}^{r-1} c_{i k} F_{k} q
$$

for the left hand-side is free of $p$. Consequently, we have:

$$
F_{i}^{\prime}(x)-F_{i} \frac{\partial \eta}{\partial y}=\sum_{k=1}^{r-1} c_{i k} F_{k}
$$

so that $\eta$ is linear in $y$ :

$$
\eta=y \varphi(x)+\chi(x)
$$

But if we set:

$$
x_{1}=x, \quad y_{1}=y \alpha(x)+\beta(x),
$$

it comes out:

$$
p+\eta q=p_{1}+\left(y \alpha^{\prime}(x)+\beta^{\prime}(x)+\alpha \eta\right) q_{1}
$$

hence when we choose $\alpha$ and $\beta$ in such a way that:

$$
\alpha^{\prime}+\alpha \varphi=0, \quad \beta^{\prime}+\alpha \chi=0
$$

which is always possible, then it comes plainly: $p+\eta q=p_{1}$. Since in addition the remaining infinitesimal transformations of the group essentially keep their form through the performed change of variables, our group then becomes visible in the shape:

$$
F_{1}(x) q, \ldots, F_{r-1}(x) q, \quad p
$$

Finally, the equation: $\left[p, F_{i} q\right]=F_{i}^{\prime} q$ now shows that the $F_{i}$ must satisfy a system of ordinary differential equations of the form:

$$
\frac{d F_{i}}{d x}=\sum_{k=1}^{r-1} c_{i k} F_{k} \quad(i=1 \cdots r-1)
$$

Here, the constants $c_{i k}$ are absolutely arbitrary, since the Jacobi identity produces no relation between the $c_{i k}$.

According to known results, $F_{1}, \ldots, F_{r-1}$ do all satisfy a certain linear homogeneous differential equation of $(r-1)$-th order with constant
|coefficients:

$$
\frac{d^{r-1} F}{d x^{r-1}}+C_{r-2} \frac{d^{r-2} F}{d x^{r-2}}+\cdots+C_{1} \frac{d F}{d x}+C_{0} F=0
$$

we hence can replace $F_{1}, \ldots, F_{r-1}$ by particular integrals of this differential equation. These particular integrals can be ordered in several, say $l>0$, systems of the form:

$$
\begin{array}{cc}
e^{\alpha_{1} x}, & x e^{\alpha_{1} x}, \ldots, x^{m_{1}} e^{\alpha_{1} x} \\
e^{\alpha_{2} x}, & x e^{\alpha_{2} x}, \ldots, x^{m_{2}} e^{\alpha_{2} x} \\
\cdot & \cdot \\
e^{\alpha_{l} x}, & x e^{\alpha_{l} x}, \ldots, x^{m_{l}} e^{\alpha_{l} x}
\end{array}
$$

where $\alpha_{1}, \ldots, \alpha_{l}$ denote constants which are all distinct one another, and where $m_{1}, \ldots, m_{l}$ are all integers $\geqslant 0$, whose sum has the value $r-1-l$. Therefore, our group has the form:
[5]

$$
\begin{array}{|c}
e^{\alpha_{k} x} q, \quad x e^{\alpha_{k} x} q, \quad x^{2} e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k} x} q, \quad p \\
(k=1,2 \cdots l ; l>0)
\end{array}
$$

We now turn to the groups:

$$
F_{1}(x) q, \ldots, F_{r}(x) q, \quad y q, \quad p+\eta(x, y) q
$$

where according to p .96 , we must assume the integer $r$ to be $>0$.
There exists an equation of the form:

$$
[p+\eta q, y q]=\left(\eta-y \frac{\partial \eta}{\partial y}\right) q=c y q+\sum_{k=1}^{r} c_{k} F_{k} q
$$

and in the same way:

$$
\left[p+\eta q, F_{i} q\right]=\left[F_{i}^{\prime}(x)-F_{i} \frac{\partial \eta}{\partial y}\right] q=C_{i} y q+\sum_{k=1}^{r} C_{i k} F_{k} q
$$

The latter equation shows that $\eta$ possesses the form: $\alpha(x)+y \beta(x)+$ $y^{2} \gamma(x)$; but when this expression is inserted in the first equation, it comes instantly:

$$
\gamma(x)=0, \quad c=0, \quad \alpha(x)=\sum_{k=1}^{r} c_{k} F_{k}(x)
$$

As a result of this, we replace the infinitesimal transformation: $p+\eta q$ by:

$$
p+\left(\eta-\sum_{k=1}^{r} c_{k} F_{k}\right) q=p+y \beta(x) q
$$

then we introduce a new $y: y_{1}=y \cdot \psi(x)$, and we obtain:

$$
p+y \beta q=p+\frac{y_{1}\left(\psi^{\prime}+\beta \psi\right)}{\psi} q_{1}
$$

hence when we make: $\psi^{\prime}+\beta \psi=0$, we obtain plainly: $p$. But since the remaining infinitesimal transformations do essentially not change through the performed change of variables, our group has now the form:

$$
F_{1}(x) q, \ldots, F_{r}(x) q, \quad p, \quad y q .
$$

Here obviously, the first $r+1$ infinitesimal transformations engender a subgroup which, according to what precedes, possesses the form [5]; as a result, we obtain the group:
[6]

$$
\begin{gathered}
e^{\alpha_{k} x} q, x e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k} x} q, \quad y q, \quad p \\
(k=1,2 \cdots l ; l>0)
\end{gathered}
$$

Now, we turn to the groups:

$$
q, \quad y q, \quad y^{2} q, \quad p+\eta(x, y) q .
$$

It comes out:

$$
\begin{aligned}
{[q, p+\eta q] } & =\frac{\partial \eta}{\partial y} q=\left(a+2 b y+3 c y^{2}\right) q \\
{[y q, p+\eta q] } & =\left(y \frac{\partial \eta}{\partial y}-\eta\right) q=\left(\alpha+\beta y+\gamma y^{2}\right) q
\end{aligned}
$$

where $a, b, c, \alpha, \beta, \gamma$ are constants. If we put in the second equation the value:

$$
\eta=\varphi(x)+a y+b y^{2}+c y^{3}
$$

issued from the first one, it comes: $c=0, \varphi(x)=$ const., hance our group is engendered by the four infinitesimal transformations:

> [7]

$$
\begin{array}{|llll|}
\hline q, & y q, & y^{2} q, & p \\
\hline
\end{array}
$$

§ 11.
The curves: $x=$ const. are transformed in two terms
We have to add to every group of the § 10 one infinitesimal transformation of the form: $x p+\eta(x, a) q$.

At first, we consider the group:

$$
p, \quad x p+\eta(x, y) q
$$

and we find:

$$
[p, x p+\eta q]=p+\frac{\partial \eta}{\partial x} q
$$

whence $\frac{\partial \eta}{\partial x}=0$, or: $\eta=\varphi(y)$. Two cases are therefore to be be distinguished; either $\varphi=0$, which gives the group:
[8] $\square$
or $\varphi \neq 0$. In the latter case we introduce $y_{1}=\psi(y)$ in place of $y$ and we obtain:

$$
x p+\varphi(y) q=x p+\varphi \psi^{\prime} q_{1}
$$

thus if we choose $\psi$ so that $\varphi \psi^{\prime}=1$, we obtain the group:

$$
\text { [9] } \quad p, x p+q \text {. }
$$

Moreover, we have to determine the groups:

$$
\begin{gathered}
e^{\alpha_{k} x} q, x e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k}} q, \quad p, x p+\eta(x, y) q \\
(k=1,2 \cdots l ; l>0)
\end{gathered}
$$

or, as we want to write for abbreviation, the groups:

$$
F_{1}(x) q, \ldots, F_{r}(x), \quad p, x p+\eta(x, y) q \quad(r>0) .
$$

The relations:

$$
\begin{aligned}
{[p, x p+\eta q] } & =p+\frac{\partial \eta}{\partial x} q=p+\sum_{k=1}^{r} a_{k} F_{k} q \\
{\left[x p+\eta q, F_{i} q\right] } & =\left(x F_{i}^{\prime}(x)-F_{i} \frac{\partial \eta}{\partial y}\right) q=\sum_{k=1}^{r} b_{i k} F_{k} q
\end{aligned}
$$

give:

$$
\frac{\partial \eta}{\partial x}=\varphi^{\prime}(x), \quad \frac{\partial \eta}{\partial y}=\psi(x)
$$

whence $\psi^{\prime}(x)=0$ and:

$$
\eta=c y+\varphi(x), \quad \varphi^{\prime}(x)=\sum_{k=1}^{r} a_{k} F_{k}(x)
$$

But if we substitute for $F_{i}$ in the transformation: $\left(x F_{i}^{\prime}-c F_{i}\right) q$ the expression: $x^{m_{k}} e^{\alpha_{k} x}$, we receive the transformation:

$$
\alpha_{k} x^{m_{k}+1} e^{\alpha_{k} x} q+\left(m_{k}-c\right) x^{m_{k}} e^{\alpha_{k} x} q,
$$

which can be contained in our group only if $\alpha_{k}$ vanishes. The $r$ infinitesimal transformations $F_{k} q$ have in consequence of that the simple shape:

$$
q, \quad x q, \quad x^{2} q, \ldots, x^{r-1} q
$$

and one has:

$$
\eta=c y+\sum_{k=0}^{r-1} \frac{a_{k+1}}{k+1} x^{k+1}+\text { const. }
$$

or, because we can take away, by virtue of the $F_{k} q$, all remaining terms, simply:

$$
\eta=c y+h x^{r} .
$$

If we now set:

$$
x_{1}=x, \quad y_{1}=y+\alpha x^{r},
$$

the totality of all infinitesimal transformations that can be deduced linearly from $q, x q, \ldots, x^{r-1} q, p$ remains wholly unchanged, but there will be:

$$
\begin{aligned}
x p+\eta q & =x p_{1}+\left\{c y+(\alpha r+h) x^{r}\right\} q_{1} \\
& =x_{1} p_{1}+\left\{c y_{1}+(h+\alpha r-\alpha c) x_{1}^{r}\right\} q_{1} .
\end{aligned}
$$

Thus if $c \neq r$, by an appropriate choice of $\alpha$, we can fulfill the equation: $h+\alpha(r-c)=0$, and we obtain the group:
[10]

$$
q, \quad x q, \ldots, x^{r-1} q, \quad p, \quad x p+c y q .
$$

If on the contrary $c=r$, we can in any case suppose that $h$ does not vanish, because otherwise we would come back to the group just found; we can therefore introdude: $x \sqrt[r]{h}$ as new $x$, and we find the group:
[11]

$$
q, \quad x q, \ldots, x^{r-1} q, \quad p, \quad x p+\left(r y+x^{r}\right) q
$$

At present, we turn to the groups:

$$
e^{\alpha_{k} x} q, x e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k} x} q, \quad y q, \quad p, x p+\eta(x, y) q
$$

or written in a shorter way:

$$
F_{1}(x) q, \ldots, F_{r}(x) q, y q, \quad p, x p+\eta q \quad(r>0) .
$$

One gets:

$$
\begin{aligned}
{[y q, x p+\eta q] } & =\left(y \frac{\partial \eta}{\partial y}-\eta\right) q=a y q+\sum_{k=1}^{r} a_{k} F_{k} q \\
{\left[F_{i} q, x p+\eta q\right] } & =\left(F_{i} \frac{\partial \eta}{\partial y}-x F_{i}^{\prime}(x)\right) q=b y q+\sum_{k=1}^{r} b_{i k} F_{k} q
\end{aligned}
$$

and from this without difficulty:

$$
a=0, \quad \eta=y \alpha(x)-\sum_{k=1}^{r} a_{k} F_{k}(x) .
$$

If one replaces $x p+\eta q$ by the transformation: $x p+y \alpha(x) q$, which is obviously allowed, one finds:

$$
\begin{aligned}
{[p, x p+y \alpha q] } & =p+y \alpha_{x} q \\
& =p+c y q+\sum_{k=1}^{r} c_{k} F_{k}(x) q,
\end{aligned}
$$

therefore all $c_{k}$ vanish and one has: $\alpha=c x+$ const., where the integration constant can just be left out. In place of $y$ and $x$, if one now introduces: $y_{1}=y e^{-c x}$ and $x_{1}=x$, then one obtains:

$$
\begin{aligned}
& p=p_{1}-c y e^{-c x} q_{1}=p_{1}-c y_{1} q_{1}, \\
& q=e^{-c x} q_{1}=e^{-c x_{1}} q_{1}, \quad y q=y_{1} q_{1},
\end{aligned}
$$

so the transformations $F_{i} q, y q$ do essentially keep their form, while:

$$
x p+\eta q=x p+c x y q
$$

is transferred to $x_{1} p_{1}$. As in the preceding case, one now yet realizes that $F_{1} q, \ldots, F_{r} q$ must have the form: $q, x q, \ldots, x^{r-1} q$, so as a result, one arrives at the group:

$$
\begin{equation*}
q, \quad x q, \ldots, x^{r-1} q, \quad y q, \quad p, \quad x p \tag{12}
\end{equation*}
$$

Finally, it remains the groups:

$$
q, \quad y q, y^{2} q, \quad p, \quad x p+\eta(x, y) q .
$$

Considering the equations:

$$
\begin{aligned}
{[q, x p+\eta q] } & =\frac{\partial \eta}{\partial y} q=\left(a+2 b y+3 c y^{2}\right) q \\
{[y q, x p+\eta q] } & =\left(y \frac{\partial \eta}{\partial y}-\eta\right) q=\left(\alpha+\beta y+\gamma y^{2}\right) q
\end{aligned}
$$

the first one gives:

$$
\eta=\varphi(x)+a y+b y^{2}+c y^{3}
$$

and the second one: $c=0, \varphi=$ const., so that $\eta$ can be set equal to zero and our group has the form:
[13]
$q, \quad y q, \quad y^{2} q, \quad p, \quad x p$.
§ 12.
The curves: $x=$ const. are transformed in three terms
We now have to add to every group found in § 11 a transformation of the form: $x^{2} p+\eta(x, y) q$.

For the group:

$$
p, \quad x p, \quad x^{2} p+\eta(x, y) q
$$

there are the equations:

$$
\begin{aligned}
{\left[p, x^{2} p+\eta q\right] } & =2 x p+\frac{\partial \eta}{\partial x} q=2 x p \\
{\left[x p, x^{2} p+\eta q\right] } & =x^{2} p+x \frac{\partial \eta}{\partial x} q=x^{2} p+\eta q
\end{aligned}
$$

whence $\frac{\partial \eta}{\partial x}$ vanishes as well as $\eta$ itself, and it remains only the group:

$$
\text { [14] } \quad p, \quad x p, \quad x^{2} p \text {. }
$$

For: $p, x p+q$, we can write: $p, x p+y q$ by introducing $e^{y}$ as new $y$; therefore, the groups of the form:

$$
p, \quad x p+y q, \quad x^{2} p+\eta(x, y) q
$$

are to be determined. We find:

$$
\begin{aligned}
{\left[p, x^{2} p+\eta q\right] } & =2 x p+\frac{\partial \eta}{\partial x} q \\
{\left[x p+y q, x^{2} p+\eta q\right] } & =x^{2} p+\left(x \frac{\partial \eta}{\partial x}+y \frac{\partial \eta}{\partial y}-\eta\right) q
\end{aligned}
$$

whence one has:

$$
\frac{\partial \eta}{\partial x}=2 y, \quad x \frac{\partial \eta}{\partial x}+y \frac{\partial \eta}{\partial y}=2 \eta
$$

that is to say: $\eta$ must possess the form: $2 x y+\varphi(y)$ and at the same time be homogeneous of second order in $x$ and in $y$, namely: $\eta=2 x y+c y^{2}$. Now, if $c$ vanishes, we introduce $\sqrt{y}$ as new $y$ and we find the group:

$$
[15]
$$

$$
\begin{array}{|c}
\hline p, \quad 2 x p+y q, \quad x^{2} p+x y q \text {. }
\end{array}
$$

If on the contrary $c$ does not vanish, we introduce $b y$ as new $y$; at the same time, $y q$ keeps its form and $y^{2} q$ is transferred to $\frac{1}{b} y^{2} q$, hence we just need
to choose $b=c$ and we obtain:
[16]

$$
p, \quad x p+y q, \quad x^{2} p+\left(2 x y+y^{2}\right) q
$$

In order to find all groups of the form:

$$
q, x q, \ldots, x^{r-1} q, \quad p, x p+c y q, x^{2} p+\eta(x, y) q \quad(r>0)
$$ we form the equations:

$$
\begin{aligned}
& {\left[q, x^{2} p+\eta q\right]=\frac{\partial \eta}{\partial y} q=\sum_{k=0}^{r-1} a_{k} x^{k} q} \\
& {\left[p, x^{2} p+\eta q\right]=2 x p+\frac{\partial \eta}{\partial x} q=2 x p+\left(2 c y+\sum_{k=0}^{r-1} b_{k} x^{k}\right) q}
\end{aligned}
$$

from which it follows:

$$
\begin{aligned}
\eta & =\varphi(x)+y \sum_{k=0}^{r-1} a_{k} x^{k} \\
& =\psi(y)+2 c x y+\sum_{k=0}^{r-1} \frac{b_{k}}{k+1} x^{k+1} .
\end{aligned}
$$

By comparing these two expressions, it comes:

$$
\eta=a_{0} y+2 c x y+\sum_{k=0}^{r} g_{k} x^{k}
$$

or, since $g_{0}, g_{1}, \ldots, g_{r-1}$ can simply be set equal to zero:

$$
\eta=a_{0} y+2 c x y+g x^{r}
$$

At present, one has:

$$
\begin{aligned}
{\left[x p+c y q, x^{2} p+\eta q\right] } & =x^{2} p+\left\{2 c x y+g(r-c) x^{r}\right\} q \\
& =x^{2} p+\eta q+\sum_{k=0}^{r-1} h_{k} x^{k} q
\end{aligned}
$$

whence:

$$
a_{0}=0, \quad g(r-c-1)=0
$$

But on the other hand, one has:

$$
\begin{aligned}
{\left[x p+c y q, x^{2} p+\eta q\right] } & =\left\{(1-r) x^{r}+2 c x^{r}\right\} q \\
& =\sum_{k=0}^{r-1} l_{k} x^{k} q
\end{aligned}
$$

whence: $r-1=2 c$, that is to say, as soon as $r$ is $>1, r-c-1$ cannot vanish and one has $g=0$. By condensing the two cases: $r>1$ and $r=1$, $g=0$, we obtain the group:
[17]

$$
\left.q, \quad x q, \ldots, x^{r-1} q, \quad p, \quad \begin{array}{c} 
\\
(r>0)
\end{array}\right)(r-1) y q, \quad x^{2} p+(r-1) x y q
$$

By contrast, in the case $r=1, g \neq 0$, we obtain the group:

$$
q, \quad p, \quad x p, \quad x^{2} p+g x q
$$

or, when we introduce $e^{\frac{y}{g}}$ as new $y$, the group:
[18]
$y q, \quad p, \quad x p, \quad x^{2} p+x y q$

For the determination of all groups:
$q, x q, \ldots, x^{r-1} q, p, x p+\left(r y+x^{r}\right) q, x^{2} p+\eta(x, y) q \quad(r>0)$,
we set up the equations:

$$
\begin{aligned}
& {\left[p, x^{2} p+\eta q\right]=2 x p+\frac{\partial \eta}{\partial x} q=2 x p+2\left(r y+x^{r}\right)+\sum_{k=0}^{r-1} a_{k} x^{k} q} \\
& {\left[q, x^{2} p+\eta q\right]=\quad \frac{\partial \eta}{\partial y} q=\sum_{k=0}^{r-1} b_{k} x^{k} q .}
\end{aligned}
$$

From these it follows:

$$
\begin{aligned}
\eta & =\varphi(x)+y \sum_{k=0}^{r-1} b_{k} x^{k} \\
& =\psi(y)+2\left(r x y+\frac{x^{r+1}}{r+1}\right)+\sum_{k=0}^{r-1} \frac{a_{k}}{k+1} x^{k+1}
\end{aligned}
$$

or by comparison of the two expressions:

$$
\eta=b_{0} y+2 r x y+a x^{r}+\frac{2 x^{r+1}}{r+1},
$$

where for reasons of brevity, we think that the superfluous terms with $x^{0}, x^{1}, \ldots, x^{r-1}$ are took away. Further, we form the equation:

$$
\begin{aligned}
{\left[x p+\left\{r y+x^{r}\right\} q, x^{2} p+\eta q\right]=x^{2} p } & +\left(2 r x y+b_{0} x^{r}+\right. \\
& \left.+\frac{r^{2}+r+2}{r+1} x^{r+1}\right) q
\end{aligned}
$$

whose right hand side must visibly take the form: $x^{2} p+\eta q$, so it comes:

$$
b_{0}=a=0, \quad \frac{r^{2}+r}{r+1}=r=0
$$

But this is impossible, because $r$ must be $>0$, and as a result, there are in general no groups of the demanded sort.

Now, we turn to the groups:

$$
q, x q, \ldots, x^{r-1} q, y q, \quad p, x p, x^{2} p+\eta(x, y) q \quad(r>0) .
$$

The equations:

$$
\left.\begin{array}{rl}
{\left[p, x^{2} p+\eta q\right]} & =2 x p+\frac{\partial \eta}{\partial x} q=2 x p+\sum_{k=0}^{r-1} a_{k} x^{k} q+\alpha y q \\
{\left[q, x^{2} p+\eta q\right]} & =\quad \frac{\partial \eta}{\partial y} q= \\
{\left[y q, x^{2} p+\eta q\right]} & =\left(y \frac{\partial \eta}{\partial y}-\eta\right) q=
\end{array} \sum_{k=0}^{r-1} b_{k} x^{k} q+\beta y q\right]
$$

show that $\eta$ takes the form: $\eta=\alpha x y$ after removal of the superfluous terms of the form: $\sum g_{k} x^{k}+h y$. Furthermore, one finds:

$$
\left[x^{r-1} q, x^{2} p+\alpha x y q\right]=(1-r) x^{r} q+\alpha x^{r} q,
$$

and this expression must vanish, because the group contains no transformation $x^{r} q$, whence $\alpha=r-1$. Thus, we have the groups:
[19]

$$
q, \quad x q, \ldots, x^{r-1} q, \quad y q, \quad p, \quad x p, \quad x^{2} p+(r-1) x y q
$$

Finally, the groups of the form:

$$
q, y q, y^{2} q, \quad p, x p, x^{2} p+\eta(x, y) q
$$

still have to be found. We receive:

$$
\begin{aligned}
{\left[q, x^{2} p+\eta q\right] } & =\frac{\partial \eta}{\partial y} q=\left(a+2 b y+3 c y^{2}\right) q \\
{\left[y q, x^{2} p+\eta q\right] } & =\left(y \frac{\partial \eta}{\partial y}-\eta\right) q=\left(\alpha+\beta y+\gamma y^{2}\right) q
\end{aligned}
$$

The first one of these equations shows that $\eta$ has the form: $\varphi(x)+a y+$ $b y^{2}+c y^{3}$, the second one that $c$ vanishes and that $\varphi(x)$ is a constant. Thus,
we obtain the group:
[20]
$q, \quad y q, \quad y^{2} q, \quad p, \quad x p, \quad x^{2} p$.

As a result, all finite imprimitive transformation groups of the plane have been reduced to certain normal forms.
§ 13.
At present, we have produced the determination of all imprimitive groups of the plane insofar as we can say: every imprimitive group of the plane is equivalent, through a point transformation of the plane, to one of the groups found in § 9-12. But the goal that we have set ourselves for the imprimitive groups on p. 84 has not yet been reached with that. What matters for us is to possess one and only one representative for every type of imprimitive group of the plane, and it is easy to see that in § 9-12, certain types of groups are represented by more than one representative.

For instance, through the transformation: $x_{1}=y, y_{1}=x$, the group [14] is transferred to the group [13]; hence both groups represent the same type. In the same way, through the transformation mentioned, the group [4] changes to one of the groups [1], the group [8] to one of the groups [2], and so on.

Furthermore, it can be shown that the arbitrary elements which appear in our groups can in part be left out, hence that the number of distinct types of groups is smaller than what it appears to be, according to the number of these arbitrary elements.

To begin with, we consider the groups [1] on p. 39 and we look for finding out what are the different types of groups that are contained among them.

To this end, we must above all determine all point transformations: $x_{1}=\alpha(x, y), y_{1}=\beta(x, y)$ through which the totality of all groups [1] remains invariant, hence through which every $r$-term group of the form:
(A)

$$
F_{1}(x) q, \ldots \ldots, F_{r}(x) q
$$

is transferred to one of the form:
(B)

$$
\mathfrak{F}_{1}(x) q_{1}, \ldots \ldots, \mathfrak{F}_{r}(x) q_{1} .
$$

Now, by introducing the new variables $x_{1}, y_{1}$, the group (A) changes to the following:

$$
\begin{equation*}
F_{k}(x) \frac{\partial \alpha}{\partial y} p_{1}+F_{k}(x) \frac{\partial \beta}{\partial y} q_{1} \quad(k=1 \cdots r), \tag{A'}
\end{equation*}
$$

where one has to think that the coefficients of $p_{1}$ and of $q_{1}$ are expressed in terms of $x_{1}$ and $y_{1}$. But the group (A') possesses the form (B) if and only if, firstly $\alpha_{y}$ vanishes identically and secondly $\beta_{y}$ is a function of $x$ alone. As a result, the most general point transformation which leaves invariant all the groups [1] writes in the following way:
(C) $\quad x_{1}=\varphi(x), \quad y_{1}=y \chi(x)+\psi(x)$,
and to be precise, through this transformation, the group (A) receives the form:
(A")

$$
F_{1}(x) \chi(x) q_{1}, \ldots \ldots, F_{r}(x) \chi(x) q_{1}
$$

where still $x$ is expressed in terms of $x_{1}$ by means of the equation: $x_{1}=$ $\varphi(x)$. From this, one sees that the function $\psi(x)$ in the transformation (C) has absolutely no influence on the form of the transformed group (A"). Consequently, we need not to consider all transformations (C), but only those for which $\psi(x)$ equals zero.

At present, we imagine that two arbitrary $r$-term groups of the form [1], say (A) and (B), are presented. For these two groups to belong to the same type, it is necessary and sufficient that they are equivalent one to the other through a transformation of the form:

$$
x_{1}=\varphi(x), \quad y_{1}=y \chi(x),
$$

hence that the group (A"), in which naturally one has still to think that $x$ is expressed in terms $x_{1}$, coincides with the group (B) for an appropriate choice of $\varphi(x)$ and $\chi(x)$. This occurs if and only if, by virtue of ( $\mathrm{C}^{\prime}$ ), $r$ relations of the form:

$$
\mathfrak{F}_{k}\left(x_{1}\right) q_{1}=\sum_{j=1}^{r} c_{k j} F_{j}(x) \chi(x) q_{1} \quad(k=1 \cdots r)
$$

hold identically, where the $c_{k j}$ denote constants whose determinant does not vanish. The question whether the two groups (A) and (B) belong to the same type therefore amounts to deciding whether the $r$ equations:
(D)

$$
\mathfrak{F}_{k}\left(x_{1}\right)=\chi(x) \sum_{j=1}^{r} c_{k j} F_{j}(x) \quad(k=1 \cdots r)
$$

can be identically satisfied, by substituting for $x_{1}$ and $\chi$ certain functions of $x$ and by choosing the constants $c_{k j}$ in such a way that their determinant does not vanish; here naturally, $x_{1}$ must be a true, arbitrary function of $x$, whereas $\chi$ can reduce to a nonzero constant.

The settlement of the question whether the equations (D) can be satisfied in the indicated way certainly presents no difficulty from the theoretical side, but in general rather many from the practical side, especially because of the occurence of the $r^{2}$ unknown constants $c_{k j}$. So it is not superluous to develope yet another method which enables to decide whether the two groups (A) and (B) belong to the same type, but which leads to success without introducing the constants $c_{k j}$.

At first, we observe that the general infinitesimal transformation:

$$
\sum_{k=1}^{r} e_{k} F_{k}(x) q=F(x) q
$$

of the group (A) can be defined by an ordinary differential equation of $r$-th order, for $F(x)$ visibly is the most general solution of a linear homogeneous differential equation of the form:

$$
\begin{equation*}
\frac{d^{r} F}{d x^{r}}+\alpha_{1}(x) \frac{d^{r-1} F}{d x^{r-1}}+\cdots+\alpha_{r-1}(x) \frac{d F}{d x}+\alpha_{r}(x) F=0 . \tag{E}
\end{equation*}
$$

In the same way, in the general infinitesimal transformation:

$$
\sum_{k=1}^{r} \mathfrak{e}_{k} \mathfrak{F}_{k}\left(x_{1}\right) q_{1}=\mathfrak{F}\left(x_{1}\right) q_{1}
$$

of the group (B), the function $\mathfrak{F}\left(x_{1}\right)$ is the most general solution of a differential equation of the form:
(G) $\quad \frac{d^{r} \mathfrak{F}}{d x_{1}^{r}}+\mathfrak{a}_{1}\left(x_{1}\right) \frac{d^{r-1} \mathfrak{F}}{d x_{1}^{r-1}}+\cdots+\mathfrak{a}_{r-1}\left(x_{1}\right) \frac{d \mathfrak{F}}{d x_{1}}+\mathfrak{a}_{r}\left(x_{1}\right) \mathfrak{F}=0$.

Now, by execution of the transformation ( $\mathrm{C}^{\prime}$ ), the general infinitesimal transformation $F(x) q$ receives the shape: $F(x) \chi(x) q_{1}$ and everything amounts to whether it is the general infinitesimal transformation of the group (B), that is to say whether: $F(x) \chi(x)$, expressed as a function of $x_{1}$, is the general solution of the differential equation (G). Thus, the two groups (A) and (B) will always belong to the same type, if and only if there is a transformation of the form:

$$
\begin{equation*}
x_{1}=\varphi(x), \quad \mathfrak{F}=F \cdot \chi(x), \tag{H}
\end{equation*}
$$

by virtue of which the differential equation (E) goes to (G)*). As soon as the two groups (E) and (G) are presented, it is theoretically not difficult to decide whether there is such a transformation. For, if one executes the transformation $(\mathrm{H})$ on the differential equation (E) and if one requires that the resulting equation should have the form $(\mathrm{G})$, then one receives for $\varphi(x)$ and $\chi(x)$ a series of ordinary differential equations, about which one
always can determine whether they are mutually compatible or not and whether they can be satistied without $\varphi$ reducing to a constant.
> *) The first one who occupied himself with the question to know under which conditions the differential equation (E) can be transferred to (G) by means of a transformation of the form (H) is Laguerre. After him, several mathematicians, notably Halphen, treated the theory of invariants of the linear differential equation (E) vis-à-vis all transformations (H). This theory has several points of contact [BERÜHRUNGSPUNKTE] with the general theory of the finite and infinite transformation groups.

It follows from what has been said that the search for all different types of $r$-term groups of the form [1] is now also reduced to another problem, namely to the problem of looking for all invariant properties that the linear differential equation of $r$-th order (E) has vis-à-vis all transformations of the form $(\mathrm{H})$. We do not want to tackle this problem, because this would lead us going too far, but we want only to observe that for its resolution, the question whether the differential equation (E) admits an infinitesimal transformation of the form: $\delta x=\xi(x) \delta t, \delta F=F \cdot \Omega(x) \delta t$ plays an important role, in which the $\xi$ is distinct from zero.

However, we want not to suppress another remark. The most general $r$-term group of the form [1] contains $r$ arbitrary parameters, whereas in the transformation ( $\mathrm{C}^{\prime}$ ), we in total only have two arbitrary functions at our disposal in order to simplify the form of this group. Hence it is clear from the beginning that amongst the $r$ arbitrary functions of the group:
(A)

$$
F_{1}(x) q, \ldots \ldots, F_{r}(x) q,
$$

we can remove two, and only two functions. In fact, when we introduce $\frac{1}{F_{1}(x)} y$ as new $y$, the group (A) is transferred to:

$$
q, \quad \frac{F_{2}(x)}{F_{1}(x)}, \ldots \ldots, \frac{F_{r}(x)}{F_{1}(x)} q
$$

and if furthermore $r$ is $>1$, we can introduce $\frac{F_{2}(x)}{F_{1}(x)}$, which surely is not just a constant, as new $x$, and we obtain a group of the form:

$$
q, \quad x q, \quad \Phi_{1}(x) q, \ldots \ldots, \Phi_{r-2}(x) q \quad(r \geqslant 2) .
$$

We hence can replace the groups [1] by the following ones:

$$
\begin{array}{|ccc}
\hline q, \quad x q, \quad \Phi_{1}(x) q, \ldots, \Phi_{r-2}(x) q \quad(r \geqslant 2) .
\end{array}
$$

Here, the $r-2$ arbitrary functions $\Phi$ are essential in a certain sens, that is to say, none amongst them can be took away by the introduction of new variables.

The gained result for the groups [1] can easily be translated into the groups [2] on p. 96, since the totality of these groups remains as well invariant through all transformations of the form ( $\mathrm{C}^{\prime}$ ). That is why we need not to halt at the question of how many different types of groups are contained amongst the groups [2]. It is only to yet be observed that we can replace the groups [2] by the ones standing below:
[2'] $q, y q \quad q, x q, \Phi_{1}(x) q, \ldots, \Phi_{r-2}(x) q, y q \quad(r \geqslant 2)$,
where the $r-2$ arbitrary functions $\Phi$ are essential in the sens indicated above.

Also the form of the groups [6] on p. 101 is yet able of a simplification [VEREINFACHUNG FÄHIG]. Namely, one can always arrange that a constant $\alpha_{k}$ vanishes. Indeed, letting all $\alpha_{k} \neq 0$, we then can introduce $y e^{-\alpha_{1} x}$ as new $y$, and as a result:

$$
e^{\alpha_{1} x} q, \quad x e^{\alpha_{1} x} q, \ldots \ldots, x^{m_{1}} e^{\alpha_{1} x} q, \quad x^{\nu} e^{\alpha_{k} x} q
$$

are transferred to:

$$
q, \quad x q, \ldots \ldots, x^{\nu} e^{\left(\alpha_{k}-\alpha_{1}\right) x} q
$$

while $y q$ remains unchanged and $p$ takes the form: $p-\alpha_{1} y q$. As a result, we can replace the groups [6] by the following ones:
[6']

$$
q, x q, \ldots, x^{m} q, e^{\alpha_{k} x} q, x e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k} x} q, y q, \quad p
$$

Finally, it must still be remarked that in the groups [5] on p. 100 and likewise in the groups [6'], one of the nonzero coefficients $\alpha_{k}$ can always be made equal to 1 , since if for instance $\alpha_{1} \neq 0$, one only needs to introduce $\alpha_{1} x$ as new $x$.

## § 14.

In the next chapter we will systematically examine what are the different types of groups belonging to the groups of the §§ 9-13, and we shall be in position to draw up a table in which for every type of group, one and only one representative will be contained. For the moment, taking into consideration the observations made in the $\S 13$, we want to content
ourselves with putting together all different types of one-, two-, three- and four-term groups. Here they are:
I. One-term groups. $\quad q$.
II. Two-term groups.
a. transitive:
$p, q$
$p, x p+y q$
b. intransitive:
$q, x q$
$q, y q$

## III. Three-term groups.

a. transitive:

b. intransitive:
$q, x q, F(x) q$
$q, x q, y q$,
$q, y q, y^{2} q$,
IV. Four-term groups.
a. transitive:


$$
\begin{gathered}
q, e^{x} q, e^{\alpha x} q, p \\
(\alpha \neq 0,1)
\end{gathered}
$$

$q, e^{x} q, x e^{x} q, p$
$q, x q, e^{x} q, p$
$q, x q, x^{2} q, p$
$q, e^{x} q, y q, p$
$q, x q, y q, p$ $q, y q, y^{2} q, p$
$q, x q, p, x p+c y q$
$q, x q, p, x p+\left(2 y+x^{2}\right) q$
$q, y q, p, x p$

$$
y q, p, x p, x^{2} p+x y q
$$

b. intransitive:

$$
\begin{array}{|l|}
\hline q, x q, F_{1}(x) q, F_{2}(x) q \\
q, x q, F(x) q, y q \\
\hline
\end{array}
$$

The remaining arbitrary constants and arbitrary functions standing in this table cannot be took away; one can easily convince oneself of that in each individual case. As a result, every type of one-, two-, three- and fourterm group of the plane is present as only one representative in our table, in the main whole.

## Chapter 4.

## Classification of the Finite Continuous Groups of Point Transformations of the Plane.

In the preceding chapter, we at first separated the primitive groups of the plane from the imprimitive ones. We demonstrated that there are only three different types of primitive groups and we set up a representative for each one of these types; however, we did not yet succeed do determine how many types of imprimitive groups there are, and we only know that for every type of imprimitive group, we possess at least one representative. It is because we have not achieved a real classification of the imprimitive groups, but only a distribution of these groups in categories which are selected in such a way that one imprimitive group can very well belong simultaneously to two of our categories.

In fact, we started from the assumption that every imprimitive group of the plane leaves invariant one family of $\infty^{1}$ curves in any case. Amongst the families of $\infty^{1}$ curves invariant by the group, we then selected any family and we reckoned the group among the first, second, third or fourth category according to whether it transformed the curves of the concerned family in zero-, one-, two- or three-terms. But now, when an imprimitive group leaves invariant two distinct families of $\infty^{1}$ curves, it can very well happen that for instance it transforms the curves of one family in oneterm, and the curves of the second family in two-terms, so that it not only belongs to the second category, but also to the third.

Consequently, we must look around for a principle of classification [NACH EINEM EINTHEILUNGSGRUNDE UMSEHEN] which would enable us to distribute in classes the imprimitive groups in such a way that every imprimitive group belongs to one, but only one of these classes. Such a principle of classification offers itself as the number of families of $\infty^{1}$ invariant curves that accompany the group. Thus, we at first seek how many families of $\infty^{1}$ curves can remain invariant by an imprimitive group of the plane.

## § 15.

Let:

$$
X_{k} f=\xi_{k}(x, y) p+\eta_{k}(x, y) q \quad(k=1 \cdots r)
$$

be an $r$-term imprimitive group and let $x_{0}, y_{0}$ be a point in general position. We form the linear homogeneous group defined on p . 88:

$$
\begin{gathered}
\mathcal{Y}_{k}=\left(\lambda_{k} x^{\prime}+\mu_{k} y^{\prime}\right) p^{\prime}+\left(\nu_{k} x^{\prime}+\rho_{k} y^{\prime}\right) q^{\prime} \\
(k=1 \cdots r-m)
\end{gathered}
$$

that indicates in which way the $\infty^{1}$ line-elements $x^{\prime}: y^{\prime}$ passing through the point $x_{0}, y_{0}$ are transformed, as soon as one only takes those transformations of the group: $X_{1} f, \ldots, X_{r} f$ which leave untouched this point.

Since we have assumed the group: $X_{1} f, \ldots, X_{r} f$ as imprimitive, the linear homogeneous group: $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ leaves at rest at least one line-element $x^{\prime}: y^{\prime}$ through the point $x_{0}, y_{0}$ (cf. p. 87). But according to the developments on p. 81 sq., different cases are still possible. Indeed, the group: $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ can leave untouched either one line-element, or two separate line-elements, or lastly every line-element through the point $x_{0}, y_{0}$; if there is only a single invariant line-element, then it can still specially happen the case that this line-element counts doubly, hence that it consists of two collapsing line-elements.

By translating to the group: $X_{1} f, \ldots, X_{r} f$ these different conceivable cases for the group $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$, we obtain what follows:

If the group: $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ fixes only one line-element through the point $x_{0}, y_{0}$, then the group: $X_{1} f, \ldots, X_{r} f$ leaves invariant only one ordinary first order differential equation:

$$
\alpha(x, y) d y-\beta(x, y) d x=0
$$

and as a result also, only a single family of $\infty^{1}$ curves $\varphi(x, y)=$ const. If especially this line element counts doubly, then the invariant differential equation and the invariant family of curves must be considered as doubly counting; the group: $X_{1} f, \ldots, X_{r} f$ then leaves at rest two coinciding, or if one want, two infinitely close families of $\infty^{1}$ curves.

When the group: $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ holds fixed two and only two separate line-elements through the point: $x_{0}, y_{0}$, then the group: $X_{1} f, \ldots, X_{r} f$ leaves invariant exactly two different ordinary differential equations of first order and hence also exactly two different families of $\infty^{1}$ curves.

Lastly, if the group: $\mathfrak{Y}_{1} f, \ldots, \mathfrak{Y}_{r-m} f$ leaves untouched every individual line-element that goes through the point $x_{0}, y_{0}$, then two cases have to be distinguished, according to the group: $X_{1} f, \ldots, X_{r} f$ being transitive or not. In the first case, each one of the $\infty^{1}$ invariant line-elements takes exactly $\infty^{2}$ positions by the prolonged group: $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$ (see p. 86), and the totality of these line elements visibly determines a first order differential equation: $\alpha(x, y) d y-\beta(x, y) d x$ invariant by the group:
$X_{1} f, \ldots, X_{r} f$; as a result, there are in total $\infty^{1}$ invariant first order differential equations:

$$
\Phi\left(x, y, \frac{d y}{d x}\right)=\text { const. }
$$

invariant by the group: $X_{1} f, \ldots, X_{r} f$, and in consequence of that, also exactly $\infty^{1}$ different invariant families of $\infty^{1}$ curves. In the second case however, each one of the $\infty^{1}$ invariant line-elements takes only $\infty^{1}$ different positions by the group: $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$, hence the $\infty^{3}$ line-elements of the plane are arranged in $\infty^{2}$ invariant families that are represented by two equations of the form:

$$
\chi\left(x, y, \frac{y^{\prime}}{x^{\prime}}\right)=\text { const., } \quad \psi\left(x, y, \frac{y^{\prime}}{x^{\prime}}\right)=\text { const.; }
$$

here, the two functions $\chi$ and $\psi$ are certainly not both free of $\frac{y^{\prime}}{x^{\prime}}$, since otherwise the group: $X_{1} f, \ldots, X_{r} f$ would leave untouched every point of the plane. In this case, there are $\infty^{\infty}$ different first order differential equations invariant by the group: $X_{1} f, \ldots, X_{r} f$, and hence also $\infty^{\infty}$ different invariant families of $\infty^{1}$ curves; as one easily sees, the invariant differential equations in question are represented by an equation of the form:

$$
\Omega\left(\chi\left(x, y, \frac{d y}{d x}\right), \psi\left(x, y, \frac{d y}{d x}\right)\right)=0
$$

where the function $\Omega$ is absolutely arbitrary and only chosen in such a way that it is not free of $\frac{d y}{d x}$.

With this, all the possible cases are exhausted. As a result, there are four different classes of imprimitive groups of the plane and of these four, the first one yet decomposes in two subclasses. These are the following:
I) An individual invariant family of $\infty^{1}$ curves.
a. This family of curves counts once.
b. This family of curves counts twice.
II) Two different invariant families of $\infty^{1}$ curves.
III) $\infty^{1}$ different invariant families of $\infty^{1}$ curves.
IV) $\infty^{\infty}$ different invariant families of $\infty^{1}$ curves.

However, we do not want now to turn to applying straight this gained classification of the imprimitive groups of the plane, to the groups found in the preceding chapter. Rather, we want to settle, once again and in a pure analytic way, the question whether there are families of $\infty^{1}$ curves which can remain invariant by a group of the plane. We do this, because
the settlement given above of this question was based on conceptual considerations which perhaps are ?? too shortly expressed ?? for ?? a number of readers, and also because on the new way, we can really write down the first order differential equations that these families of curves satisfy, and this is actually desirable.

## $\S 16$.

We imagine that an $r$-term group of the plane:

$$
X_{k} f=\xi_{k}(x, y) p+\eta_{k}(x, y) q \quad(k=1 \cdots r)
$$

is presented, and we ask for all families of $\infty^{1}$ curves invariant by this group, or, what amounts to the same (see p. 85), for all equations of the form: $\alpha(x, y) y^{\prime}-\beta(x, y) x^{\prime}=0$ which remain invariant by the prolonged group:

$$
X_{k}^{\prime} f=\xi_{k} p+\eta_{k} q+\xi_{k}^{\prime} p+\eta_{k}^{\prime} q \quad(k=1 \cdots r)
$$

Each one of the sought equations admits, aside from the infinitesimal transformations $X_{k}^{\prime} f$, still obviously also the following:

$$
U f=x^{\prime} p^{\prime}+y^{\prime} q^{\prime}
$$

Now, as one easily convinces oneself, all the expressions: $\left[X_{k}^{\prime}, U\right]$ vanish identically, but on the other hand, $U f$ surely cannot be linearly deduced from $x_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$, since otherwise, it should arise by prolongation from an infinitesimal point transformation: $\xi(x, y) p+\eta(x, y) q$, which is visibly not the case. As a result, the $r+1$ infinitesimal transformations:

$$
\begin{equation*}
X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f, U f \tag{1}
\end{equation*}
$$

are independent of each other and they engender an $(r+1)$-term group in the variables $x, y, x^{\prime}, y^{\prime}$. But our problem of determining all equations of the form: $\alpha(x, y) y^{\prime}-\beta(x, y) x^{\prime}=0$ that remain invariant by the group: $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f$ can also at this point be expressed as follows: to determine equations in the variables $x, y, x^{\prime}, y^{\prime}$ not all free of $x^{\prime}$ and $y^{\prime}$ which are admitted by the group (1). In this new form, our problem can be settled without difficulty on the basis of the developments of the Chap. 14 in Vol. I [here: ??].

We start by considering the case that the $(r+1)$-term group (1) in the variables $x, y, x^{\prime}, y^{\prime}$ is transitive. According to the rules of the mentioned chapter, we then have to examine whether all four-by-four determinants of
the matrix:

$$
\left|\begin{array}{cccc}
\xi_{1} & \eta_{1} & \xi_{1}^{\prime} & \eta_{1}^{\prime}  \tag{2}\\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\xi_{r} & \eta_{r} & \xi_{r}^{\prime} & \eta_{r}^{\prime} \\
0 & 0 & x^{\prime} & y^{\prime}
\end{array}\right|
$$

can vanish, by virtue of an equation between $x, y, x^{\prime}, y^{\prime}$ which is not free of $x^{\prime}$ and $y^{\prime}$. Now, the four-by-four determinants in question are all complete homogeneous functions of second degree in $x^{\prime}$ and $y^{\prime}$, so everything comes down to determining whether these complete homogeneous functions possess one common factor which is not free of $x^{\prime}$ and $y^{\prime}$. If there does not exist such a common factor, then the group: $X_{1} f, \ldots, X_{r} f$ leaves invariant absolutely no family of $\infty^{1}$ curves: it is primitive. On the contrary, if there is such a common factor, then different cases are to be distinguished. Indeed, it can firstly happen that this common factor is just linear in $x^{\prime}$ and $y^{\prime}$, in which case the group: $X_{1} f, \ldots, X_{r} f$ leaves invariant only a single family of $\infty^{1}$ curves, counting once. But secondly, the common factor can be of second degree in $x^{\prime}$ and $y^{\prime}$; then if it is divisible by the square of a linear homogeneous function of $x^{\prime}$ and $y^{\prime}$, the group: $X_{1} f, \ldots, X_{r} f$ leaves invariant just one, and only one family of $\infty^{1}$ curves, counting twice; if on the other hand, it is not divisible by such a square, then the group: $X_{1} f, \ldots, X_{r} f$ leaves invariant two different families of $\infty^{1}$ curves.

In order to illustrate these developments by an example, we want to apply them to the group [15] on p. 105:

$$
\begin{equation*}
p, \quad 2 x p+y q, \quad x^{2} p+x y q \tag{3}
\end{equation*}
$$

For this group, the matrix (2) goes to the determinant:

$$
\left|\begin{array}{cccc}
1 & 0 & 0 & 0 \\
2 x & y & 2 x^{\prime} & y^{\prime} \\
x^{2} & x y & 2 x x^{\prime} & x y^{\prime}+x^{\prime} y \\
0 & 0 & x^{\prime} & y^{\prime}
\end{array}\right|=-y^{2} x^{\prime 2}
$$

whence: $x^{\prime}=0$ is the only invariant equation of the required constitution and $x=$ const. the only family of curves invariant by the group (3), but this family of curves does visibly count twice.

We now come to the case where the group (1) in the variables $x, y, x^{\prime}, y^{\prime}$ is intransitive, so that the $r+1$ equations:

$$
\begin{equation*}
X_{1}^{\prime} f=0, \cdots, X_{r}^{\prime} f=0, U f=0 \tag{4}
\end{equation*}
$$

possess at least one joint solution.

Firstly, assume at the least that the group: $X_{1} f, \ldots, X_{r} f$ in the variables $x, y$ is transitive; then the equations: $X_{1}^{\prime} f=0, \ldots, X_{r}^{\prime} f=0$ can be solved with respect to $p$ and $q$, therefore the equations (4) are solvable with respect to three of the differential quotients and they possess a joint solution which necessarily has the form: $\chi\left(x, y, \frac{y^{\prime}}{x^{\prime}}\right)$ and cannot be free of $\frac{y^{\prime}}{x^{\prime}}$. As a result, the $\infty^{1}$ first order differential equations:

$$
\chi\left(x, y, \frac{d y}{d x}\right)=\text { const. }
$$

all remain invariant by the group: $X_{1} f, \ldots, X_{r} f$, and at the same time, they are the only such equations which remain invariant.

Secondly, assume that the group: $X_{1} f, \ldots, X_{r} f$ in the variables $x, y$ itself is intransitive, but that the number $r$ is larger than 1 . Then there are relations of the form:

$$
X_{k} f=\rho_{k}(x, y) \cdot X_{1} f \quad(k=2,3 \cdots r),
$$

so it ensues:

$$
\begin{gathered}
X_{k}^{\prime} f=\rho_{k} X_{1}^{\prime} f+\left(x^{\prime} \frac{\partial \rho_{k}}{\partial x}+y^{\prime} \frac{\partial \rho_{k}}{\partial y}\right)\left(\xi_{1} p^{\prime}+\eta_{1} q^{\prime}\right) \\
(k=2,3 \cdots r) .
\end{gathered}
$$

Now, because no expression:

$$
x^{\prime} \frac{\partial \rho_{k}}{\partial x}+y^{\prime} \frac{\partial \rho_{k}}{\partial y} \quad(k=2 \cdots r)
$$

can vanish identically - otherwise the infinitesimal transformations: $X_{1} f, \ldots, X_{r} f$ would not at all be independent of each other - , then the $r+1$ equations (4) can be replaced by the three equations:

$$
X_{1}^{\prime} f=0, \quad \xi_{1} p^{\prime}+\eta_{1} q^{\prime}=0, \quad x^{\prime} p^{\prime}+y^{\prime} q^{\prime}=0
$$

that are independent of each other; but since the expression: $\xi_{1} y^{\prime}-\eta_{1} x^{\prime}$ is not identically zero, these equations are equivalent to the equations:

$$
X_{1} f=0, \quad p^{\prime}=0, \quad q^{\prime}=0
$$

In other words: under the assumptions made, the equations (4) have one and only one joint solution: $\varphi(x, y)$ free of $x^{\prime}$ and $y^{\prime}$, which is nothing else but the invariant of the intransitive group: $X_{1} f, \ldots, X_{r} f$ in $x, y$. It follows from this that every equation not free of $x^{\prime}$ and $y^{\prime}$ which is admitted by the group: $X_{1}^{\prime} f, \ldots, X_{r}^{\prime} f, U f$ must be obtained by setting equal to zero all three-by-three determinants of the matrix (2). If one thinks furthermore
|that under the assumptions made, there are relations of the form:

$$
\begin{aligned}
\xi_{k}=\rho_{k} \cdot \xi_{1}, & \xi_{k}^{\prime}=\rho_{k} \xi_{1}^{\prime}+\xi_{1}\left(x^{\prime} \frac{\partial \rho_{k}}{\partial x}+y^{\prime} \frac{\partial \rho_{k}}{\partial y}\right) \\
\eta_{k}=\rho_{k} \cdot \eta_{1}, & \eta_{k}^{\prime}=\rho_{k} \eta_{1}^{\prime}+\eta_{1}\left(x^{\prime} \frac{\partial \rho_{k}}{\partial x}+y^{\prime} \frac{\partial \rho_{k}}{\partial y}\right) \\
& (k=2 \cdots r),
\end{aligned}
$$

so one realizes that by setting equal to zero all three-by-three determinants of the matrix (2), the following equations come out:

$$
\left(\xi_{1} y^{\prime}-\eta_{1} x^{\prime}\right)\left(x^{\prime} \frac{\partial \rho_{k}}{\partial x}+y^{\prime} \frac{\partial \rho_{k}}{\partial y}\right)=0 \quad(k=2,3 \cdots r)
$$

here, the nonzero factors which depend only on $x$ and $y$ are already left out. As a result, one first order differential equation invariant by the group $X_{1} f, \ldots, X_{r} f$ is:

$$
\xi_{1} d y-\eta_{1} d x=0 ;
$$

the $\infty^{1}$ integral curves of this differential equation are visibly represented by the equation: $\varphi(x, y)=$ const., and they all remain invariant. Now, whether there still is a second invariant differential equation, this depends on the behaviour of the matrix:

$$
\left|\begin{array}{lll}
\frac{\partial \rho_{2}}{\partial x} & \cdots & \frac{\partial \rho_{r}}{\partial x}  \tag{5}\\
\frac{\partial \rho_{2}}{\partial y} & \cdots & \frac{\partial \rho_{r}}{\partial y}
\end{array}\right| .
$$

If not all two-by-two determinants of this matrix vanish identically, then: $\xi_{1} d y-\eta_{1} d x=0$ is the only invariant differential equation by the group: $X_{1} f, \ldots, X_{r} f$, and to be precise, this differential equation has to be counted once. If on the contrary, all the said two-by-two determinants vanish, then two cases can occur. Indeed, either there is, aside from: $\xi_{1} d y-\eta_{1} d x=0$, yet a second, different first order differential equation, namely the following one:

$$
\frac{\partial \rho_{2}}{\partial x} d x+\frac{\partial \rho_{2}}{\partial y} d y=0
$$

or the invariant differential equation: $\xi_{1} d y-\eta_{1} d x=0$ has to be counted twice.

Lastly, the case $r=1$ still has to be dealt with. Then visibly, the equations (4) have two independent solutions in common, of which as the first one, we can even choose the invariant $\varphi(x, y)$ of the one-term group: $X_{1} f$, while the second one cannot be free of $x^{\prime}$ and $y^{\prime}$, hence has the form: $\chi\left(x, y, \frac{y^{\prime}}{x^{\prime}}\right)$. Consequently, all the first order differential equations that are invariant by the one-term group: $X_{1} f$ are represented by an equation of
the form:

$$
\chi\left(x, y, \frac{d y}{d x}\right)=\Omega(\varphi(x, y))
$$

where $\Omega$ denotes an arbitrary function.
With these words, all the results that we have gained in the preceding paragraph by means of conceptual considerations are derived in an analytic way, and at the same time, are completed in a not inessential way.

## § 17.

Now, we pass to the determination, for each imprimitive group found in the preceding chapter, of the families of $\infty^{1}$ curves that are invariant by it. Here, we could employ the general method developed just now, since it provides all first order differential equations invariant by a given group and as a result also, by integration, all invariant families of $\infty^{1}$ curves. But since all the groups that we have to consider are presented here in simple normal form, we prefer to take another, somehow shorter path.

If the family of curves: $\varphi(x, y)=$ const. admits an infinitesimal transformation $X f$, then this can occur in essentially two different ways. Either every individual curve of the family remains invariant, so the expression: $X \varphi$ vanishes identically, or the curves of the family are exchanged one another, so that: $X \varphi=\Omega(\varphi)$, where the function is not identically zero. In the second case, by introducing:

$$
\int \frac{d \varphi}{\Omega(\varphi)}
$$

as new $\varphi$, one can insure that $X \varphi$ has the value 1 .
From this, it follows that every family of curves: $\varphi(x, y)=$ const. which remains invariant by the infinitesimal transformation: $q$, satisfies either the equation: $\varphi^{\prime}(y)=0$ or the equation: $\varphi^{\prime}(y)=1$; consequently, aside from admitting the family: $x=$ const., the infinitesimal transformation $q$ also admits every family of the form: $y+\omega(x)=$ const., where it is understood that $\omega(x)$ is an arbitrary function of $x$. But as a result, all families of $\infty^{1}$ curves invariant by $q$ are found.

If one family of curves distinct from the family: $x=$ const. shall admit, apart from $q$, also another transformation of the form: $F(x) q$, then it must have the form: $y+\omega(x)=$ const. and moreover, the expression:

$$
F(x) \cdot \frac{\partial}{\partial y}(y+\omega(x))=F(x)
$$

must be a function of: $y+\omega(x)$ only. But since this cannot be true, it follows that $x=$ const. is the only family of $\infty^{1}$ curves which simultaneously admits the two infinitesimal transformations: $q, F(x) q$, and more generally, the only one which admits two independent infinitesimal transformations of the form: $F_{1}(x) q, F_{2}(x) q$.

The two infinitesimal transformations: $q$ and $y q$ both leave invariant the family of curves: $x=$ const. Every other family invariant by them two must have the form: $y+\omega(x)=$ const., and in addition, the expression:

$$
y \cdot \frac{\partial}{\partial y}(y+\omega(x))=y
$$

must be a function of $y+\omega(x)$ alone. Consequently, $\omega(x)$ is a constant and then: $x=$ const. and $y=$ const. are the only families of $\infty^{1}$ curves that are simultaneously invariant by $q$ and $y q$. In the same way, aside from: $x=$ const., there is yet only the family: $\frac{y}{F(x)}=$ const. which remains invariant by the two transformations: $F(x) q$, yq.

If the family: $y+\omega(x)=$ const. shall admit the infinitesimal transformation $p$, then $\omega^{\prime}(x)$ must be a function of $y+\omega(x)$ alone and thus be a constant. The equation:

$$
a x+b y=\text { const. }
$$

with the arbitrary parameter: $a: b$ therefore represents all families of $\infty^{1}$ curves invariant by $p$ and $q$.

Finally, there still remains a point to be taken care of. All imprimitive groups of the preceding chapter leave invariant the family of curves: $x=$ const. Now, if for a given group: $X_{1} f, \ldots, X_{r} f$, aside from the family: $x=$ const., there is no other family of invariant curves, then always, the question whether the family: $x=$ const. counts once or twice remains open. How does one settle this?

So, let: $x=$ const. be the only family of $\infty^{1}$ curves which remains invariant by a given $r$-term group: $X_{1} f, \ldots, X_{r} f$; then all transformations of this group which leave invariant a point $x_{0}, y_{0}$ in general position will transform the $\infty^{1}$ line elements through this point in such a way that only the line-element $x^{\prime}=0$ remains invariant, but no other one. Now especially, for the family of curves: $x=$ const. to remain doubly invariant, it is necessary and sufficient that the line-element: $x^{\prime}=0$ remains doubly invariant, but (cf. Chap. 2, p. 83) this happens if and only if the linear homogeneous group defined on p. 88:

$$
\left(\lambda_{k} x^{\prime}+\mu_{k} y^{\prime}\right) p^{\prime}+\left(\nu_{k} x^{\prime}+\rho_{k} y^{\prime}\right) q^{\prime} \quad(k=1 \cdots r-m)
$$

which is associated to the group: $X_{1} f, \ldots, X_{r} f$, has one of the two forms:

$$
x^{\prime} q^{\prime}+\alpha\left(x^{\prime} p^{\prime}+y^{\prime} q^{\prime}\right) ; \quad \quad x^{\prime} q^{\prime}, x^{\prime} p^{\prime}+y^{\prime} q^{\prime}
$$

where $\alpha$ means a finite, arbitrary constant. Consequently, the group: $X_{1} f, \ldots, X_{r} f$ will leave doubly invariant the family: $x=$ const. if and only if all its infinitesimal transformations, whose power series expansion, with respect to the powers of $x-x_{0}$ and $y-y_{0}$, begin with terms of first order, can be linearly reduced either to the single form:

$$
\left(x-x_{0}\right) q+\alpha\left\{\left(x-x_{0}\right) p+\left(y-y_{0}\right) q\right\}+\cdots
$$

or to one of the two forms:

$$
\left(x-x_{0}\right) q+\cdots, \quad\left(x-x_{0}\right) p+\left(y-y_{0}\right) q+\cdots
$$

here, the form of the terms of second or higher order which appear in these infinitesimal transformations is completely disregarded.

After these preliminary remarks, we want to go through, one by one, all the groups found in the preceding chapter and to determine the families of $\infty^{1}$ curves invariant by them. On this occasion, we also make use of the simplifications introduced in the § 13.

At first, for what concerns the group [1'] on p. 112, the one-term group:

$$
q
$$

leaves invariant $\infty^{\infty}$ different families of curves, namely aside from the family: $x=$ const., yet every family of the form: $y+\omega(x)=$ const.; by contrast, the groups:

$$
q, \quad x q, \quad F_{1}(x) q, \ldots, F_{r}(x) q \quad(r \geqslant 0)
$$

leave invariant only the single family: $x=$ const., but as one easily sees, it leaves it doubly invariant.

Amongst the groups [2'] on p. 113, the two-term one:

$$
q, \quad y q
$$

leaves invariant two families, namely: $x=$ const. and $y=$ const.; but the remainding ones:

$$
q, \quad x q, \quad F_{1}(x) q, \ldots, F_{r}(x) q, \quad y q \quad(r \geqslant 0)
$$

leave invariant only the family: $x=$ const. and in fact, simply invariant.
For the group [3] on p. 98:

$$
q, y q, y^{2} q
$$

there occur only the two invariant families of curves:

$$
x=\text { const. } \quad \text { and } \quad y=\text { const. }
$$

The one-term group $p$ on p .99 is equivalent to the group $q$ and hence needs not be specially taken into consideration.

We come to the groups [5] on p. 100 which have the form:

$$
\begin{gathered}
e^{\alpha_{k} x} q, \quad x e^{\alpha_{k} x} q, \ldots, \quad x^{m_{k}} e^{\alpha_{k} x} q, \quad p \\
(k=1,2 \cdots l)
\end{gathered}
$$

If such a group has more than two parameters, then it leave invariant only the family: $x=$ const., which, however, counts twice. On the other hand, if it has only two parameters, then things are completely different. Indeed, the group has the form: $e^{\alpha x} q, p$, where $\alpha$ either vanishes or may be set equal to 1 ( $c f . \mathrm{p} .113$ ). In the first case, we have the group:

$$
p, \quad q
$$

with the $\infty^{1}$ invariant families of curves: $a x+b y=$ const. In the second case we have the group: $e^{x} q, p$; we firstly bring it to the form: $q, p-y q$ by introducing $y e^{-x}$ as new $y$, and lastly we introduce $e^{-x}$ as new $x$ to obtain the group:

$$
q, \quad x p+y q .
$$

For this new group, we have at first the invariant family of curves: $x=$ const.; every other invariant family must have the form: $y+\omega(x)=$ const., and to be be precise, the expression:

$$
x \frac{\partial}{\partial x}(y+\omega)+y \frac{\partial}{\partial y}(y+\omega)=y+x \omega^{\prime}(x)
$$

must be a function of: $y+\omega(x)$ alone, so that $\omega(x)$ has the form: $\omega(x)=$ $a x+c$ and hence each one of the family of $\infty^{1}$ curves: $a x+b y=$ const. remains invariant, also here.

If the group [6'] on p. 113:

$$
\begin{gathered}
q, \quad x q, \ldots, x^{m} q, e^{\alpha_{k} x} q, \quad x e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k} x} q, y q, \quad p \\
(k=1,2 \cdots l ; l \geqslant 0)
\end{gathered}
$$

has more than three parameters, then there is only the single invariant family: $x=$ const., which besides counts just once; if on the contrary it has only three parameters, then it possesses the form:

$$
q, \quad y q, \quad p
$$

and it leaves invariant the two families: $x=$ const., $y=$ const.

The group [7] on p. 101:

$$
q, y q, y^{2} q, p
$$

leaves invariant only the two families: $x=$ const. and: $y=$ const.
The groups: $p, x p$ and: $p, x p+q$ ([8] on p .102 and [9] on p .102 ) can be left out consideration, since the first is equivalent to the group: $q, y q$, the second to the group: $q, x p+y q$.

When $r$ is $>0$, aside from: $x=$ const., there is no family of curves which remains invariant by the group [10] on p. 103:

$$
q, \quad x q, \ldots, x^{r} q, \quad p, x p+c y q
$$

and in fact, the family: $x=$ const. is to be counted once when $c$ is $\neq 1$, whereas it is to be counted twice in the case: $c=1$. If $r=0$, we have the group:

$$
q, \quad p, \quad x p+c y q
$$

for which, because of the presence of $p$ and $q$, only some invariant families of the form: $a x+b y=$ const. can appear. Which ones are really invariant amongst these families, this can be determined from the condition that:

$$
x \frac{\partial}{\partial x}(a x+b y)+c y \frac{\partial}{\partial y}(a x+b y)=a x+c b y
$$

must be a function of $a x+b y$ only. One realizes at once that in the case: $c=1$, each one of the $\infty^{1}$ families: $a x+b y=$ const. remains invariant, while in the case: $c \neq 1$, only the two families: $x=$ const. and $y=$ const. are invariant.

For the groups [11] on p. 103:

$$
q, x q, \ldots, x^{r-1} q, p, x p+\left(r y+x^{r}\right) q \quad(r>0)
$$

only the family: $x=$ const. remains invariant, and to be precise, doubly invariant in general, but only once when $r=1$.

The groups [12] on p. 104:

$$
q, x q, \ldots, x^{r} q, y q, \quad p, x p
$$

leave invariant, when $r$ is $>0$, only the family $x=$ const., but when $r=0$, we then have the group:

$$
q, y q, \quad p, x p
$$

by which, in addition, yet the family: $y=$ const. remains invariant.
The group [13] on p. 105:

$$
q, y q, y^{2} q, p, x p
$$

gives only the two families: $x=$ const. and $y=$ const..

The group [14]: $p, x p, x^{2} p$ is equivalent to the group: $q, y q, y^{2} q$ and hence is removed.

About the group [15] on p. 105:

$$
p, \quad 2 x p+y q, \quad x^{2} p+x y q
$$

we already saw on p .120 that it leaves invariant only the family: $x=$ const., but this family counts twice.

For the group [16] on p. 106:

$$
p, \quad x p+y q, \quad x^{2} p+\left(2 x y+y^{2}\right) q
$$

only the families of curves of the form: $a x+b y=$ const. can remain invariant, since the subgroup: $p, x p+y q$ leaves invariant all these families of curves, but also only them (cf. p. 126). Furthermore, the expression:

$$
x^{2} \frac{\partial}{\partial x}(a x+b y)+\left(2 x y+y^{2}\right) \frac{\partial}{\partial y}(a x+b y)=a x^{2}+2 b x y+b y^{2}
$$

must be a function of $a x+b y$ alone, which can be the case when either $b=0$ or $a=b$. Consequently, the two families of curves that are invariant by our group are: $x=$ const. and: $x+y=$ const. By still introducing $x+y$ as new $y$, we obtain from our group the following:

$$
p+q, \quad x p+y q, \quad x^{2} p+y^{2} q,
$$

with the two invariant families of curves: $x=$ const. and $y=$ const.
The group [17] on p. 107:

$$
q, \quad x q, \ldots, x^{r} q, \quad 2 x p+r y q, \quad x^{2} p+r x y q
$$

gives, when $r>0$, only the family: $x=$ const., and to be precise, doubly counting in the case $r=2$, but only once otherwise. If on the other hand $r=0$, then the group is equivalent to the group: $q, y q, y^{2} q, p$ and hence is left out.

To the group [18] on p. 107:

$$
y q, \quad p, x p, \quad x^{2} p+x y q
$$

there belongs only the single invariant family: $x=$ const., because by virtue of the presence of $p$ and $x p$, there could yet come into question the family: $y=$ const., which, however, does not admit the infinitesimal transformation: $x^{2} p+x y q$. The single invariant family: $x=$ const. counts once.

The group [19] on p. 108:

$$
q, x q, \ldots, x^{r} q, y q, \quad p, x p, x^{2} p+r x y q
$$

gives, when $r>0$, only the invariant family: $x=$ const., which counts ounce. If $r=0$, we have the group: $q, y q, p, x p, x^{2} p$ which is equivalent to the group: $q, y q, y^{2} q, p, x p$ and hence has not to be taken into consideration.

Finally, to the group [20] on p. 109:

$$
q, y q, y^{2} q, p, x p, x^{2} p
$$

belong the two invariant families: $x=$ const. and $y=$ const.
§ 18.
At present, we can at last turn to the drawing up of the table for the individual groups of the plane. For the imprimitive groups we naturally apply the classification stated in $\S 15$. Thus we obtain the

Theorem 6. Every finite continuous group of point transformations of the plane $x, y$ is equivalent, through a point transformation, to one and in general, to only one of the groups listed below:
A) Primitive groups:
$p, q, x q, x p-y q, \quad y p, x p+y q, x^{2} p+x y q, x y p+y^{2} q$

$$
p, \quad q, x q, x p-y q, \quad y p, x p+y q
$$

$$
p, q, x q, x p-y q, y p
$$

B) Imprimitive groups:
I) Groups with a single invariant family of $\infty^{1}$ curves.
a) The invariant family counts only once.

$$
q, x q, \quad p, 2 x p+y q, x^{2} p+x y q
$$



$$
q, x q, \ldots, x^{r} q, y q, \quad p, x p, x^{2} p+r x y q
$$

$$
y q, \quad p, x p, x^{2} p+x y q
$$

$$
\begin{aligned}
& q, x q, \ldots, x^{r} q, y q, \quad p, x p \\
& \text { ( } r>0 \text { ) } \\
& q, x q, \ldots, x^{r} q, \quad p, x p+c y q \\
& (r>0 ; c \neq 1) \\
& q, x q, \ldots, x^{r-1} q, \quad p, x p+\left(r y+x^{r}\right) q \\
& (r>1) \\
& \begin{array}{c}
q, x q, \ldots, x^{m} q, \quad e^{\alpha_{k} x} q, x e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k} x} q, y q, \quad p \\
\left(k=1,2 \cdots l ; l \geqslant 0 ; l+m+m_{1}+\cdots+m_{l}>0 ; \alpha_{1}=1\right)
\end{array} \\
& q, \quad x q, F_{1}(x) q, \ldots, F_{r}(x) q, y q \\
& (r \geqslant 0) \\
& \text { b) The invariant family counts twice. } \\
& q, \quad x q, \quad x^{2} q, \quad p, x p+y q, \quad x^{2} p+2 x y q \\
& p, \quad 2 x p+y q, \quad x^{2} p+x y q \\
& \begin{array}{c}
\mid q, x q, \ldots, x^{r} q, \quad p, x p+y q \\
(r>0) \\
\hline
\end{array} \\
& q, \quad p, \quad x p+(x+y) q \\
& \begin{array}{|c|}
e^{\alpha_{k} x} q, x e^{\alpha_{k} x} q, \ldots, x^{m_{k}} e^{\alpha_{k} x} q, \quad p \\
\left(\alpha_{1}\left(\alpha_{1}-1\right)=0 ; k=1,2 \cdots l ; l>0 ; l+m_{1}+\cdots+m_{l}>1\right)
\end{array} \\
& q, \quad x q, F_{1}(x) q, \ldots, F_{r}(x) q \\
& (r \geqslant 0) \\
& \text { II) Groups with two invariant families of } \infty^{1} \text { curves. }
\end{aligned}
$$

$$
\begin{aligned}
& q, y q, y^{2} q, p, x p, x^{2} p \quad p+q, x p+y q, x^{2} p+y^{2} q \\
& q, y q, y^{2} q, p, x p \quad q, y q, y^{2} q, p \quad q, y q, y^{2} q \\
& q, y q, p, x p \quad q, p, x p+c y q \quad(c \neq 0,1) \\
& q, y q, p \quad q, y q \\
& \text { III) Groups with } \infty^{1} \text { invariant families of } \infty^{1} \text { curves. } \\
& \text { IV) Groups with } \infty^{\infty} \text { invariant families of } \infty^{1} \text { curves. }
\end{aligned}
$$

In this table, the groups which leave invariant only one family of curves are ordered in such a way that the groups which transform in three terms the invariant family of curves do stand first, then the ones which transform this family in two terms do follow, and so on. The ordering of the groups which leave invariant two families of curves is similar.

Of the arbitrary parameters which appear in our table, none can be took away, for one easily convinces oneself that in each individual case, already the composition of the concerned group contains this parameter and that it cannot be removed from the composition. In the same way, none of the occuring arbitrary function can be eliminated. As a result, every type of group of the plane takes place in our table only through one representative in the main whole.

# Theorie der Transformationsgruppen 

# Abschnitt I <br> Abschnitt III, Abtheilung I 

# Sophus LIE <br> Unter Mitwirkung von Friedrich Engel <br> Translation, writing and $\mathrm{IT}_{\mathrm{E}} \mathrm{X}$ principles 

Joël Merker

## Decomposition in parts:

I. Local Lie Transformation Groups (Abschnitt I) ........................................
II.Abschnitt III, Abtheilung I . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . II.
III. Amaldi's imprimitive Lie Algebras ............................................ III.

## GENERAL STRUCTURE OF THE TEXT

## Translated parts:

- \engellie and \stopengellie are declared in the preamble. When calling these commands, one should always put a blank line before, and also a blank line after.
- $\backslash$ fboxrule appears twice in the preamble: general: \fboxrule=0.47pt, and in factor for all Engel-Lie boxes: $\backslash$ fboxrule=1pt. In fact, one may adjust frame widths in the macro engellie declared in preamble.tex.
- Some useful commands are copied from the regularly updated file samples.tex.
- Footnotes are located inside the Engel-Lie boxes just after they are called:
\starnote\{Lie, Christiania 1874\}
- In order to adjust the height of the framed gained groups in the theorems:
$\backslash$ rule[-3pt]\{0pt\}\{11pt\}.
- Headings in engellie:
$\backslash$ HEAD\{The Complete Systems.\}\{
Volume I, \, \, \,Chapter 5, \, \, \,§§\, \, \,22, \, \,23, \, \,24.\}
- However, in the environment engellie, only one call (unfortunately) of $\backslash \operatorname{HEAD}\}$ is taken account of. One should therefore gather all the concerned paragraphs in one head in factor.
- Trick: one might update the headings as soon as a commentary cuts the environment engellie.
- The declaration of a heading always just precedes the declaration of a section.
- Sections in translated text (environment engellie):
$\backslash$ sectionengellie $\{\backslash S \backslash, \backslash, \backslash, 123$.
$\backslash$ label\{S-123\}
\nopagebreak
Thus, we consider local transformation equations. . .
-Microcomments: they appear in the translated text:
$\backslash$ microcomment $\{\{\backslash$ em i.e. $\}$ the two values $+c$ and $-c\}$
"Here, one has to become aware of the fact that two equally opposed values of $c$ [i.e. the two values $+c$ and $-c$ ] always produce two infinitesimal transformations ...".
"(Vol. I, Theor. 37, p. 197 [here: see p. 153])".


## Modernized text:

- $\backslash$ modernized and $\backslash$ stopmodernized.

Here is an excerpt of the modernized text. . .

- Headings in modernized text:
\HEAD\{First Order Scalar Partial Differential Equation\}\{
E. $\backslash, \backslash, \backslash, \backslash, \backslash$, Complete Systems of Partial Differential Equations\}
- Capital letters: always present in the headings and in the titles.
- Sections in modernized text:

\sectionmodernized\{Essential parameters\}

$\backslash$ label\{A-1\}
\nopagebreak

- Two lines section in modernized text:
$\backslash$ bigsectionmodernized\{Group Composition Axiom\}\{
And Fundamental Differential Equations\}
\label\{D-2\}
\nopagebreak
- Subsections in modernized text:

\subsectionmodernized\{Concept of local Lie group\}

- Use sometimes similar (sub)sections with a shorter preliminary spacing.
- Never any subsection just after a section.
- Almost no numbering.
- Always build a short abstract after the chapter title.

Comments inserted: Usually, comment should appear just after the end of an engellie paragraph. Sometimes the comment is anticipated, just before the very next concerned paragraph. LATEXmacro declared in the preamble:
$\backslash$ COMMENT\{Translation note\}\{Two continuous transformation groups which transform .
$\backslash$ stopCOMMENT\}

- Never jump line before closing a comment to insure good position of " $\triangleleft$ ".
$\triangleright$ Translation note. Two continuous transformation groups which transform one into the other by an invertible change of coordinates, .... $\triangleleft$
$\triangleright$ Explanation. $\triangleleft$
$\triangleright$ Concept of local Lie group. $\quad \triangleleft$
$\triangleright$ Notion of isomorphy. $\quad \triangleleft$


## Terminology:

\terminology\{independent infinitesimal transformations\}

## German words:

\deutsch\{Zusammensetzung\}.

## Mathematicians's names:

\names\{Kowalewsky\}.

## Footnotes:

- Restart at each Chapter:
$\backslash$ footnotetext $\{\backslash$ baselineskip $=0.37 \mathrm{~cm}$.\}
\setcounter\{footnote\}\{0\}
- Footnotes of engellie: "*)", ""**)".


## Labels and references internal to the whole text:

- \label, \pageref, \ref: Arguments should precisely be those of the German text, for instance: SATZ-1; $408 ; 123$.
- labels always try to include a minus sign "-".
- labels never use any capital letter, except for chapter names.

Bibliography: Incoherence with \thechapter; duplication of "Bibliography". In 12pt: -2.823 cm ; 11pt: -2.36 cm .

Index: Learn how to create.

## Provisory tracks:

- Problem of mathematical understanding:
?? Mathematics??
- Problem of translation:
"?? DASELBST ??".


## Worked out texts

## Complete list of $\mathrm{ET}_{\mathrm{E}} \mathrm{X}$ files:

- preamble.tex ............................................. . begin document
- engel-lie.tex ....................... . frontmatter and total document
- I-prologue.tex . . . . . . . . . . . . . . . . . . . . . . . . . . . . Prologue for Part II
- A.tex ........................................ essentiality of parameters
- B.tex ............................................ . transformation groups
- C.tex .................................................... . . one-term groups
- D.tex . ................................................. . . complete systems
- E.tex . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . complete systems
- F.tex . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . complete systems
- H.tex ................................................... invariant families
- L.tex . .................................................. . . . the adjoint group
- M.tex ..................................................... . . projective group
- N.tex . . . . . . . . . . . . . . . . rigidity of $\mathrm{SL}_{n}(\mathbb{C}), \mathrm{GL}_{n}(\mathbb{C})$ and $\mathrm{PGL}_{n}(\mathbb{C})$.
- II-prologue.tex . ................................. . . Prologue for Part II
- III-1.tex ............................ . . translation of Chapter 1, Vol III
- III-2.tex ............................. . . translation of Chapter 2, Vol III
- III-3.tex ............................ . . translation of Chapter 3, Vol III
- III-4.tex ........................... translation of Chapter 4, Vol III
- principles.tex ..................................... . . principles of writing
- references.tex ................................................... . . references
- glossary.tex . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . memory-glossary
- idioms.tex ................................................................. idioms
- index.tex .............................................. . . project of Index
- boites-exemples.sty


## Compilations

Doubling the files: 1.tex is the worked out file, while 1-compilation.tex is the compiled file, giving rise to the viewable file 1-compilation.dvi.

List of figures: Possibly to be done.
2000 Mathematics Subject Classification:
Primary: 22E05.
Secondary: 17B05, 22E10, 22E60, 34A30, 35A30, 58 J 90.
Title: is it suitable ?
CONVENTIONS: TYPOGRAPHY, TRANSLATION, ETC

- Attention: where to introduce majuscule conventions?
- Majuscule: ISOMORPH MIT = Isomorph with.
- Majuscule: ISOMORPHISMUS = Isomorphism.
- Majuscule: Combination = Combination.
- Majuscule: Relation = Relation.
- zeroth.
- line-element.
- $n$-FACH: $n$-times, $n$-fold.
- Chap. 1, Theor. 28, Prop. 1.
-*).
- BEREICH $\Rightarrow$ region (of a space).
- Gebiete $\Rightarrow$ domain (real or complex).
- $\gtrless$.
- Statement of a theorem.
- all the $\alpha$, all the $m_{\nu}$ and the $m_{\nu \pi}$, without 's.
- Theorem I. 30 .
- Finite continuous group.
- Punktcoordinaten $=$ point-coordinates.
- Gerade = straight line.
- LINIE = line.
- $d x: d y$.
- Label: Proposition-10-p-18.
- Projective spaces: $\mathbb{C P}^{n}, \mathbb{R P}^{n}, \mathbb{K} \mathbb{P}^{n}$.
- Expand but never develope in power series.
- Point in general position.
- due in the main whole to JACOBI and Clebsch.
- neighbourhood, behaviour: correct final text.

Bibliography

## Bibliography

[1] Ackerman, M.; Hermann, R.: Sophus Lie's 1880 Transformation Group paper, Math. Sci. Press, Brookline, Mass., 1975.
[2] Amaldi, U.: Contributo alla determinazione dei gruppi finiti dello spazio ordinario, Giornale di mathematiche di Battaglini per il progresso degle studi nelle universita italiane, I: 39 (1901), 273-316.
[3] Arnol'd, V.I.: Ordinary differential equations. Translated from the Russian and edited by R.A. Silverman, MIT Press, Cambridge, Mass.-London, 1978.
[4] Arnol'd, V.I.: Dynamical systems. I. Ordinary differential equations and smooth dynamical systems, Translated from the Russian. Edited by D. V. Anosov and V. I. Arnol'd. Encyclopaedia of Mathematical Sciences, 1. Springer-Verlag, Berlin, 1988. x+233 pp.
[5] Amaldi, U.: Contributo alla determinazione dei gruppi finiti dello spazio ordinario, Giornale di mathematiche di Battaglini per il progresso degle studi nelle universita italiane, II: 40 (1902), 105-141.
[6] Bianchi, L.: Lezioni sulla teoría dei gruppi finiti di trasformazioni, Enrico Spoerri Editore, Pisa, 1918.
[7] Bluman, G.W.; Kumei, S.: Symmetries and differential equations, Applied mathematical sciences, 81, Springer-Verlag, Berlin, 1989, xiv+412 pp.
[8] Bochnak, J.; Coste, M.; Roy, M.-F.: Géométrie algébrique réelle, Ergenisse der Mathematik und ihrer Grenzgebiete (3), 12. Springer-Verlag, Berlin, x+373 pp.
[9] Campbell, J.E.: Introductory treatise on Lie's theory of finite continuous transformation groups, The Clarendon Press, Oxford, 1903.
[10] Cartan, É.: Sur les variétés à connexion projective, Bull. Soc. Math. France 52 (1924), 205241.
[11] Chirka, E.M.: Complex analytic sets, Mathematics and its applications (Soviet Series), 46. Kluwer Academic Publishers Group, Dordrecht, 1989. xx +372 pp.
[12] van den Essen, A.: Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics, 190, Birkhäuser Verlag, Basel, 2000, xviii +329 pp.
[13] Golubitski, M.: Primitive actions and maximal subgroups of Lie groups, J. Differential Geom. 7 (1972), 175-191.
[14] González López, A.; Kamran, N.; Olver, P.J.: Lie algebras of vector fields in the real plane, Proc. London Math. Soc. 64 (1992), no. 2, 339-368.
[15] Gröbner, W.: Die Lie-Reihen und ihre Anwendungen, Math. Monog. Veb Deutschen Verlag der Wissenschaften, 1960
[16] Gunning, R.: Introduction to Holomorphic Functions of Several Variables, 3 vol., Wadsworth \& Brooks/Cole, I: Function theory, xx+203 pp., II: Local theory, +218 pp, III: Homological theory, +194 pp., 1990.
[17] Hawkins, T.: Emergence of the theory of Lie groups, An essay in the history of mathematics 1869-1926, Sources and studies in the history of mathematics and physical sciences, SpringerVerlag, Berlin, 2001, xiii+564 pp.
[18] Hawkins, T.: Frobenius, Cartan, and the problem of Pfaff, Arch. Hist. Exact Sci. 59 (2005), 381-436.
[19] Hitchin, N.: Projective geometry, lecture notes, 2003, www2.maths.ox.ac.uk/~hitchin/
[20] Killing, W.: Die Zusammensetzung der stetigen endlichen Transformationsgruppen. Vierter Theil, Math. Ann. 36 (1890), 161-189.
[21] Krantz, S.G.: Function theory of several complex variables, Second Edition, The Wadsworth \& Brooks/Cole Mathematics Series, Pacific Grove, CA, 1992, xvi+557 pp.
[22] Lie, S.: Über Gruppen von Transformationen, Göttinger Nachrichten 1874 (1874), 529-542. Reprinted in Abhandlungen 5, 1-8 [3 December 1874].
[23] Lie, S.: Theorie der Transformationsgruppen III. Bestimmung aller Gruppen einer zweifach ausgedehnten Punktmannigfaltigkeit, Archiv for Mathematik 3 (1878), 93-128. Reprinted in Abhandlungen 5, 78-133.
[24] Lie, S.: Theorie der Transformationsgruppen, Math. Ann. 16 (1880), 441-528; translated in English and commented in: [1].
[25] Lie, S.: Theorie der transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von Dr. Friedrich Engel, bearbeitet von Sophus Lie, B.G. Teubner, Leipzig, 1888. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1970).
[26] Lie, S.: Theorie der transformationsgruppen. Zweiter Abschnitt. Unter Mitwirkung von Prof. Dr. Friedrich Engel, bearbeitet von Sophus Lie, B.G. Teubner, Leipzig, 1890. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1970).
[27] Lie, S.: Theorie der transformationsgruppen. Dritter und Letzter Abschnitt. Unter Mitwirkung von Prof. Dr. Friedrich Engel, bearbeitet von Sophus Lie, B.G. Teubner, Leipzig, 1893. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1970).
[28] Lie, S.: Gesammelte Abhandlungen, Band ??, Leipzig, Teubner, 192??.
[29] Malgrange, B.: Ideals of Differentiable Functions, Tata Institute of Fundamental Research Studies in Mathematics, No. 3, Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1967, vii+106 pp.
[30] Merker, J.: On the local geometry of generic submanifolds of $\mathbb{C}^{n}$ and the analytic reflection principle, Journal of Mathematical Sciences (N. Y.) 125 (2005), no. 6, 751-824.
[31] Merker, J.: Jets de Demailly-Semple d'ordres 4 et 5 en dimension 2, to appear.
[32] Milnor, J.: Dynamics in one complex variable, Annals of Mathematics Studies, 160, Princeton University Press, Princeton, NJ, Third Edition, 2006, viii+304 pp.
[33] Olver, P.J.: Applications of Lie groups to differential equations, Graduate Texts in Mathematics, 107, Springer-Verlag, Heidelberg, 1986, xxvi+497 pp.
[34] Olver, P.J.: Equivalence, Invariance and Symmetries, Cambridge University Press, Cambridge, 1995, xvi+525 pp.
[35] Rao, M.R.M.: Ordinary differential equations, theory and applications, Edward Arnold, London, 1981.
[36] Samuel, P.: Projective geometry, Undergraduate Texts in Mathematics, Readings in Mathematics, Springer-Verlag, Berlin, 1988, ix+156 pp.
[37] Sharpe, R.W.: Differential Geometry. Cartan's generalization of Klein's Erlangen program, Springer-Verlag, Berlin, 1997, xix+421 pp.
[38] Stormark, O.: Lie's structural approach to PDE systems, Encyclopædia of mathematics and its applications, vol. 80, Cambridge University Press, Cambridge, 2000, xv+572 pp.
[39] de Tannenberg, W.; Vessiot, E.: Compte Rendu et analyse de [25], Bull. Sci. Math., $2{ }^{\mathrm{e}}$ série, 13 (1889), 113-148.


[^0]:    ${ }^{1}$ - technically defined to be the zero-set of all $\rho_{\infty} \times \rho_{\infty}$ minors of the Jacobian matrix $\left(\frac{\partial \mathcal{U}_{\alpha}^{i}}{\partial a_{j}}\right)_{1 \leqslant j \leqslant r}^{\alpha \in \mathbb{N}^{n}, 1 \leqslant i \leqslant n}$.
    ${ }^{2}$ Here and in the sequel, what can be said at points of the exceptional sets $D$ would require sophisticated tools from Singularity Theory that are beyond the scope of the present work.

[^1]:    ${ }^{3}$ The birth of the theory is beautifully reinscribed in its historical perspective by T. Hawkins in [17]. There, it is explained that the Poisson-Jacobi bracket identity: $0=[[X, Y], Z]+[[Z, X], Y]+[[Y, Z], X]$ between three local vector fields has been reconsidered by Lie, after deep reflection, to be true because the totality of contact transformations leaving a function invariant forms a group, the mentioned identity issuing in Lie's views from the differentiation of a commutator relation and from group associativity.

[^2]:    ${ }^{4}$ On considers $\left[\lambda_{1}: \cdots: \lambda_{r}\right]$ as homogeneous coordinates in the projective space of dimension $r-1$.

[^3]:    ${ }^{5}$ Coordinatewise transformation rules for vector fields under a diffeomorphism will be recalled in a while.
    ${ }^{6}$ By free relocalization, one then avoids for instance the deep problem of providing a normal form for a single analytic vector field $X$ at a singular point (a question which is still

[^4]:    *) Lie, Gött. Nachr., Dec. 1874 and Math. Ann., vol. 16, the method used in the text

[^5]:    ${ }^{1}$ Note. This is a claim, to be argued presently; what happens in the limiting plane $e_{4}=0$ could have been studided before.

