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Characteristics ALE Method for the Unsteady 3D Navier-Stokes Equations with a Free Surface

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Because of its great adaptability, the Arbitrary Lagrangian Eulerian (ALE) method is often used to solve the Navier-Stokes equations with a free surface. The kinematic condition relating the normal velocity to the mesh velocity suggests a simple way to move the domain, but it leads to unstable schemes in many cases. A method to take into account the non-linearity of the free surface is presented in this paper and integrated into a Finite Method with Galerkin characteristics. A variational form of the surface tension is used to overcome the difficulty of estimating the main curvature of the surface grid. A stability estimate is then established on the global scheme, under regularity conditions on the grid.

Keywords: Domain velocity, free surface convection, surface tension, ALE, Navier-Stokes.

1. INTRODUCTION

ALE methods have been used by many authors. A general presentation of the approach is given in Donea[2]. See also Hughes[6], where free oscillations of a fluid in a container and wave propagation are solved numerically with good accuracy and stability. Both authors use upwinding techniques for the convection term of the momentum equations. Our purpose here is to treat the non-linearities by a characteristics method. Characteristics will be used three times: firstly, to define ALE quantities and establish the first order ALE formulation of the Navier-Stokes equations (4D-characteristics), secondly, to move the free surface (2D-characteristics), and finally, to take into account the convection term in the momentum equation (3D-characteristics).

The goal of the method is to compute flows which are characterized by a predominating tangential velocity. Typical examples are water jets, coating flows and, (with other physical properties), laminating processes. Because of the CFL condition (which is very restrictive in such cases, see for example Nichols[9]), most explicit approaches are not applicable. These flows are often computed by a fixed point algorithm on the surface, based on the kinematic condition. Such algorithms have been successfully applied to many physical situations (see Zienkiewicz[12], Fleury[4], Scriven[7], Di Pietro[3], d'Halewyn[5]). A non-stationary approach is proposed in this paper. Because of the great stability of the characteristics method, the algorithm which is presented here tolerates large time steps, and the stationary state, when it exists, is reached quickly.

Furthermore, the method enables us to study the non-stationary effects due to perturbations on the boundary conditions or changes in the physical properties, arising in practical stability studies in industrial processes.

2. ALE FORMULATION

For illustrative purposes, the kinematic approach is applied to a particular configuration. Nevertheless, the method is suitable for a large class of problems. All surfaces and functions are supposed to be sufficiently regular.

2.1. Notations

In this part, the subscripts indicate the domain in which a quantity is defined, but not the physical time. For example, $\mathbf{u}_\tau(\mathbf{x}, t)$ is a vector field in Ω^τ, but represents the velocity at time t. The correspondence between the different domains Ω^τ and Ω^t is given by the mapping $C(\cdot, \tau; t)$, which is introduced in Section 2.3. $\mathbf{u}(\mathbf{x}', t)$ is the velocity in the usual sense ($\mathbf{x}' \in \Omega^t$), $\mathbf{u}_\tau(\mathbf{x}, t)$ denotes $\mathbf{u}(\mathbf{x}', t)$ with $\mathbf{x}' = C(\mathbf{x}, \tau; t)$.

2.2. Problem 1

Let us consider the 3D flow illustrated by Figure 1. Ω^t is delimited by

$$\partial\Omega^t = \Gamma_- \cup \Gamma^t \cup \Gamma^t_+, \tag{1}$$

where Γ_- is the inlet (prescribed velocity), Γ^t the free surface (prescribed normal stress, no shear stress), and Γ^t_+ the free outlet (moving portion of the fluid intersecting a fixed horizontal plane II). The problem is to find Ω^t, $\mathbf{u}(x, t)$, and $p(x, t)$ for $x \in \Omega^t$, $t \in [0, T]$, such that

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = g \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{2}$$

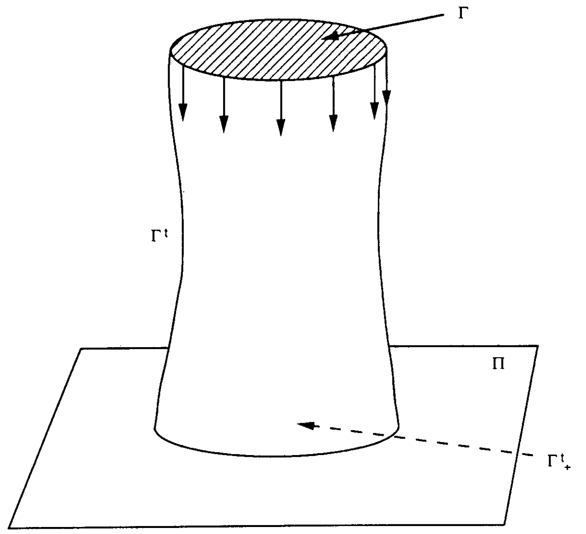

FIGURE 1 Geometry.

in the moving domain Ω^t, with boundary conditions

$$\begin{cases} \mathbf{u} = \mathbf{u}_- & \text{on } \Gamma_- \\ \nu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) - p\mathbf{n} = -p_e \mathbf{n} - \sigma(\kappa_1 + \kappa_2) & \text{on } \Gamma^t \\ \nu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})n - p\mathbf{n} = \mathbf{F}\mathbf{n} & \text{on } \Gamma^t_+, \end{cases} \tag{3}$$

where κ_1 and κ_2 are the principal curvatures of Γ^t, σ is the surface tension coefficient (between the fluid and the external gas), F is a prescribed traction, and p_e is the external pressure ($p_e = 0$ in the following). $(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})$ is the [redacted] tensor:

$$(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}) = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)_{i,j}, \tag{4}$$

with $\mathbf{u} = (u_1, u_2, u_3)$.

2.3. Domain Velocity and ALE Variables

In this section, space-time characteristics curves are used to introduce the ALE variables and the new formulation. The approach is the following: for all $t \in [0, T]$, $\mathbf{R}(\cdot, t)$ is an arbitrary normalized vector field on Γ^t, such that $\mathbf{R} \cdot \mathbf{n} > 0$ on Γ^t and, in the chosen configuration, $\mathbf{R}(\mathbf{x}, t)$ lies in II (plane containing Γ^t_+) for all $\mathbf{x} \in \Gamma^t_+ \cap \Gamma^t$. $\mathbf{R}$ is the "direction" of the boundary motion. The domain velocity $\mathbf{c}_t(\mathbf{x})$ at time t is then defined by

$$\mathbf{c}_t(\mathbf{x}) = \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{R} \cdot \mathbf{n}}\right)\mathbf{R}, \quad \forall \mathbf{x} \in \Gamma^t, \tag{5}$$

and inside the domain by (for example)

$$\Delta \mathbf{c}_t(\cdot) = 0,$$

with boundary conditions:

on Γ^t: Dirichlet condition (5),
on Γ_-: Dirichlet condition $\mathbf{c}_t = 0$,
on Γ^t_+: Dirichlet condition $\mathbf{c}^z_t = 0$ on the vertical component, Neumann conditions on the tangential components ($\partial \mathbf{c}^\gamma_t / \partial n = 0$, $\gamma = x, y$).

The 4D field $(\mathbf{c}_t(\mathbf{x}, t), 1)$ can be integrated in the physical space-time domain S corresponding to the time interval $[0, T]$. By (5), one can show that this field is tangential to the "lateral" boundary of S (boundary of S except $\Gamma^0 \times \{0\}$ and $\Gamma^T \times \{T\}$). It leads to mappings between the different Ω^t:

$$C(\cdot, t_1; t_2): \Omega^{t_1} \to \Omega^{t_2}$$

$$\mathbf{x}_1 \in \Omega^{t_1} \mapsto \mathbf{x}_2 = C(\mathbf{x}_1, t_1; t_2),$$

where $(C(\mathbf{x}_1, t_1; t), t)$ is the characteristics curve[1] from $(\mathbf{x}_1, t_1)$ to $(\mathbf{x}_2, t_2)$ in S:

$$\begin{cases} \dfrac{d}{dt}[C(\mathbf{x}_1, t_1; t), t] = [\mathbf{c}_1(C), 1] \\ C(\mathbf{x}_1, t_1; t_1) = (\mathbf{x}_1, t_1). \end{cases} \tag{6}$$

For each τ, the *ALE* velocity is then defined by

$$\mathbf{u}_\tau(\mathbf{x}, t) = \mathbf{u}(C(\mathbf{x}, \tau; t), t) \quad (\mathbf{x} \in \Omega^t, C(\mathbf{x}, \tau; t) \in \Omega^t), \tag{7}$$

[1] $C(\mathbf{x}_1, t_1; t)$ corresponds to the motion of a grid vertex which is at point $\mathbf{x}_1$ at time t_1.

which is equivalent to

$$\mathbf{u}_\tau(C(\mathbf{x}, t; \tau), t) = \mathbf{u}(\mathbf{x}, t) \quad (\mathbf{x} \in \Omega^t). \tag{8}$$

The partial derivative with respect to the time of (8) at $t = \tau$ gives

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}_\tau}{\partial t} + \left.\frac{\partial C}{\partial t}\right|_{t=\tau} \nabla \mathbf{u}_\tau$$

$$= \frac{\partial \mathbf{u}_\tau}{\partial t} - \mathbf{c}_\tau \nabla \mathbf{u}_\tau.$$

Introducing the variables $\mathbf{u}_\tau$ and p_τ, the Navier-Stokes equations at $t = \tau$ become:

$$\begin{cases} \dfrac{\partial \mathbf{u}_\tau}{\partial t} + (\mathbf{u}_\tau + \mathbf{c}_\tau) \cdot \nabla \mathbf{u}_\tau - \nu \Delta \mathbf{u}_\tau + \nabla p_\tau = g \\ \nabla \cdot \mathbf{u}_\tau = 0. \end{cases} \tag{9}$$

A first order form of the ALE Navier-Stokes equations, with t in the neighborhood of τ, will be used to justify the discretization (see Section 3.2).

Remarks In the Eulerian case ($\mathbf{u} \cdot \mathbf{n} = 0$ and domain is fixed), the curve $t \mapsto C(\mathbf{x}, \tau; t)$ would be a straight line parallel to the time axis.

As a general rule, the curve $C(\mathbf{x}, \tau; t)$ is not the pathline of any particle.

2.4. Convection of the Free Surface

2.4.1. *Introduction*

Let us consider the time-discretized problem. $k = t^{m+1} - t^m$ denotes the time step, Γ^m and $\mathbf{u}^m$ the free boundary and the velocity at time step t^m, $\mathbf{R}^m$ is the field $\mathbf{R}$ at time t^m. Formula (5) suggests a natural way to move the boundary: Γ^m and $\mathbf{u}^m$ being known, the displacement $\mathbf{d}^m$ on Γ^m could be defined by

$$\mathbf{d}^m = k\mathbf{c}^m = k\left(\frac{\mathbf{u}^m \cdot \mathbf{n}}{\mathbf{R}^m \cdot \mathbf{n}}\right)\mathbf{R}^m. \tag{10}$$

This simple method is consistent from a kinematic point of view, but it leads to a scheme which is unstable in many applications, when the velocity is essentially tangential to the free surface. Its unreliability is illustrated by the following simple example: consider the pure horizontal convection of a flat rigid body which has been locally deformed. By application of (10), the shape of the deformation will change at each iteration but its location will not move in the horizontal direction (an initially flat zone remains flat, by construction). Our purpose in this section is to establish a local motion equation which takes into account this convection. The approach is first applied to the fully continuous problem.

2.4.2. *Equation of the Motion*

In some particular cases, it is suitable to use a "non parametric" representation of the free surface, i.e.: the surface is considered as a function of two variables. For example, in the case of water waves, the upper surface is usually represented by

$$z = \Phi(x, y, t), \tag{11}$$

where z is the vertical coordinate of a surface point. The equation of the motion is then deduced from (11).

As a general rule, it is not possible to represent the whole surface by an equation of the type (11). Nevertheless, a regular surface can always be **locally** described in this way. Our purpose is to introduce a non-parametric representation of the surface in the neighborhood of each of its point, in order to establish a local equation of the motion.

For $\tau \in [0, T]$, let us consider an arbitrary point $\mathbf{x}_o \in \Gamma^\tau$. A local coordinate system with origin $\mathbf{x}_o$ is introduced:

$\mathbf{R}_o$ denotes $\mathbf{R}(\mathbf{x}_o, \tau)$,
$\mathbf{S}_1$ and $\mathbf{S}_2$ are such that $(\mathbf{R}_o, \mathbf{S}_1, \mathbf{S}_2)$ is orthonormal,

(x_1, x_2, x_3) are the coordinates of a point in the system $(\mathbf{x}_o; \mathbf{R}_o, \mathbf{S}_1, \mathbf{S}_2)$ (referential with origin $\mathbf{x}_o$ and basis vectors $\mathbf{R}_o$, $\mathbf{S}_1$ and $\mathbf{S}_2$).

As $\mathbf{R}_o \cdot \mathbf{n}(\mathbf{x}_o) > 0$, the free surface can be locally represented, in the referential $(\mathbf{x}_o; \mathbf{R}_o, \mathbf{S}_1, \mathbf{S}_2)$, by the equation

$$x_3 = \Phi(x_1, x_2, t). \tag{12}$$

Particles on the free surface at time τ remain on it, so: $(X_1, X_2, X_3)(t)$ being the trajectory of such a particle,

$$\frac{D}{Dt}(\Phi(X_1(t), X_2(t), t) - X_3(t)) = 0. \tag{13}$$

Let (U_1, U_2, V) be the velocity in the referential $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{R}_o)$. The 2D-vector (U_1, U_2) is denoted by $\mathbf{U}$. (U_1, U_2, V), velocity of a particle on the free surface, can be written as a function of x_1, x_2 and t. Equation (13) can then be written as

$$\frac{\partial \Phi}{\partial t}(x_1, x_2, t) + \mathbf{U}(x_1, x_2, t) \cdot \nabla \Phi(x_1, x_2, t)$$
$$= V(x_1, x_2, t). \tag{14}$$

This equation expresses simply the motion of the surface as a convection of the displacement according to $\mathbf{R}$, convected by the "horizontal" velocity, with a source term equal to the "vertical" velocity.

3. DISCRETIZATION IN TIME

3.1. Notations

In what follows, superscripts refer to the physical time, and subscripts to the domain in which the function is defined. For example, $\mathbf{u}_m^{m+1}$ designs the approximated velocity at time t^{m+1}, but is defined in Ω^m. If the domain corresponds to the real time, the subscript is omitted: $\mathbf{u}^m$ is the velocity at time step t^m and is defined in Ω^m (usual notation).

3.2. Navier-Stokes Equations

3.2.1. *Discretization*

$0 = t^\circ < t^1 < \cdots < t^M = T = m_T k$ are the discretization time steps ($k = t^{m+1} - t^m$). The discretization of the Eq. (9) is based on the method of characteristics (for further details about the method, see Pironneau[11]). Ω^m, $\mathbf{u}^m$, p^m and $\mathbf{c}^m$ are supposed to be known at a time step t^m. The approximation of $\mathbf{u}_{t^m}(\mathbf{x}, t^{m+1})$, *ALE* velocity at time $t^{m+1} = t^m + k$, is denoted by $\mathbf{u}_m^{m+1}$ (in the same way, p_m^{m+1} is the approximated pressure). $\mathbf{u}_m^{m+1}$ and p_m^{m+1} are calculated in the following way (the global scheme, involving the Navier-Stokes equations and movement of the free surface is presented in Section 4.3):

$$\begin{cases} \mathbf{u}_m^{m+1} - \mathbf{u}^m \circ X^m - k\nu\Delta\mathbf{u}_m^{m+1} + k\nabla p_m^{m+1} = k\mathbf{g} \\ \nabla\cdot\mathbf{u}_m^{m+1} = 0, \end{cases} \tag{15}$$

$\mathbf{u}^m \circ X^m$ denoting $\mathbf{u}^m(X^m(\mathbf{x}, t^m))$, with

$$\begin{cases} \dfrac{\partial X^m}{\partial t}(\mathbf{x}, t) = \mathbf{u}^m(X^m) - \mathbf{c}^m(X^m) \\ X^m(\mathbf{x}, t^{m+1}) = \mathbf{x}. \end{cases} \tag{16}$$

$X^m(\mathbf{x}, t)$ is the representation in the domain Ω^m of a pathline. More precisely, $[C(X^m(\mathbf{x}, t), t^m; t), t]$ is the pathline in the physical space-time domain of the particle which is at point $\mathbf{x}$ at time t^{m+1}. $X^m(\mathbf{x}, t^m)$ is then the position of this particle at time t^m.

3.2.2. *Consistency Error*

$\mathbf{u}_\tau$ and p_τ being the exact velocity and pressure at time τ, the consistency error at time t^m is defined by

$$e^m(\mathbf{x}) = \left| \frac{\mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) - \mathbf{u}_{t^m}(X^m, t^m)}{k} - \nu\Delta\mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) + \nabla p_{t^m}(\mathbf{x}, t^{m+1}) - \mathbf{g} \right|,$$

where the X^m are the characteristics convected in Ω^{t^m} by the field

$$(\mathbf{u}_{t^m}(\cdot, t^m) - \mathbf{c}_{t^m}(\cdot)).$$

Let us now verify the first order consistency. The continuous eq. (9), written at $t = \tau$, is satisfied at the first order in $|\tau - t|$, so, for t^{m+1} in the neighborhood of t^m,

$$e^m(\mathbf{x}) = \left| \frac{\mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) - \mathbf{u}_{t^m}(X^m, t^m)}{k} - \frac{\partial\mathbf{u}_{t^m}}{\partial t}(\mathbf{x}, t^{m+1}) - (\mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) - \mathbf{c}_{t^m}(\mathbf{x}))\cdot\nabla\mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) \right| + O(k),$$

and, by construction of X^m,

$$\begin{aligned} &\mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) - \mathbf{u}_{t^m}(X^m, t^m) \\ &= \mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) - \mathbf{u}_{t^m}(\mathbf{x} - (\mathbf{u}_{t^m} - \mathbf{c}_{t^m})k, t^m) + O(k^2) \\ &= \mathbf{u}_{t^m}(\mathbf{x}, t^{m+1}) - \mathbf{u}_{t^m}(\mathbf{x}, t^m) + (\mathbf{u}_{t^m} - \mathbf{c}_{t^m})\cdot\nabla\mathbf{u}_{t^m} + O(k^2) \\ &= k\frac{\partial\mathbf{u}_{t^m}}{\partial t}(\mathbf{x}, t^{m+1}) + k(\mathbf{u}_{t^m} - \mathbf{c}_{t^m})\cdot\nabla\mathbf{u}_{t^m} + O(k^2), \end{aligned}$$

so that, finally, $e^m(\mathbf{x}) = O(k)$.

3.3. Free Surface Convection

3.3.1. *Discretization*

The discretization in time of (14) at a time step t^m is done in the same way: $\mathbf{U} = (U_1, U_2)$ and V are taken constant in the time interval $[t^m, t^{m+1}]$ (explicit scheme), and the obtained equation

$$\frac{\partial\Phi}{\partial t}(x_1, x_2, t) + \mathbf{U}(x_1, x_2, t^m)\cdot\nabla\Phi(x_1, x_2, t) = V(x_1, x_2, t^m) \tag{17}$$

is solved by a method of 2D characteristics. Let us denote by $P(\mathbf{x}_o)$ the plane

$$P(\mathbf{x}_o) = \{\mathbf{x}_o + x_1\mathbf{S}_1 + x_2\mathbf{S}_2, (x_1, x_2)\in\mathbb{R}^2\}. \tag{18}$$

The boundary displacement defined at time t^m in the neighborhood of $\mathbf{x}_o\in\Gamma^m$ in the plane $P(\mathbf{x}_o)$ is denoted by $\Phi_m(\mathbf{x}_o; x_1, x_2, t)$, with $(x_1, x_2)\in\mathbb{R}^2$. The local coordinates of $\mathbf{x}_o$ in the plane $P(\mathbf{x}_o)$ are $(0, 0)$. The displacement at point $\mathbf{x}_o$ then is

$$k\mathbf{c}^m(\mathbf{x}_o) = \Phi_m(\mathbf{x}_o; 0, 0, t^{m+1})\mathbf{R}^m(\mathbf{x}_o). \tag{19}$$

It is obtained by

$$\Phi_m(\mathbf{x}_o; 0, 0, t^{m+1}) = \Phi_m(\mathbf{x}_o; \xi_1(t^m), \xi_2(t^m), t^m) + \int_{t^m}^{t^{m+1}} V(\xi_1(t), \xi_2(t), t^m)\,dt, \tag{20}$$

where $(\xi_1(t), \xi_2(t)) \in \mathbb{R}^2$ is the 2D-characteristic

$$\begin{cases} \dfrac{d(\xi_1, \xi_2)}{dt}(t) = \mathbf{U}(\xi_1(t), \xi_2(t), t^m) \\ (\xi_1, \xi_2)(t^{m+1}) = (0, 0). \end{cases} \tag{21}$$

Γ^{m+1}, approximation of $\Gamma^{t^{m+1}}$, is then given by

$$\begin{aligned} \Gamma^{m+1} &= \{\mathbf{x} + k\mathbf{c}^m(\mathbf{x}) \\ &= \mathbf{x} + \Phi_m(\mathbf{x}; 0, 0, t^{m+1})\mathbf{R}^m(\mathbf{x}), \mathbf{x} \in \Gamma^m\}. \end{aligned} \tag{22}$$

3.3.2. *Consistency Error*

The characteristics method is a classical way to solve convection equations. The first order consistency lies in the fact that, if

$$(\xi_1, \xi_2)(t^m) = (0, 0) - k\mathbf{U}(0, 0, t^m) + O(k^2), \tag{23}$$

then

$$\begin{aligned} &\Phi_m(\mathbf{x}; \xi_1(t^m), \xi_2(t^m), t^m) \\ &\quad = \Phi_m(\mathbf{x}; 0, 0, t^m) + k\mathbf{U}(0, 0, t^m) \cdot \nabla\Phi + O(k^2). \end{aligned} \tag{24}$$

From a kinematic point of view, the consistency may be established by verifying that the field $\mathbf{c}^m$ is such that Eq. (5) holds at the first order:

$$\mathbf{c}^m = \left(\frac{\mathbf{u}^m \cdot \mathbf{n}}{\mathbf{R}^m \cdot \mathbf{n}}\right)\mathbf{R}^m + O(k). \tag{25}$$

Finally, in a more general approach, it can be shown that the Hausdorff distance between the exact and the approximated surfaces is $O(k^2)$ (see Maury[8] for the 2D problem).

Remark The displacement field could be simply defined by Eq. (10). The characteristics method does not improve the order of the approximation, but as it has been shown in a similar context (in the 2D case, Maury[8]), the approximation which is made is more precise and stable. The motion of the free surface is in fact accelerated by the real acceleration of its particles and by convection. The present method takes into account the second factor.

3.3.3. *Choice of the Field* $\mathbf{R}^m$

In many cases, the choice of $\mathbf{R}^m$ is directly suggested by the physical flow itself. As far as possible, it is chosen constant in time and "nearly normal". The simplest way is to defined $\mathbf{R}^m$ as the trace on Γ^m of a constant 3D vector field. In the classical problem of water waves, for example, it is natural to choose a field $\mathbf{R}^m = \mathbf{R}^o$ along the vertical direction. For a water jet, even if the flow is not strictly axisymmetric, $\mathbf{R}$ will be radial according to the main axis. Nevertheless, for more complex geometries, the normal field can be chosen (see Maury[8]).

4. DISCRETIZATION IN SPACE

The problem corresponding to Figure 1 can be discretized in space, as illustrated in part 6. Nevertheless, we shall restrict our study to a less general case in this section, in order to establish in part 5 a stability property: the domain must be globally Lagrangian (no fluid getting in nor out of the space-time domain).

Problem 2: water in a container

The configuration is illustrated by Figure 2. The bottom wall and the immersed part of the lateral walls are denoted respectively by Γ_b and Γ_ℓ^τ, the upper surface (free surface) is Γ^τ. The external forces are limited to gravity, Γ_b and Γ_ℓ^τ are considered as sliding walls, and the free surface is submitted to

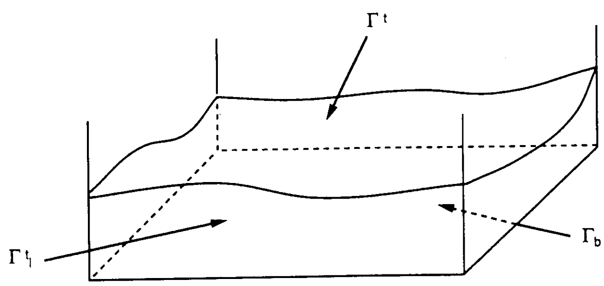

FIGURE 2 Problem 2.

external pressure and surface tension. In this case, the field $\mathbf{R}$ is chosen uniform and constant in time, equal to the vertical unit vector.

4.1. Navier-Stokes Equations

4.1.1. *Variational Formulation*

Ω^m and $\mathbf{c}^m$ being known, the discretization in time of the new problem leads to equations of the type

$$\begin{cases} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{26}$$

with the boundary conditions

$$\begin{cases} \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } \Gamma_\ell^m \cup \Gamma_b \\ \nu(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})\mathbf{n} - p\mathbf{n} = -p_e - \sigma(\kappa_1 + \kappa_2)\mathbf{n} & \text{on } \Gamma^m. \end{cases} \tag{27}$$

α is $1/k$ and $\mathbf{f}$ is the sum of the gravity and the inertia term $\mathbf{u}\circ X$ estimated at the previous time step. Let us introduce the following spaces:

$$V^m = \{\mathbf{u}\in H^1(\Omega^m), \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_b \cup \Gamma_\ell^m\},$$

$$Q^m = L^2(\Omega^m). \tag{28}$$

The variational formulation of (26)–(27) (see Pironneau[10]) is: find $(\mathbf{u}, p)\in V^m \times Q^m$ such that

$$\begin{cases} \alpha\displaystyle\int_{\Omega^m} \mathbf{u}\cdot\mathbf{w} + \frac{1}{2}\nu\int_{\Omega^m}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}):(\nabla\mathbf{w} + {}^t\nabla\mathbf{w}) \\ + \displaystyle\int_{\Omega^m} p\nabla\cdot\mathbf{w} + \sigma\int_{\Gamma^m}(\kappa_1 + \kappa_2)\mathbf{n}\cdot\mathbf{w} = \int_{\Omega^m}\mathbf{f}\cdot\mathbf{w}, \ \forall\mathbf{w}\in V \\ \displaystyle\int_{\Omega^m} q\nabla\cdot\mathbf{u} = 0, \ \forall q\in Q. \end{cases} \tag{29}$$

4.1.2. *Discretization in Space*

Let T^m be a triangulation of Ω^m. e_i is a vertex of T^m, T_k is a tetrahedron of T^m. λ^i denotes the classical P^1 basis function associated to point e_i (in each tetrahedron containing e_i, λ^i is the barycentric coordinate associated with e_i, $\lambda^i = 0$ everywhere else). μ^k, called bubble function (see Pironneau[10], p. 105), is the normalized product of the barycentric coordinates according to the 4 vertices of the tetrahedron T_k ($\mu^k = 0$ outside the interior of T_k). We introduce the following spaces:

$$V_h^m = \left\{\mathbf{u}\in V^m | \mathbf{u} = \sum_{e_i,\gamma} u_\gamma^i\lambda^i\mathbf{e}_\gamma + \sum_{T_k,\gamma} q_\gamma^k\mu^k\mathbf{e}_\gamma, \quad \gamma = x, y \text{ or } z\right\} \tag{30}$$

and

$$Q_h^m = \left\{p\in Q^m | p = \sum_{e_i} p^i\lambda^i\right\}. \tag{31}$$

A sense has first to be given to the integral involving surface tension forces.

Notations ∂T^m denotes the surface triangulation. e_i and e_j are connected vertices of ∂T^m (which will be denoted by $e_i \sim e_j$ or $i \sim j$), the length of $S_{ij} = e_ie_j$ is denoted by ℓ_{ij}. $\mathbf{t}_{ij}^\ell$ and $\mathbf{t}_{ij}^r$ are the left and right tangent vectors along the edge S_{ij} (see Figure 3). $\mathbf{t}_{ij}^\ell$ and $\mathbf{t}_{ij}^r$ are orthogonal to S_{ij} and lie in the left and right triangles containing S_{ij}, respectively.

PROPOSITION 4.1 *For all test functions w, piecewise P^1 on ∂T^m, the integral*

$$\int_{\partial T^m}(\kappa_1 + \kappa_2)\mathbf{n}\cdot\mathbf{w} \tag{32}$$

is defined and its value is

$$-\frac{1}{2}\sum_{e_i}\left(\sum_{e_j\sim e_i}\ell_{ij}(\mathbf{t}_{ij}^\ell + \mathbf{t}_{ij}^r)\right)\cdot\mathbf{w}(e_i). \tag{33}$$

Remark This proposition may be expressed as follows: the field $(\kappa_1 + \kappa_2)\mathbf{n}$ is in the dual of $P^1(\partial T^m)$ (but not in $P^1(\partial T^m)$), and its expression is

$$(\kappa_1 + \kappa_2)\mathbf{n} = \sum_{e_i}\mathbf{C}_i\cdot\delta_{e_i}, \tag{34}$$

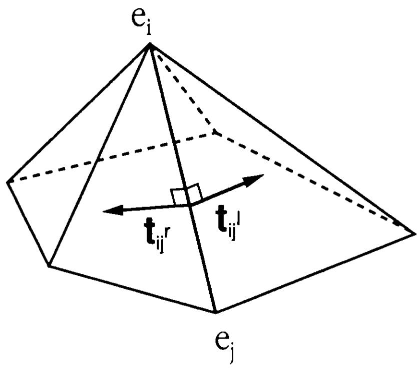

FIGURE 3 Detail of the surface grid.

where $\mathbf{C}_i$ is the "curvature vector" at point e_i and δ_{e_i} is the Dirac mass at point e_i. $\mathbf{C}_i$ is given by

$$\mathbf{C}_i = -\frac{1}{2}\sum_{e_j\sim e_i} \ell_{ij}(\mathbf{t}_{ij}^{\ell} + \mathbf{t}_{ij}^{r}). \tag{35}$$

Proof V is the zone illustrated by Figure 4. $(\mathbf{t}_1, \mathbf{t}_2)$ is almost everywhere an orthonormal tangential referential of V, with $\mathbf{t}_2 = -\mathbf{t}^r$ in V^r, $\mathbf{t}_2 = \mathbf{t}^{\ell}$ in V^{ℓ}. Let Φ be a Lipschitz function with support included in V. It is possible in this case to estimate explicitly

$$\begin{aligned}\int_V (\kappa_1+\kappa_2)\mathbf{n}\cdot\Phi &= \int_{S_1}\int_{S_2(s_1)}\left(\frac{\partial \mathbf{t}_1}{\partial s_1}+\frac{\partial \mathbf{t}_2}{\partial s_2}\right)\cdot\Phi \\ &= \int_{S_1}\int_{S_2(s_1)}\frac{\partial \mathbf{t}_2}{\partial s_2}\cdot\Phi \qquad (36)\\ &= \int_{S_1}\left(\int_{S_2(s_1)} \mathbf{t}_2\cdot\frac{\partial \Phi}{\partial s_2} + [\Phi\cdot\mathbf{t}_2]\right).\end{aligned}$$

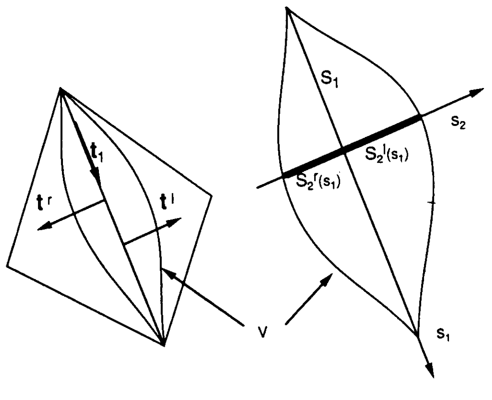

FIGURE 4 Definition of V.

$\mathbf{\Phi}$ being Lipschitz, $\partial\mathbf{\Phi}/\partial s_2$ is L^∞, so that

$$\begin{aligned}&\int_{\partial T^m}(\kappa_1+\kappa_2)\mathbf{n}\cdot\mathbf{\Phi}\\ &= -\int_{S_1}\int_{S_2^r(s_1)}\mathbf{t}^r\cdot\frac{\partial\mathbf{\Phi}}{\partial s_2} + \int_{S_1}\int_{S_2^{\ell}(s_1)}\mathbf{t}^{\ell}\cdot\frac{\partial\mathbf{\Phi}}{\partial s_2}\\ &= -\mathbf{t}^r\cdot\int_{S_1}\mathbf{\Phi}(s_1) - \mathbf{t}^{\ell}\cdot\int_{S_1}\mathbf{\Phi}(s_1)\\ &= -(\mathbf{t}^r+\mathbf{t}^{\ell})\cdot\int_{S_1}\mathbf{\Phi}(s_1). \qquad (37)\end{aligned}$$

Let us now consider $\mathbf{w}\in P^1(\partial T^m)$. $\mathbf{w}$ can be expressed as a sum of functions $\mathbf{\Phi}_{ij}$ associated to edges S_{ij} (equal to $\mathbf{w}$ on S_{ij}), and Lipschitz function $\mathbf{\Psi}_T$ vanishing on the boundaries of the triangles. The complete integral over the boundary can then be written (as $\mathbf{w}$ is Lipschitz, the contribution of the vertices is equal to 0):

$$\begin{aligned}\int_{\partial T^m}(\kappa_1+\kappa_2)\mathbf{n}\cdot\mathbf{w} &= -\sum_{i\sim j}(\mathbf{t}^r+\mathbf{t}^{\ell})\cdot\int_{S_{ij}}\mathbf{\Phi}_{ij}\\ &\quad+\sum_{T\in\partial T^m}\int_T(\kappa_1+\kappa_2)\mathbf{n}\cdot\mathbf{\Psi}_T\\ &= -\sum_{i\sim j}(\mathbf{t}^r+\mathbf{t}^{\ell})\cdot\int_{S_{ij}}\mathbf{w}\\ &= -\frac{1}{2}\sum_{i\sim j}(\mathbf{t}^r+\mathbf{t}^{\ell})\cdot(\mathbf{w}(e_i)+\mathbf{w}(e_j))\ell_{ij}\\ &= -\frac{1}{2}\sum_i \mathbf{w}(e_i)\cdot\sum_{i\sim j}\ell_{ij}(\mathbf{t}^r+\mathbf{t}^{\ell}). \qquad (38)\end{aligned}$$

Let us denote by $\Lambda(\mathbf{w})$ this last expression. The discretized variational formulation of (26)–(27) is: find $(\tilde{\mathbf{u}}, \tilde{p})\in V_h^m\times Q_h^m$ such that

$$\left\{\begin{aligned}&\alpha\int_{\Omega^m}\tilde{\mathbf{u}}\cdot\tilde{\mathbf{w}} + \tfrac{1}{2}\nu\int_{\Omega^m}(\nabla\tilde{\mathbf{u}}+{}^t\nabla\tilde{\mathbf{u}}):(\nabla\tilde{\mathbf{w}}+{}^t\nabla\tilde{\mathbf{w}})\\ &+\int_{\Omega^m}\tilde{p}\nabla\cdot\tilde{\mathbf{w}}+\sigma\Lambda(\tilde{\mathbf{w}}) = \int_{\Omega^m}\mathbf{f}\cdot\tilde{\mathbf{w}},\quad\forall\tilde{\mathbf{w}}\in V_h^m \qquad (39)\\ &\int_{\Omega^m}\tilde{q}\nabla\cdot\tilde{\mathbf{u}} = 0,\quad\forall\tilde{q}\in Q_h^m.\end{aligned}\right.$$

4.2. Free Surface

The computation of $\mathbf{\Phi}$ from (17) is done by a local projection of the grid (in the neighborhood of a boundary vertex e), onto the plane $P(e)$ containing e and normal to $\mathbf{R}$. Equation (17) is then solved by a classical 2D method of characteristics. The technique is illustrated by Figure 5. For the sake of clarity, the plane $P(e)$ has been translated along $\mathbf{R}(e)$. $(\xi_1(t^m), \xi_2(t^m))$ is denoted by $\xi(t^m)$ in the figure. The two-headed arrow represents the first term of equation (20):

$$\Phi_m(e,; \xi_1(t^m), \xi_2(t^m), t^m).$$

The second term,

$$\int_{t^m}^{t^{m+1}} V(\xi_1(t), \xi_2(t), t^m)dt,$$

is obtained by integrating V($\mathbf{R}$−velocity) along the broken line represented in bold (Figure 5).

4.3. Global Scheme

$\tilde{\Omega}^o$, $\tilde{\mathbf{u}}^o$ defined in $\tilde{\Omega}^o$, are given. $\mathbf{R}$ is equal to the vertical unit vector.

Iterations

a) $\tilde{\mathbf{c}}^m$ is defined on $\tilde{\Gamma}^m$ by the characteristics method presented in Section 4.2. $\tilde{\mathbf{c}}^m$ is then defined in $\tilde{\Omega}^m$ by

$$\begin{cases} \Delta\tilde{\mathbf{c}}^m = 0 & \text{in } \tilde{\Omega}^m \\ \tilde{\mathbf{c}}^m = \tilde{\mathbf{c}}^m|_{\tilde{\Gamma}^m} & \text{on } \tilde{\Gamma}^m \\ \dfrac{\partial\tilde{\mathbf{c}}^m}{\partial n} = 0 & \text{on } \tilde{\Gamma}_\ell^m \\ \tilde{\mathbf{c}}^m = 0 & \text{on } \tilde{\Gamma}_b. \end{cases} \tag{40}$$

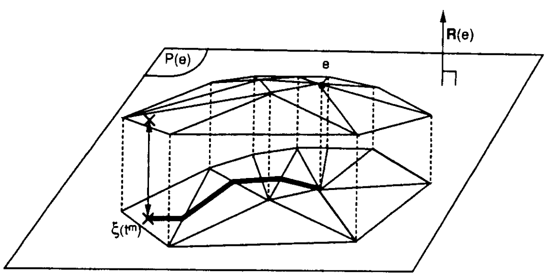

FIGURE 5 Characteristics in the plane $P(e)$.

b) $\tilde{\mathbf{u}}_m^{m+1}$ and $\tilde{p}_m^{m+1}$ are solutions of

$$\begin{cases} \displaystyle\int_{\tilde{\Omega}^m} \tilde{\mathbf{u}}_m^{m+1}\cdot\tilde{\mathbf{w}} - \int_{\tilde{\Omega}^m} \tilde{\mathbf{u}}^m \circ \tilde{X}^m \cdot \tilde{\mathbf{w}} \\ \displaystyle +\tfrac{1}{2}k\nu \int_{\tilde{\Omega}^m} (\nabla\tilde{\mathbf{u}}_m^{m+1} + {}^t\nabla\tilde{\mathbf{u}}_m^{m+1}):(\nabla\tilde{\mathbf{w}} + {}^t\nabla\tilde{\mathbf{w}}) \\ \displaystyle + k\int_{\tilde{\Omega}^m} \tilde{p}_m^{m+1}\nabla\cdot\tilde{\mathbf{w}} + k\sigma\Lambda(\tilde{\mathbf{w}}) = k\int_{\tilde{\Omega}^m} \mathbf{g}\cdot\tilde{\mathbf{w}}, \quad \forall\tilde{\mathbf{w}}\in V_h \\ \displaystyle\int_{\tilde{\Omega}^m} \tilde{q}\nabla\cdot\tilde{\mathbf{u}}_m^{m+1} = 0, \quad \forall\tilde{q}\in Q_h, \end{cases} \tag{41}$$

where $\tilde{X}^m = \tilde{X}^m(\mathbf{x}, t^m)$ is such that

$$\begin{cases} \dfrac{\partial\tilde{X}^m}{\partial t}(\mathbf{x}, t) = (\tilde{\mathbf{u}}^m - \tilde{\mathbf{c}}^m)(\tilde{X}^m) \\ \tilde{X}^m(\mathbf{x}, t^{m+1}) = \mathbf{x}. \end{cases} \tag{42}$$

c) The new domain $\tilde{\Omega}^{m+1}$ is

$$\tilde{\Omega}^{m+1} = \{\mathbf{x} + k\tilde{\mathbf{c}}^m, \mathbf{x}\in\tilde{\Omega}^m\}. \tag{43}$$

d) $\tilde{\mathbf{u}}^{m+1}$ is defined in $\tilde{\Omega}^{m+1}$ by

$$\tilde{\mathbf{u}}^{m+1}(\mathbf{x} + k\tilde{\mathbf{c}}^m) = \tilde{\mathbf{u}}_m^{m+1}(\mathbf{x}), \quad \forall\mathbf{x}\in\tilde{\Omega}^m. \tag{44}$$

5. STABILITY

In Pironneau[11], a stability property for the characteristics method applied to advection-diffusion problems is established. In Boukir[1], the stability is demonstrated for the Navier-Stokes equations with a fixed domain. The demonstration involves the error between the estimated and the exact solution, about which suitable regularity hypotheses are made. In our case, exact and approximated solutions are not defined *a priori* in the same domain, which is why such error estimations have not been established at this time. For this reason, new conditions have to be verified.

As this section deals only with space-time discretized quantities, the symbols $\sim$ are suppressed.

5.1. Introduction

5.1.1. *A New Time Discretization*

Previously, the iterative process was based on a calculation of variables at time t^{m+1} in the domain Ω^m. It is the most natural way. Nevertheless, a similar approach can be done by calculating in Ω^{m+1}: $\mathbf{c}^m$ is replaced by $\mathbf{c}^m_{m+1}$, defined in Ω^{m+1}, and the new time discretization is

$$\begin{cases} \mathbf{u}^{m+1} - \mathbf{u}^m_{m+1} \circ X^m_{m+1} - k\nu\Delta\mathbf{u}^{m+1} + k\nabla p^{m+1} = k\mathbf{g} \\ \nabla\cdot\mathbf{u}^{m+1} = 0, \end{cases} \tag{45}$$

$X^m_{m+1}(\mathbf{x})$ denoting $X^m_{m+1}(\mathbf{x}, t^m)$, with

$$\begin{cases} \dfrac{\partial X^m_{m+1}}{\partial t}(\mathbf{x}, t) = (\mathbf{u}^m_{m+1} - \mathbf{c}^m_{m+1})(X^m_{m+1}) \\ X^m_{m+1}(\mathbf{x}, t^{m+1}) = \mathbf{x}. \end{cases} \tag{46}$$

It leads to a first order scheme as well as the discretization (15). For technical reasons (the proof of proposition 5.1 is more natural), this second approach is chosen in this part.

5.1.2. *Hypotheses*

A few assumptions need to be made about the family of surface triangulations ∂T^m. Let us first introduce:

$h > 0$, $\alpha \in]0, 1[$, $q_1 > 0$ and $M > 0$, fixed numbers, $\mathbf{t}^r_{ij}$ and $\mathbf{t}^\ell_{ij}$, the tangent vectors associated to edge e_ie_j (see Fig. 3).

The hypotheses are the following:

H1 For all edges e_ie_j of ∂T^m, the length $|e_ie_j|$ is in $[\alpha h, h]$.

H2 For all triangles $\mathcal{T}$ of ∂T^m, $|\mathcal{T}|$ being the measure of $\mathcal{T}$ and $h_{\mathcal{T}}$ its diameter,

$$\frac{|\mathcal{T}|}{h^2_{\mathcal{T}}} \geqslant q_1. \tag{47}$$

Similarly, for all tetrahedra $\mathcal{K}$ in T^m with diameter $h_{\mathcal{K}}$,

$$\frac{|\mathcal{K}|}{h^3_{\mathcal{K}}} \geqslant q_1. \tag{48}$$

H3 For all edges e_ie_j, t^ℓ_{ij} and $\mathbf{t}^r_{ij}$ being tangent vectors associated to e_ie_j,

$$|\mathbf{t}^\ell_{ij} + \mathbf{t}^r_{ij}| \leqslant Mh. \tag{49}$$

H4 $k\mathbf{c}^m$ being the displacement from Ω^m to Ω^{m+1}, the time step k must be chosen such that $|I_d + k\mathbf{c}^m| > 0$. It imposes the following restriction: the time step k is such that

$$k|\mathbf{c}^m|_{W^{1,\infty}} \leqslant q_2. \tag{50}$$

Remark h is the diameter of the triangulation, α measures the uniformity of the mesh, q_1 its quality. M is related to the "curvature" of the surface mesh (in fact angles between faces). Finally, q_2 controls the distortion of the mesh between two time steps.

5.2. Stability Estimate

As the triangulation of the domain after a few time steps is *a priori* unknown, proposition 5.1 must be read in the following way: initial data, a time interval $[0, T]$, and a field $\mathbf{R}$ (direction of the displacement) having been given, a computation may be associated to any initial triangulation T^o and time step k. If **H1, H2, H3** and **H4** are verified at each time step, then the inequality (51) holds true. The field $\mathbf{u}^m$ used for the characteristics method is supposed to be projected on a space of fields with exactly vanishing divergence (it may be done in practice by using a stream function method). The volume and the area of the surface of the approximated domains are supposed to be bounded.

PROPOSITION 5.1 *m_T being the number of steps ($m_Tk = T$), there exist constants C_1, C_2 and C_3 such that*

$$|\mathbf{u}^{m_T}|_o \leqslant \exp(q_2C_3T)|\mathbf{u}^o|_o + T\left(\frac{C_2M}{hq_1^2\alpha^3} + gC_1\right), \tag{51}$$

where g is the modulus of the gravity vector ($g = |\mathbf{g}|$).

Proof One has

$$\left\{\begin{aligned}&\int_{\Omega^{m+1}} \mathbf{u}^{m+1}\cdot\mathbf{w}-\int_{\Omega^{m+1}} \mathbf{u}^m_{m+1}\circ X^m_{m+1}\cdot\mathbf{w}\\ &+\frac{1}{2}k\nu\int_{\Omega^{m+1}}(\nabla\mathbf{u}^{m+1}+{}^t\nabla\mathbf{u}^{m+1}):(\nabla\mathbf{w}+{}^t\nabla\mathbf{w})\\ &+k\int_{\Omega^{m+1}} p^{m+1}\nabla\cdot\mathbf{w}+k\sigma\Lambda(\mathbf{w})=k\int_{\Omega^{m+1}}\mathbf{g}\cdot\mathbf{w},\ \forall\mathbf{w}\in V_h.\end{aligned}\right. \tag{52}$$

$\mathbf{w}=\mathbf{u}^{m+1}$ leads to

$$\begin{aligned}\int_{\Omega^{m+1}}|\mathbf{u}^{m+1}|^2&=\int_{\Omega^{m+1}}\mathbf{u}^m_{m+1}\circ X^m_{m+1}\cdot\mathbf{u}^{m+1}\\ &-\frac{1}{2}k\nu\int_{\Omega^{m+1}}|\nabla\mathbf{u}^{m+1}+{}^t\nabla\mathbf{u}^{m+1}|^2\\ &-k\sigma\Lambda(\mathbf{u}^{m+1})+k\int_{\Omega^{m+1}}\mathbf{g}\cdot\mathbf{u}^{m+1}\\ &\leqslant|\mathbf{u}^m_{m+1}\circ X^m_{m+1}|_o|\mathbf{u}^{m+1}|_o+k\sigma|\Lambda(\mathbf{u}^{m+1})|\\ &+kg|\mathbf{u}^{m+1}|_o|\Omega^{m+1}|^{1/2}.\end{aligned} \tag{53}$$

$|\Lambda(\mathbf{u}^{m+1})|$ can be majorated:

$$\begin{aligned}&|\Lambda(\mathbf{u}^{m+1})|\\ &=\frac{1}{2}\left|\sum_{e_i\sim e_j}\ell_{ij}(\mathbf{t}^{\ell}_{ij}+\mathbf{t}^{r}_{ij})\cdot(\mathbf{u}^{m+1}(e_i)+\mathbf{u}^{m+1}(e_j))\right|\\ &\leqslant\frac{1}{2}\sum_{e_i\sim e_j}Mh^2(|\mathbf{u}^{m+1}(e_i)|+|\mathbf{u}^{m+1}(e_j)|)\\ &\leqslant\frac{1}{2}\sum_{\mathcal{T}\in\partial T^{m+1}}Mh^2\sum_{1\leqslant i\leqslant 3}|\mathbf{u}^{m+1}(e_i)|,\end{aligned} \tag{54}$$

where the e_i are the three vertices of triangle $\mathcal{T}$. Conditions $h\leqslant h_{\mathcal{T}}/\alpha$ and $|\mathcal{T}|/h^2_{\mathcal{T}}\geqslant q_1$ imply

$$|\Lambda(\mathbf{u}^{m+1})|\leqslant\frac{M}{2q_1\alpha^2}\sum_{\mathcal{T}\in\partial T^{m+1}}\left(|\mathcal{T}|\sum_{1\leqslant i\leqslant 3}|\mathbf{u}^{m+1}(e_i)|\right), \tag{55}$$

$$\begin{aligned}\left(\sum_{1\leqslant i\leqslant 3}|\mathbf{u}^{m+1}(e_i)|\right)^2&\leqslant 3\sum_{1\leqslant i\leqslant 3}|\mathbf{u}^{m+1}(e_i)|^2\\ &\leqslant 3\left(\sum_{1\leqslant i\leqslant 3}|\mathbf{u}^{m+1}(e_i)|^2+\left|\sum_{1\leqslant i\leqslant 3}\mathbf{u}^{m+1}(e_i)\right|^2\right)\\ &=3\left(2\sum_{1\leqslant i\leqslant 3}|\mathbf{u}^{m+1}(e_i)|^2\right.\\ &\left.+2\sum_{1\leqslant i<j\leqslant 3}\mathbf{u}^{m+1}(e_i)\cdot\mathbf{u}^{m+1}(e_j)\right)\\ &=36\int_{\mathcal{T}}(\mathbf{u}^{m+1}(\mathbf{x}))^2d\mathbf{x}/|\mathcal{T}|\end{aligned} \tag{56}$$

(the last equation comes from the exact integration of a P^1-field on $\mathcal{T}$). Finally, we get

$$\begin{aligned}|\Lambda(\mathbf{u}^{m+1})|&\leqslant\frac{C_2M}{q_1\alpha^2}\sum_{\mathcal{T}\in\partial T^{m+1}}|\mathcal{T}|^{1/2}\left|\int_{\mathcal{T}}(\mathbf{u}^{m+1}(\mathbf{x}))^2d\mathbf{x}\right|^{1/2}\\ &\leqslant\frac{C_2M}{q_1\alpha^2}\left(\sum_{\mathcal{T}\in\partial T^{m+1}}|\mathcal{T}|\right)^{1/2}\left(\int_{\mathcal{T}}(\mathbf{u}^{m+1}(\mathbf{x}))^2d\mathbf{x}\right)^{1/2}\\ &=\frac{C_2M}{q_1\alpha^2}|\partial T^{m+1}|^{1/2}\left(\int_{\mathcal{T}}(\mathbf{u}^{m+1}(\mathbf{x}))^2d\mathbf{x}\right)^{1/2}\\ &\leqslant\frac{C_2'M}{q_1\alpha^2}|\mathbf{u}^{m+1}|_1\end{aligned} \tag{57}$$

$$\leqslant\frac{C_2'M}{hq_1^2\alpha^3}|\mathbf{u}^{m+1}|_o. \tag{58}$$

Let us now bound the term $\int_{\Omega^{m+1}}(\mathbf{u}^m_{m+1}\circ X^m_{m+1})^2$. The change of variables

$$\mathbf{x}\mapsto\mathbf{y}\in\Omega^m,\quad\mathbf{y}=\mathbf{x}-k\mathbf{c}^m_{m+1}, \tag{59}$$

with Jacobian

$$\begin{aligned}|I_d-k\nabla\mathbf{c}^m_{m+1}|^{-1}(\mathbf{x})&=1+k\nabla\cdot\mathbf{c}^m(\mathbf{y})+O(k^2)|\mathbf{c}^m|_{1,\infty}\\ &=1+k\nabla\cdot\mathbf{c}^m(\mathbf{y})+O(k)q_2,\end{aligned} \tag{60}$$

gives

$$\begin{aligned}&\int_{\Omega^{m+1}}(\mathbf{u}^m_{m+1}\circ X^m_{m+1})^2\\ &=\int_{\Omega^m}(\mathbf{u}^m\circ X^m)^2(1+k\nabla\cdot\mathbf{c}^m(\mathbf{y})+O(k)q^2).\end{aligned} \tag{61}$$

A second change of variables is made along the characteristics $y \mapsto X^m$. The Jacobian J is

$$J = \exp\left(\int_{t^m}^{t^{m+1}} \nabla\cdot(\mathbf{u}^m - \mathbf{c}^m)\circ X^m\right)$$

$$= \exp\left(-\int_{t^m}^{t^{m+1}} \nabla\cdot\mathbf{c}^m\circ X^m\right)$$

$$= 1 - k\nabla\cdot\mathbf{c}^m + O(k^2)|\mathbf{c}^m|_{1,\infty}$$

$$= 1 - k\nabla\cdot\mathbf{c}^m + O(k)q_2. \tag{62}$$

Finally, it holds

$$\int_{\Omega^{m+1}} (\mathbf{u}^m_{m+1}\circ X^m_{m+1})^2 \leqslant \int_{\Omega^m} (\mathbf{u}^m)^2(1 + kq_2C_3). \tag{63}$$

Equations (53), (58) and (63) lead to the inequality (constants have been modified)

$$|\mathbf{u}^{m+1}|_o \leqslant |\mathbf{u}^m|_o(1 + kq_2C_3) + k\sigma\frac{C_2M}{q_1^2\alpha^3} + kgC_1, \tag{64}$$

so that, by summing from 0 to m_T and using classical majorations,

$$|\mathbf{u}^{m_T}|_o \leqslant \exp(q_2C_3T)|\mathbf{u}^o|_o + T\left(\frac{C_2M}{hq_1^2\alpha^3} + gC_1\right). \tag{65}$$

6. NUMERICAL APPLICATIONS

6.1. Impinging of a Jet on a Plane

The geometry illustrated in Figure 1 can be easily treated by this method, by choosing a radial field **R**. To illustrate the robustness of the method, a more complicated case has been chosen. The physical conditions are similar, but the fluid is now falling on a horizontal plane. The fluid is supposed to slip on the plane and to get out freely through the vertical cylindrical slab delimiting the domain. The stress coefficient of the horizontal plane is 0 (perfect slip condition). The calculations were done in two cases. The initial geometry for both cases consists in two coaxial cylinders (see Figure 6). The initial velocity is taken equal to 0. The topology of the mesh remains unchanged during the iterations. r being the distance to the vertical axis and z the vertical coordinate, **R** is radial for $z > 2$, vertical for $r > 2$, and it points at the circle defined by $z = 2$ and $r = 2$ otherwise. **R** is represented in a half cross section in Figure 7. **R** is axisymmetric and constant in time. The surface tension coefficient σ is 0.01 for the first case, and 0.005 for the second one.

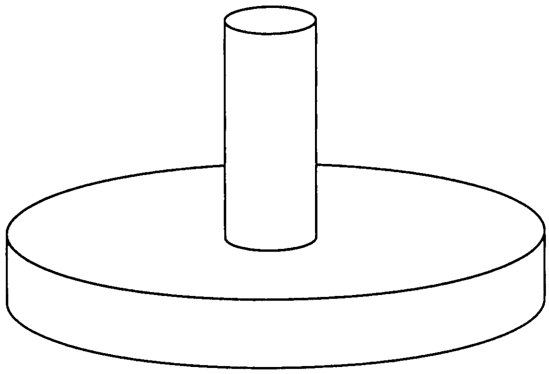

FIGURE 6 Initial domain.

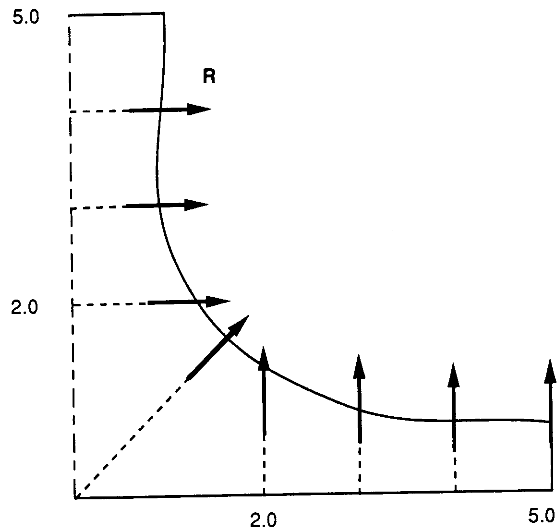

FIGURE 7 Field **R**.

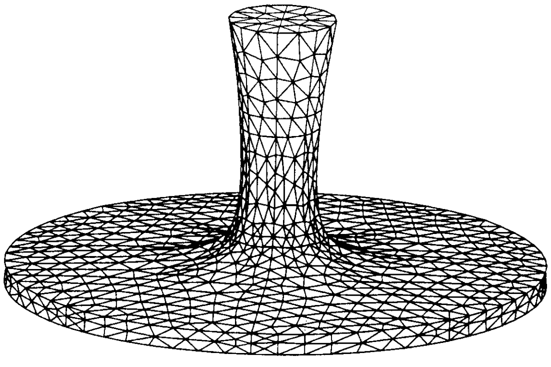

FIGURE 8 Final grid.

6.1.1. *Coating Flow*

In this case the Reynolds number Re is 1, and a stationary state is obtained after a few steps. Figures 8 and 9 represent respectively the grid and the velocity field (seen from the side) when the state is stationary (the grid does not move any more).

6.1.2. *Unstable Flow*

In the second case, Re = 100. The flow is not stationary, but the circular "wave" which is observed in the real flow is found in the model. Figure 10, 11 and 12 show the flow at time 1.5 (after 200 time iterations).

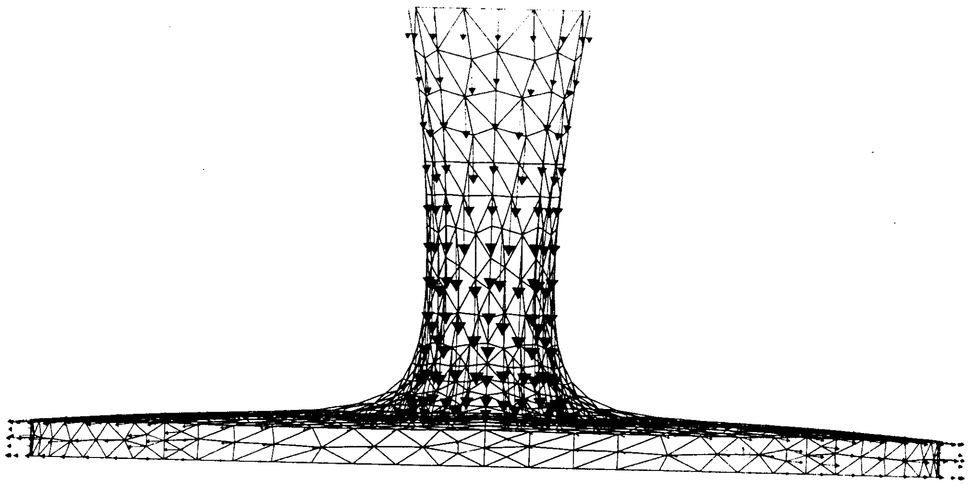

FIGURE 9 Velocity, stationary state.

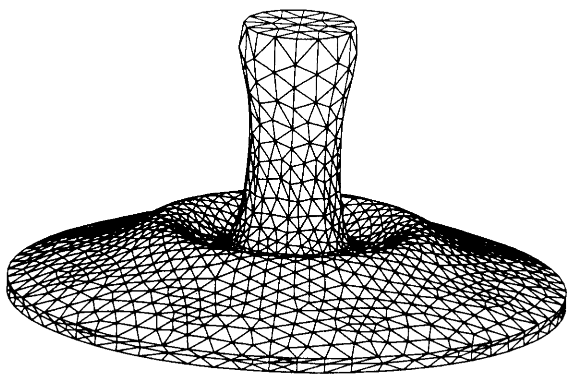

FIGURE 10 Moving grid, case 2.

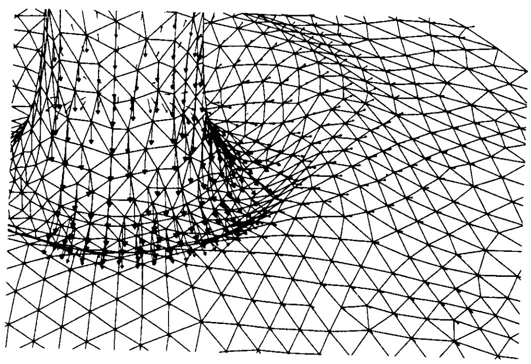

FIGURE 11 Detail of the grid.

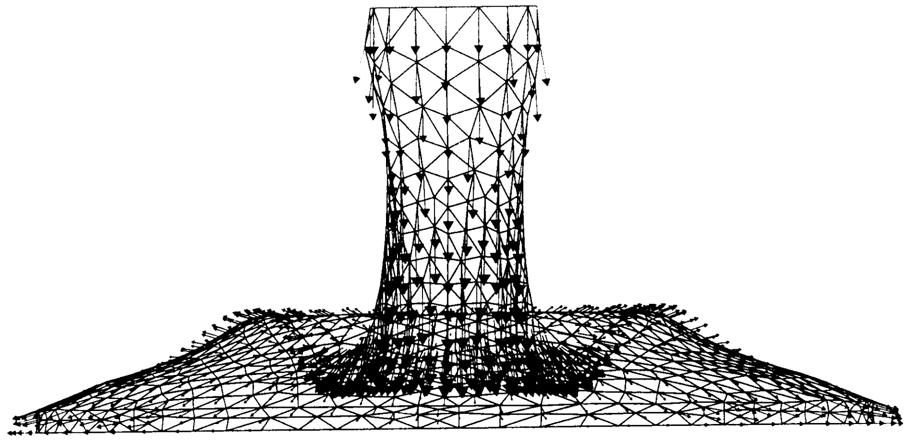

FIGURE 12 Side view, grid and velocity.

7. CONCLUSION

Particular attention has been paid in this paper to convection phenomena. The characteristics method confirms its stability in the domain of free surface flows. An important point is that, even in the case of very low Reynolds numbers (usually considered as Stokes flows), the non-linearity of the momentum equation and the non-linearity of the free surface have to be taken into account. This remark particularly concerns flows which are characterized by a predominating tangential velocity.

The second important feature is the control of the surface grid. It lacks at this time estimates relating the calculated velocity to the "regularity" of the resulting new surface grid. Such estimations would provide a stability property based only on hypotheses about the initial configuration.

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