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TITRE DE LA THÈSE :

Surfaces de Veech arithmétiques en genre deux :
disques de Teichmüller, groupes de Veech et constantes de Siegel–Veech

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Introduction générale

L'introduction générale et les trois chapitres qui la suivent peuvent être lus indépendamment ; chacun a sa propre table des matières, sa pagination et sa bibliographie.

L'introduction générale est écrite en français et reprend le contexte, les définitions utiles, et les résultats.

Les chapitres sont rédigés comme des articles, en anglais (les deux premiers, écrits avec Pascal Hubert, sont acceptés pour publication).

Contenu de l'introduction générale

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Présentation générale

Les espaces de modules de différentielles abéliennes et quadratiques sont des espaces très riches : leur étude mêle analyse complexe, géométrie algébrique, dynamique, combinatoire et théorie des nombres.

Ces espaces sont fibrés au-dessus des espaces de modules de surfaces de Riemann, et sont munis d'une action naturelle du groupe $\mathrm{SL}(2,\mathbf{R})$.

Les projections des orbites de cette action dans les espaces de modules de surfaces de Riemann jouent le rôle de géodésiques complexes pour la métrique de Teichmüller.

L'un des problèmes les plus importants à l'heure actuelle dans l'étude de la géométrie et de la dynamique de Teichmüller est de classifier les sous-variétés $\mathrm{SL}(2,\mathbf{R})$-invariantes dans les espace des modules de différentielles abéliennes et quadratiques.

L'espoir d'un analogue de la théorie de Ratner est devenu particulièrement fort depuis les récents résultats de Calta et de McMullen qui ont obtenu des résultats de classification en genre deux pour l'action de $\mathrm{SL}(2,\mathbf{R})$ entier, et d'Eskin–Marklof–Morris qui ont obtenu des résultats pour l'action de sous-groupes unipotents sur certains sous-espaces d'espaces de modules en genre supérieur.

Les sous-variétés invariantes les plus simples, les orbites fermées, sont les orbites des différentielles abéliennes dont le stabilisateur est un réseau. Elles se projettent dans l'espace des modules des surfaces de Riemann sur des courbes algébriques.

Un point de vue fructueux pour étudier l'action de $\mathrm{SL}(2,\mathbf{R})$ est de voir les différentielles abéliennes comme des surfaces de translation. Veech a initié l'étude des surfaces de translation dont le stabilisateur est un réseau, recherchant en particulier des réseaux non arithmétiques. Depuis, on appelle groupe de Veech le stabilisateur d'une surfac de translation sous l'action de $\mathrm{SL}(2,\mathbf{R})$, et surfaces de Veech celles dont le stabilisateur est un réseau.

Les surfaces de translation dites « à petits carreaux » sont les points rationnels des espaces de modules de différentielles abéliennes ; en particulier elles y sont réparties de façon dense. Gutkin et Judge ont montré que ce sont exactement les surfaces dont les groupes de Veech sont des réseaux arithmétiques, ce qui leur vaut l'appellation de surfaces de Veech arithmétiques, et fait de leurs orbites les orbites fermées les plus simples.

Cette thèse est une étude détaillée de ces orbites dans l'espace de modules $\mathcal{H}(2)$ des différentielles abéliennes sur des surfaces de genre deux avec un zéro double.

On s'intéresse en particulier aux trois problèmes suivants :

– nombre et géométrie des disques de Teichmüller (ou $\mathrm{SL}(2, \mathbf{R})$-orbites) des surfaces à nombre fixé de carreaux ;
– problème de congruence pour leurs groupes de Veech (stabilisateurs pour l'action de $\mathrm{SL}(2, \mathbf{R})$)
– comportement asymptotique des constantes de Siegel–Veech des orbites de surfaces à petits carreaux lorsque le nombre de carreaux tend vers l'infini.

Les trois chapitres traitent chacun d'un de ces trois problèmes. Ils reprennent des articles tels qu'ils ont été rédigés pour publication, et peuvent donc être lus indépendamment.

Le premier problème est traité dans un article écrit avec Pascal Hubert, accepté pour publication dans *Israel Journal of Mathematics* sous le titre "Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$" et dont le résultat principal est que lorsque n est premier $\geqslant 5$, il y a exactement deux disques de Teichmüller de surfaces à n carreaux dans $\mathcal{H}(2)$. Ce résultat a été généralisé par C. McMullen au comptage des disques de Teichmüller de surfaces de Veech de tous discriminants dans $\mathcal{H}(2)$.

Le deuxième problème est traité dans un article également écrit avec Pascal Hubert, et accepté pour publication dans *International Mathematics Research Notices* sous le titre "Noncongruence subgroups in $\mathcal{H}(2)$". Le résultat principal est que les groupes de Veech des surfaces à petits carreaux de la strate $\mathcal{H}(2)$ ne sont pas des groupes de congruence, sauf pour les surfaces à trois carreaux (le seul cas qui était compris jusqu'à il y a peu, si bien qu'on pensait que tous les groupes de Veech de surfaces à petits carreaux étaient des groupes de congruence).

Le troisième problème fait l'objet d'un travail réalisé seul, qui n'est pas encore soumis pour publication. Le résultat obtenu concerne la convergence des constantes de Siegel–Veech des surfaces à petits carreaux de la strate $\mathcal{H}(2)$ vers les constantes génériques de la strate lorsque le nombre de carreaux tend vers l'infini.

L'introduction générale commence (§ 0) par un énoncé des résultats plus précis que la brève description qui précède. Elle donne ensuite (§§ 1–3) un aperçu de la théorie où s'inscrivent les travaux exposés dans cette thèse. La § 4 entre dans le vif du sujet. Les §§ 5–7 complètent la présentation des chapitres 1, 2, 3 avec des exemples et des esquisses de démonstrations. On donne enfin (§ 8) des éléments de bibliographie.

On a annexé à l'introduction générale quelques développements complémentaires : sur les formes quadratiques, sur les comptages de surfaces à petits carreaux et sur les formes quasi-modulaires.

0. Résultats exposés dans la thèse

Nous énonçons ici les résultats principaux de cette thèse, en renvoyant au corps de l'introduction générale et des chapitres pour les définitions utiles.

0.1. Résultats du chapitre 1. Dans le premier chapitre, on s'attache à distinguer les disques de Teichmüller de surfaces à petits carreaux.

Le résultat principal est :

Théorème 1. *Les surfaces à n petits carreaux dans la strate $\mathcal{H}(2)$ forment, pour n premier,*
– *si $n = 3$, un seul disque de Teichmüller,*
– *si $n \geqslant 5$, deux disques de Teichmüller.*

(Les surfaces à petits carreaux dans $\mathcal{H}(2)$ ont au moins 3 carreaux.)

Ce résultat est étendu par McMullen [**Mc4**] de la façon suivante :

Théorème (McMullen). *Les surfaces à n petits carreaux dans la strate $\mathcal{H}(2)$ forment*
– *pour $n = 3$ ou n pair, un seul disque de Teichmüller,*
– *pour n impair $\geqslant 5$, deux disques de Teichmüller.*

La généralisation de McMullen va plus loin puisqu'elle traite toutes les surfaces de Veech de la strate $\mathcal{H}(2)$. À chaque surface de Veech de la strate $\mathcal{H}(2)$ on peut associer un *discriminant* $D \geqslant 5$ qui correspond à l'*ordre* (sous-anneau de l'anneau des entiers d'un corps quadratique totalement réel) qui agit par multiplication réelle sur sa jacobienne ; les discriminants sont des entiers congrus à 0 ou 1 modulo 4, et aux surfaces à petits carreaux correspondent des discriminants carrés. McMullen montre en fait :

Théorème (McMullen). *Les surfaces de Veech dans la strate $\mathcal{H}(2)$ forment : pour $D \equiv 1 \pmod 8$, $D \neq 9$, deux disques de Teichmüller ; pour les autres discriminants, un seul disque de Teichmüller.*

Un autre résultat du chapitre 1 est la présentation d'un invariant qui distingue les disques de Teichmüller de surfaces à n carreaux dans la strate $\mathcal{H}(2)$. Il s'agit du nombre de points de Weierstrass entiers, qui peut être soit 1 soit 3 lorsque n est impair.

Ce résultat est lui aussi étendu par McMullen qui exprime cet invariant comme une parité de structure spin, et montre qu'il distingue les disques de Teichmüller de surfaces de Veech de discriminant D de $\mathcal{H}(2)$.

Définition/notation. Nous appelons A_n et B_n les disques de Teichmüller correspondant respectivement à 1 et 3 points de Weierstrass entiers (pour $n = 3$ il n'y a pas de disque B).

Nous montrons également que les disques de Teichmüller des surfaces à petits carreaux peuvent avoir un *genre non nul*, et même aussi grand que l'on veut.

Théorème 2. *Les disques de Teichmüller A_n et B_n ont un genre qui croît asymptotiquement comme $\frac{3}{16}\frac{n^3}{12}$.*

Le fait d'avoir un genre positif implique que les groupes de Veech ne sont pas engendrés par des unipotents, contrairement aux exemples classiques de Veech.

Cela donne également des difféomorphismes pseudo-anosovs qui ne sont pas engendrés par des twists de Dehn, contrairement à la construction classique de Thurston.

Par exemple, on obtient un disque de Teichmüller de genre 1 pour les surfaces à 8 carreaux.

Voir § 5 un tableau donnant le genre et quelques autres renseignements sur les disques de Teichmüller de surfaces à n carreaux pour les petites valeurs de n.

Un autre résultat du premier chapitre est la croissance asymptotique du volume des disques de Teichmüller de surfaces à nombre premier de carreaux dans $\mathcal{H}(2)$.

Ce volume coïncide à un facteur près avec le nombre de surfaces à petits carreaux primitives que contient le disque de Teichmüller.

Des comptages exacts sont donnés sous forme de conjecture :

Conjecture 1. *Pour n impair $\geqslant 3$, le nombre de surfaces à petits carreaux primitives dans les disques A_n et B_n est respectivement*

$$\frac{3}{16}(n-1)n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right) \text{ et } \frac{3}{16}(n-3)n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right),$$

où les produits sont sur les diviseurs premiers de n.

Cette conjecture renvoie à des résultats sur des propriétés de quasi-modularité de certaines fonctions de comptages.

Une partie des résultats du chapitre 2 dépendent de cette conjecture ; on développe le thème des comptages et de la quasi-modularité dans la section 10 de l'introduction générale.

0.2. Résultats du chapitre 2. Dans le deuxième chapitre, on étudie les groupes de Veech des surfaces à petits carreaux.

Si l'on se restreint aux surfaces à petits carreaux primitives (ce qui revient à prendre « les bons carreaux », voir § 4.4), ces groupes sont des sous-groupes de $\mathrm{SL}(2, \mathbf{Z})$.

Pour les surfaces à trois carreaux, le groupe est assez facile à calculer, et c'est un sous-groupe de congruence de niveau 2 de $\mathrm{SL}(2, \mathbf{Z})$.

Il est donc assez naturel de se demander si c'est le cas des groupes de Veech des autres surfaces à petits carreaux, d'autant plus que d'autres sous-groupes de congruence de $\mathrm{SL}(2, \mathbf{Z})$ ont été trouvés comme groupes de Veech de surfaces à petits carreaux dans d'autres strates que $\mathcal{H}(2)$ par Schmoll [**Schmo**].

Quelques sous-groupes de $\mathrm{SL}(2, \mathbf{Z})$ qui ne sont pas des sous-groupes de congruence ont également été décelés ; la remarque en est faite au chapitre 1, ainsi que par Schmithüsen [**Schmi**].

Le résultat démontré au chapitre 2 est le suivant :

Théorème 3. *Les groupes de Veech des surfaces à $n \geqslant 4$ carreaux primitives sont, pour tous les n pairs et pour tous les n impairs pour lesquels la conjecture* 1 *est vraie, des sous-groupes de* $\mathrm{SL}(2, \mathbf{Z})$ *qui ne sont pas des sous-groupes de congruence.*

0.3. Résultats du chapitre 3. Dans le troisième chapitre, on s'intéresse aux géodésiques fermées sur les surfaces de translation de la strate $\mathcal{H}(2)$. Ces géodésiques fermées forment des cylindres.

Pour les surfaces à petits carreaux (et plus généralement pour les surfaces de Veech), on sait que le nombre de tels cylindres formés de géodésiques fermées simples de longueur n'excédant pas L croît asymptotiquement comme $c\pi L^2$, où c est une constante qui dépend de la surface considérée.

On sait également que pour chaque composante connexe de strate d'espace de modules de différentielles abéliennes, il y a une constante c telle que presque toute surface de chaque strate vérifie la même propriété avec cette constante c.

Ces constantes s'appellent constantes de Siegel–Veech des (composantes connexes de) strates.

Cependant les constantes particulières des surfaces de Veech ne coïncident pas avec celles des strates dans lesquelles elles se trouvent.

Le résultat du chapitre 3 est que les constantes des surfaces à petits carreaux permettent cependant de retrouver celles de la strate $\mathcal{H}(2)$.

Théorème 4. *Soit une suite S_n de surfaces à petits carreaux dans $\mathcal{H}(2)$, chacune étant pavée par un nombre premier p_n de petits carreaux, avec $p_n \to \infty$. Alors les constantes de Siegel–Veech des surfaces S_n tendent vers celle de la strate $\mathcal{H}(2)$.*

1. Espaces de modules

Le terme “espace de modules” est souvent utilisé de préférence à “espace de paramètres” dans des contextes où on cherche à décrire des objets géométriques à une certaine équivalence près.

Par exemple, on considère le module d'un cylindre (rapport de sa hauteur à sa circonférence) lorsque l'on s'intéresse à sa forme sans se préoccuper de sa taille. L'espace des modules des cylindres est l'ensemble des réels strictement positifs.

Un exemple plus instructif est l'espace des modules des tores.

1.1. Tores. Ici encore, on cherche à décrire la forme d'un tore (de dimension réelle 2, muni d'une structure complexe) sans tenir compte de sa taille.

Un tore peut être défini comme un quotient $\mathbf{C}/\Lambda$ où Λ est un réseau de $\mathbf{C}$. Considérons comme équivalents des tores correspondants à des réseaux obtenus l'un à partir de l'autre par rotation et dilatation. On peut alors, étant donné un tore, le considérer comme un quotient $\mathbf{C}/\Lambda$ où Λ est un réseau de $\mathbf{C}$ de base $(1, \tau)$. Deux paramètres τ et τ' décrivent des tores équivalents lorsqu'ils diffèrent d'un entier, lorsqu'ils sont opposés ou lorsque l'un est l'inverse de l'autre. On peut donc supposer $|\operatorname{Re}\tau| \leqslant 1/2$, $\operatorname{Im}\tau > 0$, et $|\tau| > 1$. Ceci dessine un domaine dans le demi-plan supérieur, situé entre les droites verticales d'abscisses $-1/2$ et $1/2$, et en-dehors du cercle de rayon 1 centré à l'origine. Certains points de ce domaine doivent encore être identifiés : les demi-droites verticales d'abscisses $-1/2$ et $1/2$ par translation horizontale, et les deux moitiés de l'arc de cercle qui borde le domaine inférieurement par l'application $z \mapsto -1/z$. On peut décrire globalement les identifications faites sur le bord du domaine : elles sont faites par réflexion par rapport à son axe de symétrie (vertical).

On appelle espace des modules des tores l'espace obtenu après ces identifications. Sa topologie est celle d'une sphère privée d'un point. Sa géométrie est plus riche, elle est héritée de la métrique hyperbolique du demi-plan supérieur. Les deux points correspondants à i et à $e^{i2\pi/3}$ représenent le tore carré et le tore hexagonal, qui ont des automorphismes d'ordre 4 et 6 respectivement ; ce sont des points coniques d'angles respectifs π et 2π dans l'espace des modules des tores.

Cet espace de modules, très classique, porte le nom de *surface modulaire* ou de *courbe modulaire* (suivant que l'on préfère le point de vue réel ou complexe). À cause des points coniques, ce n'est pas tout à fait une variété ; on dit que c'est un orbifold.

Remarque. On peut voir l'espace des modules des tores comme le quotient $\mathbf{C}^\times \times \mathrm{GL}(2,\mathbf{Z})\backslash\mathrm{GL}(2,\mathbf{R})$.

On peut également le voir comme $\mathrm{PSL}(2,\mathbf{Z})\backslash\mathbf{H}$, où $\mathrm{PSL}(2,\mathbf{Z})$ agit par homographies sur le demi-plan supérieur $\mathbf{H} = \{z \in \mathbf{C},\ \mathrm{Im}\, z > 0\}$.

1.2. Surfaces de Riemann de genre g. On ne considère ici que des surfaces orientées compactes et sans bord.

L'espace des modules des surfaces de Riemann de genre g décrit l'espace des structures complexes dont on peut doter une surface compacte de genre g, à équivalence biholomorphe près.

On s'intéresse ici au cas où le genre est au moins deux (le genre un correspond aux tores, vus plus haut).

Comme dans le cas des tores, cet espace est presque une variété complexe, mais pas tout à fait. Teichmüller a eu l'idée de considérer une relation d'équivalence plus restrictive sur les structures complexes, en introduisant un marquage (difféomorphisme depuis une surface de référence) et en ne considérant deux surfaces comme équivalentes que lorsqu'elles sont biholomorphement équivalentes via une application qui se traduit sur la surface de référence par un difféomorphisme homotope à l'identité ; cette relation d'équivalence plus fine donne un espace plus gros, qui est une variété complexe de dimension $3g-3$, homéomorphe à une boule ouverte ; on l'appelle espace de Teichmüller de genre g et on le note $\mathcal{T}_g$.

L'espace des modules qui nous intéresse en est un quotient par le *groupe modulaire* (de Teichmüller) de genre g, noté Mod_g. Ce groupe est lui-même le quotient du groupe des difféomorphismes d'une surface de genre g préservant l'orientation par le groupe des difféomorphismes homotopes à l'identité. C'est un groupe discret, qui agit sur $\mathcal{T}_g$ par isométries, proprement et discontinûment. Cependant l'action n'est pas libre, ce qui donne lieu dans le quotient à des singularités. Ainsi l'espace des modules n'est pas une variété complexe mais seulement un orbifold.

On notera ici $\mathcal{M}_g$ l'espace des modules des surfaces de Riemann de genre g (il est également parfois noté $\mathcal{R}_g$).

Remarque. Dans le cas du genre 1, l'espace de Teichmüller est le demi-plan supérieur $\mathbf{H}$. et le groupe modulaire est $\mathrm{PSL}(2,\mathbf{Z})$; ceci éclaire la remarque de la section 1.1. La dimension de $\mathcal{M}_g$ n'est donnée par la formule $3g-3$ qu'à partir du genre deux.

1.3. Différentielles abéliennes. On appelle différentielle abélienne une 1-forme holomorphe sur une surface de Riemann.

Les différentielles abéliennes sur une surface de Riemann de genre g donnée forment un espace vectoriel de dimension g.

La définition de l'espace de Teichmüller $\Omega\mathcal{T}_g$ et de l'espace des modules $\Omega\mathcal{M}_g$ des différentielles abéliennes sur des surfaces de Riemann de genre g mime celle de $\mathcal{T}_g$ et $\mathcal{M}_g$; les relations d'équivalence sur les surfaces de Riemann considérées pour définir $\mathcal{T}_g$ et $\mathcal{M}_g$ sont étendues à des différentielles abéliennes en demandant qu'une différentielle soit le tiré en arrière de l'autre par le difféomorphisme biholomorphe qui rend les surfaces de Riemann équivalentes.

Sur les tores. Deux différentielles abéliennes sur un même tore coïncident à un facteur multiplicatif complexe près. La différentielle abélienne standard est la forme $\mathrm{d}z$ (la forme $\mathrm{d}z$ de $\mathbf{C}$ passe au quotient $\mathbf{C}/\Lambda$ car c'est un quotient par des translations $z \mapsto z + \lambda$, $\lambda \in \mathbf{C}$, qui ne modifient pas la forme $\mathrm{d}z$).

L'espace de Teichmüller des différentielles abéliennes sur les tores est donc le fibré tangent à l'espace de Teichmüller des tores. De même pour les espaces de modules (en un point conique, il faut faire un peu attention pour définir l'espace tangent : on peut le définir comme quotient par un groupe fini de rotations de l'espace tangent à l'espace de Teichmüller en un point qui se projette sur le point conique).

En genre supérieur. Dès que le genre est au moins deux, $\Omega\mathcal{T}_g$ n'est plus le fibré tangent à $\mathcal{T}_g$, car leurs dimensions complexes respectives sont $4g - 3$ et $3g - 3$. Il s'agit cependant toujours d'un fibré au-dessus de $\mathcal{T}_g$, de fibre $\mathbf{C}^g$.

Une différentielle abélienne sur une surface de genre $g > 1$ a nécessairement des zéros, dont les ordres ont pour somme $2g - 2$.

Stratification. Cela donne lieu à une décomposition de $\Omega\mathcal{T}_g$ et $\Omega\mathcal{M}_g$ en *strates* correspondant aux différentes possibilités pour les ordres des zéros. On note $\Omega\mathcal{T}_g(k_1, \ldots, k_n)$ et $\Omega\mathcal{M}_g(k_1, \ldots, k_n)$ les strates correspondant à des zéros d'ordres k_1, …, k_n. La somme des k_i valant $2g-2$, la notation est parfois allégée en supprimant le genre g de la notation.

Chaque strate est un orbifold complexe de dimension $2g-1+n$. La strate correspondant à $2g - 2$ zéros simples s'appelle strate *principale*, sa dimension est $4g - 3$. La strate correspondant à un unique zéro d'ordre $2g - 2$ s'appelle la strate *minimale*, sa dimension est $2g$.

Genre deux. En genre deux, on a $2g-2 = 2$; il y a donc uniquement deux strates, la strate principale (correspondant à la partition $(1, 1)$) et la strate minimale (correspondant à la partition (2)).

Les travaux décrits dans cette thèse portent principalement sur la strate minimale.

Différentielles abéliennes normées. On définit la *norme* d'une différentielle abélienne ω sur une surface de Riemann X comme la quantité $\frac{1}{2}\int_X \omega \wedge \overline{\omega}$. On sera amené à considérer les différentielles abéliennes de norme 1. Leurs espaces de Teichmüller et de modules sont notés $\Omega_{(1)}\mathcal{T}_g$ et $\Omega_{(1)}\mathcal{M}_g$.

Notations. On utilise aussi les notations $\mathcal{H}_g$ et $\mathcal{H}(k_1, \ldots, k_n)$ pour $\Omega\mathcal{M}_g$ et $\Omega\mathcal{M}_g(k_1, \ldots, k_n)$. L'utilisation de la lettre H vient de ce que les différentielles abéliennes sont les 1-formes holomorphes.

2. Surfaces de translation

2.1. Définition à partir des différentielles abéliennes. Considérons une différentielle abélienne ω sur une surface de Riemann X.

Il existe un système de coordonnées complexes z sur X, compatible avec sa structure complexe, tel qu'hors des zéros de ω on puisse écrire ω comme $\mathrm{d}z$, et qu'au voisinage d'un zéro d'ordre k on puisse écrire ω comme $z^k \mathrm{d}z$.

Ce système de coordonnées est facile à décrire : il suffit, au voisinage d'un point P_0 de X, de repérer la position d'un point P par la coordonnée $z(P) = \int_{P_0}^{P} \omega$.

Cela donne des cartes dans $\mathbf{C}$ formant un atlas dont les changements de cartes sont de la forme $z \mapsto z + c$, autrement dit des translations.

Plutôt que de choisir des cartes ouvertes, on peut alors choisir des cartes polygonales, avec des recollements le long de côtés de ces polygones, par translation ; les zéros ne sont pas gênants si on les prend pour sommet des cartes polygonales.

L'angle total autour d'un zéro d'ordre k sera $(k+1)2\pi$ dans les cartes. Cela correspond au fait qu'à un facteur constant près, $z^k \mathrm{d}z$ est $\mathrm{d}(z^{k+1})$.

Une surface de translation peut être définie comme un assemblage fini de polygones euclidiens, les recollements se faisant par translation.

À partir d'une surface de translation définie de cette façon, on peut retrouver une différentielle abélienne.

Les notions de surface de translation et de différentielle abélienne sont équivalentes ; voir [**Ma3**]. Dans la suite, on utilise les notations S ou (X, ω) en s'autorisant à passer de l'une à l'autre suivant le point de vue envisagé.

La norme d'une différentielle abélienne n'est autre que l'aire de la surface de translation correspondante.

Enfin, il est parfois utile de marquer des points sur une surface de translation, ce qui consiste à introduire des singularités coniques artificielles (d'angle 2π).

Remarque. On pourrait définir les surfaces de translation "à rotation près" et considérer que la définition donnée plus haut fournit une surface de translation avec en plus le choix d'une direction particulière. Cependant en pratique la notion définie plus haut, et équivalente à celle de différentielle abélienne, est plus utile.

2.2. Action de $\mathrm{GL}(2, \mathbf{R})$**.** Étant donnée une surface de translation, on peut faire agir $\mathrm{GL}(2, \mathbf{R})$ sur ses cartes (en utilisant la structure naturelle d'espace vectoriel réel de dimension 2 de $\mathbf{C}$).

L'action de $\mathrm{GL}(2, \mathbf{R})$ respecte le parallélisme et l'égalité des longueurs sur des parallèles, si bien que les identifications par translations sur les transformées des cartes restent possible.

Ceci permet de définir une action de $\mathrm{GL}(2, \mathbf{R})$ sur les surfaces de translation. En pratique, il est plus intéressant de considérer l'action de $\mathrm{GL}^+(2, \mathbf{R})$. Cette action passe au quotient en une action sur chaque espace de modules $\Omega\mathcal{M}_g$, et sur chaque strate $\Omega\mathcal{M}_g(k_1, \ldots, k_n)$.

Lorsqu'on transforme une surface de translation par une matrice A de $\mathrm{GL}(2, \mathbf{R})$, son aire est multipliée par $|\det A|$.

Le groupe $\mathrm{SL}(2, \mathbf{R})$ agit donc sur les espaces de modules et les strates de différentielles abéliennes normées.

2.3. Mesure invariante. Masur et Veech ont montré indépendamment [**Ma2, Ve82**] qu'il existe une mesure naturelle sur les espaces de modules, qui donne un volume fini aux strates de différentielles abéliennes normées. Cette mesure est invariante pour l'action de $\mathrm{SL}(2, \mathbf{R})$, et les composantes ergodiques des strates sont leurs composantes connexes. Celles-ci ont été classifiées par Kontsevich et Zorich [**KoZo**].

La topologie et la géométrie des espaces de modules de différentielles abéliennes posent encore de nombreuses questions.

Le problème majeur consiste à décrire toutes les sous-variétés invariantes (par l'action de $\mathrm{SL}(2, \mathbf{R})$)fermées, en particulier les adhérences d'orbites, et les mesures invariantes.

L'orbite d'un point générique d'une strate $\mathcal{H}(k_1, \ldots, k_n)$ est dense dans la composante connexe où il se trouve.

Il n'y a pas d'orbites compactes, mais il existe des orbites fermées, celles pour lesquelles les stabilisateurs sont des réseaux.

2.4. Surfaces de Veech.

Théorème (Veech [**Ve89**]). *Si le stabilisateur sous l'action de* $\mathrm{SL}(2,\mathbf{R})$ *d'une surface de translation est un réseau, alors dans chaque direction le flot directionnel sur cette surface est soit complètement périodique, soit uniquement ergodique.*

On dit qu'une surface de translation satisfait l'alternative de Veech si dans chaque direction le flot directionnel sur cette surface est soit complètement périodique, soit uniquement ergodique.

On peut reformuler ce théorème en disant que les surfaces de Veech satisfont l'alternative de Veech. La réciproque est un problème ouvert.

Ce théorème est complété par un résultat qui indique que la propriété d'avoir un groupe de Veech réseau est préservée par certains revêtements. On introduit d'abord deux notions : un revêtement de translation est un revêtement qui se traduit dans les cartes par des translations, et il est dit équilibré si chaque image et chaque antécédent de singularité conique est une singularité conique. On rappelle également que deux groupes sont dits commensurables s'ils partagent un sous-groupe d'indice fini dans chacun.

Théorème (Gutkin–Judge, Vorobets). *Si S' est un revêtement de translation équilibré de S, alors leurs groupes de Veech sont commensurables.*

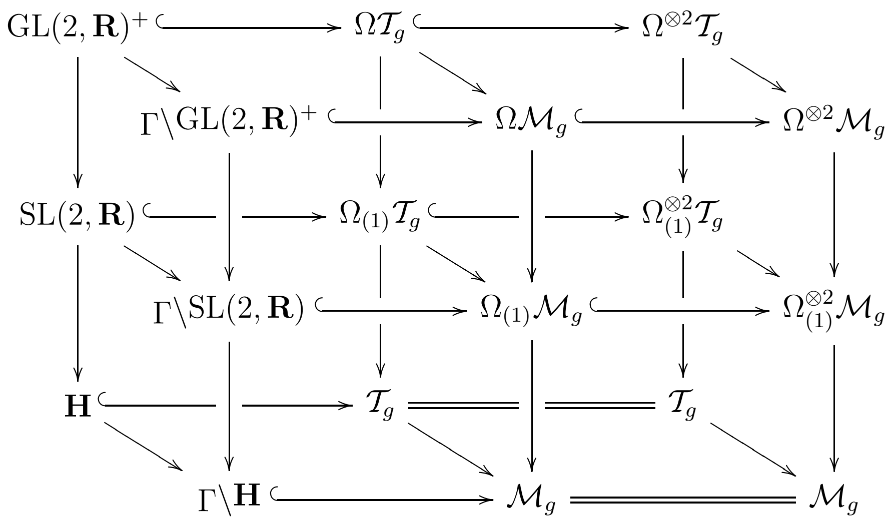
Remarque. Il y a une notion plus faible de commensurabilité, qui étend la relation de commensurabilité définie ci-dessus par conjugaison. Lorsque l'on a besoin de distinguer les deux notions, on dit parfois *commensuré* pour commensurable au sens fort.

3. Vue d'ensemble des espaces

Soit (X,ω) un point de $\Omega\mathcal{T}_g$. Sa $\mathrm{GL}(2,\mathbf{R})^+$-orbite dans $\Omega\mathcal{T}_g$ est isométrique à $\mathrm{GL}(2,\mathbf{R})^+$. Cette orbite elle-même, et ses projections dans $\Omega_{(1)}\mathcal{T}_g$, $\Omega\mathcal{M}_g$, $\Omega_{(1)}\mathcal{M}_g$, sont parfois appelées disque de Teichmüller (c'est la projection dans $\mathcal{T}_g$, isométrique à $\mathcal{H}$, qui est un vraiment un disque). Par abus, on note également (X,ω) l'image dans $\Omega\mathcal{M}_g$ de ce point de $\Omega\mathcal{T}_g$. On note $\mathrm{SL}(X,\omega)$ le stabilisateur de ce point dans $\Omega\mathcal{M}_g$. Lorsque le stabilisateur $\Gamma = \mathrm{SL}(X,\omega)$ est un réseau, la projection de l'orbite de (X,ω) dans $\mathcal{M}_g$ est une courbe de Teichmüller, isométrique à $\Gamma\backslash\mathbf{H}$.

Le paragraphe précédent peut être répété en remplaçant partout Ω par $\Omega^{\otimes 2}$. Cependant, dans cette thèse, on ne s'intéresse qu'aux différentielles abéliennes, et on laisse de côté les différentielles quadratiques.

Le diagramme suivant représente différents espaces considérés dans cette introduction.



Les espaces de différentielles quadratiques sur la face de droite du diagramme ne sont décrits qu'en annexe, car ils ne sont pas étudiés dans cette thèse.

4. Surfaces à petits carreaux

Comme on l'a vu, une surface de translation peut être munie d'un atlas de translation, à cartes polygonales.

En utilisant pour cartes des carrés horizontaux de même aire, on obtient une surface de translation « à petits carreaux ». Ce sont ces surfaces que nous désignerons par le vocable « surfaces à petits carreaux ».

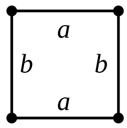
Dans une acception plus large, l'appellation « surface à petits carreaux » pourrait désigner toute surface ayant un atlas formé de carrés identiques, comme par exemple un cube (la surface d'un cube).

Pour le cube, certaines des identifications font nécessairement intervenir des rotations d'angle $\pi/2$. L'angle conique en chacun des sommets d'un cube est $3\pi/2$. Remarquons que par contre, l'angle autour d'un point situé sur une arête est 2π : la particularité des points situés sur les arêtes n'est qu'un artifice du plongement du cube dans l'espace ; lorsqu'on considère un patron plan du cube, les points situés sur les arêtes ne se distinguent pas des points situés sur les faces.

L'intérêt de l'étude des surfaces à petits carreaux dans ce sens plus large sort du cadre de cette thèse, aussi nous n'y reviendrons plus. Nous renvoyons le lecteur intéressé à l'article [**Wi**].

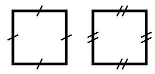
4.1. Exemples. Examinons les surfaces (de translation) à petit nombre de carreaux.

4.1.1. *Un carreau.* Le premier exemple de surface à petits carreaux est le tore carré, formé d'un seul carreau dont on identifie les côtés opposés par translation comme indiqué ci-contre. C'est la seule surface à un carreau.

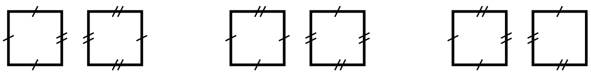


4.1.2. *Deux carreaux.* Les possibilités se diversifient très peu.

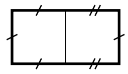
Les identifications indiquées ci-contre donnent une surface non connexe, composée de deux tores à un carreau.



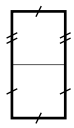
Il y a trois autres possibilités d'identifications de côtés :



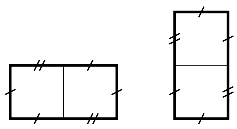
Pour la surface représentée par les deux carrés de gauche, on a « croisé » les identifications de côtés horizontaux. On préfère représenter cette surface comme ci-contre.



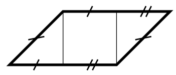
Pour la surface représentée par les deux carrés du centre, ce sont les identifications de côtés verticaux qui ont été croisées. On préfère la représenter comme un rectangle de largeur 1 et de hauteur 2.



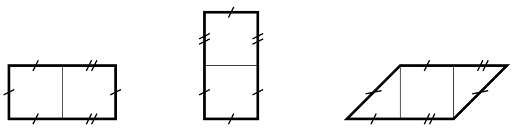
Pour la surface représentée par les deux carrés de droite, à la fois les identifications de côtés horizontaux et de côtés verticaux. On pourrait la représenter d'une des deux façons ci-contre...



On privilégie (arbitrairement) les cylindres de trajectoires fermées horizontales; une représentation sous forme de parallélogramme horizontal permet de simplifier encore les identifications de côtés horizontaux.



Finalement, les surfaces (connexes) à deux carreaux sont trois tores :



4.1.3. *Trois carreaux.* Avec trois carreaux, on peut fabriquer quatre tores de translation différents, mais on peut également fabriquer trois surfaces de $\mathcal{H}(2)$ (genre deux, une singularité conique d'angle 6π).

4.1.4. *Quatre carreaux.* Avec quatre carreaux, on peut fabriquer sept tores de translation, neuf surfaces de $\mathcal{H}(2)$, et quatre surfaces de $\mathcal{H}(1,1)$.

4.2. Caractérisation. Le théorème dû a Gutkin–Judge et à Vorobets, cité § 2.4, indique, puisque le groupe de Veech du tore standard est $\mathrm{SL}(2,\mathbf{Z})$, que les groupes de Veech des surfaces à petits carreaux sont commensurables à $\mathrm{SL}(2,\mathbf{Z})$ (de tels groupes sont appelés *arithmétiques*).

Le théorème suivant indique que cela caractérise les surfaces à petits carreaux.

THÉORÈME (Gutkin–Judge [**GuJu**]). *Les surfaces à petits carreaux sont exactement les revêtements de translation du tore ramifiés au-dessus d'un seul point, et ce sont exactement les surfaces de translation ayant un groupe de Veech commensurable à* $\mathrm{SL}(2, \mathbf{Z})$.

4.3. Coordonnées.

4.3.1. *Permutations.* La définition des surfaces à petits carreaux comme assemblage de carrés horizontaux identiques avec identifications de côtés par translations suggère d'utiliser des permutations. Pour une surface formée de n carrés, numérotés de 1 à n, les identifications bord droit–bord gauche définissent une permutation de S_n, de même que les identifications bord haut–bord bas.

Ainsi, une surface à petits carreaux peut être définie par deux permutations.

Cependant, un assemblage de carrés selon deux permutations ne définit pas toujours une surface connexe ; de plus, le problème de déterminer si deux couples de permutations définissent la même surface est assez délicat, de même que le problème de définir des coordonnées canoniques en ces termes.

Nous préfèrerons donc une autre description.

4.3.2. *Cylindres horizontaux.* Soit $S = (X, \omega)$ une surface à petits carreaux.

Sur l'ensemble C des carreaux de S, on peut définir l'application d qui à un carreau associe celui situé à sa droite sur S. Pour chaque carreau c, il existe un entier $w \geqslant 1$, inférieur ou égal au nombre de carreaux de S, tel que $d^w(c) = c$. Le bord droit de $d^{w-1}(c)$ est identifié au bord gauche de c.

Ainsi toutes les lignes (géodésiques) horizontales menées à partir des points situés dans l'intérieur des carrés sont périodiques et font partie de bandes de géodésiques horizontales périodiques ; on peut voir ces bandes comme des cylindres euclidiens ouverts $\mathbf{R}/w\mathbf{Z} \times]0, h[$, avec w et h entiers.

En plus des géodésiques horizontales périodiques, S peut contenir des géodésiques horizontales singulières, qui relient deux singularités coniques de S (éventuellement confondues).

Ces liaisons géodésiques entre selles seront appelées *liens de selles horizontaux* (en anglais *horizontal saddle connections*). Plus généralement, les liaisons géodésiques entre selles sont appelées liens de selles. L'intégrale de ω le long d'un lien de selle est appelée vecteur de lien de selles ; c'est une *période relative* de la forme ω. Le sous-groupe de $\mathbf{Z}^2$

engendré par les vecteurs de liens de selles de ω s'appelle le *réseau des périodes* de ω.

On peut doter l'ensemble des surfaces à petits carreaux de coordonnées reflétant leur décomposition en cylindres de géodésiques horizontales périodiques. On appellera ces cylindres les cylindres horizontaux de S (même s'ils mériteraient d'être appelés verticaux, puisque leurs cercles sont horizontaux).

4.3.3. *Cas de la strate* $\mathcal{H}(2)$. Dans la strate $\mathcal{H}(2)$, une surface à petits carreaux compte au maximum deux cylindres.

En effet la singularité conique d'angle 6π, doit apparaître trois fois sur des bords hauts de cylindres, et trois fois sur des bords bas, et au moins une fois sur chaque bord haut et sur chaque bord bas de cylindre. Cela limite a priori le nombre de cylindres à trois.

Supposons que S ait trois cylindres ; on a donc une seule apparition de la selle sur chaque bord haut et sur chaque bord bas de cylindre. Pour former une surface connexe à partir des trois cylindres, on doit identifier leurs bords hauts et bas dans un certain ordre cyclique. Les largeurs des cylindres doivent donc être les mêmes ; mais ces identifications forment alors un tore, sur lesquels les points qui étaient censés être la selle ne sont pas identifiés ; on n'est donc pas dans la strate $\mathcal{H}(2)$.

4.3.4. *Surfaces à un cylindre.* Pour former une surface de $\mathcal{H}(2)$ à partir d'un seul cylindre, la selle doit apparaître trois fois sur son bord haut et trois fois sur son bord bas. Ceci donne trois liens de selles en haut et trois en bas ; si on les identifie en conservant leur ordre cyclique, on forme un tore, mais si on les identifie en inversant leur ordre cyclique, on forme bien une surface de $\mathcal{H}(2)$.

4.3.5. *Surfaces à deux cylindres.* Un des cylindres doit avoir la selle présente une fois sur son bord haut et une fois sur son bord bas, l'autre deux fois et deux fois. (Si c'était une fois–deux fois et deux fois–une fois, les longueurs des liens de selle ne permettraient pas des identifications, car la circonférence d'un cylindre euclidien est la même en haut et en bas.) Le cylindre qui n'a la selle qu'une fois en haut et en bas a donc un seul lien de selles sur son bord haut et sur son bord bas. Ces liens de selles ne sont pas identifiés entre eux (cela formerait un tore), donc ils sont identifiés avec des liens de selle sur le bord bas et sur le bord haut du deuxième cylindre. Il reste un lien de selle sur chacun des bords (haut et bas) du deuxième cylindre, qui doivent être identifiés.

Ces identifications forment bien une surface de $\mathcal{H}(2)$.

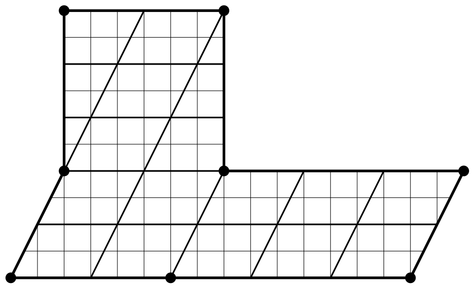
Remarque. Voir une autre approche en partant des diagrammes de séparatrices au chapitre 1. La description dans les paragraphes qui précèdent m'a été inspirée par Thierry Monteil.

4.4. Primitivité. On dit qu'une surface à petits carreaux est primitive si son réseau des périodes est $\mathbf{Z}^2$.

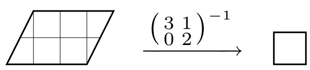
On peut compléter le théorème de Gutkin et Judge en précisant que le groupe de Veech d'une surface à petits carreaux primitive est toujours un sous-groupe de $\mathrm{SL}(2, \mathbf{Z})$.

Une surface de genre au moins deux à nombre premier de petits carreaux est toujours primitive.

Voici un exemple de surface à 96 carreaux qui n'est pas primitive. En effet, elle peut être pavée par des parallélogrammes d'aire 6 « à coordonnées entières » ; ce sont les plus gros possibles et on peut les considérer comme « les bons carreaux ».

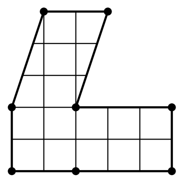


On trouve facilement une matrice de $\mathrm{GL}^+(2, \mathbf{R})$ qui transforme ces parallélogrammes en carrés unités.



En faisant agir cette matrice sur notre surface de départ, on obtient une surface à 16 carreaux primitive de la même $\mathrm{GL}^+(2, \mathbf{R})$ orbite.

En choisissant un autre parallélogramme à 6 carreaux, on aurait obtenu une autre surface à 16 carreaux de la même $\mathrm{SL}(2, \mathbf{Z})$-orbite.



5. Disques de Teichmüller

5.1. Esquisse de démonstration du théorème 1. L'invariant décrit plus haut (§ 0.1) permet de montrer, pour un nombre de carreaux $n \geqslant 5$ impair, qu'il y a au moins deux orbites. Il reste à montrer qu'il n'y en a que deux.

Le schéma de démonstration est le suivant : dans un premier temps, on montre que chaque surface de $\mathcal{H}(2)$ à nombre premier n de petits carreaux a dans son orbite une surface à petits carreaux à un seul cylindre.

Ensuite on montre que chaque surface à un seul cylindre a dans son orbite une surface à un seul cylindre bordé par des liens de selles de longueurs soit $(1, 1, n-2)$ soit $(1, 2, n-3)$.

Pour ces deux étapes, on utilise les « mouvements élémentaires » consistant à faire agir les générateurs $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ et $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ de $\mathrm{SL}(2, \mathbf{Z})$.

5.2. Quelques données numériques. On donne dans le tableau ci-dessous les caractéristiques géométriques des disques de Teichmüller de surfaces à petits carreaux de $\mathcal{H}(2)$ pour un nombre de carreaux n compris entre 3 et 40. On y donne pour chaque disque le nombre de surfaces à petits carreaux qu'il contient, le nombre de points elliptiques, le nombre de pointes (ou cusps), et le genre.

n impair

n	$\#A_n$	e	c	g	$\#B_n$	e	c	g
3	3	1	2	0				
5	18	-	5	0	9	1	3	0
7	54	2	10	0	36	-	8	0
9	108	-	16	2	81	3	14	0
11	225	3	26	6	180	-	26	3
13	378	-	37	14	315	3	39	7
15	504	4	42	21	432	-	42	16
17	864	-	60	43	756	4	68	29
19	1215	5	72	65	1080	-	84	49
21	1440	-	80	81	1296	8	88	63
23	2178	6	98	132	1980	-	120	106
25	2700	-	126	163	2475	5	148	132
27	3159	9	124	200	2916	-	148	170
29	4410	-	157	290	4095	7	199	241
31	5400	8	174	362	5040	-	224	309
33	5760	-	190	386	5400	12	230	333
35	7344	8	232	495	6912	-	280	437
37	9234	-	245	648	8721	9	323	564
39	9576	12	246	673	9072	-	310	602

n pair

n	$\#E_n$	e	c	g
4	9	1	3	0
6	36	-	8	0
8	108	2	17	1
10	216	-	30	4
12	360	4	38	11
14	648	-	60	25
16	1008	4	76	46
18	1296	-	88	65
20	1944	4	124	100
22	2700	-	148	152
24	3168	8	150	188
26	4536	-	206	276
28	5616	8	246	344
30	6048	-	240	385
32	8640	8	300	569
34	10368	-	356	687
36	11016	12	340	746
38	14580	-	420	1006
40	16416	8	458	1138

6. Groupes de Veech

Les surfaces à petits carreaux dont le calcul du groupe de Veech est élémentaire sont le tore, les surfaces à trois carreaux, et quelques exemples construits spécialement pour avoir comme groupe de Veech le groupe $\mathrm{SL}(2, \mathbf{Z})$ entier.

Dans le cas du tore, on trouve $\mathrm{SL}(2, \mathbf{Z})$, et dans le cas des surfaces à trois carreaux de $\mathcal{H}(2)$ on trouve un sous-groupe d'indice 3 de $\mathrm{SL}(2, \mathbf{Z})$ qui contient le sous-groupe de congruence principal de niveau 2 comme sous-groupe d'indice 2.

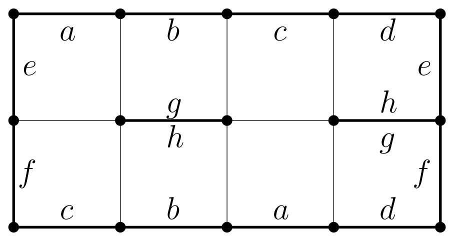
Voir la section 2.3 du chapitre 3 pour la définition et quelques propriétés des groupes de congruence.

Schmoll a calculé les groupes de Veech des tores à plusieurs points marqués [**Schmo**]. Il trouve des groupes de congruence.

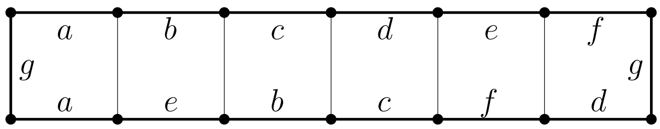
Cependant, alors qu'on aurait pu penser que tous les groupes de Veech de surfaces à petits carreaux sont des groupes de congruence, il n'en est rien.

Au chapitre 2, on montre (en admettant la conjecture de quasi-modularité des comptages par orbites), qu'en fait parmi les surfaces à petits carreaux de la strate $\mathcal{H}(2)$, celles à trois carreaux sont les seules qui ont pour groupe de Veech un groupe de congruence.

6.1. Groupes de Veech particuliers. Herrlich et Möller ont trouvé plusieurs surfaces dont le groupe de Veech est $\mathrm{SL}(2, \mathbf{Z})$ entier ; l'une d'elle, à 8 carreaux, est présentée dans [**HS**]. Il s'agit d'une surface de la strate $\mathcal{H}(1, 1, 1, 1)$ (genre 3). En voici une représentation :



En genre 2, toutes les surfaces sont hyperelliptiques, et ont donc $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ dans leur groupe de Veech. Dès le genre 3, cela cesse d'être vrai. Voici une surface non-hyperelliptique de $\mathcal{H}(4)$ (genre 3) qui n'a pas $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ dans son groupe de Veech.



6.2. Sous-groupes « de non-congruence ». Le groupe $\mathrm{SL}(2, \mathbf{Z})$ possède une famille de sous-groupes remarquables, qui sont les sous-groupes de congruence, définis à partir des congruences d'entiers de la façon suivante.

Pour tout entier $m > 1$, la réduction modulo m définit un morphisme d'anneaux de $\mathrm{SL}(2, \mathbf{Z})$ dans $\mathrm{SL}(2, \mathbf{Z}/m\mathbf{Z})$. Le noyau de ce morphisme, constitué des matrices « congrues à l'identité modulo m », est appelé *sous-groupe de congruence principal de niveau* m, et noté $\Gamma(m)$. On a alors, pour $m_1 \mid m_2$, $\Gamma(m_2) \subset \Gamma(m_1)$. L'indice $[\mathrm{SL}(2, \mathbf{Z}) : \Gamma(m)]$ vaut $\frac{1}{2}m^3 \prod_{p|m}(1 - \frac{1}{p^2})$ (le produit est sur les diviseurs premiers de m). Tout sous-groupe de $\mathrm{SL}(2, \mathbf{Z})$ contenant un sous-groupe de congruence principal est dit *de congruence*, et son niveau est défini comme le niveau du plus gros sous-groupe de congruence principal qu'il contient.

Il y a une autre définition du niveau d'un sous-groupe d'indice fini de $\mathrm{SL}(2, \mathbf{Z})$, comme plus petit multiple commun des largeurs de ses pointes. Un théorème de Wohlfahrt indique qu'un sous-groupe d'indice fini de $\mathrm{SL}(2, \mathbf{Z})$ de niveau m est de congruence si et seulement si il contient le sous-groupe de congruence principal de niveau m.

Wohlfahrt utilise ce théorème pour montrer que certains groupes d'indice finis ne sont pas de congruence en montrant que leur indice ne divise pas celui du groupe de congruence principal de même niveau.

Pour les groupes de Veech des surfaces à petits carreaux de $\mathcal{H}(2)$, cet argument ne s'applique pas, car leur indice divise toujours celui du sous-groupe de congruence principal de même niveau.

Un argument plus fin de Kühnlein utilisé par Schmithüsen [**Schmi**] dans un cas particulier permet par contre de conclure.

7. Constantes de Siegel–Veech

Il s'agit comme on l'a dit dans la § 0.3 des constantes qui apparaissent dans les asymptotiques quadratiques des comptages de cylindres de géodésiques simples fermées sur les surfaces à petits carreaux.

Un résultat d'Eskin et Masur indique que dans chaque composante connexe de strate d'espace de modules de différentielles abéliennes, *presque toutes* les surfaces partagent les mêmes constantes.

Ce n'est pas le cas de *toutes* les surfaces ; en effet les constantes correspondantes pour les surfaces à petits carreaux sont différentes.

Cependant, on peut se demander si les constantes des surfaces à petits carreaux d'une composante connexe de strate ont une limite quand le nombre de carreaux tend vers l'infini.

Au chapitre 3, on établit que c'est en effet le cas, en se restreignant (pour des raisons techniques) aux surfaces à nombre premier de carreaux.

La méthode consiste à distinguer, dans l'asymptotique du nombre de points entiers primitifs de $\mathbf{Z}^2$ dans un disque de rayon L, la proportion qui correspond à des directions associées à un cusp particulier du disque de Teichmüller.

Pour la surface à trois carreaux en L sans twists, on sait que les points entiers primitifs (a, b) avec a et b impairs correspondent au cusp à 1 cylindre et ceux pour lesquels a ou b est pair au cusp à 2 cylindres.

Dans le cas général, on ne sait pas déterminer immédiatement à quel cusp correspond une direction donnée sur une surface, mais les proportions asymptotiques sont celles des largeurs de cusps.

Le résultat du chapitre 3 pourrait être obtenu comme conséquence de théorèmes « de Ratner » (classification de mesures invariantes par les actions de groupes unipotents), mais de tels théorèmes, dont on suppose qu'ils sont démontrables pour les espaces de modules de différentielles abéliennes et quadratiques comme pour les espaces homogènes, ne sont pas encore démontrés, bien que des progrès aient été faits en ce sens.

8. Élements de bibliographie

Pour clore cette introduction générale, nous indiquons quelques textes introductifs qui complètent notre introduction dans différentes directions. Concernant la théorie de Teichmüller, citons le livre *The complex analytic theory of Teichmüller spaces* de Nag [**N**]. Concernant les surfaces plates, et leur lien avec les billards rationnels, le survol “Rational billiards and flat structures” de Masur et Tabachnikov [**MT**]. Sur les surfaces de Veech, les notes de cours “An introduction to Veech surfaces” de Hubert et Schmidt fournissent une introduction ainsi qu’un état de l’art du sujet [**HS**].

Annexes

9. Différentielles quadratiques et d'ordre supérieur

9.1. Différentielles quadratiques. Pour compléter la présentation des divers espaces introduits à la section 1, nous mentionnons brièvement les différentielles quadratiques. Notons qu'il n'en sera pas question dans le corps de la thèse.

Définition locale. Étant donnée une surface de Riemann X, une différentielle quadratique sur X peut être définie par la données sur chaque carte (U, z) d'une fonction méromorphe f de la variable z, avec la condition suivante de recollement lorsque des cartes (U_1, z_1) et (U_2, z_2) ont une intersection :

$$\frac{f_2(z_2(z_1))}{f_1(z_1)} \times \left(\frac{\mathrm{d}z_2(z_1)}{\mathrm{d}z_1}\right)^2 = 1.$$

On s'intéresse plus particulièrement aux différentielles quadratiques dont les pôles, s'il y en a, sont simples. Ceci assure l'intégrabilité de la différentielle. Notons qu'en admettant les pôles simples, on permet dans chaque genre une infinité de types combinatoires.

On restreint parfois l'attention aux différentielles quadratiques holomorphes. Sur une surface de Riemann donnée, l'espace de ces différentielles est le carré tensoriel de l'espace des différentielles abéliennes sur cette surface de Riemann.

Bien entendu, le carré d'une différentielle abélienne est une différentielle quadratique, mais toute différentielle quadratique n'est pas le carré d'une différentielle abélienne, même si elle n'a que des zéros pairs.

9.2. Surfaces de demi-translation. De même qu'on peut réaliser les différentielles abéliennes comme surfaces de translation, on peut réaliser les différentielles quadratiques holomorphes ou à pôles simples en utilisant des cartes polygonales dont les recollements se font le long de côtés, soit par des translations, soit par des symétries centrales (rotations d'angle π). On parle de surfaces de demi-translation. Un zéro d'ordre k d'une différentielle quadratique correspond à une singularité conique d'angle $(k+2)\pi$ sur la surface de demi-translation. Les points coniques d'angle π correspondent aux pôles simples.

Il est à noter que pour chaque genre g, l'espace de Teichmüller des différentielles quadratiques holomorphes peut être considéré comme le fibré cotangent de l'espace de Teichmüller des surfaces de Riemann.

9.3. Différentielles d'ordre supérieur. On pourrait considérer des différentielles cubiques, quartiques, quintiques… k-tiques.

On aurait à nouveau une interprétation en termes de surfaces plates (cartes polygonales et identifications par translation ou rotation d'angle multiple de $2\pi/k$), mais sans plus pouvoir définir d'action de $\mathrm{SL}(2, \mathbf{R})$ sur ces espaces : l'action sur les cartes n'est pas compatible avec les identifications par rotations d'angle $2\pi/k$ pour $k > 2$.

10. Comptages

Une jolie propriété de certains comptages de surfaces à petits carreaux est la quasi-modularité de leurs fonctions génératrices.

10.1. Formes quasi-modulaires. On définit ici les formes quasi-modulaires de façon algébrique ; on renvoie à [**MR**] pour une définition qui indique ce qu'est la quasi-modularité.

On considère pour tout entier pair $k \geqslant 2$ les séries d'Eisenstein définies pour $\operatorname{Im} z > 0$ et $q = e^{2i\pi z}$ par

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geqslant 1} \sigma_{k-1}(n) q^n,$$

où B_k désigne le k-ième nombre de Bernoulli, k-ième dérivée en 0 de $t/(e^t - 1)$, et pour tous entiers $m \geqslant 0$ et $n \geqslant 1$, $\sigma_m(n) = \sum_{d|n} d^m$.

On définit alors pour tout entier naturel pair k les *formes quasi-modulaires de poids pur k* comme les combinaisons linéaires des $E_2^a E_4^b E_6^c$ tels que $2a + 4b + 6c = k$. Notons que

$$E_2 = 1 - 24 \sum_{n \geqslant 1} \sigma_1(n) q^n, \quad E_4 = 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n,$$

$$\text{et} \quad E_6 = 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n.$$

Par extension (et abus de langage), on convient d'appeler forme quasi-modulaire tout élément de l'algèbre engendrée par E_2, E_4 et E_6. Cette algèbre est graduée par le poids. Les formes quasi-modulaires de poids pur, ou homogènes, ont une propriété de quasi-modularité ; les autres, dites inhomogènes ou de poids mélangés, n'en ont plus.

On appelle coefficients de Fourier d'une forme quasi-modulaire les coefficients de son développement en puissances de $q = e^{2i\pi z}$.

10.2. Problème de Hurwitz. Le problème de Hurwitz consiste à compter les revêtements ramifiés de surfaces de Riemann en fixant une surface de base et un type de ramification. Notons que le type de ramification fixe lui-même le genre de la surface revêtante. On convient de pondérer les comptages par l'inverse du nombre d'automorphismes de chaque revêtement.

Avant de restreindre notre attention au cas des revêtements du tore, signalons qu'un exposé très détaillé de ce sujet se trouve dans l'introduction de la thèse de Zvonkine [**Zv**].

Cas du tore. Dans le cas particulier où la surface de base est un tore (i.e. est de genre 1), ce comptage par degré fait apparaître pour chaque type de ramification une forme quasi-modulaire comme fonction génératrice. Cela a été démontré par Dijkgraaf [**Di**] et Kaneko–Zagier [**KaZa**] pour le cas des revêtements à ramification simples au-dessus de points distincts, et par Eskin–Okounkov [**EsOk**] pour les types de ramifications quelconques.

Ce résultat est énoncé comme le théorème C dans l'introduction du chapitre 2.

Ramifications simples. Dans le cas des ramifications simples (i.e. d'indice de ramification 2) au-dessus de points distincts, on obtient une forme quasi-modulaire de poids pur $6g-6$ où g est le genre des surfaces revêtantes. De plus ces formes quasi-modulaires sont elles-mêmes engendrées par une série génératrice que l'on peut relier à une fonction theta de Jacobi généralisée. (Voir Dijkgraaf [**Di**], Kaneko–Zagier [**KaZa**].)

Autres types de ramifications. Pour les autres types de ramification, on obtient toujours une forme quasi-modulaire mais de poids mélangés $\leqslant 6g-6$. (Voir Eskin–Okounkov [**EsOk**].)

10.3. Genre deux. Détaillons le cas du genre deux. Il y a trois types combinatoires possibles.

Pour deux points de ramification simples au-dessus de points distincts, on obtient une forme quasi-modulaire de poids pur 6 :

$$\frac{1}{5184}E_2^3 - \frac{1}{8640}E_2E_4 - \frac{1}{12960}E_6.$$

Pour deux points de ramification simples au-dessus du même point, on obtient une forme de poids mélangés 0, 2, 4, 6 :

$$-\frac{1}{90} + \frac{1}{72}E_2 - \frac{1}{288}E_2^2 + \frac{1}{1440}E_4 + \frac{1}{10368}E_2^3 - \frac{1}{17280}E_2E_4 - \frac{1}{25920}E_6.$$

Pour un point de ramification double, on obtient une forme de poids mélangés 0, 2, 4 (pas de poids 6) :

$$\frac{9}{640} - \frac{1}{64}E_2 + \frac{1}{384}E_2^2 - \frac{1}{960}E_4.$$

Ce dernier cas correspond à la strate $\mathcal{H}(2)$, et le précédent à la strate $\mathcal{H}(1,1)$.

Voir en appendice de cette introduction un développement sur la détermination d'une forme quasi-modulaire génératrice à partir d'un nombre fini de comptages.

10.4. Comptages et primitivité. Les surfaces à petits carreaux non primitives sont celles dont le réseau des périodes est plus petit que $\mathbf{Z}^2$. Autrement dit, leur revêtement sur le tore standard factorise par un tore plus gros.

Le nombre de tore à d carreaux est $\sigma(d)$. En effet, considérons un tore à d carreaux. On peut trouver une base de son réseau des périodes de la forme $((r,0),(d/r,t))$ pour un r divisant d, et un $t \in \{0,\dots,r-1\}$. Une telle base est unique pour un tore à petits carreaux donné ; et tout tel couple de vecteurs définit un tore à petits carreaux. Pour chaque r divisant d, il y a r possibilités pour le twist t, d'où le comptage $\sum_{r|d} r = \sigma(d)$.

Dans une strate $\mathcal{H}(\alpha)$ donnée, le comptage total $h_n(\alpha)$ et le comptage des surfaces primitives $h_n^{\mathrm{P}}(\alpha)$ sont donc reliés par :

$$h_n(\alpha) = \sum_{d|n} \sigma(d) h_n^{\mathrm{P}}(\alpha).$$

Strate $\mathcal{H}(2)$. Comme on l'a dit plus haut, les comptages de surfaces à petits carreaux dans $\mathcal{H}(2)$ sont engendrés par une forme quasi-modulaire. Connaissant les premiers comptages, on peut déterminer cette forme :

$$\frac{9}{640} - \frac{1}{64}E_2 + \frac{1}{384}E_2^2 - \frac{1}{960}E_4,$$

que l'on peut linéariser, sachant que $E_2^2 = 12\,\mathrm{D}\,E_2 + E_4$, comme :

$$\frac{9}{640} - \frac{1}{64}E_2 + \frac{1}{32}\,\mathrm{D}\,E_2 + \frac{1}{640}E_4 = \frac{1}{640}(9 - 10E_2 + 20\,\mathrm{D}\,E_2 + E_4).$$

De là on tire une formule pour le nombre de surfaces de $\mathcal{H}(2)$ à n carreaux, primitives ou non :

$$h_n(2) = \frac{1}{640}(9 - 10(n-2)\sigma_1(n) + \sigma_3(n)).$$

10.5. Comptages par orbites. L'un des résultats importants du chapitre 1 est l'existence d'un invariant, le nombre de points de Weierstrass entiers, qui distingue les orbites de surfaces à petits carreaux primitives dans $\mathcal{H}(2)$.

Cet invariant a été reformulé par McMullen [**Mc4**] comme une parité de structure spin.

Dans [**EMS**], on trouve deux formules pour le nombre de surfaces à n carreaux primitives dans $\mathcal{H}(2)$:

$$h_n^{\mathrm{P}}(2) = \sum_{r|n} \mu(r) \left(\sum_{\substack{h_1 w_1 + h_2 w_2 = n/r \\ h_1 \wedge h_2 = 1 \\ w_1 < w_2}} r w_1 w_2 + \frac{1}{3} \sum_{l_1 + l_2 + l_3 = n/r} n \right),$$

$$h_n^{\mathrm{P}}(2) = \frac{3}{8}(n-2)n^2 \sum_{r|n} \frac{\mu(r)}{r^2} = \frac{3}{8}(n-2)n^2 \prod_{p|n} (1 - \frac{1}{p^2}).$$

La première formule vient du paramétrage des surfaces à deux cylindres et à un cylindre de $\mathcal{H}(2)$. Eskin–Masur–Schmoll indiquent que la deuxième peut se déduire de la série génératrice quasi-modulaires.

Répartition par orbites pour n impair. Pour n impair, on conjecture que la répartition par orbites est la suivante :

$$h_n^{\mathrm{P}}(2, \mathrm{A}) = \frac{3}{16}(n-1)n^2 \prod_{p|n} (1 - \frac{1}{p^2}),$$

$$h_n^{\mathrm{P}}(2, \mathrm{B}) = \frac{3}{16}(n-3)n^2 \prod_{p|n} (1 - \frac{1}{p^2}).$$

Remarque. Pour le prouver, ils suffirait de calculer la différence $h_n^{\mathrm{P}}(2, \mathrm{A}) - h_n^{\mathrm{P}}(2, \mathrm{B})$.

11. Formes quasi-modulaires et nombres premiers

Les fonctions arithmétiques σ_k ($k \geqslant 0$) qui à tout entier associent la somme des puissances k-ièmes de ses diviseurs ($\sigma_k(n) = \sum_{d|n} d^k$), peuvent être confondues avec des polynômes si on ne regarde que leurs valeurs aux nombres premiers. En effet, pour p premier, $\sigma_k(p) = p^k + 1$.

Cela explique que certains comptages de surfaces à petits carreaux soient polynomiaux en restriction aux nombres premiers de carreaux. Voir quelques polynômes en section 8 du chapitre 2.

Cela a également pour conséquence de rendre impossible la détermination de formes quasi-modulaires à partir de leurs coefficients de Fourier de rangs premiers.

Nous intercalons ici un petit développement sur ce thème, en remerciant Mike Roth et Emmanuel Royer pour des conversations enrichissantes autour de ces idées.

Interpolation. Sachant que les comptages sont engendrés par une forme quasi-modulaire de poids maximal $\leqslant 6g-6$, une façon d'obtenir l'expression de cette forme dans la base des $E_2^a E_4^b E_6^c$, $2a+4b+6c \leqslant 6g-6$, est, connaissant les comptages pour un nombre suffisant de degrés, d'interpoler avec les coefficients de Fourier de cette base.

Cependant, il peut arriver qu'un mauvais choix de degrés ne permette pas d'interpoler.

En particulier, il faut toujours utiliser le degré 0 (pour lequel le nombre de revêtements est 0 quel que soit le type de ramification envisagé) pour trouver le coefficient de 1 (poids 0).

On peut également utiliser le degré 1, pour lequel le nombre de revêtements est 0 si $g > 1$.

Par ailleurs, les comptages étant souvent plus faciles à réaliser pour les degrés premiers, on aimerait utiliser les degrés 0, 1, et un nombre convenable de degrés premiers pour réaliser l'interpolation.

Relation. Malheureusement, un tel choix ne permet jamais l'interpolation en poids maximal $6g-6$, même pour $g = 2$. En effet les coefficients de Fourier des éléments de la base $(1, E_2, E_2^2, E_4, E_2^3, E_2E_4, E_6)$ des formes quasi-modulaires de poids $\leqslant 6$ satisfont, en restriction à 0, 1, et aux entiers premiers, une relation de dépendance linéaire :

Lemme 2. *La forme quasi-modulaire (de poids mélangés* 0, 2, 4, 6*)*

$$f = -396 + 360E_2 - 30E_2^2 + 66E_4 + 5E_2^3 - 15E_2E_4 + 10E_6$$

a tous ses coefficients de Fourier de rang 0, 1 *ou* p *premier nuls (et seulement ceux-là).*

Cela découle du lemme suivant :

Lemme 3. *Pour tous entiers naturels distincts* k *et* ℓ*, la fonction*

$$g_{k,\ell} : \begin{array}{ccc} \mathbf{N}^* & \to & \mathbf{N} \\ n & \mapsto & (n^\ell+1)\sigma_k(n) - (n^k+1)\sigma_\ell(n) \end{array}$$

est nulle exactement aux entiers premiers et en 1.

qui a pour corollaire, en notant D l'opérateur différentiel $q\dfrac{d}{dq} = \dfrac{1}{2i\pi}\dfrac{d}{dz}$ et

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n\geqslant 1} \sigma_{k-1}(n)q^n = -\frac{B_k}{2k}E_k :$$

Corollaire 4. *Pour toute paire d'entiers impairs distincts k et ℓ, la forme quasi-modulaire (inhomogène)*

$$f_{k,\ell} = (D^\ell + 1)G_{k+1} - (D^k + 1)G_{\ell+1}$$

a ses coefficients de Fourier nuls exactement en 1 et aux entiers premiers.

Remarque. L'opérateur D préserve la propriété

$$\widehat{f}(n) = 0 \text{ et } n \neq 0 \iff n = 1 \text{ ou } p \text{ premier.}$$

Question. Que peut-on dire de l'ensemble des formes quasi-modulaires (homogènes ou non) qui satisfont cette propriété ?

Remarque. Le coefficient de Fourier de rang 0 peut être ajusté sans modifier les autres par l'ajout d'un terme constant (i.e. de poids 0).

Preuve du lemme 2. On peut linéariser la forme quasi-modulaire f : en utilisant

$$E_2^2 - E_4 = 12DE_2 \quad \text{et} \quad E_2^3 - 3E_2E_4 + 2E_6 = 72D^2E_2$$

on peut écrire $f = -36h$ où $h = 11 - E_4 - 10(D^2 - D + 1)E_2$.

On a alors $\widehat{h}(0) = 0$ et pour $n \geqslant 1$

$$\widehat{h}(n) = 240[(n^2 - n + 1)\sigma_1(n) - \sigma_3(n)].$$

On est ramené au lemme 3 en multipliant par $n + 1$.

On aurait également pu se ramener au corollaire 4 en faisant agir l'opérateur $D + 1$ sur h. □

Preuve du lemme 3. La nullité de $g_{k,\ell}(n)$ pour $n = 1$ ou $n = p$ premier est évidente, car $\sigma_k(1) = 1$ et $\sigma_k(p) = p^k + 1$ pour p premier.

Montrons que $g_{k,\ell}$ n'est pas nulle aux autres entiers.

On utilise la notation $\sum^*_{d|n}$ pour désigner $\sum_{\substack{d|n \\ d\neq 1,n}}$; cette somme contient au moins un terme dès que $n \neq 1$ et n n'est pas premier.

Pour un tel n :

$$\begin{aligned}
g_{k,\ell}(n) &= (1 + n^\ell)(\sum_{d|n} d^k) - (1 + n^k)(\sum_{d|n} d^\ell) \\
&= (1 + n^\ell)(n^k + 1 + \sum_{d|n}{}^* d^k) - (1 + n^k)(1 + n^\ell + \sum_{d|n}{}^* d^\ell) \\
&= (1 + n^\ell)(\sum_{d|n}{}^* d^k) - (1 + n^k)(\sum_{d|n}{}^* d^\ell) \\
&= \sum_{d|n}{}^* [(1 + n^\ell)d^k - (1 + n^k)d^\ell]
\end{aligned}$$

En supposant $k < \ell$, on écrit alors

$$\begin{aligned} g_{k,\ell}(n) &= \sum_{d|n}^{*} d^k[(1+n^\ell) - (1+n^k)d^{\ell-k}] \\ &= \sum_{d|n}^{*} d^k[1 + n^\ell - d^{\ell-k} - n^k d^{\ell-k}] \end{aligned}$$

Or si $d|n$ et $d \neq n$, on a $d \leqslant n/2$, donc

$$\begin{aligned} 1 + n^\ell - d^{\ell-k} - n^k d^{\ell-k} &\geqslant 1 + n^\ell - (n/2)^{\ell-k} - n^k (n/2)^{\ell-k} \\ &\geqslant 1 + n^\ell - \frac{n^{\ell-k}}{2^{\ell-k}} - \frac{n^\ell}{2^{\ell-k}} \geqslant 1 + n^\ell - \frac{n^\ell}{2^{\ell-k}} - \frac{n^\ell}{2^{\ell-k}} \\ &\geqslant 1 + n^\ell - \frac{n^\ell}{2^{\ell-k-1}} \geqslant 1 \end{aligned}$$

Les termes de la somme $\sum^*$ sont donc tous strictement positifs, et la somme contient au moins un terme. Ceci achève la démonstration. $\square$

Retour aux revêtements du tore en genre deux. La forme quasi-modulaire génératrice du comptage par degré des revêtements du tore à deux ramifications simples au-dessus du même point est inhomogène, de poids mélangés 0, 2, 4, 6 :

$$-\frac{1}{90} + \frac{1}{72}E_2 - \frac{1}{288}E_2^2 + \frac{1}{1440}E_4 + \frac{1}{10368}E_2^3 - \frac{1}{17280}E_2E_4 - \frac{1}{25920}E_6.$$

Pour ce type de ramification, les revêtements de degré premier n'ont pas d'automorphismes non triviaux, ainsi la pondération n'influe pas sur leur comptage. En revanche, pour tout degré non premier, il existe des revêtements avec automorphismes non triviaux.

Le lemme 2 indique donc qu'on ne pourrait pas déterminer la forme quasi-modulaire génératrice des comptages de $\mathcal{H}(1,1)$ par interpolation des coefficients de Fourier en n'utilisant que des degrés où l'on peut compter sans se soucier de la pondération par l'inverse du nombre d'automorphismes du revêtement.

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Chapitre 1

Disques de Teichmüller

Ce chapitre est l'article écrit avec Pascal Hubert, accepté pour publication dans *Israel Journal of Mathematics* sous le titre “Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$” et dont le résultat principal est que lorsque n est premier $\geqslant 5$, il y a exactement deux disques de Teichmüller de surfaces à n carreaux dans $\mathcal{H}(2)$. Ce résultat a été généralisé par C. McMullen au comptage des disques de Teichmüller de surfaces de Veech de tous discriminants dans $\mathcal{H}(2)$.

PRIME ARITHMETIC TEICHMÜLLER DISCS IN $\mathcal{H}(2)$

PASCAL HUBERT AND SAMUEL LELIÈVRE

Abstract. It is well-known that Teichmüller discs that pass through "integer points" of the moduli space of abelian differentials are very special: they are closed complex geodesics. However, the structure of these special Teichmüller discs is mostly unexplored: their number, genus, area, cusps, etc.

We prove that in genus two all translation surfaces in $\mathcal{H}(2)$ tiled by a prime number $n > 3$ of squares fall into exactly two Teichmüller discs, only one of them with elliptic points, and that the genus of these discs has a cubic growth rate in n.

Keywords: Teichmüller discs, square-tiled surfaces, Weierstrass points.
MSC: 32G15 (37C35 30F30 14H55 30F35)

Contents

1. Introduction

In his fundamental paper of 1989, Veech studied the finite-volume Teichmüller discs. Translation surfaces with such discs, called Veech

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surfaces, enjoy very interesting dynamical properties: their directional flows are either completely periodic or uniquely ergodic. An abundant literature exists on Veech surfaces: Veech [Ve89, Ve92], Gutkin–Judge [GuJu1, GuJu2], Vorobets [Vo], Ward [Wa], Kenyon–Smillie [KeSm], Hubert–Schmidt [HuSc00, HuSc01], Gutkin–Hubert–Schmidt [GuHuSc], Calta [Ca], McMullen [Mc]...

The simplest examples of Veech surfaces are translation covers of the torus (ramified over a single point), called square-tiled surfaces. They are those translation surfaces whose stabilizer in $\mathrm{SL}(2,\mathbf{R})$ is arithmetic (commensurable with $\mathrm{SL}(2,\mathbf{Z})$), by a theorem of Gutkin and Judge. These surfaces (and many more!) were introduced by Thurston [Th] and studied on the dynamical aspect by Gutkin [Gu], Veech [Ve87] and Gutkin–Judge [GuJu1, GuJu2]. Square-tiled surfaces can be viewed as the "integer points" of the moduli spaces of holomorphic 1-forms. The asymptotic number of integer points in a large ball was used by Zorich [Zo] and Eskin–Okounkov [EsOk] to compute volumes of strata of abelian differentials.

It was known for years that Teichmüller discs passing through these integer points in the moduli space are very special: they are closed (*complex*) geodesics. Despite enormous interest to invariant submanifolds (especially to the simplest ones: those of complex dimension one), absolutely nothing was known about the structure of these special Teichmüller discs: about their number, genus, area, cusps, etc. It was neither known which "integer points" belong to the same Teichmüller disc.

1.1. **Main results.** In this paper, we study square-tiled surfaces in the stratum $\mathcal{H}(2)$. This stratum is the moduli space of holomorphic 1-forms with a unique (double) zero on a surface of genus two. For surfaces tiled by a prime number of squares, we show:

Theorem 1.1. *For any prime $n \geqslant 5$, the $\mathrm{SL}(2,\mathbf{R})$-orbits of n-square-tiled surfaces in $\mathcal{H}(2)$ form two Teichmüller discs $D_A(n)$ and $D_B(n)$.*

Theorem 1.2. *$D_A(n)$ and $D_B(n)$ can be seen as the unit tangent bundles to orbifold surfaces with the following asymptotic behavior:*
- *genus $\sim c\,n^3$, with $c_A = c_B = (3/16)(1/12)$,*
- *area $\sim c\,n^3$, with $c_A = c_B = (3/16)(\pi/3)$,*

- *number of cusps* $\sim c\,n^2$, *with* $c_A = 1/24$ *and* $c_B = 1/8$,
- *number of elliptic points* $O(n)$, *one of them having none.*

Proposition 1.3. *All these discs arise from L-shaped billiards.*

Our results are extended by McMullen [Mc2] to describe the repartition into different orbits of *all* Veech surfaces in $\mathcal{H}(2)$. In particular, the invariant introduced in § 4.2 also determines orbits in the non-prime case.

1.2. **Side results.** We find the following as side results of our study:

- **One-cylinder directions.**

Proposition 1.4. *All surfaces in* $\mathcal{H}(2)$ *tiled by a prime number of squares have one-cylinder directions i.e. directions in which they decompose into one single cylinder.*

- **Discs without elliptic points.** During some time, the search for new Veech surfaces focused on examples arising from billiards in rational-angled polygons. Angles of the billiard table not multiples of the right angle lead to elliptic elements in the Veech group. Billiards with all angles multiples of the right angle have however recently been studied, especially L-shaped billiards (see [Mc]).

- **Discs of (arbitrary high) positive genus.** When a Veech group has positive genus, the subgroup generated by its parabolic elements has infinite index, and cannot be a lattice. This implies that the naive algorithm which consists in finding parabolic elements in the Veech group cannot lead to obtain the *whole* group not even up to finite index.

The surfaces arising from billiards in the regular polygons, studied by Veech in [Ve92], have genus tending to infinity, and one could probably show that the genus of their Veech groups also tends to infinity, though Veech does not state this explicitly.

Our examples give families of Teichmüller discs of arbitrarily high genus, the translation surfaces in these discs staying in genus two.

- **Noncongruence subgroups.** Since we deal with families of subgroups of $\mathrm{SL}(2, \mathbf{Z})$, it is natural to check whether they belong to the well-known family of congruence subgroups. Appendix A provides an example of a Veech group that is a non-congruence subgroup of $\mathrm{SL}(2,\mathbf{Z})$. Another example was given by G. Schmithüsen [Schmi]. A detailed discussion of the congruence problem in this setting will appear in [HL].

- **Deviation from the mean order.**

Proposition 1.5. *The number of n-square-tiled surfaces in $\mathcal{H}(2)$ for prime n is asymptotically $1/\zeta(4)$ times the mean order of the number of n-square-tiled surfaces in $\mathcal{H}(2)$.*

1.3. **Methods.** We parametrize square-tiled surfaces in $\mathcal{H}(2)$ by using separatrix diagrams as in [KoZo], [Zo] and [EsMaSc]. These coordinates bring the study of Teichmüller discs of n-square-tiled surfaces down to a combinatorial problem.

We want to describe the $\mathrm{SL}(2,\mathbf{Z})$ orbits of these surfaces. Using the fact that $\mathcal{H}(2)$ is a hyperelliptic stratum, the combinatorial representation of Weierstrass points allows us to show there are at least two orbits for odd $n \geqslant 5$. Showing there are only two is done for prime n in a combinatorial way, by a careful study of the action of generators of $\mathrm{SL}(2,\mathbf{Z})$ on square-tiled surfaces.

For the countings, we use generating functions.

1.4. **Related works.** Our counting results are very close to the formulae in [EsMaSc]. Eskin–Masur–Schmoll calculate Siegel–Veech constants for torus coverings in genus two. In $\mathcal{H}(2)$, these calculations are based on counting the square-tiled surfaces with a given number of squares. The originality of our work is to count square-tiled surfaces disc by disc.

There are also analogies with Schmoll's work [Schmo]. He computes the explicit Veech groups of tori with two marked points and the quadratic asymptotics for theses surfaces. Some of the methods he uses are intimately linked to those used in our work. The Veech groups he exhibits are all congruence subgroups.

A computer program allows to give all the geometric information on Teichmüller discs of square-tiled surfaces in $\mathcal{H}(2)$. Schmithüsen [Schmi] has a program to compute the Veech group of any given square-tiled surface. She also found positive genus discs as well as noncongruence Veech groups. Möller [Mö] computes algebraic equations of some square-tiled surfaces and of their Teichmüller curves.

1.5. **Acknowledgements.** We thank Anton Zorich for stating questions and some conjectures. We thank the Institut de Mathématiques de Luminy and the Max-Planck-Institut für Mathematik for excellent welcome and working conditions. We thank Joël Rivat, Martin Schmoll and other participants of the conference 'Dynamique dans l'espace de Teichmüller et applications aux billards rationnels' at CIRM in 2003. We thank Martin Möller and Gabriela Schmithüsen for comments on a previous version of this paper, circulated under the title "Square-tiled surfaces in $\mathcal{H}(2)$".

2. Background

2.1. **Translation surfaces, Veech surfaces.** Let S be an oriented compact surface of genus g. A translation structure on S consists in a set of points $\{P_1, \ldots, P_n\}$ and a maximal atlas on $S \setminus \{P_1, \ldots, P_n\}$ with translation transition functions.

A holomorphic 1-form ω on S induces a translation structure by considering its natural parameters, and its zeros as points $P_1, \ldots, P_n$. All translation structures we consider are induced by holomorphic 1-forms. Slightly abusing vocabulary and notation, we refer to a translation surface (S, ω), or sometimes just S or ω.

A translation structure defines: a complex structure, since translations are conformal; a flat metric with cone-type singularities of angle $2(k_i + 1)\pi$ at order k_i zeros of the 1-form; and directional flows $\mathcal{F}_\theta$ on S for $\theta \in \left]-\pi, \pi\right]$.

Orbits of the flows $\mathcal{F}_\theta$ meeting singularities (backward, resp. forward) are called (outgoing, resp. incoming) separatrices in the direction θ. Orbits meeting singularities both backward and forward are called **saddle connections**; the integrals of ω along them are the associated **connection vectors**.

Define the singularity type of a 1-form ω to be the unordered tuple $\sigma = (k_1, \ldots, k_n)$ of orders of its zeros (recall $k_1 + \ldots + k_n = 2g - 2$, all $k_i > 0$). The singularity type is invariant by orientation-preserving diffeomorphisms. The moduli space $\mathcal{H}_g$ of holomorphic 1-forms on S is the quotient of the set of translation structures by the group $\mathrm{Diff}^+(S)$ of orientation-preserving diffeomorphisms. $\mathcal{H}_g$ is stratified by singularity types, the strata are denoted by $\mathcal{H}(\sigma)$.

$\mathrm{SL}(2,\mathbf{R})$ acts on holomorphic 1-forms: if ω is a 1-form, $\{(U,f)\}$ the translation structure given by its natural parameters, and $A \in \mathrm{SL}(2,\mathbf{R})$, then $A\cdot\omega = \{(U, A\circ f)\}$. As is well known, this action (to the left) commutes with that (to the right) of $\mathrm{Diff}^+(S)$ and preserves singularity types. Each stratum $\mathcal{H}(\sigma)$ thus inherits an $\mathrm{SL}(2,\mathbf{R})$ action. The dynamical properties of this action have been extensively studied by Masur and Veech [Ma, Ve82, etc.].

From the behavior of the $\mathrm{SL}(2,\mathbf{R})$-orbit of ω in $\mathcal{H}(\sigma)$ one can deduce properties of directional flows $\mathcal{F}_\theta$ on the translation surface (S,ω). The Veech dichotomy expressed below is a remarkable illustration of this.

Call **affine diffeomorphism** of (S,ω) an orientation-preserving homeomorphism f of S such that the following three conditions hold
- f keeps the set $\{P_1,\dots,P_n\}$ invariant;
- f restricts to a diffeomorphism of $S\setminus\{P_1,\dots,P_n\}$;
- the derivative of f computed in the natural charts of ω is constant.

The derivative can then be shown to be an element of $\mathrm{SL}(2,\mathbf{R})$.

Affine diffeomorphisms of (S,ω) form its affine group $\mathrm{Aff}(S,\omega)$, their derivatives form its Veech group $V(S,\omega) < \mathrm{SL}(2,\mathbf{R})$, a noncocompact fuchsian group. The Veech group is the stabilizer of (S,ω) for the action of $\mathrm{SL}(2,\mathbf{R})$ on $\mathcal{H}_g$. Veech showed that the derivation map $\mathrm{Aff}(S,\omega)\to V(S,\omega)$ is finite-to-one. We show (Proposition 4.4) that in $\mathcal{H}(2)$ it is one-to-one.

Theorem (Veech dichotomy). *If $V(S,\omega)$ is a lattice in $\mathrm{SL}(2,\mathbf{R})$ (i.e. $\mathrm{vol}\big(V(S,\omega)\backslash\mathrm{SL}(2,\mathbf{R})\big) < \infty$) then for each direction θ, either the flow $\mathcal{F}_\theta$ is uniquely ergodic, or all orbits of $\mathcal{F}_\theta$ are compact and S decomposes into a finite number of cylinders of commensurable moduli.*

Cylinder decompositions are further discussed in § 2.3. Translation surfaces with lattice Veech group are called Veech surfaces.

2.2. **Square-tiled surfaces, lattice of periods.** A translation covering is a map $f\colon (S_1,\omega_1)\longrightarrow(S_2,\omega_2)$ of translation surfaces that
- is topologically a ramified covering;
- maps zeros of ω_1 to zeros of ω_2;
- is locally a translation in the natural parameters of ω_1 and ω_2.

Translation covers of the standard torus marked at the origin are the simplest examples of Veech surfaces. Such surfaces are tiled by squares.

We call them square-tiled. The Gutkin–Judge theorem states:

Theorem (Gutkin–Judge)**.** *A translation surface (S, ω) is square-tiled if and only if its Veech group $V(S, \omega)$ shares a finite-index subgroup with* $\mathrm{SL}(2, \mathbf{Z})$*.*

Translation surfaces with such (arithmetic) Veech groups have also been called arithmetic; another name for them is origamis. A proof of Gutkin and Judge's theorem, very different from the original, is given in appendix C.

The subgroup of $\mathbf{R}^2$ generated by connection vectors is the lattice of relative periods of (S, ω), denoted by $\Lambda(\omega)$.

Lemma 2.1. *A translation surface (S, ω) is square-tiled if and only if $\Lambda(\omega)$ is a rank 2 sublattice of $\mathbf{Z}^2$.*

Proof. If (S, ω) is square-tiled, connection vectors are obviously integer vectors, so they span a sublattice of $\mathbf{Z}^2$. Conversely, let

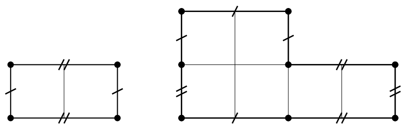
$$\begin{array}{rcl} f\colon (S,\omega) & \to & \mathbf{R}^2/\Lambda(\omega), \\ z & \mapsto & \int_{z_0}^{z} \omega \bmod \Lambda(\omega), \end{array} \qquad \text{where } z_0 \text{ is a given point of } (S,\omega).$$

The integral is well-defined modulo the lattice of absolute periods; f is a fortiori well-defined. Since f is holomorphic and onto, it is a covering. Since relative periods are integer-valued, it is clear that zeros of ω project to the origin. So, given a point $P \neq 0$ on the torus, preimages of P are all regular points, so P is not a branch point. Hence the covering is ramified only above the origin. Composing f with the covering $g\colon\ \mathbf{R}^2/\Lambda(\omega) \longrightarrow \mathbf{R}^2/\mathbf{Z}^2$, we see (S, ω) is square-tiled. ☐

A square-tiled surface (S, ω) is called primitive if $\Lambda(\omega) = \mathbf{Z}^2$.

Lemma 2.2. *Let (S, ω) be an n-square-tiled surface of genus $g > 1$. If n is prime then $\Lambda(\omega) = \mathbf{Z}^2$.*

Proof. Lemma 2.1 shows that (S, ω) is a ramified cover of $\mathbf{R}^2/\Lambda(\omega)$. Let d be the degree of the covering. Then $n = d \cdot [\mathbf{Z}^2 : \Lambda(\omega)]$. So obviously if n is prime then $\Lambda(\omega) = \mathbf{Z}^2$. ☐



Note that $\Lambda(\omega)$ is not always $\mathbf{Z}^2$, as shown by the examples in the figure. On the left, a torus T with lattice of periods $2\mathbf{Z} \times \mathbf{Z}$ and Veech group generated by $\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 1/2 & 1\end{smallmatrix}\right)$. On the right, a genus 2 cover of T, with $\Lambda(\omega) = 2\mathbf{Z} \times \mathbf{Z}$ and Veech group generated by $\left(\begin{smallmatrix}1 & 4\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & -2\\ 1/2 & 0\end{smallmatrix}\right)$.

The following lemma was explained to us first by Martin Schmoll then by Anton Zorich.

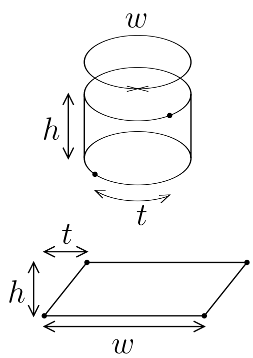
Lemma 2.3. *Let (S, ω) be a square-tiled surface, then $V(S, \omega)$ is a subgroup in $V(\mathbf{R}^2/\Lambda(\omega)\,, \mathrm{d}z)$. In particular, if (S, ω) is primitive, then $V(S, \omega) < \mathrm{SL}(2, \mathbf{Z})$.*

Proof. Let ϕ:
$$\begin{array}{ccc} V(S,\omega) & \to & V(\mathbf{R}^2/\Lambda(\omega)\,, \mathrm{d}z), \\ A = df,\ f \in \mathrm{Aff}(S,\omega) & \mapsto & A. \end{array}$$

The only difficulty is to show that ϕ is well-defined i.e. that any element A in $V(S, \omega)$ preserves $\Lambda(\omega)$. Since any element of the affine group maps a connection to a connection, hence A maps a connection vector to a connection vector (i.e. an element in $\Lambda(\omega)$). □

Remark. As shown by the examples above, there are Veech groups of square-tiled surfaces which are *not* subgroups of $\mathrm{SL}(2, \mathbf{Z})$.

2.3. **Cylinders of square-tiled surfaces.** A square-tiled surface decomposes into maximal horizontal cylinders, bounded above and below by unions of saddle connections, each of which appears once on the top of a cylinder and once on the bottom of a cylinder. Gluing the cylinders alongs these saddle connections builds back the surface.



A cylinder on a translation surface is isometric to $\mathbf{R}/w\mathbf{Z} \times [0, h]$, for some h and w.

Convention. We refer to these dimensions as height and width respectively, whether the 'horizontal direction of the cylinder' coincides with the horizontal direction of the surface or not.

An additional twist parameter t is needed, measuring the distance along the 'horizontal direction of the cylinder' between some (arbitrary) reference points on the bottom and top of the cylinder, for instance some ends of saddle connections.

2.4. **Action of** $\mathrm{SL}(2, \mathbf{Z})$ **on square-tiled surfaces.**

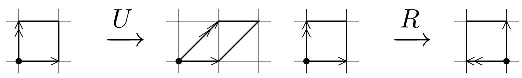
Lemma 2.4. *The* $\mathrm{SL}(2, \mathbf{Z})$*-orbit of a primitive* n*-square-tiled surface is the set of primitive* n*-square-tiled surfaces in its* $\mathrm{SL}(2, \mathbf{R})$*-orbit.*

Proof. $\mathrm{SL}(2, \mathbf{Z})$ preserves $\mathbf{Z}^2$ $(= \Lambda(\omega)$ if (S, ω) is primitive square-tiled) and hence the property of being primitive square-tiled. Conversely, if (S, ω) is primitive square-tiled and $(S_1, \omega_1) = A{\cdot}(S, \omega)$ is square-tiled for some $A \in \mathrm{SL}(2, \mathbf{R})$, then $\Lambda(\omega_1) = A{\cdot}\Lambda(\omega)$ means A preserves $\mathbf{Z}^2$, so $A \in \mathrm{SL}(2, \mathbf{Z})$. □

Remark. The number of squares, n, is preserved by $\mathrm{SL}(2, \mathbf{R})$ because it is the area of the surface. Consequently $\mathrm{SL}(2, \mathbf{Z}) \cdot (S, \omega)$ is finite.

Notation. Denote by $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ the standard generators of $\mathrm{SL}(2, \mathbf{Z})$, and by $\mathcal{U} = \langle U \rangle = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbf{Z}\}$ the subgroup generated by U.

Here is the action of U and R on squares.



The action on square-tiled surfaces is obtained by applying the same to all square tiles. The new horizontal cylinder decomposition is then recovered by cutting and gluing (see example in § 3.4).

2.5. **Hyperelliptic surfaces, Weierstrass points.** Recall that a Riemann surface X of genus g is hyperelliptic if there exists a degree 2 meromorphic function on X. Such a function induces a holomorphic involution on X. This involution has $2g + 2$ fixed points called Weierstrass points. The set of these points is invariant by all automorphisms of the complex structure. A translation surface is called hyperelliptic if the underlying Riemann surface is hyperelliptic.

Hyperelliptic translation surfaces have been studied by Veech. He showed [Ve95] that in genus g they are obtained from centrosymmetric polygons with $4g$ or $4g + 2$ sides by pairwise identifying opposite sides.

The hyperelliptic involution is in these coordinates the reflection in the center of the polygon; the Weierstrass points are the center of the polygon, the midpoints of its sides, and the vertices (identified into one point) in the $4g$ case (in the $4g{+}2$ case the vertices are indentified into two points exchanged by the hyperelliptic involution).

2.6. **Cusps.** Let Γ be a fuchsian group. A parabolic element of Γ is a matrix of trace 2 (or -2). A point of the boundary at infinity of $\mathbf{H}^2$ is parabolic if it is fixed by a parabolic element of Γ. A cusp is a conjugacy class under Γ of primitive parabolic elements (primitive meaning not powers of other parabolic elements of Γ).

Recall that a lattice admits only a finite number of cusps.

Geometrically, each cusp in $\Gamma\backslash\mathbf{H}^2$ has, for some positive λ called its **width**, neighborhoods isometric to the quotients of the strips $\{z \in \mathbf{C} \colon 0 < |\operatorname{Re} z| < \lambda,\ \operatorname{Im} z > M\}$ by the translation $z \mapsto z + \lambda$, for large M.

On a Veech surface (S, ω), any 'periodic' direction is fixed by a parabolic element of the Veech group. Conversely the eigendirection of a parabolic element in the Veech group is a 'periodic' direction. We call such directions parabolic. Thus parabolic limit points of $V(S, \omega)$ are cotangents of parabolic directions.

When (S, ω) is a square-tiled surface, the set of parabolic limit points is $\mathbf{Q}$. Cusps are therefore equivalence classes of rationals under the homographic action of $V(S, \omega)$. The following lemma gives a combinatorial description of cusps for a square-tiled surface.

Lemma 2.5 (Zorich)**.** *Let (S, ω) be a primitive n-square-tiled surface and $E = \mathrm{SL}(2, \mathbf{Z}) \cdot (S, \omega)$ the set of n-square-tiled surfaces in its orbit. The cusps of (S, ω) are in bijection with the $\mathcal{U}$-orbits of E.*

Proof. Denote by $\mathcal{C}$ the set of cusps of (S, ω).

Let $\varphi\colon \begin{array}{ccccc} \mathrm{SL}(2,\mathbf{Z}) & \xrightarrow{f} & \mathbf{Q} & \xrightarrow{\pi} & \mathcal{C}, \\ A & \mapsto & A^{-1}\infty & \mapsto & A^{-1}\infty \bmod V(S,\omega). \end{array}$

Note that ∞ corresponds to the horizontal direction in (S, ω) because the projective action is the action on co-slopes and not on slopes. $A^{-1}\infty$ corresponds to the direction on (S, ω) that is mapped by A to the horizontal direction of $A \cdot (S, \omega)$.

φ pulls down as $\psi\colon \begin{array}{ccc} E & \to & \mathcal{C}, \\ A \cdot (S,\omega) & \mapsto & A^{-1}\infty \bmod V(S,\omega). \end{array}$

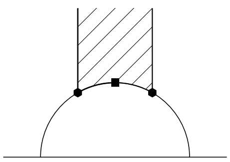
ψ is well-defined: if $A \cdot (S, \omega) = B \cdot (S, \omega)$, then $\exists P \in V(S)$, $B = AP$, so setting $A^{-1}\infty = \alpha$, $B^{-1}\infty = \beta$, we have $\beta = B^{-1}\infty = (B^{-1}A)A^{-1}\infty = P^{-1}\alpha$, so α and β correspond to the same cusp. Further, ψ is surjective because f is. Indeed, $\forall \alpha = p/q$, $\exists A \in \mathrm{SL}(2, \mathbf{Z})$ s.t. $A^{-1}\infty = \alpha$. (The orbit of ∞ under $\mathrm{SL}(2, \mathbf{Z})$ is $\mathbf{Q}$.)

Recall that the stabilizer of ∞ for the action of $\mathrm{SL}(2,\mathbf{Z})$ is $\mathcal{U}$. If $\psi(S_1,\omega_1) = \psi(S_2,\omega_2)$, where $(S_1,\omega_1) = A\cdot(S,\omega)$ and $(S_2,\omega_2) = B\cdot(S,\omega)$, then $\varphi(A) = \varphi(B)$.

Let $\alpha = f(A) = A^{-1}\infty$ and $\beta = f(B) = B^{-1}\infty$. Since α and β correspond to the same cusp, $\exists P \in V(S)$ s.t. $\beta = P\alpha$. So $\infty = A\alpha = AP^{-1}\beta = AP^{-1}B^{-1}\infty$ which implies $AP^{-1}B^{-1} \in \mathcal{U}$ i.e. $\exists U^k \in \mathcal{U}$ s.t. $AP^{-1} = U^k B$ i.e. $AP^{-1}\cdot(S,\omega) = A\cdot(S,\omega) = U^k B\cdot(S,\omega)$, so that (S_1,ω_1) and (S_2,ω_2) are in the same $\mathcal{U}$-orbit.

Conversely: if $(S_2,\omega_2) = U^k(S_1,\omega_1)$ with $U^k \in \mathcal{U}$, and $(S_2,\omega_2) = B\cdot(S,\omega)$ and $(S_1,\omega_1) = A\cdot(S,\omega)$, then $\psi(S_2,\omega_2) = B^{-1}\infty = A^{-1}U^{-k}\infty = A^{-1}\infty = \psi(S_1,\omega_1)$. □

2.7. **Elliptic points.** Recall that in a fuchsian group Γ, any elliptic element has finite order and is conjugate to a rational rotation.



A fixed point in $\mathbf{H}^2$ of an elliptic element of Γ is called elliptic. Its projection to the quotient $\Gamma\backslash\mathbf{H}^2$ is a cone point, with a curvature default. For instance the modular surface $\mathrm{SL}(2,\mathbf{Z})\backslash\mathbf{H}^2$ has two cone points, of angles π and $2\pi/3$.

Suppose that Γ is the Veech group of a translation surface and has an elliptic point. By applying a convenient element of $\mathrm{SL}(2,\mathbf{R})$, we can suppose that this point is i. The corresponding elliptic element is a rational rotation. The translation surfaces which project to i have this rotation in their Veech group. This roughly means that they have an apparent symmetry. At the Riemann surface level, the rotation is an automorphism of the complex structure (it modifies the vertical direction but not the metric). For genus 1, the cone point i (resp. $e^{i\pi/3}$) of the modular surface corresponds to the square (resp. hexagonal) torus, which has a symmetry of projective order 2 (resp. 3).

One should note that the translation surfaces obtained from rational polygonal billiards always have elliptic elements in their Veech group: writing the angles of a simple polygon as $(k_1\pi/r,\dots,k_q\pi/r)$, with k_1, …, k_q, r coprime, the covering translation surface is obtained by gluing $2r$ copies by symmetry. The rotation of angle $2\pi/r$ is in the Veech group (this rotation is minus the identity if $r = 2$). Many explicit calculations of lattice Veech groups make use of this remark (see [Ve89], [Vo], [Wa]). Our method is completely different.

2.8. **The Gauss–Bonnet Formula.** Let Γ be a finite-index subgroup of $\mathrm{SL}(2,\mathbf{Z})$ containing $-\mathrm{Id}$. The quotient of $\Gamma\backslash\mathrm{SL}(2,\mathbf{R})$ is the unit tangent bundle to an orbifold surface with cusps S_Γ. Algebraic information on the group is related to the geometry of the surface.

Let d be the index $[\mathrm{SL}(2,\mathbf{Z}):\Gamma]$ of Γ in $\mathrm{SL}(2,\mathbf{Z})$, e_2 (resp. e_3) the number of conjugacy classes of elliptic elements of order 2 (resp. 3) of Γ, e_∞ the number of conjugacy classes of cusps of Γ.

Then the surface S_Γ has hyperbolic area $d\frac{\pi}{3}$, e_2 cone points of angle π, e_3 cone points of angle $\frac{2\pi}{3}$, e_∞ cusps, and its genus g is given by:

The Gauss–Bonnet Formula. $g = 1 + d/12 - e_2/4 - e_3/3 - e_\infty/2.$

3. Specific Tools

In this section we give specific properties of the stratum $\mathcal{H}(2)$, and a combinatorial coordinate system for square-tiled surfaces in $\mathcal{H}(2)$.

3.1. **Hyperellipticity.** First recall that any genus 2 Riemann surface is hyperelliptic. Given a genus 2 Riemann surface X and its hyperelliptic involution τ, any 1-form ω on X satisfies $\tau^*\omega = -\omega$.

In the moduli space of holomorphic 1-forms of genus 2, $\mathcal{H}(2)$ is the stratum of 1-forms with a degree 2 zero (a cone point of angle 6π).

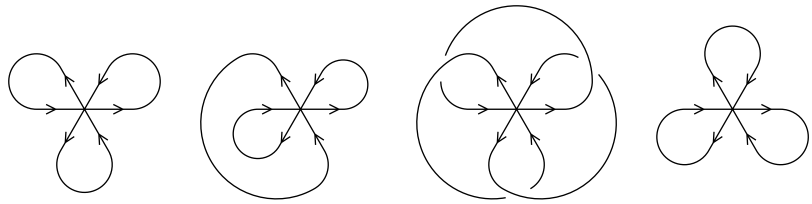
As said in § 2.5, any translation surface in $\mathcal{H}(2)$ can be represented as a centro-symmetric octagon. The six Weierstrass points are the center of the polygon, the middles of the sides and the cone-type singularity. The position of the Weierstrass points in a surface decomposed into horizontal cylinders is described in § 5.1.

3.2. **Separatrix diagrams.** Forms in $\mathcal{H}(2)$ have a single degree 2 zero, geometrically a cone point of angle 6π, with three outgoing separatrices and three incoming ones in any direction.

Recall that the horizontal direction of a square-tiled surface is completely periodic; the horizontal separatrices are saddle connections. The combinatorics of these connections is called a separatrix diagram in [KoZo]. The surface is obtained from this diagram by gluing cylinders along the saddle connections.

Each outgoing horizontal separatrix returns to the saddle making an angle π, 3π or 5π with itself. Four separatrix diagrams are combinatorially possible (up to rotation by 2π around the cone point);

they correspond to return angles (π, π, π), $(\pi, 3\pi, 5\pi)$, $(3\pi, 3\pi, 3\pi)$, $(5\pi, 5\pi, 5\pi)$:



There is no consistent way of gluing cylinders along the saddle connections of the first and last diagrams to obtain a translation surface.

The second diagram is possible with the condition that the saddle connections that return with angles π and 5π have the same length; this diagram corresponds to surfaces with two cylinders. The third diagram corresponds to surfaces with one cylinder, with no restriction on the lengths of the saddle connections.

3.3. **Parameters for square-tiled surfaces in $\mathcal{H}(2)$.** Here we give complete combinatorial coordinates for square-tiled surfaces in $\mathcal{H}(2)$. See figures in § 5.1.

Notation. We use $\wedge$ for greatest common divisor, and $\vee$ for least common multiple.

3.3.1. *One-cylinder surfaces.* A one-cylinder surface is parametrized by the height of the cylinder, the lengths of the three horizontal saddle connections (a triple of integers up to cyclic permutation), and the twist parameter. If all three horizontal saddle connections have the same length, the twist parameter is taken to be less than that length; otherwise, less than the sum of the three lengths.

For primitive surfaces, the height is 1, and the lengths of the three horizontal saddle connections add up to the area n of the surface.

The horizontal saddle connections appear in some (cyclic) order on the bottom of the cylinder, and in reverse order on the top.

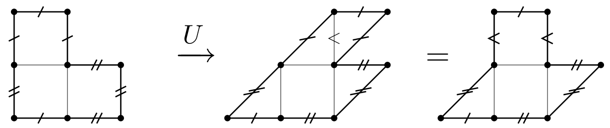
3.3.2. *Two-cylinder surfaces.* Labeling the horizontal saddle connections according to their return angles, call them γ_π, $\gamma_{3\pi}$, $\gamma_{5\pi}$. Call ℓ_1 the common length of γ_π and $\gamma_{5\pi}$, and ℓ_2 the length of $\gamma_{3\pi}$. One cylinder is bounded below by γ_π and above by $\gamma_{5\pi}$; the other one is bounded below by $\gamma_{5\pi}$ and $\gamma_{3\pi}$, and above by γ_π and $\gamma_{3\pi}$.

A two-cylinder surface is determined by the heights h_1, h_2 and widths $w_1 = \ell_1$, $w_2 = \ell_1 + \ell_2 > w_1$ of the cylinders as well as two twist parameters t_1, t_2 satisfying $0 \leqslant t_1 < w_1$, $0 \leqslant t_2 < w_2$. The area of the surface is $h_1 w_1 + h_2 w_2 = n$. For primitive surfaces, $h_1 \wedge h_2 = 1$. For prime n, in addition, $\ell_1 \wedge \ell_2 = 1$, and (P) $\ell_1 \wedge h_2 = 1$.

3.4. **Action of** $\mathrm{SL}(2, \mathbf{Z})$**.** The action of R (rotation by $\pi/2$) does not preserve separatrix diagrams in general. The horizontal cylinder decomposition of $R \cdot S$ is the vertical cylinder decomposition of S.

U is the primitive parabolic element in $\mathrm{SL}(2, \mathbf{Z})$ that preserves the horizontal direction. Its action preserves separatrix diagrams, as well as heights h_i and widths w_i of horizontal cylinders C_i, and only changes twist parameters t_i to $(t_i + h_i) \bmod w_i$.

Here is an example of how U acts on a surface.



For prime n, given a cyclically ordered 3-partition (a, b, c) of n, all one-cylinder surfaces with bottom sides of lengths a, b, c (up to cyclic permutation) are in the same $\mathcal{U}$-orbit, or cusp (see Lemma 2.5).

The following lemma describes U-orbits of two-cylinder surfaces in $\mathcal{H}(2)$ by giving their sizes and canonical representatives.

Lemma 3.1. *Let S be a primitive two-cylinder n-square-tiled surface in $\mathcal{H}(2)$ with parameters h_i, w_i, t_i ($i = 1, 2$). Then the cardinality of its U-orbit (its **cusp width**) is*

$$\mathrm{cw}(S) = \frac{w_1}{w_1 \wedge h_1} \vee \frac{w_2}{w_2 \wedge h_2} \qquad (= \tfrac{w_1}{w_1 \wedge h_1} \times \tfrac{w_2}{w_2 \wedge h_2} \text{ for prime } n).$$

The surface S' with $h'_i = h_i$, $w'_i = w_i$, and $t'_i = t_i \bmod (w_i \wedge h_i)$ is a "canonical" representative of the U-orbit of S. Each surface thus has a unique representative with $0 \leqslant t'_i < w_i \wedge h_i$.

Proof. Observe that $U^k \cdot S$ has widths w_i, heights h_i, and twist parameters $(t_i + kh_i) \bmod w_i$. So for $U^k \cdot S$ to coincide with S, the integer k must be a multiple of $\frac{w_i}{w_i \wedge h_i}$ for each i. The cusp width is the least such positive k, the least common multiple of $\frac{w_1}{w_1 \wedge h_1}$ and $\frac{w_2}{w_2 \wedge h_2}$. The second part is a simple application of the Chinese remainder theorem. □

4. Results

This section expands the results summarized in the introduction, detailed proofs are postponed to the next sections. Additional conjectures appear in § 8.

4.1. **Two orbits.**

Theorem 1.1 can be reformulated as:

Proposition 4.1. *Given a prime $n \geqslant 5$, the primitive n-square-tiled surfaces in $\mathcal{H}(2)$ fall into two $\mathrm{SL}(2,\mathbf{Z})$ orbits.*

The idea for proving this is first to give an invariant which takes two different values, thus proving that there are at least two orbits (see § 4.2 below, and § 5.1), then prove that there are exactly two orbits by showing that each orbit contains a one-cylinder surface (see § 5.2), and that all one-cylinder surfaces with the same invariant are indeed in the same orbit (§ 5.3).

We will call these orbits A and B.

Remark. An extension of this result in some components of higher-dimensional strata is presented in appendix B.

4.2. **Invariant.**

We present a geometric invariant that can easily be computed for any primitive square-tiled surface in $\mathcal{H}(2)$ (for instance presented in its decomposition into horizontal cylinders.)

The Weierstrass points of a surface in $\mathcal{H}(2)$ are
- the saddle (6π-angle cone point),
- and five regular points.

Lemma 4.2. *The number of integer Weierstrass points of a primitive square-tiled surface is invariant under the action of $\mathrm{SL}(2,\mathbf{Z})$.*

By integer point we mean a vertex of the square tiling. The proof of the lemma is obvious, since $\mathrm{SL}(2,\mathbf{Z})$ preserves $\mathbf{Z}^2$.

Proposition 4.3. *Primitive n-square-tiled surfaces in $\mathcal{H}(2)$ have*
- *for $n = 3$, exactly 1 integer Weierstrass point,*
- *for even n, exactly 2,*
- *for odd n, either 1 or 3 (both values occur).*

Martin Möller pointed out to us that this invariant also appears in [Ka, § 2, formula (6)] in algebraic geometric language; Kani's normalized covers correspond to our orbit B. This invariant is also mentioned in [Mö, Remark 3.4].

4.3. **Elliptic affine diffeomorphisms.**

Proposition 4.4. *A translation surface in $\mathcal{H}(2)$ has no nontrivial translation in its affine group. Hence the derivation from its affine group to its Veech group is an isomorphism.*

Proposition 4.5. *A translation surface in $\mathcal{H}(2)$ can have no elliptic element of order* 3 *in its Veech group.*

Lemma 4.6. *Any R-invariant Veech surface in $\mathcal{H}(2)$ can be represented as a R-invariant octagon.*

Proposition 4.7. *For any given prime n, there exist R-invariant n-square-tiled $\mathcal{H}(2)$ surfaces. All of them have the same invariant, namely, A if $n \equiv -1$ [4] and B if $n \equiv 1$ [4].*

Remark. This proposition implies the following interesting fact: there are finite-covolume Teichmüller discs with no elliptic points. This differs from the billiard case which has been the main source of explicit examples of lattice Veech groups.

4.4. **Countings.** The asymptotic number of square-tiled surfaces in $\mathcal{H}(2)$ of area bounded by N is given in [Zo] (see also [EsOk] and [EsMaSc]) to be $\zeta(4)\frac{N^4}{24}$ for one-cylinder surfaces and $\frac{5}{4}\zeta(4)\frac{N^4}{24}$ for two-cylinder surfaces. The mean order for the number of square-tiled surfaces of area exactly n is therefore $\zeta(4)\frac{n^3}{6}$ for one-cylinder surfaces and $\frac{5}{4}\zeta(4)\frac{n^3}{6}$ for two-cylinder surfaces.

The following proposition, from which Theorem 1.2 follows, states that for prime n, there are in fact asymptotics for these numbers, which are $\zeta(4)$ times smaller than the mean order.

Proposition 4.8. *For prime n, there are $O(n)$ elliptic points, and the following countings and asymptotics hold for surfaces and cusps, according to the number of cylinders and to the orbit.*

surfaces:

	1-*cyl*	2-*cyl*	*all*
A	$\frac{n(n-1)(n+1)}{24}$	$\sim \frac{7}{8}\frac{n^3}{6}$	$\sim \frac{9}{8}\frac{n^3}{6}$
B	$\frac{n(n-1)(n-3)}{8}$	$\sim \frac{3}{8}\frac{n^3}{6}$	$\sim \frac{9}{8}\frac{n^3}{6}$
all	$\frac{n(n-1)(n-2)}{6}$	$\sim \frac{5}{4}\frac{n^3}{6}$	$\sim \frac{9}{4}\frac{n^3}{6}$

cusps:

	1-*cyl*	2-*cyl*	*all*
A	$\frac{(n-1)(n+1)}{24}$	$o(n^{3/2+\varepsilon})$	$\sim \frac{n^2}{24}$
B	$\frac{(n-1)(n-3)}{8}$	$o(n^{3/2+\varepsilon})$	$\sim \frac{n^2}{8}$
all	$\frac{(n-1)(n-2)}{6}$	$o(n^{3/2+\varepsilon})$	$\sim \frac{n^2}{6}$

This proposition gives more detail than Theorem 1.2 by distinguishing one-cylinder and two-cylinder cusps and surfaces. Proposition 1.4 and Proposition 1.5 are corollaries of this proposition.

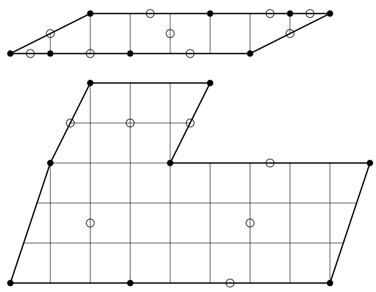
Remarks. Orbits A and B have asymptotically the same size (same number of square-tiled surfaces). However orbit B has asymptotically three times as many one-cylinder surfaces as orbit A.

In each orbit the proportion of two-cylinder cusps is asymptotically negligible; however it is not the case for the proportion of two-cylinder surfaces. This shows that the average width of the two-cylinder cusps grows faster than n. (One-cylinder cusps all have width n.)

5. Proof of main theorem (two orbits)

In this section we first prove Proposition 4.3, then Proposition 4.1.

Convention for figures. In all figures, we represent a square-tiled surface S in $\mathcal{H}(2)$ by a fundamental octagonal domain. S is obtained by identifying pairs of parallel sides of same length; all vertices (black dots) get identified to the saddle. Circles are sometimes used to indicate the other Weierstrass points.



Except in § 6.3.2, the octagon reflects horizontal cylinders: nonhorizontal sides are identified by horizontal translations. On one-cylinder surfaces, the horizontal sides on the top and on the bottom of the cylinder are identified in opposite cyclic order. Two-cylinder surfaces are represented with the cylinder of least width on top of the widest one, to the left. Its top side is glued to the leftmost side under the bottom cylinder. The remaining two sides, to the right on the top and bottom of the bottom cylinder, are identified with each other.

5.1. Two values of the invariant.

Here we prove Proposition 4.3, about the possible values of the number of integer Weierstrass points of a primitive square-tiled surface in $\mathcal{H}(2)$.

Recall that the hyperelliptic involution turns the cylinders upside-down. We deduce the position of Weierstrass points (see figure).

The saddle is always an integer Weierstrass point. We discuss the case of the remaining five, depending on the parity of the parameters.

Under the hyperelliptic involution:

• saddle connections that bound a cylinder both on its top and on its bottom are mapped to themselves with reversed orientation, so that their middpoint is fixed: it is a Weierstrass point, integer when the length of the saddle connection is even.

• the core circle of a cylinder, also mapped to itself with orientation reversed, has two antipodal fixed points. If the cylinder has odd height, none of them is integer. When the height is even and the width odd, one of them is integer. When the height and width are even, either both or none is integer, depending on the parity of the twist parameter.

5.1.1. *One-cylinder case.* The core of the (height 1) cylinder contains two non-integer Weierstrass points. The remaining three are the midpoints of the horizontal connections (whose lengths add up to n).

If n is odd, it splits into either 3 odd lengths (no integer Weierstrass point), or 1 odd and 2 even lengths (2 integer Weierstrass points). For $n = 3$ all lengths are 1 (hence odd); for greater odd n both cases occur.

If n is even, two lengths are odd and one even (if all were even, the surface could not be primitive). This completes the one-cylinder case.

5.1.2. *Two-cylinder case.* We use parameters h_1, h_2, w_1, w_2, t_1, t_2 introduced above. We also use ℓ_1 and ℓ_2 to denote the lengths of the horizontal saddle connections. We then have:

$\ell_1 = w_1$, $\ell_1 + \ell_2 = w_2$, $n = w_1 h_1 + w_2 h_2 = h_1 \ell_1 + h_2(\ell_1 + \ell_2)$ $(*)$.

• **Odd** n. If ℓ_2 is even, the corresponding Weierstrass point is integer. Because n is odd, equation $(*)$ implies that ℓ_1 is odd, thus both cylinders have odd widths, and still by $(*)$ one of the heights must be even. The corresponding cylinder has one integer Weierstrass point on its core line. The total number of integer Weierstrass points is then 3.

If ℓ_2 is odd, the corresponding Weierstrass point is non-integer; if ℓ_1 is odd (resp. even), then w_2 is even (resp. odd), thus by $(*)$ h_1 (resp. h_2) has to be odd, meaning the top (resp. bottom) cylinder contains two non-integer Weierstrass points. The two Weierstrass points in the bottom (resp. top) cylinder are integer if h_2 is even and t_2 is odd (resp. if h_1 and t_1 are even), non-integer otherwise (see figure above). The value of the invariant is accordingly 3 or 1.

For $n = 3$, $\ell_1 = \ell_2 = 1$; for greater odd n both values do occur.

• **Even** n**.** Recall that primitivity implies $h_1 \wedge h_2 = 1$. In particular at least one of them is odd.

If both heights are odd, the Weierstrass points inside the cylinders are non-integer, and because $n = (h_1 + h_2)\ell_1 + h_2\ell_2$ is even, ℓ_2 has to be even, so the last Weierstrass point is integer, and the invariant is 2.

If h_1 is odd and h_2 even, then, by $(*)$, ℓ_1 has to be even. Then if ℓ_2 is odd, the corresponding Weierstrass point is non-integer, one of the Weierstrass points inside the bottom cylinder is integer, and the invariant is 2. If ℓ_2 is even, the corresponding Weierstrass point is integer, and t_2 has to be odd for the surface to be primitive, hence the remaining Weierstrass points are non-integer, and the invariant is 2.

The last case to consider is when h_1 is even and h_2 odd. If ℓ_1 is odd, then so is ℓ_2 (by $(*)$), so one Weierstrass point in the top cylinder is integer, and the invariant is 2. If ℓ_1 is even, then ℓ_2 is also even by $(*)$. The corresponding Weierstrass point is integer, and t_1 is odd for primitiveness. Thus all Weierstrass points inside cylinders are non-integer, and the invariant is 2.

This completes the two-cylinder case, and Proposition 4.3 is proved.

• **Summary of two-cylinder case.** For future reference, we sum up the case study above in a table giving the invariant for odd n according to the parity of h_1, h_2, ℓ_1, ℓ_2 (recall that $w_1 = \ell_1$ and $w_2 = \ell_1 + \ell_2$).

h_1	h_2	ℓ_1	ℓ_2	invariant
0	1	1	0	3
1	0	1	0	3
0	1	0	1	t_1 odd: 1; t_1 even: 3
1	0	1	1	t_2 odd: 3; t_2 even: 1
1	1	0	1	1
1	1	1	1	1

Table for odd n case.

The other combinations of parities of the parameters cannot happen for odd n and primitive surfaces.

Note that for even n we concluded that the invariant is 2 for all primitive surfaces.

5.2. **Reduction to one cylinder.**

Proposition 5.1. *Each orbit contains a one-cylinder surface. Equivalently, each surface has a direction in which it decomposes in one single cylinder.*

A baby version of this proposition is the following lemma.

Lemma 5.2. *A two-cylinder surface of height* 2 *tiled by a prime number of squares has one-cylinder directions.*

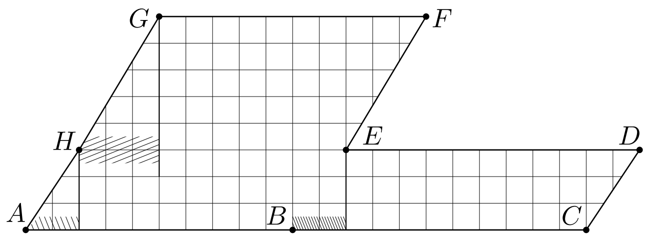
Proof of the lemma. Consider a surface made of two cylinders, both of height 1. Since n is prime, the two widths are relatively prime. By acting by U, the twists can be set to any values (see Lemma 3.1). Set the top twist to 0 and the bottom twist to 1. Then by considering the vertical flow, we get a one-cylinder surface. □

We prove the proposition by induction on the height of the surface: given a two-cylinder surface, we show that its orbit contains a surface of strictly smaller height.

Consider a two-cylinder square-tiled surface S in $\mathcal{H}(2)$, with a prime number of square tiles. By acting by U we can move to the canonical representative of the same cusp (see Lemma 3.1), so we will assume $t_i < w_i$, $i = 1, 2$.

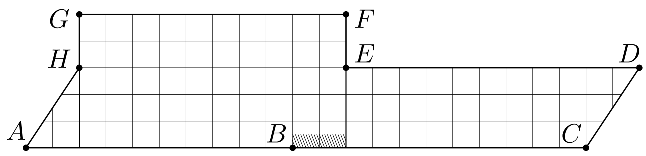
We split our study into four cases according to which twists are zero.

Case 1. Both twists are nonzero.



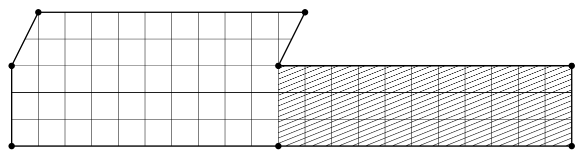
Call h_1, h_2 the heights and t_1, t_2 the twists of the horizontal cylinders of S. Consider the rotated surface RS. If RS consists of one horizontal cylinder, we are done. Otherwise, it has two horizontal cylinders, which are the vertical cylinders of S, and fill S. Looking to the right of A, H, and B, we see all vertical cylinders of S. The vertical cylinder to the right of A has height at most t_2, that to the right of B also, and that to the right of H at most t_1. So one of the vertical cylinders has heights at most t_2, and the other one has height at most t_1. The sum of their heights is hence at most $t_1 + t_2$, so it is less than $h_1 + h_2$.

Case 2. The bottom twist is nonzero but the top twist is zero.



In this case the same vertical cylinder is to the right of A and H. If the vertical separatrix going down from H ends in B, there is only one vertical cylinder (one horizontal cylinder for the rotated surface RS); if not, it necessarily crosses the shaded region to the right of B, so there are two vertical cylinders, and the sum of their heights is at most t_2 (the twist of the bottom cylinder of S), hence less than the height of the bottom cylinder of S.

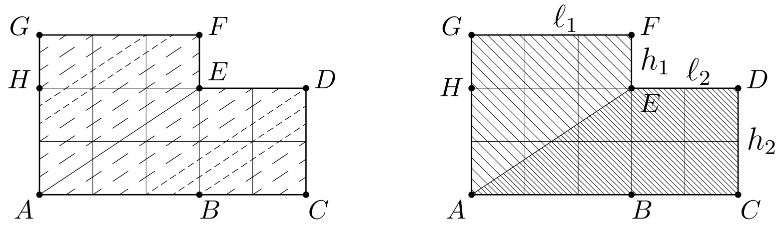
Case 3. The bottom twist is zero but the top twist is nonzero.



Act by R; this rotates S by $\pi/2$. The rotated surface $R \cdot S$ has two cylinders: a top cylinder, corresponding to the side part of S (shaded on the figure), with twist 0, and a bottom cylinder cylinder of height at most t_1, which we assumed to be less than h_1. The surface in the same cusp with least nonnegative twists also has top twist 0, so if it has bottom twist 0, conclude by case 4, otherwise apply case 2 to obtain a surface of height less than h_1.

Case 4. The twist parameters are both zero. In this case we end the induction by jumping to a one-cylinder surface directly:

Lemma 5.3. *The diagonal direction for the "base rectangle" of an L surface tiled by a prime number of squares is a one-cylinder direction.*



Proof. The ascending diagonal $[AE]$ of the base rectangle of our L surface cuts it into two zones. Note that $[AE]$ has no other integer point than A and E by (P) of § 3.3.2.

The other two saddle connections parallel to $[AE]$ start from B and H and end in F and D. We want to prove that the one starting from H ends in F and the one issued from B ends in D, meaning each saddle connection returns with angle 3π.

Set the origin in A or E and consider coordinates modulo $\ell_1\mathbf{Z}\times h_2\mathbf{Z}$.

Follow a saddle connection parallel to $[AE]$ from integer point to integer point. While it winds in a same zone, the coordinates of the integer points it reaches remain constant modulo $\ell_1\mathbf{Z} \times h_2\mathbf{Z}$. Changing zone has the following effects for the coordinates of the next integer point:

- from the upper to the lower zone: decrease y by h_1 modulo h_2;
- from the lower to the upper zone: decrease x by ℓ_2 modulo ℓ_1.

Zone changes have to be alternated. Once inside a zone with the right coordinates modulo $\ell_1\mathbf{Z} \times h_2\mathbf{Z}$, a separatrix reaches the top right corner of the zone with no more zone change.

So we want to prove that starting from B, in the lower zone with coordinates $(0, 0)$, and adding in turn $(-\ell_2, 0)$ and $(0, -h_1)$, coordinates $(\ell_2, 0)$ (point D) will be reached before $(0, h_1)$ (point H).

After k changes from lower to upper zone and k changes from upper to lower zone, the coordinates are final if $k \equiv -1\ [\ell_1]$ and $k \equiv 0\ [h_2]$; that is, if k is $h_2(\ell_1 - 1)$. After $k + 1$ changes from lower to upper zone and k changes from upper to lower zone, the coordinates are final if $k \equiv 0\ [\ell_1]$ and $k \equiv 0\ [h_2]$, which means k is $h_2 \cdot \ell_1$. So the separatrix parallel to $[AE]$ starting from B reaches D. □

5.3. **Linking one-cylinder surfaces of each type.** We call a surface type A (resp. B) if it has 1 (resp. 3) integer Weierstrass points.

Recall that a primitive one-cylinder surface in $\mathcal{H}(2)$ has height one, hence it is determined by the cyclically ordered lengths of the three saddle connections on the bottom of this cylinder (which add up to n), and by a twist parameter.

The repeated action of U can set the twist parameter to any of its n possible values, so for the purpose of linking surfaces of the same type by $\mathrm{SL}(2,\mathbf{Z})$ action, we may already consider surfaces with the same cyclically ordered partition (a,b,c) as equivalent (allowing implicit U-action). We will call them (a,b,c) surfaces.

Partitions into three odd numbers correspond to type A; partitions into two even numbers and one odd number correspond to type B.

We will first show that any one-cylinder surface has a $(1,*,*)$ surface in its orbit; then we will show that $(1,b,c)$ surfaces with b and c odd are in the orbit of a $(1,1,n-2)$ surface, proving all type A surfaces to be in one orbit; then that $(1,2a,2b)$ surfaces are in the orbit of a $(1,2,n-3)$ surface, proving all type B surfaces to be in one orbit.

Consider a rational-slope direction on a square-tiled surface S; this direction is completely periodic. Say it is given by a vector $(p,q) \in \mathbf{Z}^2$, with $p \wedge q = 1$. For any $(u,v) \in \mathbf{Z}^2$ such that $\det\left(\begin{smallmatrix} p & u \\ q & v \end{smallmatrix}\right) = 1$ our surface can be seen as tiled by parallelograms of sides (p,q), (u,v), whose vertices are the vertices of the square tiling.

These parallelograms are taken to unit squares by $M = \left(\begin{smallmatrix} p & u \\ q & v \end{smallmatrix}\right)^{-1} \in \mathrm{SL}(2,\mathbf{Z})$. We call $M \cdot S$ "the surface seen in direction (p,q)" on S.

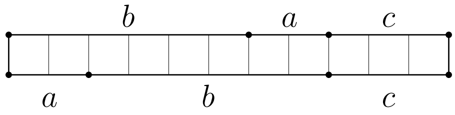
Consider a saddle connection σ on S in direction (p,q); the corresponding saddle connection on $M \cdot S$ is horizontal with an integer length equal to the number of integer points (vertices of the square tiling) σ reaches on S. Abusing vocabulary we also call this the length of σ.

A saddle connection returns at an angle of 3π if and only if it has a Weierstrass point in its middle. If two saddle connections in a given direction return with angle 3π then so does the third, and that direction is one-cylinder; thus two saddle connection lengths give the third.

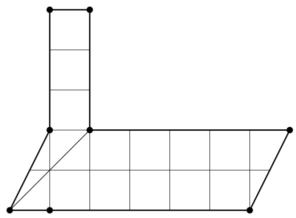
5.3.1. *First step: any one-cylinder surface has a* $(1,*,*)$ *surface in its orbit.* To show this, we prove that an (a,b,c) surface has a $(\delta,k\delta,\gamma)$ surface in its orbit, where $\delta \mid a \wedge b$. Then because n is prime we have

$\gamma \wedge \delta = 1$, hence applying the argument a second time with γ and δ in place of a and b shows that there is a $(1, *, *)$ one-cylinder surface in the orbit of the surface we started with.

The proof is as follows. Consider the (a, b, c) surface S having saddle connections of lengths a, b, c on the bottom, b, a, c on the top.



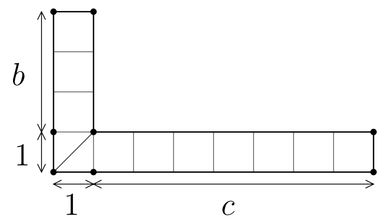
RS has two cylinders, the top one of height c and width 1, and the bottom one of height $d = a \wedge b$ and width $\frac{a+b}{d}$, and some twist t.



Now the direction $(1+t, d)$ is a $(\delta, k\delta, \gamma)$ one-cylinder direction with $\delta = (1+t) \wedge d$. Note that $k = \frac{a+b}{d} - 1$, and that $\gamma \wedge \delta = 1$.

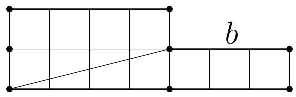
So by applying this procedure twice we see that any surface has a $(1, *, *)$ one-cylinder surface in its orbit.

5.3.2. *End of proof for type A surfaces.* There only remains to link any $(1, b, c)$ surface, where b and c are odd, to a $(1, 1, n-2)$ surface.



Consider the L surface with arms of width 1 and lengths b and c.

Apply U^2 to set the bottom twist to 2. Then rotate by applying R, and obtain a surface with two cylinders of height 1. By applying a convenient power of U the twists can be made both 0.



In the diagonal direction of the base rectangle of this new L surface, we see a $(1, 1, n-2)$ surface.

5.3.3. *End of proof for type B surfaces.* Here we take the one-cylinder surface with the partition $(1, 2, n-3)$ as the reference surface, and prove by steps that any type B surface has it in its orbit.

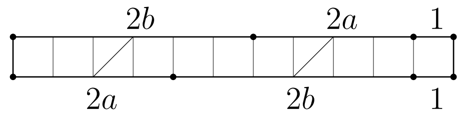
To do this, we first show that any one-cylinder surface has a one-cylinder surface with a $(1, 2a, 2b)$ partition in its orbit. This is done by the first step explained above.

Then we link

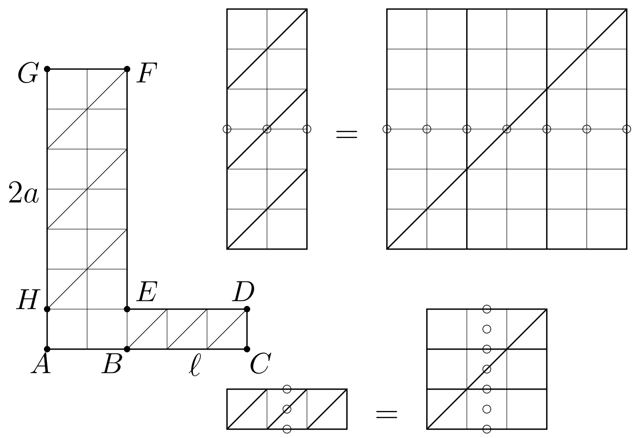
- $(1, 2a, 2b)$ where $a \neq b$ with $(d, 2d, *)$, then with $(1, 2, n-3)$;
- $(1, 2a, 2b)$ where $a = b$ with $(2, 2, n-4)$, then with $(1, 2, n-3)$.

  - Linking $(1, 2a, 2b)$ with $(1, 2, *)$ when $a \neq b$.

Without loss of generality, suppose $a < b$. Consider the one-cylinder surface with saddle connections of lengths $2a$, $2b$, 1 on the bottom and $2b$, $2a$, 1 on the top.



In the direction $(b-a, 1)$ there is a connection between two integer Weierstrass points, so in this direction we see a two-cylinder surface. Its top cylinder has height $2a$ and width 2 and its bottom cylinder has height 1 and with $2+\ell$ for some ℓ.

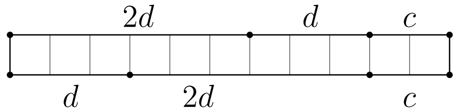


In certain directions, the separatrix issued from H winds around the horizontal cylinder $HEGF$. In particular, in any direction (k, a), $k \in \mathbf{N}$, it will run into a Weierstrass point (and into a saddle after twice the distance).

Likewise, in appropriate directions, the separatrix issued from B winds around the vertical cylinder $BCDE$. In particular, in any direction $(\ell/2, k/2)$ (equivalently ℓ, k), $k \in \mathbf{N}$, it will run into a Weierstrass point (and into a saddle after twice the distance).

Consider therefore the direction (ℓ, a). In this direction we get a $(d, 2d, *)$ one-cylinder surface, where $d = a \wedge \ell$.

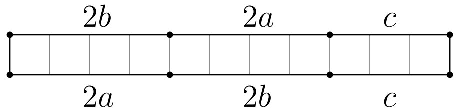
Now there only remains to link $(d, 2d, *)$ with $(1, 2, *)$, which is easily done: consider the one-cylinder surface with saddle connections d, $2d$, c on the bottom and $2d$, d, c on the top;



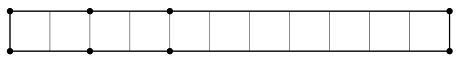
in the $(d, 1)$ direction we get a $(1, 2, *)$ one-cylinder surface.

- Linking $(1, 2a, 2b)$ with $(1, 2, *)$ when $a = b$.

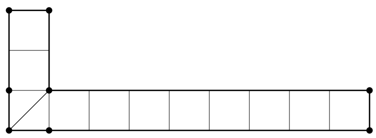
Consider the one-cylinder surface with saddle connections of length $2a$, $2b$, c on the bottom and $2b$, $2a$, c on the top.



In the direction $(a, 1)$ we see a $(2, 2, *)$ one-cylinder surface.

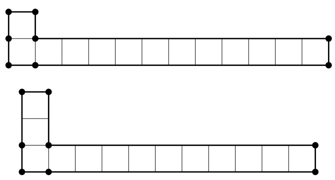


On this surface, in the direction $(2, 1)$, we have a two-cylinder surface with its top cylinder of height 2 and width 1, and its bottom cylinder of height 1. Acting by U we can set the twist parameters to 0.



Then in the direction $(1, 1)$ we see a $(1, 2, n-3)$ one-cylinder surface.

5.4. **L-shaped billiards.** L-shaped billiards give rise to L-shaped translation surfaces by an unfolding process; any L-shaped translation (with zero twists) surface is the covering translation surface of an L-shaped billiard.



Fix some prime $n > 3$, and consider the two-cylinder surfaces S_1 and S_2, both having $h_2 = 1$, $w_1 = 1$ and $t_1 = t_2 = 0$, and S_1 having $h_1 = 1$, $w_2 = n - 1$ and S_2 having $h_2 = 2$, $w_2 = n - 2$. The picture on the side represents S_1 and S_2 for $n = 13$.

For each n, S_1 and S_2 belong to orbit A and B respectively, and arise from L-shaped billiards. This proves Proposition 1.3.

6. Proof of results about elliptic points

Some constructions in this section are inspired by [Ve95].

6.1. **Translations.** Here we prove Proposition 4.4.

Suppose a surface $S \in \mathcal{H}(2)$ has a nontrivial translation f in its affine group. f fixes the saddle and induces a permutation on outgoing horizontal separatrices. Let ε be smaller than the length of the shortest saddle connection of S, and consider the three points at distance ε from the saddle on the three separatrices in a given direction. f cannot fix any of these points, otherwise it would be the identity of S, but it fixes the set of these points, hence it induces a cyclic permutation on them. This implies that except for the saddle, which is fixed, all f-orbits have size 3. However the set of regular Weierstrass points is also fixed (since the translation f is an automorphism of the underlying Riemann surface), and has size 5. This is a contradiction.

6.2. **Elliptic points of order** 3**.** Here we prove Proposition 4.5.

Suppose a surface S in $\mathcal{H}(2)$ has an elliptic element of projective order three in its Veech group. Since the hyperelliptic involution has order 2, S has in fact an elliptic element of order 6 in its Veech group. Conjugate by $\mathrm{SL}(2, \mathbf{R})$ to a surface that has the rotation by $\pi/3$ (hereafter denoted by r) in its Veech group.

Considering Proposition 4.4, we denote by r the corresponding affine diffeomorphism.

The set of Weierstrass points is preserved by r. The saddle being fixed, the remaining five Weierstrass points are setwise fixed, so at least two of them are also fixed. Consider one Weierstrass point that is fixed, call it W. Consider the shortest saddle connections through W. They come by triples making angles $\pi/3$.

Take one such triple, consider the corresponding regular hexagon (which has these saddle connections as its diagonals).

We can take this hexagon as a building block for a polygonal fundamental domain of the surface. Consider a pair of opposite sides of this hexagon; they cannot be identified, since the rotational symmetry would imply other identifications and mean we have a torus.

Hence, these sides and the diagonal parallel to them are three saddle connections in the same direction. So this is a completely periodic direction, and we want to see two cylinders in this direction. This would imply identifying two opposite sides, which we have excluded.

6.3. **Elliptic elements of order 2.**

6.3.1. *Proof of Lemma* 4.6. Here, inspired by [Ve95], we give a convenient representation for R-invariant Veech surfaces in $\mathcal{H}(2)$: a fundamental octagon which is R-invariant. Consider a Veech surface in $\mathcal{H}(2)$ that has R in its Veech group; denote also by R the corresponding affine diffeomorphism.

The set of Weierstrass points is fixed by R (as by any affine diffeomorphism). The saddle being fixed, at least one of the remaining 5 Weierstrass points must be fixed.

Consider such a point and the shortest saddle connections through this point. They come by orthogonal pairs. Take one such pair. Consider the square having this pair of saddle connections as diagonals. Without loss of generality, consider the sides of the square as horizontal and vertical.

This square is the central piece of our fundamental domain. Other than the corners (the saddle) and the center, there are no Weierstrass point inside this square or on its edges.

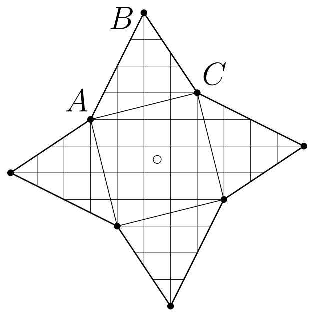
Consider the horizontal sides of our square. These sides are saddle connections so they define a completely periodic direction on the surface.

These sides are not identified, otherwise by R-symmetry the other two would also be and we would have a torus. So this is a two-cylinder direction and our two sides bound the short cylinder in this direction. This short cylinder lies outside the square and can be represented as a parallelogram with its "top-left" corner in the vertical strip defined by the square (i.e. with a "reasonable" twist).

By R-symmetry there also is such a parallelogram in the other direction. To make the picture more symmetric each parallelogram can be cut into two triangles, glued to opposite sides of the square. Thus we get a representation of the surface as an octagon with (parallel) opposite sides identified. Note that the four remaining Weierstrass points are the middle of the sides of this octagon.

6.3.2. *Proof of Proposition* 4.7. Represent the surface as above: an octagon made of a square and four triangles glued to its sides. All vertices lie on integer points.

Let ABC be one of the triangles, labeled clockwise so that AC is a side of the square.



Let (p, q) be the coordinates of $\overrightarrow{AC}$ and (r, s) those of $\overrightarrow{AB}$. The area of the surface is then $p^2 + q^2 + 2(ps - qr)$.

If n is prime then p and q have to be relatively prime, and of different parity. Then $p^2 + q^2 \equiv 1$ [4]. The center of the square lies at the center of a square of the tiling. The condition for two Weierstrass points to lie on integer points is for $(ps - rq)$ to be even.

We conclude by observing that n is 1 (resp. 3) modulo 4 when $(ps - rq)$ is even (resp. odd).

7. Proof of countings

Here we establish the countings and estimates of Proposition 4.8.

7.1. **One-cylinder cusps and surfaces.** For prime $n > 3$, one-cylinder n-square-tiled cusps in $\mathcal{H}(2)$ are in 1-1 correspondence with cyclically ordered 3-partitions of n.

Ordered 3-partitions (a, b, c) of n are in 1-1 correspondence with pairs of distinct integers $\{\alpha, \beta\}$ in $\{1, \ldots, n-1\}$: assuming $\alpha < \beta$, the correspondence is given by $a = \alpha$, $a + b = \beta$, $a + b + c = n$. So there are C_{n-1}^2 ordered 3-partitions of n. Ordered 3-partitions of n being in 3-1 correspondence with cyclically ordered 3-partitions, there are $\frac{1}{3}C_{n-1}^2 = \frac{(n-1)(n-2)}{6}$ cyclically ordered 3-partitions of n.

Thus there are $\frac{(n-1)(n-2)}{6}$ one-cylinder cusps of n-square-tiled translation surfaces in $\mathcal{H}(2)$.

Those in orbit A are those with 3 odd parts $2a - 1$, $2b - 1$, $2c - 1$; these are in 1-1 correspondence with cyclically ordered partitions a, b, c, of $\frac{n+3}{2}$. Their number is hence $\frac{1}{6}(\frac{n+3}{2} - 1)(\frac{n+3}{2} - 2) = \frac{(n+1)(n-1)}{24}$.

The remaining ones are in orbit B, their count is hence the difference, $\frac{(n-1)(n-3)}{8}$.

All one-cylinder cusps discussed here have width n (n possible values of the twist parameter), so the counts of one-cylinder surfaces are n times the corresponding cusp counts.

7.2. **Two-cylinder surfaces.** The total number of two-cylinder n-square-tiled surfaces (n prime) is

$$S(n) = \sum_{a,b,k,\ell} k\ell,$$

where the sum is over $a, b, k, \ell \in \mathbf{N}^*$ such that $k < \ell$ and $ak + b\ell = n$.

This follows from the parametrization in § 3.3.2; the letters a, b, k, ℓ used here correspond to the parameters h_1, h_2, w_1, w_2 there, and the summand is the number of possible values of the twist parameters, given the heights and widths of the two cylinders.

We want the asymptotic for this quantity as n tends to infinity, n prime. In order to find this, we consider the sum as a double sum: the sum over a and b of the sum over k and ℓ.

Write $S(n) = \sum_{a,b} S_{a,b}(n)$, where $S_{a,b}(n) = \sum_{k,\ell} k\ell$.

We study the inner sum by analogy with a payment problem: how many ways are there to pay n units with coins worth a and b units?

This problem is classically solved by the use of generating series: denote the number of ways to pay by $s_{a,b}(n)$; then

$s_{a,b}(n) = \mathrm{Card}\{(k,\ell) \in \mathbf{N}^2 : ak + b\ell = n\} = \sum_{k,\ell\in\mathbf{N}:ak+b\ell=n} 1.$

Now notice that $\sum_{k=0}^{\infty} z^{ak} \sum_{\ell=0}^{\infty} z^{b\ell} = \sum_{n=0}^{\infty} s_{a,b}(n) z^n$, and deduce that the number looked for is the n-th coefficient of the power series expansion of the function $\frac{1}{1-z^a}\frac{1}{1-z^b}$.

We turn back to our real problem, $S_{a,b}(n) = \sum_{k,\ell\in\mathbf{N}^*:ak+b\ell=n,k<\ell} k\ell$.

We want to show that $S(n) \sim cn^3$ for prime n. For this we will use the dominated convergence theorem: we show that $S_{a,b}(n)/n^3$ has a limit $c_{a,b}$ when n tends to infinity with a and b fixed, and that $S_{a,b}(n)/n^3$ is bounded by some $g_{a,b}$ such that $\sum_{a,b} g_{a,b} < \infty$, to conclude that $S(n)/n^3 = \sum_{a,b} S_{a,b}(n)/n^3$ tends to $c = \sum_{a,b} c_{a,b}$, which means $S(n) \sim cn^3$.

The dominated convergence is proved as follows.

Write $S_{a,b}(n) = \sum_{k,h\in\mathbf{N}^*:(a+b)k+bh=n} k(k+h)$ by introducing $h = \ell - k$. Then split the sum into $\sum k^2$ and $\sum kh$. Write

$$S'_{a,b}(n) = \sum_{k,h\in\mathbf{N}^*,\ (a+b)k+bh=n} k^2/n^3 \leqslant \sum_{k\in\mathbf{N}^*,\ h\in\mathbf{Q},\ (a+b)k+bh=n} k^2/n^3$$

$$S''_{a,b}(n) = \sum_{k,h\in\mathbf{N}^*,\ (a+b)k+bh=n} kh/n^3 \leqslant \sum_{k\in\mathbf{N}^*,\ h\in\mathbf{Q},\ (a+b)k+bh=n} kh/n^3$$

(in the sums on the right-hand side, h has been allowed to be a rational instead of an integer.) Hence

$$S'_{a,b}(n) \leqslant \frac{1}{(a+b)^3}\Big[\frac{a+b}{n}\sum_{k=1}^{\lfloor n/(a+b)\rfloor}\big(\frac{a+b}{n}k\big)^2\Big]$$

$$S''_{a,b}(n) \leqslant \frac{1}{(a+b)^2 b}\Big[\frac{a+b}{n}\sum_{k=0}^{n/(a+b)}\big(\frac{a+b}{n}k\big)\big(1-\frac{a+b}{n}k\big)\Big]$$

The expressions in brackets, Riemann sum approximations to the integrals $\int_0^1 x^2\mathrm{d}x$ and $\int_0^1 x(1-x)\mathrm{d}x$, are uniformly bounded by 1.

Now notice that $\sum_{a,b}\frac{1}{(a+b)^3}$ and $\sum_{a,b}\frac{1}{(a+b)^2 b}$ are convergent. This ends the dominated convergence argument.

We can now investigate the limit. For ease of calculation, we drop the condition $k < \ell$. We take care of it by writing $\sum_{k,\ell} = 2\sum_{k<\ell} + \sum_{k=\ell}$. For prime n, $k = \ell$ implies that they are both equal to 1. The sum for $k = \ell$ is hence equal to $n-1$, and we will not need to take it into account since the whole sum will grow as n^3.

Denote by $\tilde{S}(n,a,b)$ the sum over all k and ℓ.

Notice that $\sum_{k=0}^{\infty} kz^{ak}\sum_{\ell=0}^{\infty}\ell z^{b\ell} = \sum_{n=0}^{\infty}\tilde{S}(n,a,b)z^n$.

$\tilde{S}(n,a,b)$ is therefore the n-th coefficient of the power series expansion of the function $f_{a,b} = \frac{z^a}{(1-z^a)^2}\frac{z^b}{(1-z^b)^2}$.

To determine this coefficient, decompose $f_{a,b}$ into partial fractions. This function has poles at a-th and b-th roots of 1. Since n is prime, we are only interested in relatively prime a and b, for which the only common root of 1 is 1 itself, which is hence a 4-th order pole of $f_{a,b}$, while other poles have order 2.

The n-th coefficient of the power series expansion of $f_{a,b}$ is a polynomial of degree 3 in n, whose leading term is $c_{a,b}\frac{n^3}{6}$, where $c_{a,b}$ is the coefficient of $\frac{1}{(1-z)^4}$ in the decomposition of $f_{a,b}$ into partial fractions. This coefficient is computed to be $\frac{1}{a^2b^2}$.

We want the sum over relatively prime a and b. We relate it to the sum over all a and b by sorting the latter according to $d = a \wedge b$.

$$\sum_{a,b} \frac{1}{a^2b^2} = \sum_{d} \sum_{a,b,\ a\wedge b=d} \frac{1}{a^2b^2} = \sum_{d} \frac{1}{d^4} \sum_{a,b,\ a\wedge b=1} \frac{1}{a^2b^2}.$$

By observing that $\sum_{a,b} \frac{1}{a^2b^2} = (\sum_a \frac{1}{a^2})^2 = \zeta(2)^2 = \frac{\pi^4}{36}$ and that $\sum_d \frac{1}{d^4} = \zeta(4) = \frac{\pi^4}{90}$ we get that the sum $\sum_{a,b,\ a\wedge b=1} \frac{1}{a^2b^2}$ is equal to $5/2$. Divide by 2 to get back to $k < \ell$, and find that $S(n) \sim \frac{5}{4}\frac{n^3}{6}$.

7.3. **Two-cylinder surfaces by orbit.** Two-cylinder surfaces for which both heights are odd are in orbit A; those for which both widths are odd are in orbit B; half of the remaining ones are in orbit A, and half in B; the factor one half comes from the conditions on the twists. (See the table in § 5.1.)

First compute the asymptotic for **odd heights**. Write

$$S^{\text{oh}}(n) = \sum_{\substack{a,b,k,\ell \\ ak+b\ell=n \\ a,b \text{ odd} \\ a\wedge b=1 \\ k<\ell}} k\ell.$$

Then $S^{\text{oh}}(n) \sim \frac{1}{2}\widetilde{S}^{\text{oh}}(n)$ where $\widetilde{S}^{\text{oh}}(n)$ is the same sum without the condition $k < \ell$. The dominated convergence works as previously.

For odd a and b such that $a \wedge b = 1$,

$$\widetilde{S}^{\text{oh}}_{a,b}(n) = \sum_{\substack{k,\ell \\ ak+b\ell=n}} k\ell \sim \frac{1}{a^2b^2} \cdot \frac{n^3}{6}.$$

We need to sum over relatively prime odd a and b. Using the same trick as previously, write

$$\sum_{a,b \text{ odd}} \frac{1}{a^2b^2} = \sum_{d \text{ odd}} \sum_{\substack{a,b \text{ odd} \\ a\wedge b=d}} \frac{1}{a^2b^2} = \sum_{d \text{ odd}} \frac{1}{d^4} \sum_{\substack{a,b \text{ odd} \\ a\wedge b=1}} \frac{1}{a^2b^2}.$$

Now

$$\sum_{a,b \text{ odd}} \frac{1}{a^2b^2} = \Big(\sum_{a \text{ odd}} \frac{1}{a^2}\Big)^2 = ((1-1/2^2)\zeta(2))^2 = \frac{9}{16}\cdot\frac{\pi^4}{36}$$

and

$$\sum_{d \text{ odd}} \frac{1}{d^4} = (1-1/2^4)\zeta(4) = \frac{15}{16}\cdot\frac{\pi^4}{90}$$

so

$$\sum_{\substack{a,b \text{ odd}\\ a\wedge b=1}} \frac{1}{a^2b^2} = 3/2.$$

We deduce that $S^{\mathrm{oh}}(n) \sim \frac{3}{4}\frac{n^3}{6}$ (the condition $k<\ell$ is responsible for a factor $1/2$).

Similarly compute the asymptotic for **odd widths**. Write

$$S^{\mathrm{ow}}(n) = \sum_{\substack{a,b,k,\ell\\ ak+b\ell=n\\ k,\ell \text{ odd}\\ a\wedge b=1\\ k<\ell}} k\ell.$$

For fixed a and b with $a\wedge b=1$, put

$$\widetilde{S}^{\mathrm{ow}}_{a,b}(n) = \sum_{\substack{k,\ell \text{ odd}\\ ak+b\ell=n}} k\ell.$$

Notice that $\sum_{k \text{ odd}} kz^{ak} \sum_{\ell \text{ odd}} \ell z^{b\ell} = \sum \widetilde{S}^{\mathrm{ow}}_{a,b}(n) z^n$.

Because $\sum kz^k = \frac{z}{(1-z)^2}$, $\sum 2kz^{2k} = \frac{2z^2}{(1-z^2)^2}$, and the difference is $\sum (2k+1)z^{2k+1} = \frac{z(1+z^2)}{(1-z^2)^2}$.

$\widetilde{S}^{\mathrm{ow}}_{a,b}(n)$ is now the n-th coefficient of the power series expansion of $\frac{z^a(1+z^{2a})}{(1-z^{2a})^2}\cdot\frac{z^b(1+z^{2b})}{(1-z^{2b})^2}$. When $a\wedge b=1$, this rational function has two order 4 poles at 1 and -1 and its other poles have order 2; the coefficients of $\frac{1}{(1-z)^4}$ and $\frac{1}{(1+z)^4}$ in its decomposition into partial fractions are respectively $\frac{1}{4a^2b^2}$ and $\frac{(-1)^{a+b}}{4a^2b^2}$.

Because n is odd, and k and ℓ are odd, a and b have to have different parities, so $a+b$ is odd. So $\frac{1}{4a^2b^2} + \frac{(-1)^{a+b}(-1)^n}{4a^2b^2} = \frac{1}{2a^2b^2}$.

Now

$$\sum_{\substack{a\wedge b=1\\ a\not\equiv b\ [2]}} \frac{1}{a^2b^2} = \sum_{a\wedge b=1} \frac{1}{a^2b^2} - \sum_{\substack{a\wedge b=1\\ a,b \text{ odd}}} \frac{1}{a^2b^2} = 5/2 - 3/2 = 1.$$

The condition $k < \ell$ brings a factor $1/2$, thus we get $S^{\text{ow}}(n) \sim (1/4)(n^3/6)$.

The remaining surfaces are those for which heights as well as widths of the cylinders have mixed parities. The asymptotic for this "**even-odd**" part is computed as the difference between the total sum and the odd-widths and odd-heights sums.

Write $S(n) = S^{\text{oh}}(n) + S^{\text{ow}}(n) + S^{\text{eo}}(n)$. We already know that $S(n) \sim \frac{5}{4} \cdot \frac{n^3}{6}$, $S^{\text{oh}}(n) \sim \frac{3}{4} \cdot \frac{n^3}{6}$, and $S^{\text{ow}}(n) \sim \frac{1}{4} \cdot \frac{n^3}{6}$. So the even-odd part has asymptotics $S^{\text{eo}}(n) \sim \frac{1}{4} \cdot \frac{n^3}{6}$.

Putting pieces together, the number of n-square-tiled two-cylinder surfaces of type A, n prime, is equivalent to $(3/4 + 1/8)(n^3/6) = (7/8)(n^3/6)$. For type B, we get $(1/4 + 1/8)(n^3/6) = (3/8)(n^3/6)$.

7.4. **Two-cylinder cusps.** For n prime, the number of two-cylinder cusps (in both orbits) is given by

$$S(n) = \sum_{\substack{a,b,k,\ell \in \mathbf{N}^*\\ ak+b\ell=n\\ k<\ell}} (a \wedge k)(b \wedge \ell).$$

(see counting of two-cylinder surfaces in § 7.2 and discussion of cusps in § 3.4.)

Remark. For nonprime n, the number of two-cyl cusps is less than $S(n)$ defined as above, so the bound found here is still valid.

$S(n)$ is less than

$$\widetilde{S}(n) = \sum_{\substack{a,b,k,\ell \in \mathbf{N}^*\\ ak+b\ell=n}} (a \wedge k)(b \wedge \ell).$$

where the condition $k < \ell$ is dropped.

We will show that for any $\varepsilon > 0$, $\widetilde{S}(n) \ll_{n\to\infty} n^{3/2+\varepsilon}$.

This will imply that the number of two-cylinder cusps of n-square-tiled surfaces is sub-quadratic, thus negligible before the (quadratic) number of one-cylinder cusps in each orbit.

$$\widetilde{S}(n) = \sum_{\substack{A,B,u,v\in\mathbf{N}^* \\ Au^2+Bv^2=n}} uvf(A)f(B), \quad \text{where } f(m) = \sum_{\substack{rs=m \\ r\wedge s=1}} 1.$$

Note that $f(m) \leqslant \mathrm{d}(m) \ll m^\varepsilon$, where $\mathrm{d}(m)$ is the number of divisors of m. The factors $f(A)f(B)$ therefore contribute less than an n^ε.

$$\sum_{\substack{A,B,u,v\in\mathbf{N}^* \\ Au^2+Bv^2=n}} uv = \sum_u u \sum_{A\leqslant n/u^2} \Big(\sum_{v^2|n-Au^2} v \Big).$$

The sum in parentheses has less than $\mathrm{d}(n - Au^2)$ summands, each of which is bounded by $\sqrt{n - Au^2}$, so

$$\widetilde{S}(n) \ll n^{1/2+2\varepsilon} \sum_u n/u \ll n^{3/2+3\varepsilon}.$$

□

We thank Joël Rivat for contributing this estimate [Ri].

7.5. **Elliptic points.** The discussion in § 6.3.2 implies that their number is less than the number of integer-coordinate vectors in a quarter of a circle of radius $\sqrt{n}$, so it is $O(n)$.

8. Strong numerical evidence

Martin Schmoll pointed out to us that the number of primitive n-square-tiled surfaces in $\mathcal{H}(2)$ is given in [EsMaSc] to be

$$\frac{3}{8}(n-2)n^2 \prod_{p|n}(1 - \frac{1}{p^2}).$$

By [Mc2], for even n all these surfaces are in the same orbit, and for odd $n \geqslant 5$ they fall into two orbits. So Eskin, Masur and Schmoll's formula gives the cardinality of the single orbit for even n, and the sum of the cardinalities of the two orbits for odd n.

Conjecture 8.1. *For odd n, the cardinalities of the orbits are given by the following functions:*
orbit A: $\frac{3}{16}(n-1)n^2 \prod_{p|n}(1 - \frac{1}{p^2})$,
orbit B: $\frac{3}{16}(n-3)n^2 \prod_{p|n}(1 - \frac{1}{p^2})$.

These formulae give degree 3 polynomials when restricted to prime n, for which Theorem 1.2 gives the leading term. These polynomials are expressed in the table below.

	one-cylinder	two-cylinder	all
A	$\frac{1}{24}(n^3 - n)$	$\frac{1}{48}(7n^3 - 9n^2 - 7n + 9)$	$\frac{3}{16}(n^3 - n^2 - n + 1)$
B	$\frac{1}{8}(n^3 - 4n^2 + 3n)$	$\frac{1}{16}(n^3 - n^2 - 9n + 9)$	$\frac{3}{16}(n^3 - 3n^2 - n + 3)$
all	$\frac{1}{6}(n^3 + 3n^2 + 2n)$	$\frac{1}{24}(5n^3 - 6n^2 - 17n + 18)$	$\frac{3}{8}(n^3 - 2n^2 - n + 2)$

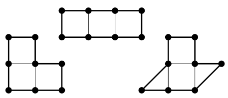
On the other hand, the counting functions for two-cylinder cusps are not polynomials.

Conjecture 8.2. *For prime n, the number of elliptic points is $\lfloor \frac{n+1}{4} \rfloor$.*

This conjecture is valid for the first thousand odd primes.

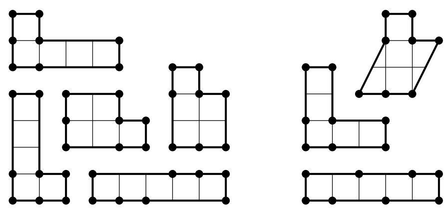
Appendix A. $n = 3$ and $n = 5$

$n = 3$**.** For $n = 3$, we have the following three surfaces.



If we call S_1 the one-cylinder surface, and S_2 and S_3 the two-cylinder surfaces, the generators of $\mathrm{SL}(2, \mathbf{Z})$ act as follows: $US_1 = S_1$, $US_2 = S_3$, $US_3 = S_2$, $RS_1 = S_3$, $RS_2 = S_2$, $RS_3 = S_1$. So there is only one orbit, containing $d = 3$ surfaces, the number of cusps is $c = 2$, the number of elliptic points (R-invariant surfaces) is $e = 1$, so the genus is $g = 0$ by the Gauss-Bonnet formula.

$n = 5$**.** For $n = 5$, we have 27 surfaces forming 8 cusps, a representative of which appears on the following picture.



Computing the $\mathrm{SL}(2, \mathbf{Z})$ action shows that they fall into two orbits, orbit A being made of the surfaces on the left and orbit B of those on the right.

The data for orbit A is $d = 18$ surfaces, $c = 5$ cusps, $e = 0$ elliptic point, so the genus is $g = 0$ by the Gauss-Bonnet formula.

The data for orbit B is $d = 9$ surfaces, $c = 3$ cusps, $e = 1$ elliptic point, so the genus is $g = 0$ by the Gauss-Bonnet formula.

By inspection of the congruence subgroups of genus 0 of $\mathrm{SL}(2, \mathbf{Z})$ (see for example [CuPa]), the stabilizers of orbits A and B are non-congruence subgroups of $\mathrm{SL}(2, \mathbf{Z})$.

Appendix B. Hyperelliptic components of other strata

For all hyperelliptic square-tiled surfaces, one can count the number of Weierstrass points with integer coordinates. This provides an invariant for the action of $\mathrm{SL}(2, \mathbf{Z})$ on square-tiled surfaces in all hyperelliptic components of strata of moduli spaces of abelian differentials.

The strata with hyperelliptic components are $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$, for $g > 1$.

Proposition B.1. *In $\mathcal{H}(2g-2)^{\mathrm{hyp}}$ and $\mathcal{H}(g-1, g-1)^{\mathrm{hyp}}$, for large enough odd n there are at least g orbits containing one-cylinder surfaces.*

This is proved by the following reasoning.

Completely periodic surfaces in $\mathcal{H}(2g-2)$ or $\mathcal{H}(g-1, g-1)$, for $g > 1$, have respectively $2g-1$ and $2g$ saddle connections.

For one-cylinder primitive surfaces (necessarily of height 1), the lengths of the saddle connections add up to n, and the Weierstrass points are two points on the circle at half-height of this cylinder (these do not have integer coordinates), the saddle in the $\mathcal{H}(2g-2)^{\mathrm{hyp}}$ case, and the midpoints of the saddle connections that bound the cylinder (these have integer coordinates for exactly those saddle connections of even length).

If n is odd, the sum of the lengths is odd. So the number of odd-length saddle connections has to be odd, and is between 1 and $2g-1$. There are g possibilities for that. Since the value of the invariant is the number of even-length saddle connections, it can take g different values.

Appendix C. The theorem of Gutkin and Judge

Theorem (Gutkin–Judge)**.** *(S, ω) has an arithmetic Veech group if and only if (S, ω) is parallelogram-tiled.*

Up to conjugating by an element of $\mathrm{SL}(2,\mathbf{R})$, it suffices to show:

Theorem. *(S,ω) is a square-tiled surface if and only if $V(S,\omega)$ is commensurable to* $\mathrm{SL}(2,\mathbf{Z})$.

(i.e. these two groups share a common subgroup of finite index in each.)

Remark. In this theorem, the size of the square tiles is not assumed to be 1. One can always act by a homothety to make this true, and we will suppose that in the proof of the direct way of this theorem.

C.1. **A square-tiled surface has an arithmetic Veech group.** Consider a square-tiled surface (S,ω), and its lattice of periods $\Lambda(\omega)$. By Lemma 2.3, $V(S,\omega) < V(\mathbf{R}^2/\Lambda(\omega)\ ,\mathrm{d}z)$.

Case 1. Let us first assume that $\Lambda(\omega) = \mathbf{Z}^2$, i.e. (S,ω) is a primitive square-tiled surface.

Lemma 2.4 implies that $\mathrm{SL}(2,\mathbf{Z})$ acts on the set E of square-tiled surfaces contained in its $\mathrm{SL}(2,\mathbf{R})$-orbit. The set E is finite and the stabilizer of this action is $V(S,\omega)$. The class formula then implies that $V(S,\omega)$ has finite index in $\mathrm{SL}(2,\mathbf{Z})$.

Case 2. Suppose that $\Lambda(\omega)$ is a strict sublattice of $\mathbf{Z}^2$. Consider $P_1,\ldots,P_k$ the preimages of the origin on S. Denote by $\mathrm{Aff}_{P_1,\ldots,P_k}$ the stabilizer of the set of these points in the affine group of (S,ω), and $V(P_1,\ldots,P_k)$ the associated Veech group. The translation surface $(S,\omega,\{P_1,\ldots,P_k\})$ where $\{P_1,\ldots,P_k\}$ are artificially marked is a primitive square-tiled surface. From Case 1 above, its Veech group $V(P_1,\ldots,P_k)$ is therefore a lattice contained in the discrete group $V(S,\omega)$, hence of finite index in this group.

Thus $V(P_1,\ldots,P_k)$ is a finite-index subgroup in both $V(S,\omega)$ and $\mathrm{SL}(2,\mathbf{Z})$.

C.2. **A surface with an arithmetic Veech group is square-tiled.** This part is inspired by ideas of Thurston [Th] and Veech [Ve89, §9], and appeared in [Hu, appendix B].

Let S be a translation surface with an arithmetic Veech group Γ.

If Γ is commensurable to $\mathrm{SL}(2,\mathbf{Z})$ only in the wide sense, we move to the case of strict commensurability. This conjugacy on Veech groups is obtained by $\mathrm{SL}(2,\mathbf{R})$ action on surfaces.

We prove the following propositions.

Proposition C.1. *A group* Γ *commensurable with* $\mathrm{SL}(2,\mathbf{Z})$ *contains two elements of the form* $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix}1 & 0\\ n & 1\end{smallmatrix}\right)$ *for some* $m, n \in \mathbf{N}^*$.

Proposition C.2. *If the Veech group* Γ *of a translation surface* S *contains two elements of the form* $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix}1 & 0\\ n & 1\end{smallmatrix}\right)$ *for some* $m, n \in \mathbf{N}^*$, *then* S *is square-tiled.*

Proposition C.1 follows from the following lemma.

Lemma C.3. *If* $H \leq G$ *is a finite-index subgroup then every* $g \in G$ *of infinite order has a power in* H.

Proof of the lemma. If H has finite index there is a partition of G into a finite number of classes modulo H. The powers of g, in countable number, are distributed in these classes, so there exist distinct integers i and j such that g^i and g^j are in the same class, and then $g^{j-i} \in H$. □

Apply this lemma to $G = \mathrm{SL}(2,\mathbf{Z})$ and H the common subgroup to G and Γ, of finite index in both G and Γ, and $g = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ or $g = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$.

We now prove Proposition C.2.

Since $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) \in \Gamma$, the horizontal direction is parabolic, so S decomposes into horizontal cylinders C_i^{h} of rational moduli. Replacing $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$ with one of its powers if necessary, suppose it fixes the boundaries of these cylinders. This means their moduli are multiples of $1/m$. Calling w_i^{h}, h_i^{h} the widths and heights of these cylinders, we have relations $h_i^{\mathrm{h}}/w_i^{\mathrm{h}} = k_i/m$ for some integers k_i.

By a similar argument, since $\left(\begin{smallmatrix}1 & 0\\ n & 1\end{smallmatrix}\right) \in \Gamma$, the vertical direction is also parabolic, and S decomposes into vertical cylinders C_j^{v} of rational moduli $h_j^{\mathrm{v}}/w_j^{\mathrm{v}} = k'_j/n$ for some integers k'_j.

Combining these two decompositions yields a decomposition of S into rectangles of dimensions $h_j^{\mathrm{v}} \times h_i^{\mathrm{h}}$ (these rectangles are the connected components of the intersections of the horizontal and vertical cylinders). Here we keep on with the convention of § 2.3 about heights and widths of cylinders.

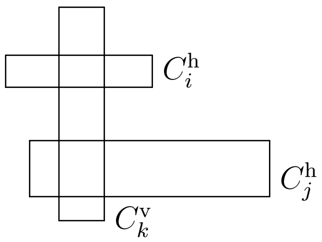
What we want to show is that these rectangles have rational dimensions (up to a common real scaling factor), in order to prove that S is a covering of a square torus; indeed, if the rectangles are such, then they can be divided into equal squares, so we obtain a covering of a square torus. Since singular points of S lie on the edges both of horizontal and of vertical cylinders, they are at corners of rectangles and hence of squares of the tiling, so that the covering is ramified over only one point.

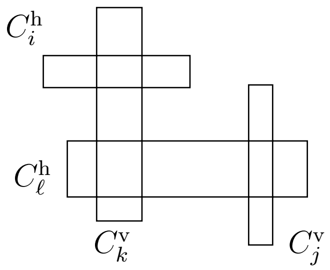
Because the cylinders in the decompositions above are made up of these rectangles, we have $w_i^{\mathrm{h}} = \sum m_{ij} h_j^{\mathrm{v}}$ and $w_j^{\mathrm{v}} = \sum n_{ji} h_i^{\mathrm{h}}$, where $m_{ij}, n_{ji} \in \mathbf{N}$.

Combining equations, $mh_i^h = \sum k_i m_{ij} h_j^v$ and $nh_j^v = \sum k'_j n_{ji} h_i^h$.

Then, setting $X^{\mathrm{h}} = (h_i^{\mathrm{h}})$, $X^{\mathrm{v}} = (h_j^{\mathrm{v}})$, $M = (k_i m_{ij})_{ij}$, $N = (k'_j n_{ji})_{ji}$, we have $mX^{\mathrm{h}} = MX^{\mathrm{v}}$ and $nX^{\mathrm{v}} = NX^{\mathrm{h}}$, so that $MNX^{\mathrm{h}} = mnX^{\mathrm{h}}$ and $NMX^{\mathrm{v}} = nmX^{\mathrm{v}}$.

M, N and their products are matrices with nonnegative integer coefficients. In view of applying the Perron–Frobenius theorem, we show that MN and NM have powers with all coefficients positive.





This results from the connectedness of S and the following observation: $M_{ij} \neq 0$ if and only if C_i^{h} and C_j^{v} intersect; $(MN)_{ij} \neq 0$ if and only if there exists a cylinder C_k^{v} which intersects both C_i^{h} and C_j^{h}, as in the picture; more generally the element i, j of a product of alternately M and N matrices is nonzero if and only if there exists a corresponding sequence of alternately horizontal and vertical cylinders such that two successive cylinders intersect. So MN and NM do have powers with all coefficients positive.

X^{h} (resp. X^{v}) is an eigenvector for the eigenvalue nm of the square matrix MN (resp. NM). By the Perron–Frobenius theorem, there exists a unique eigenvector associated with the real positive eigenvalue nm for the matrix NM (resp. MN). Since both matrices have rational coefficients and the eigenvalue is rational, there exist eigenvectors with rational coefficients. Up to scaling, they are unique by the Perron–Frobenius theorem. This allows to conclude that X^{h} is a multiple of a vector with rational coordinates. From the equation $nX^{\mathrm{v}} = NX^{\mathrm{h}}$, we then conclude that the rectangles have rational moduli and can be tiled by identical squares. This completes the proof of the theorem.

C.3. **A corollary.** The following result of [GuHuSc] arises as a corollary of § C.1 and Proposition C.2.

Corollary C.4. *If a subgroup* $\Gamma < \mathrm{SL}(2, \mathbf{Z})$ *contains two elements* $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ *and has infinite index in* $\mathrm{SL}(2, \mathbf{Z})$*, then* Γ *cannot be realized as the Veech group of a translation surface.*

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Chapitre 2

Groupes de Veech

Ce chapitre est l'article écrit avec Pascal Hubert, accepté pour publication dans *International Mathematics Research Notices* sous le titre "Non-congruence subgroups in $\mathcal{H}(2)$". Le résultat principal est que les groupes de Veech des surfaces à petits carreaux de la strate $\mathcal{H}(2)$ ne sont pas des groupes de congruence, sauf pour les surfaces à trois carreaux (le seul cas qui était compris jusqu'à il y a peu, si bien qu'on pensait que tous les groupes de Veech de surfaces à petits carreaux étaient des groupes de congruence).

NONCONGRUENCE SUBGROUPS IN $\mathcal{H}(2)$

PASCAL HUBERT AND SAMUEL LELIÈVRE

Abstract. We study the congruence problem for subgroups of the modular group that appear as Veech groups of square-tiled surfaces in the minimal stratum of abelian differentials of genus two.

Keywords: congruence problem, Veech group, square-tiled surfaces

Contents

1. Introduction

Let ω be a holomorphic 1-form on a compact Riemann surface X. If there exists a branched covering $f : X \to \mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, ramified only over the origin of $\mathbf{T}^2$, such that $f^*(dz) = \omega$, the flat surface $(X, |\omega|)$ is tiled by squares whose vertices project to the origin of the torus, and (X, ω) is called a **square-tiled (translation) surface**.

In each genus g, square-tiled surfaces are the integer points of the moduli space $\mathcal{H}_g = \Omega\mathcal{M}_g$ of holomorphic 1-forms on Riemann surfaces of genus g. This space is stratified by the combinatorial type of zeros, and each stratum is a complex orbifold endowed with an action of $\mathrm{SL}(2, \mathbf{R})$. Orbits for this action are called Teichmüller discs.

The main problem in dynamics in Teichmüller spaces is to understand this $\mathrm{SL}(2, \mathbf{R})$-action, and to obtain Ratner-like classification results for its orbit closures and its invariant closed submanifolds.

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The first step is to determine as many invariant closed submanifolds as possible. The simplest of them are closed orbits. These are the orbits of translation surfaces with finite-covolume stabilisers, called Veech surfaces because of Veech's pioneering work [Ve]. These Teichmüller discs project to geodesically embedded curves, called Teichmüller curves, in the moduli space $\mathcal{M}_g$ of complex curves of genus g. These curves are uniformised by the stabiliser of the corresponding $\mathrm{SL}(2, \mathbf{R})$-orbit.

Square-tiled surfaces are Veech surfaces. They already appeared in Thurston's work on the classification of surface diffeomorphisms, see [FLP, exposé 13]. Nevertheless up to recently their Teichmüller discs have been little discussed, due to the difficulty of proving precise statements about them. The only classical result is Gutkin and Judge's theorem [GuJu] which states that the corresponding stabilisers are arithmetic (commensurable to $\mathrm{SL}(2, \mathbf{Z})$). Very recently the Teichmüller discs of square-tiled surfaces were studied into more detail, see [HL], [Mc4], [Mö], [Schmi].

A square-tiled surface (X, ω) is called **primitive** if the lattice of relative periods of ω is $\mathbf{Z}^2$ (in other words the covering $(X, \omega) \to (\mathbf{T}^2, dz)$ does not factor through a bigger torus). In this case, the stabiliser, denoted by $\mathrm{SL}(X, \omega)$, is a (finite-index) subgroup of $\mathrm{SL}(2, \mathbf{Z})$.

In order to give the most accurate description of Teichmüller discs of square-tiled surfaces, we investigate these subgroups. In the theory of subgroups of $\mathrm{SL}(2, \mathbf{Z})$, a natural and important question is the congruence problem. This question is the central object of this paper: we give a negative answer in the stratum $\mathcal{H}(2) = \Omega\mathcal{M}_2(2)$ of 1-forms on genus 2 surfaces having one double zero.

Recent results about square-tiled surfaces in $\mathcal{H}(2)$. The discrete orbit $\mathrm{SL}(2, \mathbf{Z}) \cdot (X, \omega)$ of a primitive square-tiled surface (X, ω) consists of all the primitive square-tiled surfaces in its Teichmüller disc $\mathrm{SL}(2, \mathbf{R}) \cdot (X, \omega)$; indeed, $\mathrm{SL}(2, \mathbf{Z})$ acts on primitive square-tiled surfaces, preserving the number of squares. Understanding the Teichmüller discs or the discrete orbits of primitive square-tiled surfaces is therefore equivalent. We will use the following result about the discrete orbits of primitive square-tiled surfaces in $\mathcal{H}(2)$.

Theorem A. *Primitive n-square-tiled surfaces in the stratum $\mathcal{H}(2)$ form: one orbit A_3 if $n = 3$; two orbits A_n and B_n if n is odd $\geqslant 5$; one orbit C_n if n is even.*

This was shown for prime n in [HL], and conjectured for arbitrary n; the conjecture was proved in full generality in [Mc4].

Let Γ_{A_n}, Γ_{B_n} and Γ_{C_n} denote the stabilisers of these orbits.

Remark. The indices of the groups Γ_{A_n}, Γ_{B_n}, Γ_{C_n} in $\mathrm{SL}(2,\mathbf{Z})$ are the cardinalities a_n, b_n, c_n of the discrete orbits A_n, B_n, C_n.

Eskin–Masur–Schmoll [EsMaSc] give a formula for the number of primitive n-square-tiled surfaces in $\mathcal{H}(2)$:

Theorem B. *The number of primitive n-square-tiled surfaces in $\mathcal{H}(2)$ is $\frac{3}{8}(n-2)n^2\prod_{p|n}(1-\frac{1}{p^2})$.*

Remark. Throughout this paper, the letter p always denotes prime numbers; in particular, $\prod_{p|n}$ is the product over prime divisors of n.

This formula gives c_n (and a_3) when there is one orbit and $a_n + b_n$ when there are two. We conjectured in [HL]:

Conjecture 1. *For odd $n \geqslant 5$, a_n and b_n are given by:*

$$a_n = \frac{3}{16}(n-1)n^2\prod_{p|n}(1-\frac{1}{p^2}), \quad b_n = \frac{3}{16}(n-3)n^2\prod_{p|n}(1-\frac{1}{p^2}).$$

Statement of results. In this paper, we show:

Theorem 1. *For all even $n \geqslant 4$, Γ_{C_n} is a noncongruence subgroup. For all odd $n \geqslant 5$ satisfying Conjecture 1, Γ_{A_n} and Γ_{B_n} are noncongruence subgroups.*

Remark. Conjecture 1 is proved up to $n = 10000$ by an explicit combinatorial computer calculation.

Corollary 1.1. *Under Conjecture 1, the only primitive square-tiled surfaces in $\mathcal{H}(2)$ whose stabiliser is a congruence subgroup are those tiled with 3 squares.*

Corollary 1.2. *Under Conjecture 1, of all the Teichmüller curves embedded in $\mathcal{M}_2$ that come from orbits in $\mathcal{H}(2)$, only one is uniformised by a congruence subgroup of $\mathrm{SL}(2,\mathbf{Z})$.*

Remark. For $n = 3$, Γ_{A_3} is the level 2 congruence subgroup Θ generated by $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$, named after its link to the Jacobi Theta function.

Link with the Hurwitz problem. An essential ingredient in our proof of Theorem 1 is the knowledge of the indices in $\mathrm{SL}(2,\mathbf{Z})$ of Γ_{A_n}, Γ_{B_n} and Γ_{C_n} (given by Theorem B and Conjecture 1).

Since these indices are the cardinalities of the discrete orbits A_n, B_n and C_n, finding these numbers is a variant of Hurwitz's problem, which consists in counting the number of branched covers of a fixed combinatorial type (number and multiplicity of ramification points) and fixed degree of a Riemann surface S. A very detailed survey of this subject can be found in the introduction of Zvonkine's thesis [Zv].

When S is the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ (or more generally an elliptic curve), it can be endowed with the 1-form dz. Hurwitz's problem amounts to counting the number of coverings (with fixed combinatorial type) $f : (X, \omega) \to (\mathbf{T}^2, dz)$ where $\omega = f^*(dz)$. For a fixed combinatorial type c, denote by $h_{n,c}$ the number of such coverings, weighted by the inverse of their number of automorphisms.

We have the following fundamental theorem:

Theorem C. *For any combinatorial type, the generating series* $F_c(z) = \sum_{h=1}^{\infty} h_{n,c}\, q^n$*, where* $q = e^{2i\pi z}$*, is a quasi-modular form of maximal weight* $6g - 6$.

This theorem was first proved in the case of simple ramifications by Dijkgraaf [Di] and Kaneko–Zagier [KaZa]; the general proof relies on results of Bloch–Okounkov [BlOk], see [EsOk].

The quasi-modular form is explicitated by Kani [Ka] and by Eskin–Masur–Schmoll [EsMaSc] in particular cases. Some generalisations are proved by Eskin–Okounkov–Pandharipande [EsOkPa].

Note also that the asymptotics of the countings of square-tiled surfaces of bounded area serve to compute the volumes of strata (see [Zo], [EsOk]).

Acknowledgements. We thank Gabriela Schmithüsen for the inspiration and useful discussions, and Giovanni Forni for encouraging this research.

2. Background

2.1. **Square-tiled surfaces, action of** $\mathrm{SL}(2, \mathbf{Z})$**, cusps.** We recall here some tools used in [HL], to which we refer for more detail.

The modular group $\Gamma(1) = \mathrm{SL}(2, \mathbf{Z})$ acts on primitive square-tiled surfaces, preserving the number of squares tiles. Indeed, the property of having $\mathbf{Z}^2$ as lattice of relative periods is $\mathrm{SL}(2, \mathbf{Z})$-invariant.

Given a primitive square-tiled surface (X, ω), its stabiliser $\mathrm{SL}(X, \omega)$ is a finite-index subgroup of $\mathrm{SL}(2, \mathbf{Z})$, therefore the curve $\mathrm{SL}(X, \omega)\backslash\mathbf{H}$ is a branched cover of the modular curve $\mathrm{SL}(2, \mathbf{Z})\backslash\mathbf{H}$, and the degree of the cover is the index of $\mathrm{SL}(X, \omega)$ in $\mathrm{SL}(2, \mathbf{Z})$.

The modular group is generated by any two matrices among $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Denote by $\mathcal{U}$ the subgroup generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Cusps. The cusps of $\mathrm{SL}(X, \omega)\backslash\mathbf{H}$ are classified combinatorially by the following lemma.

Lemma 2.1 (Zorich)**.** *Let* (X, ω) *be a primitive square-tiled surface. There is a* 1-1 *correspondence between the set of cusps of* $\mathrm{SL}(X, \omega)\backslash\mathbf{H}$ *and the* $\mathcal{U}$*-orbits of* $\mathrm{SL}(2, \mathbf{Z})\cdot(X, \omega)$.

Any square-tiled surface decomposes into horizontal cylinders, which are also square-tiled, and bounded by unions of saddle connections of integer lengths. This provides a way to give coordinates for square-tiled surfaces in each stratum (see below for the stratum $\mathcal{H}(2)$).

The action of the generators $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbf{Z})$ is easily seen in these coordinates: $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ exchanges the horizontal and vertical directions; $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ only changes the twists.

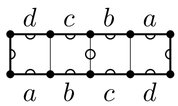
The width of a cusp is given by the cardinality of the corresponding $\mathcal{U}$-orbit. If the horizontal cusp has width ℓ, the primitive parabolic in the horizontal direction is $\left(\begin{smallmatrix}1&\ell\\0&1\end{smallmatrix}\right)$. Considering how the cylinders behave under the action of $\mathcal{U}$, we get the following lemma.

Lemma 2.2. *If a primitive square-tiled surface decomposes into horizontal cylinders c_i of height h_i and width w_i, then its (horizontal) cusp width equals the least common multiple of the $\frac{w_i}{h_i \wedge w_i}$, possibly divided by some factor.*

Notation. Here, and in the sequel, $a \wedge b$ denotes the greatest common divisor of two integers a and b.

The following example illustrates the case of division by a factor.

This surface is in $\mathcal{H}(1,1)$ and has a nontrivial translation by the vector $(2,0)$; though it is made of one cylinder of height 1 and width 4, its cusp width is only 2.



In the stratum $\mathcal{H}(2)$ on which we will focus from now on, this situation does not occur.

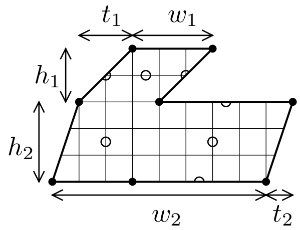
2.2. **Square-tiled surfaces in $\mathcal{H}(2)$.** The stratum $\mathcal{H}(2)$ has recently received much attention ([EsMaSc], [Ca], [Mc1, Mc3, Mc4], [HL]).

Square-tiled surfaces in $\mathcal{H}(2)$ are of two types [Zo], the one-cylinder ones and the two-cylinder ones. The corresponding coordinates are: for one-cylinder surfaces, one height, three lengths of saddle connections and one twist parameter; for two-cylinder surfaces, one height, width and twist for each cylinder.

Theorem A says that for each odd $n \geqslant 5$, primitive n-square-tiled surfaces are in two orbits A_n and B_n. These orbits are distinguished by a simple invariant, the number of integer Weierstrass points (i.e. Weierstrass points located at vertices of the square tiles). A surface is in A_n if it has one integer Weierstrass point, in B_n if it has three.

The coordinates for square-tiled surfaces in $\mathcal{H}(2)$ were used in [Zo], in [EsMaSc] and in [HL] where the position of Weierstrass points was also discussed and the invariant introduced. This invariant was independently expressed in terms of divisors by Kani [Ka]. McMullen [Mc4] expressed it as the parity of a spin structure.

Notation. Denote by $S(h_1, h_2, w_1, w_2, t_1, t_2)$ the two-cylinder surface with cylinders c_i of height h_i, width w_i and twist t_i, with $w_1 < w_2$.



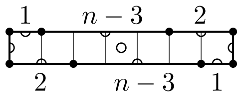
The figure shows a fundamental polygon for $S(2, 3, 3, 8, 2, 1)$; the surface is obtained from this polygon by identifying pairs of parallel sides of same lengths. We indicate the double zero by black dots and the other Weierstrass points by circles. The same conventions hold for all pictures in this paper.

Let us give some examples of square-tiled surfaces in $\mathcal{H}(2)$.

First, some one-cylinder surfaces of particular interest.

Lemma 2.3. *For each $n \geqslant 4$, there is a primitive n-square-tiled surface which is one-cylinder both horizontally and vertically.*

The one-cylinder surface with saddle connections of lengths 1, $n-3$, 2 on the top and 2, $n-3$, 1 on the bottom has this property.



Corollary 2.4. *The stabiliser of this surface contains $\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ n & 1\end{smallmatrix}\right)$.*

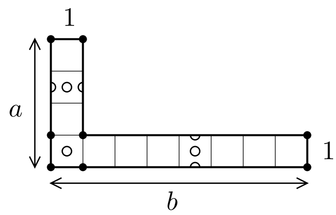
Indeed, one-cylinder cusps have width n.

Remark. When n is odd, the surface described above is in orbit B_n.

Some two-cylinder surfaces also deserve special attention.

Notation. For a and $b \geqslant 2$, denote by $\mathrm{L}(a, b)$ the surface $S(a-1, 1, 1, b, 0, 0)$. This surface is a primitive square-tiled surface tiled by $n = a+b-1$ squares. This surface has cusp width b and vertically a.

When n is odd, this surface is in A_n if a and b are even, in B_n if a and b are odd.



2.3. **Congruence subgroups; level of a subgroup.** The material in this section is classical, and can be found in [Ra].

For any integer $m > 1$, consider the natural projection $\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/m\mathbf{Z})$. This projection is a group homomorphism. Its kernel is called the **principal congruence subgroup of level** m, and denoted by $\Gamma(m)$. It consists in all matrices congruent to $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ modulo m. This is consistent with the notation $\Gamma(1)$ for $\mathrm{SL}(2, \mathbf{Z})$.

Lemma 2.5. *For any m, $[\Gamma(1) : \Gamma(m)] = m^3 \prod_{p|m}(1 - \frac{1}{p^2})$.*

Corollary 2.6. *If $m \wedge m' = 1$, then $[\Gamma(m) : \Gamma(mm')] = [\Gamma(1) : \Gamma(m')]$.*

Any group Γ containing some $\Gamma(m)$ is called a **congruence subgroup**, and its **level** is defined to be the least m such that $\Gamma(m) \subset \Gamma$ (i.e. the level of the largest principal congruence subgroup it contains).

Remark. A principal congruence subgroup is a normal subgroup of $\Gamma(1)$. Hence being a congruence subgroup is invariant by conjugation in $\mathrm{SL}(2, \mathbf{Z})$; the level is also invariant.

There is a more general notion of level, due to Wohlfahrt [Wo]. The **level** of a finite-index subgroup of $\mathrm{SL}(2, \mathbf{Z})$ is the least common multiple of its cusp widths. Wohlfahrt proved that for congruence subgroups, it coincides with the previous definition, and that:

Lemma 2.7 (Wohlfahrt [Wo])**.** *A finite-index subgroup of level ℓ is a congruence subgroup if and only if it contains the principal congruence subgroup of level ℓ.*

2.4. **Quasi-modular forms.** As said in the introduction, the generating function for the weighted countings of surfaces tiled by n squares is a quasi-modular form.

The numbers $h_{n,c}$ of surfaces tiled by n squares in a given stratum, and the numbers $h^{\mathrm{P}}_{n,c}$ of primitive ones, are related by

$$h_{n,c} = \sum_{d|n} \sigma(n/d) h^{\mathrm{P}}_{d,c},$$

where $\sigma(k) = \sum_{d|k} d$ is the sum of divisors of k. This is because the number of tori tiled by n squares is $\sigma(n)$.

In addition, we note that in $\mathcal{H}(2)$, the coverings have no automorphisms, hence the weighted and unweighted countings are the same.

Conjecture 2. *In $\mathcal{H}(2)$, the countings for odd n according to the invariant are generated by a quasi-modular form.*

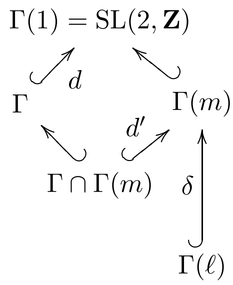
Theorem B is mentioned in [EsMaSc] as a consequence of the quasi-modularity. Likewise, Conjecture 1 would follow from Conjecture 2.

3. Strategy for the proof of Theorem 1

We build on the proof by Schmithüsen [Schmi] that the stabiliser of a 4-square-tiled surface in $\mathcal{H}(2)$ is a noncongruence subgroup, based on an idea of Stefan Kühnlein.

3.1. **Sufficient conditions for noncongruence.**

Let Γ be a subgroup of $\Gamma(1)$ of finite index d and level ℓ. For any divisor m of ℓ, consider the finite-index inclusions represented on the figure.



Two remarks. First, if Γ projects surjectively to $\mathrm{SL}(2, \mathbf{Z}/m\mathbf{Z})$, one can conclude by observing the two exact sequences below that $d' = d$, where $d' = [\Gamma(m) : \Gamma \cap \Gamma(m)]$ and $d = [\Gamma(1) : \Gamma]$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma(m) & \longrightarrow & \Gamma(1) & \longrightarrow & \mathrm{SL}(2,\mathbf{Z}/m\mathbf{Z}) & \longrightarrow & 1 \\ \| & & \uparrow d' & & \uparrow d & & \| & & \| \\ 1 & \longrightarrow & \Gamma\cap\Gamma(m) & \longrightarrow & \Gamma & \longrightarrow & \mathrm{SL}(2,\mathbf{Z}/m\mathbf{Z}) & \longrightarrow & 1 \end{array}$$

Second, if Γ is a congruence subgroup, and hence by Lemma 2.7 contains $\Gamma(\ell)$, then $\Gamma(\ell)$ is contained in $\Gamma\cap\Gamma(m)$ and the indices satisfy $[\Gamma(m) : \Gamma(\ell)] = [\Gamma(m) : \Gamma\cap\Gamma(m)]\cdot[\Gamma\cap\Gamma(m) : \Gamma(\ell)]$, which implies $d' \mid \delta$.

Combining these two remarks, we get the following sufficient condition for noncongruence, which was used by Schmithüsen [Schmi].

Proposition 3.1 (Kühnlein)**.** *If Γ is a subgroup of $\Gamma(1)$ of finite index d and level ℓ and there exists a divisor m of ℓ for which*
- *Γ projects surjectively to $\mathrm{SL}(2, \mathbf{Z}/m\mathbf{Z})$, and*
- *the index $\delta = [\Gamma(m) : \Gamma(\ell)]$ is not a multiple of d,*

then Γ is not a congruence subgroup.

Remark. Suppose Γ contains two matrices $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ k' & 1\end{smallmatrix}\right)$. If m is an integer relatively prime to both k and k', then k and k' are invertible modulo m so some powers of $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ k' & 1\end{smallmatrix}\right)$ project to $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ in $\mathrm{SL}(2, \mathbf{Z}/m\mathbf{Z})$, hence the projection $\Gamma \to \mathrm{SL}(2, \mathbf{Z}/m\mathbf{Z})$ is surjective.

This extra remark yields the following sufficient condition for noncongruence.

Proposition 3.2. *If a subgroup $\Gamma \subset \Gamma(1)$ of finite index d contains two matrices $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ k' & 1\end{smallmatrix}\right)$ and if its level ℓ has a divisor m relatively prime to both k and k', such that the index $\delta = [\Gamma(m) : \Gamma(\ell)]$ is not a multiple of d, then Γ is not a congruence subgroup.*

3.2. **Strategy.** Consider an orbit A_n, B_n or C_n, with n as in Theorem 1. Its stabiliser Γ_{A_n}, Γ_{B_n} or Γ_{C_n} is defined only up to conjugation in $\mathrm{SL}(2,\mathbf{Z})$; the representatives of the conjugacy class are the stabilisers of the (square-tiled) surfaces in the orbit. The index and level are preserved by conjugation in $\mathrm{SL}(2,\mathbf{Z})$.

Choice. Let S be a (square-tiled) surface in an orbit A_n, B_n or C_n, and Γ be its stabiliser.

Notation. Denote by d the index of Γ and by ℓ its level. Consider the prime factor decompositions $n = \prod p^\nu$ and $\ell = \prod p^\lambda$, where ν and λ can denote a different integer for each prime p.

Choice. Choose some $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ k' & 1\end{smallmatrix}\right)$ in Γ, for instance k and k' could be taken to be the horizontal and vertical cusp widths of S.

Notation. Following [Mc4], if a and b are two integers, denote by $a//b$ the greatest divisor of a that is prime to b. If $a = \prod p^\alpha$ is the prime factor decomposition of a, we have $a//b = \prod_{p\nmid b} p^\alpha = a/\prod_{p|b} p^\alpha$.

Choice. Choose $m = \ell//kk' = \ell/\prod_{p|kk'} p^\lambda$.

Notation. Denote by δ the index of $\Gamma(\ell)$ in $\Gamma(m)$.

By construction m is a divisor of ℓ, relatively prime to both k and k'. In view of applying Proposition 3.2, there remains only to check that d does not divide δ. Since m is also relatively prime to ℓ/m, by Corollary 2.6, $\delta = (\ell/m)^3 \prod_{p|\ell/m}(1 - \frac{1}{p^2})$.

Remark. If a is an integer and $a = \prod p^\alpha$ is its prime factor decomposition, one can rewrite $a^r \prod_{p|a}(1 - \frac{1}{p^2})$ as $\prod_{p|a} p^{r\alpha-2}(p^2-1)$. Hence
- $\delta = \prod_{p|kk'} p^{3\lambda-2}(p^2-1)$, and
- $d = f(n)\prod_{p|n} p^{2\nu-2}(p^2-1)$, where $f(n)$ is one of $\frac{3}{16}(n-1)$, $\frac{3}{16}(n-3)$, $\frac{3}{8}(n-2)$, according to whether orbit A_n, B_n or C_n is under consideration.

In order to complete the proof, there merely remains to describe how to apply our strategy.

For this we need the levels of Γ_{A_n}, Γ_{B_n} and Γ_{C_n}; we give them in § 4.

The last three sections then describe, in each orbit, good choices of a surface S, values of k and k', and, keeping the notations (d, ℓ, ν, λ, m, δ) introduced here (and consistent with those in § 3.1), show that d does not divide δ.

4. The level of Γ_{A_n}, Γ_{B_n} and Γ_{C_n}

As said above, the stabiliser of an $\mathrm{SL}(2,\mathbf{Z})$-orbit of square-tiled surfaces is defined up to conjugacy in $\mathrm{SL}(2,\mathbf{Z})$, but its level is well-defined.

Proposition 4.1. *The groups Γ_{A_n}, Γ_{B_n} and Γ_{C_n} have levels:*

$$\operatorname{lev}\Gamma_{A_n} = d_n, \quad \operatorname{lev}\Gamma_{B_n} = d_n/4, \quad \operatorname{lev}\Gamma_{C_n} = d_n,$$

where $d_n = \operatorname{lcm}(1,2,3,\dots,n)$.

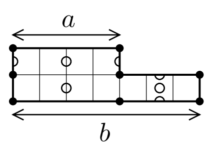
Remark. The prime factor decomposition of d_n is $\prod_{p\leqslant n} p^\tau$ where the exponents τ are the integers such that $p^\tau \leqslant n < p^{\tau+1}$.

The remainder of this section is devoted to proving the proposition.

First recall that the level of $\Gamma \subset \mathrm{SL}(2,\mathbf{Z})$ is defined as the least common multiple of the amplitudes of the cusps of Γ. When Γ is the stabiliser of a primitive square-tiled surface S, its cusp widths are equivalently the horizontal cusp widths of the surfaces in the $\mathrm{SL}(2,\mathbf{Z})$-orbit of S.

Recall also Lemma 2.2. If S is tiled by n squares, the widths of its cylinders are at most n, so the level of Γ divides $\operatorname{lcm}(1,2,3,\dots,n)$.

Orbit C_n (for even n) contains one-cylinder surfaces, which have cusp width n, and, for all a and b such that $a+b=n+1$ and $2 \leqslant a, b \leqslant n-1$, two-cylinder surfaces $\mathrm{L}(a,b)$, which have cusp width b. Hence, the level of Γ_{C_n} is a multiple of, and therefore equals, $\operatorname{lcm}(1,2,3,\dots,n)$.



Orbit A_n (for odd n) contains one-cylinder surfaces, which have cusp width n, and, for all a and b such that $a+b=n$ and $1 \leqslant a < b \leqslant n-1$, two-cylinder surfaces with two cylinders of height 1 and widths a and b, which have cusp width $\operatorname{lcm}(a,b)$. Hence, the level of Γ_{A_n} is a multiple of, and therefore equals, $\operatorname{lcm}(1,2,3,\dots,n)$.

Orbit B_n (for odd n) contains one-cylinder surfaces, which have cusp width n, and, for all odd a and b such that $a+b=n+1$ and $2 \leqslant a, b \leqslant n-1$, two-cylinder surfaces $\mathrm{L}(a,b)$, which have cusp width b. Hence, the level of Γ_{B_n} is a multiple of $\operatorname{lcm}(1,3,5,\dots,n)$.

Since $\operatorname{lcm}(1,2,3,\dots,n)$ is a power of 2 times $\operatorname{lcm}(1,3,5,\dots,n)$, there remains only to determine the power of 2 in the level of Γ_{B_n}, i.e. the maximal power of 2 that can arise as a divisor of $\frac{w}{h\wedge w}$ for the height h and the width w of a cylinder of a surface of B_n.

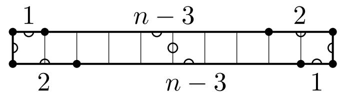
Let τ be the integer such that $2^\tau < n < 2^{\tau+1}$.

There is at least one two-cylinder surface $S(h_1, 2, w_1, 2^{\tau-1}, t_1, t_2)$ with odd t_2 in B_n; such a surface satisfies $\frac{w_2}{h_2\wedge w_2} = 2^{\tau-2}$.

Suppose a surface S in B_n has even cusp width $k = 2^t{\cdot}q$ with q odd. Then S has two cylinders, and by the discussion in [HL, § 5.1], one cylinder has even width w and even height h, while the other has odd height h' and odd width w'. Since $k = \operatorname{lcm}(\frac{w}{w\wedge h}, \frac{w'}{w'\wedge h'})$ and $\frac{w'}{w'\wedge h'}$ is odd, 2^t divides $\frac{w}{h\wedge w}$. But $h \geqslant 2$ and since $n = hw + h'w'$, $w < n/h \leqslant n/2 < 2^\tau$, so $\frac{w}{h\wedge w} \leqslant w/2 < 2^{\tau-1}$. Therefore $t \leqslant \tau - 2$.

5. Noncongruence of Γ_{C_n} for even $n \geqslant 4$

5.1. **Case when $n-2$ is not a power of 2.** We take S to be the one-cylinder surface with saddle connections of lengths 1, $n-3$, 2 on the top and 2, $n-3$, 1 on the bottom.



As a one-cylinder surface it has cusp width $k = n$ and since its vertical direction is also one-cylinder, its vertical cusp width k' is also n, so Γ contains $\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ n & 1\end{smallmatrix}\right)$.

Recall that Γ has index $d = \frac{3}{8}(n-2)\prod_{p|n} p^{2\nu-2}(p^2-1)$.

Choosing $m = \ell//n = \ell/\prod_{p|n} p^\lambda$ leads to $\delta = \prod_{p|n} p^{3\lambda-2}(p^2-1)$.

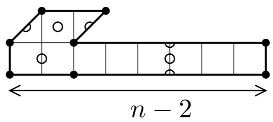
So d divides δ if and only if $3(n-2)$ divides $2^3 \cdot \prod_{p|n} p^{3\lambda-2\nu}$.

Since $n \wedge (n-2) = 2$, the assumption that $n-2$ is not a power of 2 implies it has some (odd) prime factors that do not divide n.

Hence d does not divide δ, so Γ cannot be a congruence subgroup.

5.2. **Case when $n-2$ is a power of 2.** The case $n = 4$ is known from [Schmi]. It can also be treated as above, since the index of Γ_{C_4} is $d = 9$ and, taking S and m as above, $\delta = 2^4 \cdot 3$.

From now on assume $n > 4$.



We take $S = S(1, 1, 1, n-2, 1, 0)$.

Note that this requires that $n-2 > 2$, which is why the case $n = 4$ was dealt with separately.

This surface has horizontal cusp width $n-2$ and vertical cusp width 4, so the stabiliser Γ contains $\left(\begin{smallmatrix}1 & n-2\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 4 & 1\end{smallmatrix}\right)$.

Recall that Γ has index $d = \frac{3}{8}(n-2)\prod_{p|n} p^{2\nu-2}(p^2-1)$.

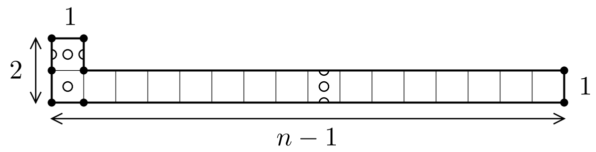
Choosing $m = \ell//2 = \ell/2^\lambda$ leads to $\delta = 2^{3\lambda-2}(2^2-1) = 2^{3\lambda-2} \cdot 3$.

Since n is even, it has $p = 2$ as a prime factor, which gives 3 as p^2-1, so 3^2 divides d.

Hence d does not divide δ, so Γ cannot be a congruence subgroup.

6. Noncongruence of Γ_{A_n} for odd $n \geqslant 5$

6.1. **Case when $n-1$ is a power of 2.** Take $S = \mathrm{L}(2, n-1)$.



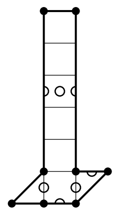
Its cusp width is $n-1$ $(= 2^\lambda)$ and its vertical cusp width is 2, so its stabiliser Γ contains $\left(\begin{smallmatrix} 1 & n-1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.

Here $d = \frac{3}{16}(n-1)\prod_{p|n} p^{2\nu-2}(p^2-1)$.

The choice of $m = \ell//2 = \ell/2^\lambda$ leads to $\delta = 2^{3\lambda-2}{\cdot}3$.

If n is a power of 3, then 3^3 divides d; otherwise n has some (odd) prime factor $p \neq 3$, for which $p^2 - 1 = (p-1)(p+1)$ is a multiple of 3, so that 3^2 divides d. Therefore d does not divide δ and Γ is not a congruence subgroup.

6.2. **Case when $n-1$ is not a power of 2.** Here we take the surface $S = S(n-2, 1, 1, 2, 0, 1)$. This surface is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right){\cdot}\,\mathrm{L}(n-1, 2)$.



The cusp width of S is 2, and S has one vertical cylinder, hence vertical cusp width n. So Γ contains $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$.

Here $d = \frac{3}{16}(n-1)\prod_{p|n} p^{2\nu-2}(p^2-1)$.

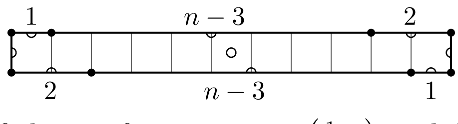
The choice of $m = \ell//2n = \ell/(2^\lambda \prod_{p|n} p^\lambda)$ leads to $\delta = 2^{3\lambda-2}{\cdot}3{\cdot}\prod_{p|n} p^{3\lambda-2}(p^2-1)$.

It follows that d divides δ if and only if $(n-1)$ divides $2^{3\lambda+2}{\cdot}\prod_{p|n} p^{3\lambda-2\nu}$.

Since n is not some 2^k+1, $n-1$ has odd prime factors; these do not divide n, so d does not divide δ and Γ is not a congruence subgroup.

7. Noncongruence of Γ_{B_n} for odd $n \geqslant 5$

7.1. **A proof for most cases.** Consider the one-cylinder surface S having saddle connections of lengths 1, $n-3$, 2 on the top and 2, $n-3$, 1 on the bottom.



The stabiliser of this surface contains $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$.

Here $d = \frac{3}{16}(n-3)\prod_{p|n} p^{2\nu-2}(p^2-1)$.

The choice of $m = \ell//n$ leads to $\delta = \prod_{p|n} p^{3\lambda-2}(p^2-1)$.

Thus d divides δ if and only if $3(n-3)$ divides $16\prod_{p|n} p^{3\lambda-2\nu}$.

Call an odd $n \geqslant 5$ "bad" if $3(n-3)$ divides $16\prod_{p|n} p^{3\lambda-2\nu}$.

As we are about to see, this is very rare, so that for "most" odd $n \geqslant 5$, d does not divide δ.

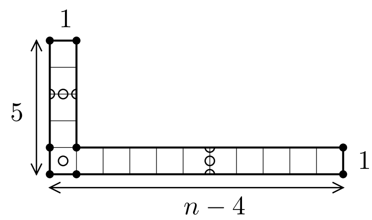
7.2. **The bad case.** If n is such that $3(n-3)$ divides $16\prod_{p|n} p^{3\lambda-2\nu}$,
- $n-3$ is not a multiple of 2^5;
- n is a multiple of 3 (and hence $(n-3) \wedge n = 3$);
- all odd prime factors of $n-3$ divide n.

Combining these three remarks, we see the bad case is when $n-3$ is of the form $2^r{\cdot}3^s$ with $1 \leqslant r \leqslant 4$ and $1 \leqslant s$.

Thus the bad case consists of the four sequences $n_{r,s} = 2^r{\cdot}3^s + 3$ for $r = 1$ to 4 and $s \geqslant 1$, which have exponential growth, hence zero density.

In particular, the discussion in § 7.1 proves the noncongruence of Γ_{B_n} when n is out of these four sequences.

7.3. **First bad cases.** Here we examine the first element of each of the four sequences, i.e. $n \in \{9, 15, 27, 51\}$. We include the second element of the first sequence, i.e. $n = 21$.



Take $S = \mathrm{L}(5, n-4)$. Its horizontal cusp width is $n-4$ and its vertical cusp width is 5, so its stabiliser Γ contains $\left(\begin{smallmatrix}1 & n-4\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 5 & 1\end{smallmatrix}\right)$.

Here $d = \frac{3}{16}(n-3)n^2\prod_{p|n}(1-\frac{1}{p^2})$.

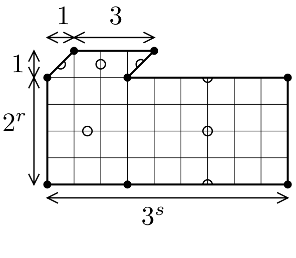
Choosing $m = \ell /\!/ 5(n-4)$ leads to $\delta = \prod_{p|5(n-4)} p^{3\lambda-2}(p^2-1)$.

The values of d and δ for $n \in \{9, 15, 21, 27, 51\}$ are:

n	9	15	21	27	51
d	3^4	$2^4{\cdot}3^3$	$2^4{\cdot}3^4$	$2^2{\cdot}3^6$	$2^8{\cdot}3^4$
δ	$2^3{\cdot}3{\cdot}5$	$2^6{\cdot}3^2{\cdot}5^2{\cdot}11$	$2^8{\cdot}3^3{\cdot}5{\cdot}17$	$2^7{\cdot}3^2{\cdot}5^4{\cdot}11{\cdot}23$	$2^8{\cdot}3^2{\cdot}5^4{\cdot}23{\cdot}47$

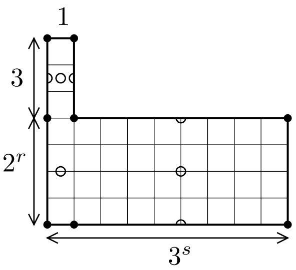
In each case, we see by observing the power of 3 in d and δ that d does not divide δ.

7.4. **Remaining bad cases.** Here we will consider two surfaces S_1 and S_2 in orbit B_n, and for each S_i find some k_i and k'_i such that $\left(\begin{smallmatrix}1 & k_i\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ k'_i & 1\end{smallmatrix}\right)$ are in the stabiliser Γ_i of S_i ($i \in \{1, 2\}$). The groups Γ_1 and Γ_2, being conjugate, have the same index d in $\Gamma(1)$ and the same level ℓ. Using $m_i = \ell /\!/ k_i k'_i$ will yield a δ_i for each $i \in \{1, 2\}$ and we will show that d cannot divide both δ_1 and δ_2, implying that Γ_{B_n} is not a congruence subgroup.



Take $S_1 = S(1, 2^r, 3, 3^s, 1, 0)$.

Its stabiliser contains $\left(\begin{smallmatrix}1 & k_1\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ k_1' & 1\end{smallmatrix}\right)$, with $k_1 = 3^s$ and $k_1' = 2^r{\cdot}(2^r{\cdot}3 + 3)$. Note that here k_1' is not the exact vertical cusp width of S_1, but a multiple of it. For $r = 1, 2, 3, 4$, the value of k_1' is respectively $2{\cdot}3^2$, $2^2{\cdot}3{\cdot}5$, $2^3{\cdot}3^3$, $2^4{\cdot}3{\cdot}17$.



Take $S_2 = S(3, 2^r, 1, 3^s, 0, 0)$.

Its stabiliser contains $\left(\begin{smallmatrix}1 & k_2\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ k_2' & 1\end{smallmatrix}\right)$, with $k_2 = 3^s$ and $k_2' = 2^r{\cdot}(2^r + 3)$; again k_2' is not the exact vertical cusp width, but a multiple of it. It equals $2{\cdot}5$, $2^2{\cdot}7$, $2^3{\cdot}11$, $2^4{\cdot}19$, respectively for $r = 1, 2, 3, 4$.

Recall that $n = 2^r{\cdot}3^s + 3$, with $s \geqslant 2$.

Here, $d = \frac{3}{16}(n-3)n^2 \prod_{p|n}(1 - \frac{1}{p^2})$. Since 3^2 divides $(n-3)$, it does not divide n. Hence we can rewrite $d = 2^{r-1}{\cdot}3^{s+1+2\nu-2} \prod_{p|\frac{n}{3}} p^{2\nu-2}(p^2-1)$.

The choice of $m_i = \ell // k_i k_i'$ leads to $\delta_i = \prod_{p|k_i k_i'} p^{3\lambda-2}(p^2-1)$.

Given the values of $p^2 - 1$ for $p \in \{2, 3, 5, 7, 11, 17, 19\}$ (cf. table),

p	2	3	5	7	11	17	19
p^2-1	3	2^3	$2^3{\cdot}3$	$2^4{\cdot}3$	$2^3{\cdot}3{\cdot}5$	$2^5{\cdot}3^2$	$2^3{\cdot}3^2{\cdot}5$

the prime factors of δ_1 and δ_2 for each $r \in \{1, 2, 3, 4\}$ are:

r	1	2	3	4
δ_1	2, 3	2, 3, 5	2, 3	2, 3, 17
δ_2	2, 3, 5	2, 3, 7	2, 3, 5, 11	2, 3, 5, 19

If d divides δ_1 and δ_2, we deduce that $\prod_{p|\frac{n}{3}} p^{2\nu-2}(p^2-1)$ can have only 2 and 3 as prime factors. If this is the case, then n has no square factor, and, by Lemma 7.1 (postponed to the end of the section), its prime factors are in $\{3, 5, 7, 17\}$. The integers of the form $3{\cdot}5^a{\cdot}7^b{\cdot}17^c$ with $a, b, c \in \{0, 1\}$ are 3, 15, 21, 51, 105, 255, 357, 885. The only bad ones are 15, 21, and 51, and these were dealt with in § 7.3.

To complete the proof of Theorem 1, there remains only to prove:

Lemma 7.1. *If p is prime and $p^2 - 1$ has no other prime factors than 2 and 3, then $p \in \{2, 3, 5, 7, 17\}$.*

This follows from the fact that 8 and 9 are the only two consecutive nontrivial powers, a famous long-standing conjecture that was recently proved by Mihăilescu [Mi].

Theorem D (Catalan's Conjecture)**.** *The equation*

$$x^u - y^v = 1, \; x > 0, \; y > 0, \; u > 1, \; v > 1$$

has no other integer solution than $x^u = 3^2, \; y^v = 2^3$.

Proof of the lemma. By Catalan's Conjecture, consecutive powers of 2 and 3 are: (1, 2); (2, 3); (3, 4); (8, 9). Suppose $(p-1)(p+1)$ has no other prime factors than 2 and 3. If p is odd, then exactly one of $p-1$, $p+1$ is a multiple of 4, and the other one is $2{\cdot}3^\alpha$. If $\frac{p-1}{2} = 3^\alpha$, then either $\alpha = 0$, and $p = 3$, or $\alpha = 1$, and $p = 7$. If $\frac{p+1}{2} = 3^\alpha$, then either $\alpha = 1$, and $p = 5$, or $\alpha = 2$, and $p = 17$. □

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Chapitre 3

Constantes de Siegel–Veech

Ce chapitre concerne les constantes de Siegel–Veech des orbites de surfaces à petits carreaux de la strate $\mathcal{H}(2)$. On y montre que lorsque nombre de carreaux tend vers l'infini en restant premier, ces constantes tendent vers des constantes génériques associées à la strate. Il semble que la convergence lorsque le nombre de carreaux tend vers l'infini sans forcément rester premier soit vraie aussi, mais nous n'avons pas encore réussi à adapter nos calculs pour le montrer.

SIEGEL–VEECH CONSTANTS IN $\mathcal{H}(2)$

SAMUEL LELIÈVRE

Abstract. Abelian differentials on Riemann surfaces can be seen as translation surfaces, which are flat surfaces with cone-type singularities. Closed geodesics for the associated flat metrics form cylinders, whose number under a given maximal length generically has quadratic asymptotics.

Siegel–Veech constants are coefficients of these quadratic growth rates, and coincide for almost all surfaces in each moduli space of translation surfaces. Square-tiled surfaces are some specific translation surfaces whose Siegel–Veech do not equal the generic ones.

It is an interesting question whether, as n tends to infinity, the Siegel–Veech constants of square-tiled surfaces with n tiles tend to the generic constants of the ambient moduli space. Here we prove that it is the case in the moduli space $\mathcal{H}(2)$ of translation surfaces of genus two with one singularity.

Contents

Date: 1 December 2004.

1. Introduction

1.1. **Geodesics on the torus.** On the standard torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, the number $N(L)$ of families of simple closed geodesics of length not exceeding L is well-known to grow quadratically in L, with

$$N(L) \sim \frac{1}{2\zeta(2)} \cdot \pi L^2$$

which is one half of the asymptotic for the number of primitive lattice points in a disc of radius L. The factor one half comes from counting unoriented rather than oriented geodesics.

By convention, the corresponding *Siegel–Veech constant* is

$$c = \frac{1}{2\zeta(2)}$$

(note that it is the coefficient of πL^2 and not of L^2).

Marking the origin of the torus (i.e. artificially considering it as a singularity or saddle), the number of geodesic segments joining the saddle to itself, of length at most L, coincides with the number of families of simple closed geodesics.

1.2. **Geodesics on translation surfaces.** It is a standard fact that Abelian differentials on Riemann surfaces can be seen as translation surfaces.

On translation surfaces of genus $\geqslant 2$, countings of closed or singular geodesics, similar to those we just described for the torus, can be made.

There, the countings of saddle connections and of families of simple closed geodesics do not coincide, but their growth rates remain quadratic.

Masur proved [Ma88, Ma90] that for every translation surface, there exist positive constants c and C such that the counting functions of saddle connections and of maximal cylinders of closed geodesics satisfy

$$c \cdot \pi L^2 \leqslant N_{\text{cyl}}(L) \leqslant N_{\text{sc}}(L) \leqslant C \cdot \pi L^2$$

for large enough L.

Veech [Ve] proved that on a square-tiled surface (and on any Veech surface) there are in fact *exact quadratic asymptotics* and Gutkin and Judge [GuJu] gave another proof of that. Another proof for the upper quadratic bounds for $N_{\text{cyl}}(L)$ and $N_{\text{sc}}(L)$ was given by Vorobets [Vo].

Eskin and Masur [EM] gave yet another one, and proved that for each connected component of each stratum of each moduli space of normalized abelian (or quadratic) differentials, there are constants c_{sc} and c_{cyl} such that *almost every surface* in the component has $N_{\text{sc}}(L) \sim c_{\text{sc}}\pi L^2$ and $N_{\text{cyl}}(L) \sim c_{\text{cyl}}\pi L^2$.

It is an interesting open problem whether *all* translation surfaces have quadratic growth rates for cylinders of closed geodesics.

The particular constants for many Veech surfaces have been computed explicitly by Veech [Ve], Vorobets [Vo], Gutkin–Judge [GuJu], Schmoll [Schmo], Eskin–Masur–Schmoll [EMS]. The generic constants for the connected components of the strata were computed by Eskin, Masur and Zorich in [EMZ] for the case of abelian differentials.

The particular constants for Veech surfaces usually do not coincide with the generic constants of the strata where they live.

There is also another subtle difference between Veech surfaces and generic surfaces. Define cylinders as **regular** if their boundary components both consist of a single saddle connection. In any connected component of stratum in genus $\geqslant 2$, the counting functions of irregular cylinders are generically subquadratic (in fact a generic surface has no irregular cylinders), while on Veech surfaces they have quadratic asymptotics.

What we will prove however is that individual quadratic constants either for *regular* cylinders or for all cylinders on square-tiled surfaces of the stratum $\mathcal{H}(2)$ (translation surfaces of genus 2 with one singularity) converge as the number of squares tends to infinity to the generic constants of $\mathcal{H}(2)$. See Theorem 1 in § 1.4 for a precise statement.

1.3. **Ratner theory for moduli spaces of abelian differentials.** Analogs of Ratner's theorems classifying invariant measures for the action of unipotent one-parameter groups on homogeneous spaces are expected to hold on strata of the moduli spaces of abelian differentials; the results we prove here could be deduced from such theorems; for the time being, they reinforce the expectation that they do hold.

Some Ratner-like theorems for moduli spaces of abelian differentials have recently been obtained, but do not allow to obtain Theorem 1.

The works of Calta [Ca] and McMullen [Mc] provide a classification of invariant measures in $\mathcal{H}(2)$, albeit for the action of the whole $\mathrm{SL}(2, \mathbf{R})$ and not of unipotent one-parameter subgroups of $\mathrm{SL}(2, \mathbf{R})$.

Eskin, Masur and Schmoll [EMS] have results for the action of unipotent groups on subspaces of $\mathcal{H}(1, 1)$.

Eskin, Marklof and Morris [EMWM] have results for the action of unipotent groups on certain moduli spaces of abelian differentials in genus larger than 2.

1.4. **In the stratum $\mathcal{H}(2)$.** In this paper, we are concerned with the stratum $\mathcal{H}(2)$ consisting of abelian differentials in genus 2 with a double zero, or translation surfaces of genus 2 with one singularity (of angle 6π). We prove:

Theorem 1. *Consider a sequence S_n of area 1 surfaces in $\mathcal{H}(2)$ such that each surface S_n is tiled by some prime number p_n of square tiles, with $p_n \to \infty$. Then the Siegel–Veech constants for cylinders of closed geodesics on the surfaces S_n tend to $\frac{10}{3} \cdot \frac{1}{2\zeta(2)}$, the generic Siegel–Veech constant of $\mathcal{H}(2)$ for cylinders of closed geodesics. Moreover, the Siegel–Veech constants for regular cylinders also tend to the generic constant, while the Siegel–Veech constants for irregular cylinders tend to 0.*

Remark. We believe that the assumption that the number of squares tiling the surfaces is prime is unnecessary, but we have not yet been able to adapt the calculations to show the convergence of Siegel–Veech constants in the case of nonprime numbers of tiles.

The proof of the theorem relies on fine estimates presented in § 3.1.

1.5. **Acknowledgements.** The author wishes to thank Anton Zorich for guiding him into this problem, Pascal Hubert, Joël Rivat and Emmanuel Royer for useful conversations, and Cécile Dartyge and Gérald Tenenbaum who helped him with the estimates in § 3.1.

2. Preliminaries

2.1. The stratum $\mathcal{H}(2)$.

2.1.1. *Orbits of square-tiled surfaces.* By a theorem of McMullen [Mc2], in $\mathcal{H}(2)$, for $n > 3$, primitive n-square-tiled surfaces form one orbit E_n if n is even, and two orbits A_n and B_n if n is odd (see [HL1] for the prime n case). Slightly abusing notation, we use the same notation A_n, B_n, E_n for the discrete orbits and for the Teichmüller discs. A formula for the cardinality of E_n (even n) and for the sum of the cardinalities of A_n and B_n is given in [EMS], which in particular results in the asymptotic

$$\frac{3}{8} n^3 \prod_{p|n} (1 - \frac{1}{p^2}).$$

Formulas for the separate countings of A_n and B_n are conjectured in [HL1], which would yield the asymptotics (proved there for prime n):

$$\frac{3}{16} n^3 \prod_{p|n} (1 - \frac{1}{p^2}).$$

Some algebraic properties of the Veech groups are discussed in [HL2].

2.1.2. *Cusps.* Square-tiled surfaces in the stratum $\mathcal{H}(2)$ decompose into either one or two horizontal cylinders, and can be given as coordinates the heights, widths and twist parameters of these cylinders, see [HL1]. Here we are interested in *regular* cylinders of closed geodesics, which exist only in *two-cylinder* decompositions (in one-cylinder decompositions, the unique cylinder has three saddle connections on each boundary component).

The decompositions into cylinders provide a way to parametrise square-tiled surfaces (by the heights, widths and twist parameters of their cylinders). These parameters are very convenient to describe the action of $\mathcal{U} = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbf{Z}\}$; it only changes the twist parameters.

The cusps of an $\mathrm{SL}(2,\mathbf{R})$-orbit of square-tiled surfaces, can be identified with the $\mathcal{U}$-orbits of square-tiled surfaces in it, and each cusp has a standard representative (see [HL1, Lemma 3.1]).

In particular, two-cylinder cusps are parametrised by the heights h_i, the widths w_i, and twists parameters t_i of their cylinders ($i \in \{1, 2\}$). A two-cylinder cusp has cusp width $\mathrm{cw}(\mathcal{C}) = \frac{w_1}{h_1 \wedge w_1} \vee \frac{w_2}{h_2 \wedge w_2}$, where $h \wedge w$ denotes the greatest common divisor of h and w, and $a \vee b$ denotes the least common multiple of a and b.

Remark. When the number of tiles is prime, this simplifies to $\mathrm{cw}(\mathcal{C}) = w_1 w_2$.

2.2. **Siegel–Veech constants of cusps.** In the case of the torus, counting families of simple closed geodesics amounts to counting primitive points of $\mathbf{Z}^2$. In this sense, when counting simple closed geodesics of a square-tiled surface of higher genus, we are counting certain multiples of those of the torus.

On a square-tiled surface, as on the torus, the directions which define a decomposition in cylinders of closed geodesics correspond to primitive integer vectors. Better than that, given a primitive square-tiled surface S, each primitive integer vector $(a, b) \in \mathbf{Z}^2$ corresponds to a cusp of the $\mathrm{SL}(2,\mathbf{R})$-orbit of S. Recall that these cusps correspond to $\mathcal{U}$-orbits of square-tiled surfaces in the $\mathrm{SL}(2,\mathbf{R})$-orbit of S.

Here is how to recover the cusp from the primitive integer vector. Since a and b are coprime, by Bezout's theorem, there exist integers c and d such that $ad - bc = 1$. Geometrically, this means (a, b) and (c, d) form an oriented basis of the lattice $\mathbf{Z}^2$. The surface S is tiled by the unit area parallelograms defined by (a, b) and (c, d). Transforming these parallelograms into squares gives a new square-tiled surface. This is done by a linear transformation sending (a, b) to $(1, 0)$ and (c, d) to $(0, 1)$, in other words by applying the matrix $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)^{-1} \in \mathrm{SL}(2,\mathbf{Z})$.

Of course c and d are not unique, but the various choices of (c,d) give square-tiled surfaces $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)^{-1} \cdot S$ which belong to the same cusp. Quick check: different choices of (c,d) differ by integer multiples of (a,b); accordingly $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} d-kb & -c+ka \\ -b & a \end{smallmatrix}\right)$.

If we count the primitive integer vectors in a ball of radius L which correspond to directions in which S decomposes in cylinders of closed geodesics, we get the same counting function as for the torus.

One thing we could do is to count the primitive integer vectors in a ball of radius L corresponding to a given cusp. The proportion of directions going to different cusp is proportional to their width; see [EMZ, §§ 3.3–3.4 and § 7]. Thus, the asymptotics for each cusps are given by:

$$\frac{\text{width of the cusp}}{\text{sum of the cusp widths of the orbit}} \times \frac{1}{\zeta(2)} \cdot \pi L^2.$$

This is not exactly what we want to count, since we do not want to count the primitive vectors a multiple of which is the holonomy of a cylinder of closed geodesics, but the multiples themselves.

If the corresponding cusp is two-cylinder, with widths w_1, w_2, we want to count the direction not when $\|(a,b)\| < L$ but when $w_1 \cdot \|(a,b)\| < L$.

So the counting for this cusp will have asymptotics

$$\frac{\text{width of the cusp}}{\text{sum of the cusp widths of the orbit}} \cdot \frac{1}{w_1^2} \times \frac{1}{\zeta(2)} \cdot \pi L^2.$$

As a consequence, denoting by D the $\mathrm{SL}(2,\mathbf{Z})$-orbit of S, the counting function for regular cylinders of simple closed geodesics on S has the same asymptotics as

$$\sum_{\text{2-cyl cusps } \mathcal{C}} \frac{\mathrm{cw}(\mathcal{C})}{\# D} \frac{1}{w_1^2} \frac{1}{2\zeta(2)} \pi L^2.$$

If S is a primitive n-square-tiled surface, when we normalize S to area 1, we introduce a factor n in the above asymptotics.

So the asymptotics for the counting function of regular cylinders of simple closed geodesics on a unit area primitive square-tiled surface in an $\mathrm{SL}(2,\mathbf{Z})$-orbit D is given by

$$c(D)\pi L^2$$

and we can write $c(D) = \widetilde{c}(D) \cdot \frac{1}{2\zeta(2)}$ with

$$\widetilde{c}(D) = \frac{n}{\# D} \sum_{\text{2-cyl cusps } \mathcal{C} \text{ of } D} \frac{1}{w_1^2} \mathrm{cw}(\mathcal{C}).$$

3. Asymptotics for a large prime number of squares

Consider some prime n, and an orbit $D_n = A_n$ or B_n. Each cusp is parametrised by some parameters w_1, w_2, h_1, h_2, and twist parameters. By the remark at the end of § 2.1.2, the cusp width is just $w_1 w_2$.

Renaming w_1, w_2, h_1, h_2 as a, b, h, y respectively, the sum over the cusps becomes:

$$\widetilde{c}(D_n) = \frac{n}{\#D_n} \sum_{a,b,h,y} \frac{ab}{a^2}$$

where the sum is over positive integers a, b, h, y satisfying: $a < b$, $ah + by = n$, parity conditions for D_n.

3.1. A simpler sum. Since $\#D_n$ is, for prime n, asymptotically $\frac{3}{16}n^3$, we first replace $\frac{n}{\#D_n}$ by $\frac{1}{n^2}$.

Second, we momentarily drop the parity conditions; we will reintroduce them in the following subsections.

Last, we drop the condition $a < b$; we will explain later why this does not change the asymptotic.

So we first consider the following simplified sum:

$$S(n) = \sum_{a \geqslant 1} \frac{1}{a^2} \sum_{b \geqslant 1} \sum_{\substack{h \geqslant 1,\, y \geqslant 1 \\ ah+by=n}} \frac{ab}{n^2}.$$

Denote the sum over b by $S(n,a)$. Introducing the variable $m = by$,

$$S(n,a) = \sum_{\substack{1 \leqslant m \leqslant n-a \\ m \equiv n\ [a]}} \sum_{b \mid m} \frac{ab}{n^2} = \frac{a}{n^2} \cdot F(n-a, n, a)$$

where

$$F(x,k,q) = \sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv k\ [q]}} \sum_{b \mid m} b.$$

The following asymptotics hold for $F(x,k,q)$, $S(n,a)$ and $S(n)$.

Lemma 1. *For $k \wedge q = 1$, and $x \to \infty$,*

$$F(x,k,q) = \frac{x^2}{q} \cdot \frac{\pi^2}{12} \prod_{p \mid q} (1 - \frac{1}{p^2}) + O_q(x \log x).$$

Lemma 2.

$$S(n,a) \xrightarrow[n \text{ prime}]{n \to \infty} \frac{\pi^2}{12} \prod_{p \mid a} (1 - \frac{1}{p^2}).$$

Lemma 3. $S(n) \xrightarrow[n \text{ prime}]{n\to\infty} \dfrac{5}{4}$.

Proof of Lemma 1. If m is prime to k, denote by $\overline{m}$ the integer in $\{0,\dots,q-1\}$ such that $\overline{m}m \equiv 1\ [q]$, and by $u = u(m,k,q)$ the integer in $\{0,\dots,q-1\}$ such that $u \equiv \overline{m}k\ [q]$; error terms depend on q.

$$\begin{aligned}
F(x,k,q) &= \sum_{\substack{1\leqslant md\leqslant x\\ md\equiv k\ [q]}} d\\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \sum_{\substack{1\leqslant d\leqslant x/m\\ d\equiv \overline{m}k\ [q]}} d\\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \sum_{\substack{1\leqslant d\leqslant x/m\\ d\equiv u\ [q]}} d\\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \sum_{1\leqslant u+\lambda q\leqslant x/m} (u+\lambda q)\\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \Big(\Big(\sum_{1\leqslant \lambda\leqslant \frac{1}{q}(\frac{x}{m}-u)} \lambda q\Big) + O(\frac{x}{m})\Big)\\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \big(\frac{1}{2}q(\frac{x}{qm})^2 + O(\frac{x}{m}) + O(1)\big)\\
&= \frac{x^2}{2q} \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \frac{1}{m^2} + O(x\log x)
\end{aligned}$$

To sum only over the integers m with $m \wedge q = 1$, we can sum over all m with a factor $\mu(m\wedge q)$, so that all terms cancel out except the ones we want.

$$\begin{aligned}
F(x,k,q) &= \frac{x^2}{2q} \sum_{d|q} (\frac{\mu(d)}{d^2} \sum_{m\leqslant x/d} \frac{1}{m^2}) + O(x\log x)\\
&= \frac{x^2}{2q} \sum_{d|q} \frac{\mu(d)}{d^2} (\frac{\pi^2}{6} + O(1/x)) + O(x\log x)\\
&= \frac{x^2}{q} \cdot \frac{\pi^2}{12} \prod_{p|q} (1-\frac{1}{p^2}) + O(x\log x).
\end{aligned}$$

□

Proof of Lemma 2. Lemma 2 follows immediately from Lemma 1 by a dominated convergence argument (similar arguments were used in [HL1, § 7]. □

Proof of Lemma 3. Lemma 3 is a consequence of Lemma 2 by the following observation.

$$\sum_{a\geqslant 1}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) = \prod_p(1+\sum_{\nu\geqslant 1}p^{-2\nu}(1-p^{-2\nu})) = \prod_p(1+p^2)$$
$$= \prod_p\frac{1-p^{-4}}{1-p^{-2}} = \frac{\zeta(2)}{\zeta(4)} = \frac{\pi^2/6}{\pi^4/90} = \frac{15}{\pi^2}$$

□

3.2. **Sums with specified parities.** We introduce sub-sums of $S(n)$ for specified parities of the parameters.

The observation we just made will need to be completed by the following one.

$$\sum_{\substack{a\geqslant 1\\ a \text{ even}}}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) = \sum_{\substack{a\geqslant 1\\ a \text{ odd}}}\frac{1}{4a^2}\frac{3}{4}\prod_{p|a}(1-\frac{1}{p^2}) + \sum_{\substack{a\geqslant 1\\ a \text{ even}}}\frac{1}{4a^2}\prod_{p|a}(1-\frac{1}{p^2})$$

so that $\sum_{\substack{a\geqslant 1\\ a \text{ odd}}}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) = \frac{12}{\pi^2}$ and $\sum_{\substack{a\geqslant 1\\ a \text{ even}}}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) = \frac{3}{\pi^2}$.

3.2.1. *Odd widths.* We now consider the sum over odd a and b:

$$S^{\text{ow}}(n) = \sum_{\substack{a\geqslant 1\\ a \text{ odd}}}\frac{1}{a^2}\sum_{\substack{b\geqslant 1\\ b \text{ odd}}}\sum_{\substack{h\geqslant 1, y\geqslant 1\\ ah+by=n}}\frac{ab}{n^2}.$$

We proceed as for the sum $S(n)$: putting

$$F^{\text{ow}}(x,k,q) = \sum_{\substack{1\leqslant m\leqslant x\\ m\equiv k\ [q]}}\sum_{\substack{b|m\\ b \text{ odd}}} b \text{ and } S^{\text{ow}}(n,a) = \frac{a}{n^2}\cdot F^{\text{ow}}(n-a,n,a),$$

$$S^{\text{ow}}(n) = \sum_{\substack{a\geqslant 1\\ a \text{ odd}}}\frac{1}{a^2}S^{\text{ow}}(n,a).$$

The following asymptotics hold for $F^{\text{ow}}(x,k,q)$, $S^{\text{ow}}(n,a)$ and $S^{\text{ow}}(n)$.

Lemma 4. *For odd q, odd k, and $x\to\infty$,*

$$F^{ow}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).$$

For odd a,

$$S^{ow}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{24}\prod_{p|a}(1-\frac{1}{p^2}).$$

Finally,

$$S^{ow}(n) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{1}{2}.$$

Proof.

$$\begin{aligned}
F^{\mathrm{ow}}(x,k,q) &= \sum_{t\geqslant 0}\sum_{\substack{1\leqslant m\leqslant x/2^t\\ 2^t m\equiv k\ [q]\\ m\equiv 1\ [2]}}\sum_{b|m} b\\
&= \sum_{t\geqslant 0}\left(\frac{(x/2^t)^2}{2q}\frac{\pi^2}{12}\prod_{p|2q}(1-\frac{1}{p^2}) + O((x/2^t)\log(x/2^t))\right)\\
&= \frac{x^2}{q}\frac{1}{1-\frac{1}{4}}\frac{\pi^2}{24}(1-\frac{1}{2^2})\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x)\\
&= \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x)
\end{aligned}$$

□

3.2.2. *Odd heights.* We now consider the sum over odd h and y:

$$S^{\mathrm{oh}}(n) = \sum_{a\geqslant 1}\frac{1}{a^2}\sum_{b\geqslant 1}\sum_{\substack{h\geqslant 1, y\geqslant 1\\ h,\ y \text{ odd}\\ ah+by=n}}\frac{ab}{n^2}.$$

Proceeding as previously, we are led to introduce

$$F^{\mathrm{oh}}(x,k,q) = \sum_{\substack{1\leqslant m\leqslant x\\ m\equiv k+q\ [2q]}}\sum_{\substack{b|m\\ m/b \text{ odd}}} b \text{ and } S^{\mathrm{oh}}(n,a) = \frac{a}{n^2}\cdot F^{\mathrm{oh}}(n-a,n,a),$$

$$\text{and to write } S^{\mathrm{oh}}(n) = \sum_{a\geqslant 1}\frac{1}{a^2}S^{\mathrm{oh}}(n,a).$$

The following asymptotics hold for $F^{\mathrm{oh}}(x,k,q)$, $S^{\mathrm{oh}}(n,a)$ and $S^{\mathrm{oh}}(n)$.

Lemma 5. *For even q, odd k, and $x\to\infty$,*

$$F^{oh}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).$$

For odd q, odd k, and $x \to \infty$,

$$F^{oh}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{32}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).$$

For even a,

$$S^{oh}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{24}\prod_{p|a}(1-\frac{1}{p^2}).$$

For odd a,

$$S^{oh}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{32}\prod_{p|a}(1-\frac{1}{p^2}).$$

Finally,

$$S^{oh}(n) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{1}{2}.$$

Proof. For even q and odd k:

$$\begin{aligned} F^{\mathrm{oh}}(x,k,q) &= \sum_{\substack{1\leqslant m\leqslant x \\ m\equiv k+q\ [2q]}} \sum_{b|m} b \\ &= \frac{x^2}{2q}\frac{\pi^2}{12}\prod_{p|2q}(1-\frac{1}{p^2}) + O(x\log x) \\ &= \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x). \end{aligned}$$

For odd q and odd k:

$$\begin{aligned} F^{\mathrm{oh}}(x,k,q) &= \sum_{t\geqslant 1} \sum_{\substack{1\leqslant m\leqslant x/2^t \\ 2^t m\equiv k+q\ [2q] \\ m \text{ odd}}} \sum_{b|m} 2^t b \\ &= \sum_{t\geqslant 1} 2^t \sum_{\substack{1\leqslant m\leqslant x/2^t \\ 2^{t-1} m\equiv \frac{k+q}{2}\ [q] \\ m \text{ odd}}} \sum_{b|m} b \\ &= \sum_{t\geqslant 1} 2^t \frac{(x/2^t)^2}{2q}\frac{\pi^2}{12}\prod_{p|2q}(1-\frac{1}{p^2}) + O(x\log x) \\ &= \sum_{t\geqslant 1} \frac{1}{2^t}\frac{x^2}{q}\frac{\pi^2}{24}(1-\frac{1}{2^2})\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x) \\ &= \frac{x^2}{q}\frac{\pi^2}{32}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x). \end{aligned}$$

□

3.2.3. *Mixed parities.* Dealing with the even-odd sums as above would be most cumbersome; this is fortunately not necessary. Indeed, since $S(n) = S^{\mathrm{ow}}(n) + S^{\mathrm{oh}}(n) + S^{\mathrm{eo}}(n)$, and we know the limits of $S(n)$, $S^{\mathrm{ow}}(n)$ and $S^{\mathrm{oh}}(n)$ when n tends to infinity staying prime, we have:

$$S^{\mathrm{eo}}(n) \xrightarrow[\substack{n\to\infty \\ n \text{ prime}}]{} \frac{1}{4}.$$

3.3. **Asymptotics for orbits A and B.**

We end by showing that the limit we obtained is unchanged by adding a condition $a < b$.

Indeed, since $\#\{(h,y)\colon h \geqslant 1,\ y \geqslant 1,\ ah + by = n\} \leqslant n$, the sum $\displaystyle\sum_{b=1}^{a} \sum_{\substack{h\geqslant 1,\, y\geqslant 1 \\ ah+by=n}} \frac{ab}{n^2}$ is $O(1/n)$, where the constant of the O depends on a.

This also shows that the constants for irregular cylinders tend to 0.

Putting things together, $\widetilde{c}(A_n)$ and $\widetilde{c}(B_n)$ have the same asymptotics as $S^{\mathrm{A}}(n) = \frac{16}{3}(S^{\mathrm{oh}}(n) + \frac{1}{2}S^{\mathrm{eo}}(n))$ and $S^{\mathrm{B}}(n) = \frac{16}{3}(S^{\mathrm{ow}}(n) + \frac{1}{2}S^{\mathrm{eo}}(n))$, so they both tend to $\frac{10}{3}$.

4. Concluding remarks

Numerical evidence suggests that the convergence to the generic constants of the stratum occurs not only for prime n but for general n; however this involves some complications in the calculations and we have not yet been able to develop this.

A similar study for the quadratic constants that appear in the counting of saddle connections could also be made. There one has to take into consideration both one-cylinder and two-cylinder cusps, and some interesting phenomena can be observed: numerical calculations suggest that the sum of the contributions of one-cylinder and two-cylinder cusps has a limit, but separate countings for one-cylinder cusps do not have a limit for general n; their asymptotics have fluctuations involving the prime factors of n.

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Surfaces de Veech arithmétiques en genre deux : disques de Teichmüller, groupes de Veech et constantes de Siegel-Veech

Sur les espaces de modules de différentielles abéliennes existe une action naturelle de $\mathrm{SL}(2, \mathbf{R})$. Ses orbites, appelées disques de Teichmüller, se projettent dans les espaces de modules de surfaces de Riemann sur des géodésiques complexes. En tirant en arrière la forme $\mathrm{d}\,z$ du tore standard par des revêtements ramifiés au-dessus d'un seul point, on obtient les surfaces à petits carreaux, points entiers des espaces de modules de différentielles abéliennes.

Nous étudions en détail les disques de Teichmüller des points entiers de l'espace des modules des différentielles abéliennes en genre deux avec un zéro double : nombre de disques de Teichmüller pour chaque nombre de carreaux, et leur géométrie ; propriétés algébriques des stabilisateurs (sous-groupes de $\mathrm{SL}(2, \mathbf{Z})$ qui ne sont pas de congruence) ; comportement asymptotique des constantes de Siegel-Veech (coefficients des taux de croissance quadratiques des géodésiques fermées) lorsque le nombre de carreaux tend vers l'infini.

Arithmetic Veech surfaces in genus two: Teichmüller discs, Veech groups and Siegel-Veech constants

On the moduli spaces of abelian differentials exists a natural action of $\mathrm{SL}(2, \mathbf{R})$. Its orbits, called Teichmüller discs, project in the moduli spaces of Riemann surfaces to complex geodesics. Pulling back the form $\mathrm{d}\,z$ of the standard torus by coverings branched over a single point, one obtains the square-tiled surfaces, integer points of the moduli spaces of abelian differentials.

We study in detail the Teichmüller discs of integer points of the moduli space of abelian differentials in genus two with a double zero: number of Teichmüller discs for each number of square tiles, and their geometry; algebraic properties of the stabilisers (subgroups of $\mathrm{SL}(2, \mathbf{Z})$ which are not congruence subgroups); asymptotic behaviour of the Siegel-Veech constants (coefficients of the quadratic growth rates of closed geodesics) when the number of tiles tends to infinity.