QUASIMODULAR FORMS WITH FOURIER COEFFICIENTS ZERO AT PRIMES

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ABSTRACT. Quasimodular forms of mixed weights up to six are not determined by their Fourier coefficients at ranks 0, 1 and primes. This note explains this fact and some generalizations, and a link to the problem of counting torus coverings of genus 2.

Keywords: quasimodular forms, prime numbers, Hurwitz problem, torus coverings.

1. Quasimodular forms

Quasimodular forms are holomorphic functions defined on the upper half-plane and satisfying a quasimodularity property, i.e. a certain behaviour under composition with modular transformations (Möbius transformations with integer coefficients). We refer to [MR] for a precise definition along these lines, and will use a simpler one here.

For the purposes of this note, we consider for each even integer $k \geqslant 2$, the Eisenstein series of weight k, defined for $\operatorname{Im} z > 0$ and $q = e^{2i\pi z}$ by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geqslant 1} \sigma_{k-1}(n) q^n,$$

where B_k denotes the k-th Bernoulli number, k-th derivative in 0 of $t \mapsto t/(e^t - 1)$, and where for all integers $m \geqslant 0$ and $n \geqslant 1$, $\sigma_m(n) = \sum_{d|n} d^m$.

We then define for each even integer $k \geqslant 0$ the quasimodular forms of weight k as linear combinations of the products $E_2^a E_4^b E_6^c$ such that $2a + 4b + 6c = k$. Note that

$$E_2 = 1 - 24 \sum_{n \geqslant 1} \sigma_1(n) q^n, \quad E_4 = 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n,$$

$$\text{and} \quad E_6 = 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n.$$

We decide to call quasimodular form any element of the algebra generated by E_2, E_4 and E_6 (which are algebraically independent).

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This algebra is graded by weights. Only pure weight quasimodular forms have the quasimodularity property mentioned above.

We call Fourier coefficients of a quasimodular form f the coefficients of its development in powers of $q = e^{2i\pi z}$, and denote them by $\widehat{f}(n)$.

The observation at the base of this paper is that in order to determine a quasimodular form of maximal weight at most 6, it is not enough to know its Fourier coefficients at ranks 0, 1 and primes. This comes from a linear dependence relation between the Fourier coefficients at these ranks of the basis elements $1, E_2, E_2^2, E_4, E_2^3, E_2E_4, E_6$ of quasimodular forms of weights up to six.

Lemma 1. *The quasimodular form (of mixed weights* 0, 2, 4, 6*)*

$$f = -396 + 360E_2 - 30E_2^2 + 66E_4 + 5E_2^3 - 15E_2E_4 + 10E_6$$

has all its Fourier coefficients zero at 0, 1 *and primes (and positive at all other ranks).*

This is a consequence of the following Lemma.

Lemma 2. *For any distinct nonnegative integers k and ℓ, the function*

$$g_{k,\ell} : \begin{array}{ccc} \mathbf{N}^* & \to & \mathbf{N} \\ n & \mapsto & (n^\ell + 1)\sigma_k(n) - (n^k + 1)\sigma_\ell(n) \end{array}$$

is zero exactly at primes and in 1 *(and has constant sign elsewhere).*

A corollary is, denoting by D the differential operator $q\dfrac{d}{dq} = \dfrac{1}{2i\pi}\dfrac{d}{dz}$ and setting

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n\geqslant 1} \sigma_{k-1}(n)q^n = -\frac{B_k}{2k}E_k :$$

Corollary 1. *For any two distinct nonnegative odd integers k and ℓ, the quasimodular form*

$$f_{k,\ell} = (D^\ell + 1)G_{k+1} - (D^k + 1)G_{\ell+1}$$

has its Fourier coefficients zero exactly at 1 *and at primes.*

Remark. The operator D preserves the property

$$(\widehat{f}(n) = 0 \text{ and } n \neq 0) \iff (n = 1 \text{ or } n \text{ is prime}).$$

Question. What can be said of the set of quasimodular forms which satisfy this property?

Remark. The Fourier coefficient at 0 can be adjusted independently of others by adding a constant term (i.e. a weight 0 component).

Proof of Lemma 1. The quasimodular form f can be linearized: using

$$E_2^2 - E_4 = 12DE_2 \quad \text{and} \quad E_2^3 - 3E_2E_4 + 2E_6 = 72D^2E_2,$$

one can write $f = -36h$, where $h = 11 - E_4 - 10(D^2 - D + 1)E_2$.

Then $\widehat{h}(0) = 0$ and for $n \geqslant 1$

$$\widehat{h}(n) = 240[(n^2 - n + 1)\sigma_1(n) - \sigma_3(n)].$$

Lemma 1 is then seen to follow from Lemma 2 by multiplying by $n+1$.

We could also see it as a corollary of Corollary 1 by applying the operator $D+1$ to h. $\square$

Proof of Lemma 2. That $g_{k,\ell}(n)$ is zero for $n = 1$ or n prime is obvious, because $\sigma_k(1) = 1$ and $\sigma_k(p) = p^k + 1$ for p prime.

Let us show that $g_{k,\ell}$ is nonzero (and constant sign) at other integers.

We use the notation $\sum_{d|n}^{*}$ to denote $\sum_{\substack{d|n \\ d\neq 1,n}}$; this sum contains at least one term as soon as $n \neq 1$ and n is nonprime.

For such an n:

$$\begin{aligned} g_{k,\ell}(n) &= (1+n^\ell)(\sum_{d|n} d^k) - (1+n^k)(\sum_{d|n} d^\ell) \\ &= (1+n^\ell)(n^k + 1 + \sum_{d|n}{}^{*} d^k) - (1+n^k)(1+n^\ell + \sum_{d|n}{}^{*} d^\ell) \\ &= (1+n^\ell)(\sum_{d|n}{}^{*} d^k) - (1+n^k)(\sum_{d|n}{}^{*} d^\ell) \\ &= \sum_{d|n}{}^{*} [(1+n^\ell)d^k - (1+n^k)d^\ell] \end{aligned}$$

Assume $k < \ell$. Then:

$$\begin{aligned} g_{k,\ell}(n) &= \sum_{d|n}{}^{*} d^k[(1+n^\ell) - (1+n^k)d^{\ell-k}] \\ &= \sum_{d|n}{}^{*} d^k[1 + n^\ell - d^{\ell-k} - n^k d^{\ell-k}] \end{aligned}$$

Now if $d|n$ and $d \neq n$, we have $d \leqslant n/2$, so

$$\begin{aligned} 1 + n^\ell - d^{\ell-k} - n^k d^{\ell-k} &\geqslant 1 + n^\ell - (n/2)^{\ell-k} - n^k (n/2)^{\ell-k} \\ &\geqslant 1 + n^\ell - \frac{n^{\ell-k}}{2^{\ell-k}} - \frac{n^\ell}{2^{\ell-k}} \geqslant 1 + n^\ell - \frac{n^\ell}{2^{\ell-k}} - \frac{n^\ell}{2^{\ell-k}} \\ &\geqslant 1 + n^\ell - \frac{n^\ell}{2^{\ell-k-1}} \geqslant 1 \end{aligned}$$

The terms of the sum $\sum^*$ are all positive, and the sum contains at least one term. This ends the proof. $\square$

2. Link with the Hurwitz problem.

The Hurwitz problem consists in counting branched coverings of Riemann surfaces, fixing a base surface and a ramification type. Note that the ramification type determines the genus of the covering surface. It is convenient to weight the countings by the inverse of the number of automorphisms of each covering.

Case of the torus. In the particular case when the base surface is a torus (i.e. has genus 1), this degree-by-degree counting involves for each ramification type a quasimodular form as a generating function. This was proved by Dijkgraaf [Di] and Kaneko–Zagier [KaZa] for the case of coverings with simple ramifications over distinct points and by Eskin–Okounkov [EsOk] in full generality.

Simple ramifications. In the case of simple ramifications over distinct points, the generating function is a quasimodular form of pure weight $6g-6$ where g is the genus of the covering surfaces. In addition, these quasimodular forms are generated by a series in several variables that can be related to a generalized Jacobi Theta function. See [Di], [KaZa].

Other ramification types. For other ramification types, the generating function is still a quasimodular form, but of mixed weights with maximal weight $\leqslant 6g-6$. See [EsOk].

Interpolation. Knowing the countings are generated by a quasimodular form of maximal weight $\leqslant 6g-6$, one way to obtain this form in the basis of the $E_2^a E_4^b E_6^c$, $2a+4b+6c \leqslant 6g-6$, is, knowing the countings for a sufficient number of degrees, to interpolate with the Fourier coefficients of the basis elements. However, not any choice of degrees allows for interpolation.

In particular, one has to use degree 0 (for which the number of coverings is zero whatever the ramification type) in order to find the coefficient of 1 ($= E_2^0 E_4^0 E_6^0$), i.e. of weight 0.

One can also use degree 1, for which the number of coverings is zero as well if $g > 1$.

Furthermore, the countings being often easier for prime degrees, one would like to use degrees 0, 1 and a convenient number of prime degrees to perform the interpolation. Unfortunately, this turns out to never allow for interpolation in maximal weight $\leqslant 6g-6$, even for genus 2.

References

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