Landau Lectures

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p-adic cohomologies: history, and new developments

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Plan

0. Introduction
1. Review of crystalline cohomology
2. Old and new on the de Rham-Witt complex
3. Liftings mod p^2, de Rham-Witt and derived de Rham complexes

0. Introduction

$X/\mathbf{C}$ proper, smooth scheme (proper $\Leftrightarrow X^{\mathrm{an}} = X(\mathbf{C})$ compact).
Example: $X = V(f) \subset \mathbf{P}^n_{\mathbf{C}}$, $f \in \mathbf{C}[x_0, \cdots, x_n]$ homogeneous, $\deg(f) > 0$, with $z \in \mathbf{C}^{n+1} - \{0\} \Rightarrow \exists i, \partial f/\partial x_i(z) \neq 0$.

For $n \in \mathbf{Z}$:

Betti cohomology: $H^n(X^{\mathrm{an}}, \mathbf{C})$

de Rham cohomology: $H^n_{dR}(X/\mathbf{C}) := H^n(X, \Omega^\bullet_{X/\mathbf{C}})$,

$$\Omega^\bullet_{X/\mathbf{C}} := (\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbf{C}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N_{X/\mathbf{C}})$$

($N = \dim(X)$) the de Rham complex

Poincaré lemma: exactness of the sequence

$$0 \to \mathbf{C}_{X^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{d} \Omega^1_{X^{\mathrm{an}}/\mathbf{C}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N_{X^{\mathrm{an}}/\mathbf{C}} \to 0$$

and Serre's GAGA

$$H^j(X, \Omega^i_{X/\mathbf{C}}) \xrightarrow{\sim} H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}/\mathbf{C}})$$

imply comparison isomorphism

$$H^n(X^{\mathrm{an}}, \mathbf{C}) \xrightarrow{\sim} H^n_{dR}(X/\mathbf{C}). \tag{1}$$

Moreover: Hodge decomposition (Deligne, 1968, for X proper, not necessarily projective):

$$H^n(X^{\mathrm{an}}, \mathbf{C}) = \oplus_{i+j=n} H^{i,j} = \oplus_{i+j=n} H^j(X, \Omega^i).$$

In particular, Hodge-to-de Rham spectral sequence degenerates at E_1:

$$E_1^{ij} = H^j(X, \Omega^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/\mathbf{C}).$$

Remark. Liftings mod p^2 + Cartier isomorphism $\Rightarrow$ algebraic proof (Deligne-I, 1987).

Assume now that $X/\mathbf{C}$ has a proper smooth model $\mathcal{X}$ over $S = \mathrm{Spec}(\mathbf{Z}[1/m])$, for some $m \geq 1$: $X = \mathcal{X}_{\mathbf{C}}$.

For $p \in \mathrm{Spec}(\mathbf{Z}[1/m])$ (i.e. $p \nmid m$), $\mathcal{X}_p := \mathcal{X} \otimes \mathbf{F}_p$ is smooth: good reduction of $\mathcal{X}_{\mathbf{Q}}$ at p.

Example: $X = V(f) \subset \mathbf{P}^n_{\mathbf{C}}$ as above, with $f \in \mathbf{Z}[1/m][x_0, \cdots, x_n]$ (s. t. for all $p \nmid m$, and all geometric points $z \neq 0$ of $\mathcal{X}_p$, $\exists i, \partial f/\partial x_i(z) \neq 0$).

By

$$H^n(X^{\mathrm{an}}, \mathbf{C}) = H^n(X^{\mathrm{an}}, \mathbf{Q}) \otimes \mathbf{C},$$

$$H^n_{\mathrm{dR}}(X/\mathbf{C}) = H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Q}}/\mathbf{Q}) \otimes \mathbf{C},$$

and the comparison isomorphism (1)

$$H^n(X^{\mathrm{an}}, \mathbf{C}) \xrightarrow{\sim} H^n_{dR}(X/\mathbf{C})$$

get the (highly transcendental) period isomorphism

$$H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Q}}/\mathbf{Q}) \otimes \mathbf{C} \xrightarrow{\sim} H^n(X^{\mathrm{an}}, \mathbf{Q}) \otimes \mathbf{C}. \tag{2}$$

Problems:

(a) How about replacing

- $\mathbf{Q}$ by $\mathbf{Q}_\ell$ (ℓ prime)?
- $\mathcal{X}$ by $\mathcal{X}_{\mathbf{Q}_p}$ (p prime)?

(b) How about integral variants of (2)?

Problem (a): replacing $\mathbf{Q}$ by $\mathbf{Q}_\ell$, $\mathcal{X}$ by $\mathcal{X}_{\mathbf{Q}_p}$

$\mathbf{Q} \mapsto \mathbf{Q}_\ell$: inputs from étale cohomology:

$$H^n(X^{\mathrm{an}}, \mathbf{Q}) \otimes \mathbf{Q}_\ell = H^n(X_{\mathrm{et}}, \mathbf{Q}_\ell) = H^n(\mathcal{X}_{\overline{\mathbf{Q}},\mathrm{et}}, \mathbf{Q}_\ell),$$

Galois group $G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on

$$H^n(\mathcal{X}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) := H^n(\mathcal{X}_{\overline{\mathbf{Q}},\mathrm{et}}, \mathbf{Q}_\ell).$$

For $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ chosen, decomposition group $G_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts through

$$H^n(\mathcal{X}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_\ell) \xrightarrow{\sim} H^n(\mathcal{X}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell).$$

Recall

$$\mathbf{Q}_p \to \mathbf{Q}_{p,\mathrm{ur}} \to \overline{\mathbf{Q}}_p,$$

Inertia $I_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p,\mathrm{ur}}) \subset G_{\mathbf{Q_p}}$

$$\mathrm{Gal}(\mathbf{Q}_{p,\mathrm{ur}}/\mathbf{Q}_p) \xrightarrow{\sim} \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p) = \widehat{\mathbf{Z}}$$

(generator: arithmetic Frobenius $\sigma : x \mapsto x^p$).

- Assume first $p \nmid m$ (hence $\mathcal{X}_p$ smooth).

Two cases:

(i) $\ell \neq p$. Then, we have a Galois equivariant isomorphism (wrt $G_{\mathbf{Q}_p} \twoheadrightarrow G_{\mathbf{Q}_p}/I_p = \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)(= \widehat{\mathbf{Z}}))$

$$H^n(\mathcal{X}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_\ell) \xrightarrow{\sim} H^n(\mathcal{X}_p \otimes \overline{\mathbf{F}}_p, \mathbf{Q}_\ell),$$

which implies that I_p acts trivially, i.e. $G_{\mathbf{Q}_p}$ acts through $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$

Action well understood by the Weil conjectures (Grothendieck, Deligne). In particular:

Zeta function

$$Z(\mathcal{X}_p, t) := \exp(\sum_{n\geq 1} \sharp(\mathcal{X}_p(\mathbf{F}_{p^n}))\frac{t^n}{n}),$$

given by

$$Z(\mathcal{X}_p, t) = \prod_{0\leq i\leq 2d} P_i(t)^{(-1)^{i+1}},$$

$$P_i(t) := \det(1 - Ft, H^i(\mathcal{X}_p \otimes \overline{\mathbf{F}}_p, \mathbf{Q}_\ell))$$

($d = \dim(X)$, $F = \sigma^{-1}$ the geometric Frobenius), and

$$P_i(t) \in \mathbf{Z}[t],$$

independent of ℓ, with inverse roots Weil numbers of weight i.

(ii) $\ell = p$. I_p no longer acts trivially on $H^n(\mathcal{X}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_\ell)$.

Example : $X = \mathbf{P}^1_{\mathbf{C}}$.
Here, can take $m = 1$, $\mathcal{X} := \mathbf{P}^1_{\mathbf{Z}}$ (good reduction of $\mathbf{P}^1_{\mathbf{Q}}$ everywhere).

Recall:

$$\mathbf{Q}_\ell(1) := \mathbf{Q}_\ell \otimes \mathbf{Z}_\ell(1), \ \mathbf{Z}_\ell(1) := \varprojlim \mu_{\ell^n}(\overline{\mathbf{Q}}_p)$$

($\xrightarrow{\sim} \mathbf{Q}_\ell$ non canonicallly), with Galois action: $gz = z^{\chi(g)}$,
$\chi : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathbf{Z}_\ell^*$ the cyclotomic character, and for $r \in \mathbf{Z}$

$$\mathbf{Q}_\ell(r) := (\mathbf{Q}_\ell(1))^{\otimes r}$$

Whether $\ell \neq p$ or $\ell = p$, we have

$$H^2(\mathbf{P}^1_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_\ell) = \mathbf{Q}_\ell(-1).$$

But: • If $\ell \neq p$, $\mu_{\ell^n}(\overline{\mathbf{Q}}_p) \subset \mathbf{Q}_{p,\mathrm{ur}}$, I_p acts trivially, and

$$\det(1 - Ft, H^2(\mathbf{P}^1_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)) = \det(1 - Ft, \mathbf{Q}_\ell(-1)) = 1 - pt.$$

• If $\ell = p$, $\mu_{\ell^n}(\overline{\mathbf{Q}}_p) \not\subset \mathbf{Q}_{p,\mathrm{ur}}$, I_p acts nontrivially (with wild ramification).

Miracle: action of $G_{\mathbf{Q}_p}$ on $H^n(\mathcal{X}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ related to de Rham cohomology:

$$H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Q}_p}/\mathbf{Q}_p) \leftrightarrow H^n(\mathcal{X}_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p),$$

by p-adic Hodge theory. How can it be? NO Galois action on LHS!

LHS has hidden extra structure, especially an action of Frobenius, coming from crystalline cohomology:

$$H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Q}_p}/\mathbf{Q}_p) = H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Z}_p}/\mathbf{Z}_p) \otimes \mathbf{Q}_p,$$

and, actually, $H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Z}_p}/\mathbf{Z}_p)$ is uniquely determined by the special fiber $\mathcal{X}_p$:

$H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Z}_p}/\mathbf{Z}_p) = H^*(\mathcal{X}_p/\mathbf{Z}_p)$ (RHS:= crystalline cohomology)

The (absolute) Frobenius endomorphism F of $\mathcal{X}_p$, though it does not, in general, lift to $\mathcal{X}_{\mathbf{Z}_p}$ defines an isogeny

$$\varphi : H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Z}_p}/\mathbf{Z}_p) \to H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Z}_p}/\mathbf{Z}_p)$$

(i.e., $\varphi \otimes \mathbf{Q}_p$ an isomorphism), which, together with the Hodge filtration on $H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Q}_p}/\mathbf{Q}_p)$, enables to recover $H^n(\mathcal{X}_{\overline{\mathbf{Q}}_p,\mathrm{et}}, \mathbf{Q}_p)$ via Fontaine's rings and p-adic period isomorphisms (Fontaine's C_{cris} conjecture).

Study of action of Frobenius on crystalline cohomology of proper, smooth varieties over perfect fields led to the theory of de Rham-Witt complexes.

Problem (a) (replacing $\mathbf{Q}$ by $\mathbf{Q}_\ell$, $\mathcal{X}$ by $\mathcal{X}_{\mathbf{Q}_p}$), cont'd.

• Assume now $p \mid m$, i.e. $p \notin \mathrm{Spec}(\mathbf{Z}[1/m])$ (possibly bad reduction of $\mathcal{X}_{\mathbf{Q}}$ at p).

(i) $\ell \neq p$. Far less well understood than in the good reduction case. Big open problems: weight monodromy conjecture (despite great advances by Scholze), Serre's and Serre-Tate's conjectures of independence of ℓ for characteristic polynomials of Frobenius (or elements of the Weil group) on various ℓ-adic cohomology groups of the geometric generic fiber $\mathcal{X}_{\overline{\mathbf{Q}}_p}$.

(ii) $\ell = p$. No obvious crystalline cohomology groups in sight (except in the case of semistable reduction, using log geometry), but still get comparison theorems (p-adic period isomorphisms) (Hodge-Tate conjecture, and Fontaine's C_{dR} conjecture).

Problem (b): Integral comparison.

Assume $p \nmid m$ (hence $\mathcal{X}_p$ smooth). Relations between

$$H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Z}_p}/\mathbf{Z}_p) \leftrightarrow H^n(\mathcal{X}_{\overline{\mathbf{Q}}_p}, \mathbf{Z}_p)(\xrightarrow{\sim} H^n(X^{\mathrm{an}}, \mathbf{Z}) \otimes \mathbf{Z}_p)$$

are the object of the (recent) integral p-adic Hodge theory and prismatic cohomology (Bhatt-Morrow-Scholze, Bhatt-Scholze). In particular,

$$\mathrm{lg}(H^n_{\mathrm{dR}}(\mathcal{X}_{\mathbf{Z}_p}/\mathbf{Z}_p)_{\mathrm{tors}}) \geq \mathrm{lg}(H^n(X^{\mathrm{an}}, \mathbf{Z})_{p-\mathrm{tors}})$$

Variants for $p \mid m$, semistable reduction case, due to Cesnavicius-Koshikawa.

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1. Review of crystalline cohomology

May 1966: Grothendieck's letter to Tate: crystals, crystalline site.
Sources of inspiration:

- Dieudonné theory (p-divisible groups) (Tate, Serre-Tate, Oda)
- de Rham cohomology: - Gauss-Manin connection
- Monsky-Washnitzer's work on formal cohomology

Talks at IHES, fall of 1966; notes by Coates-Jussila

Crystalline cohomology developed in Berthelot's thesis (SLN 407, 604 pp.), and later by many authors (Katz, Mazur, Ogus, etc.)

Notation

k: perfect field of characteristic $p > 0$, $W = W(k)$,
$W_n = W_n(k) = W/p^nW$, $K = \mathrm{Frac}(W)$
$\sigma : W \xrightarrow{\sim} W$ the Frobenius automorphism
$(\sigma(a_0, \cdots, a_n, \cdots) = (a_0^p, \cdots, a_n^p, \cdots))$

Definitions

• For X/k, crystalline site $(X/W_n)_{\mathrm{crys}}$ with objects: thickenings $X \supset U \hookrightarrow U'$,
U open in X, U'/W_n, with PD (= divided powers) on the ideal of $U \hookrightarrow U'$, compatible with that on (p),

morphisms: obvious, covering families: $(U_i \hookrightarrow U_i') \to (U \hookrightarrow U')$ such that (U_i') = Zariski cover of U'.

structural sheaf: $\mathcal{O} = \mathcal{O}_{X/W_n}$: $(U \hookrightarrow U') \mapsto \mathcal{O}_{U'}$.

PD: *a priori* mysterious additional structure, motivated by acyclicity of dR complex of PD-algebras $W < t_1, \cdots, t_d >$.

- crystalline cohomology

$H^i(X/W_n) := H^i((X/W_n)_{\mathrm{crys}}, \mathcal{O})$,
$R\Gamma(X/W) := R\varprojlim R\Gamma(X/W_n)$, $H^i(X/W) = H^i R\Gamma(X/W)$

Main properties

- Weil cohomology

X/k proper smooth $\Rightarrow H^*(X/W) = \oplus H^i(X/W)$ is a finitely generated W-algebra
$X/k \mapsto H^*(X/W) \otimes K$ ($K = \mathrm{Frac}(W)$) is a Weil cohomology theory filling the gap at p among ℓ-adic cohomology theories $H^*(X_{\overline{k}}, \mathbf{Q}_\ell)$ ($\ell \neq p$, $\overline{k}$ = algebraic closure of k)
(i.e.: Künneth, Poincaré duality, cycle class). Moreover:

$$\dim_K(H^i(X/W) \otimes K) = \dim_{\mathbf{Q}_\ell}(H^i(X_{\overline{k}}, \mathbf{Q}_\ell)).$$

- Relation with de Rham cohomology

X/k lifted to Z/W proper and smooth $\Rightarrow$ canonical iso

$$H^*(X/W) \xrightarrow{\sim} H^*_{\mathrm{dR}}(Z/W).$$

• Slopes of Frobenius

For X/k, by functoriality, absolute Frobenius F_X of X ($a \mapsto a^p$ on $\mathcal{O}_X$) defines σ-linear endomorphism

$$\varphi : H^*(X/W) \to H^*(X/W).$$

Assume X/k proper, smooth.

Then $H^*(X/W)$ is finitely generated over W, and φ is an isogeny, i.e., $\varphi \otimes K$ is an isomorphism (if X is of pure dimension d, there exists a σ^{-1}-linear endomorphism v of $H^*(X/W)$ such that $\varphi v = v\varphi = p^d$). Hence, for each $n \in \mathbf{Z}$,

$$(H^n(X/W), \varphi)$$

is an F-crystal.

For $k = \mathbf{F}_q$, $q = p^\nu$, Katz-Messing:

$$\det(1 - \varphi^\nu t, H^*(X/W) \otimes K) = \det(1 - \mathrm{Frob}\, t, H^*(X_{\overline{k}}, \mathbf{Q}_\ell))$$

($\ell \neq p$, Frob = relative Frobenius of X/k).

In general, Dieudonné-Manin decomposition into slopes:

$$H^n(X/W) \otimes K = \oplus_{\lambda \in \mathbf{Q}} H^n_\lambda,$$

where H^n_λ = isoclinic component of pure slope λ (i.e., over $K(\overline{k}) := \mathrm{Frac}(W(\overline{k}))$, H^n_λ decomposes into $\oplus K(\overline{k})_\sigma[F]/(F^s - p^r)$, $\lambda = r/s$).

Newton polygon Nwt_n: slope λ, horizontal length $\dim(H^n_\lambda)$.

Main result (solution of Katz conjecture):

Theorem (Mazur-Ogus, 1972) X/k proper, smooth $\Rightarrow$

$$\mathrm{Nwt}_n \geq \mathrm{Hdg}_n,$$

where Hdg_n = Hodge polygon, slope i, horizontal length $h^{i,n-i}$, $h^{i,j} := \dim_k H^j(X, \Omega^i_{X/k})$.

Remarks: (a) by Katz-Messing, for $k = \mathbf{F}_q$, $\mathrm{Nwt}_n \geq \mathrm{Hdg}_n$ implies estimates on p-adic valuations of eigenvalues of Frob on $H^n(X_{\overline{k}}, \mathbf{Q}_\ell)$, and Chevalley-Warning type congruences on $\sharp X(\mathbf{F}_{q^m})$ for smooth complete intersections in the projective space: for $X = V(a_1, \cdots, a_r) \subset \mathbf{P}_k^{d+r}$,

$$\sharp X(\mathbf{F}_{q^m}) \equiv \sharp \mathbf{P}^d(\mathbf{F}_{q^m}) \bmod q^{cm},$$

where

$$c = \sup(0, \mathrm{ceiling}(\frac{d+r+1-\sum a_i}{\sup(a_i)})).$$

(generalizations by Katz (1971), Esnault-Katz (2005)).

(b) If $X = \mathcal{X}_k$ for a proper, smooth $\mathcal{X}/\mathcal{O}_L$, $[L:K] < \infty$, then, stronger inequality:

$$\mathrm{Nwt}_n \geq \mathrm{Hdg}_n(\mathcal{X}_L).$$

Follows from Berthelot-Ogus comparison th. (1983):
$H^n(X/W) \otimes L \xrightarrow{\sim} H^n_{\mathrm{dR}}(\mathcal{X}_L/L)$,
and C_{cris} comparison th. ($\Rightarrow$ weak admissibility of $H^n_{\mathrm{dR}}(\mathcal{X}_L/L)$)
(Tsuji et al., 1997 - ...).

Towards the de Rham-Witt complex

Grothendieck's questions in letter to Barsotti, May, 1970.
Rewrite $H^n(X/W) \otimes K$ as

$$H^n(X/W) \otimes K = \oplus_{i \in \mathbf{Z}} (H^n(X/W) \otimes K)_{[i,i+1)},$$

where

$$(H^n(X/W) \otimes K)_{[i,i+1)} = \oplus_{\lambda \in [i,i+1)} H^n_\lambda.$$

- Cohomological interpretation for $(H^n(X/W) \otimes K)_{[i,i+1)}$?
- $(H^n(X/W) \otimes K)_{[i,i+1)} \otimes (K, p^{-i}\sigma) = F$-(iso)crystal of slopes $\in [0,1)$

$\leftrightarrow$ (by Cartier theory) formal p-divisible group G^{ij}.

Dimension of G^{ij}? Height of G^{ij}?

dRW theory brings answers to these questions.

2. Old and new on the de Rham-Witt complex

2.1. Old

X/k smooth. For $\dim(X) < p$, $p > 2$, Bloch (1974) constructs the so-called complex of typical curves

$$C = (C^0 \to C^1 \to \cdots \to C^{\dim(X)} \to 0),$$

a complex of abelian sheaves on X_{zar}, with $C^0 = W\mathcal{O}_X$, C^i equipped with F, V satisfying $FV = VF = p$, $dF = pFd$, and extending the usual F, V on $W\mathcal{O}_X$, with the property that, for X/k proper and smooth, there is a natural isomorphism

$$H^*(X, C) \xrightarrow{\sim} H^*(X/W),$$

with φ on $H^*(X/W)$ given by the endomorphism $p^\bullet F$ of C. Moreover he shows that $\dim H^j(X, C^i) \otimes K < \infty$, and

$$H^j(X, C^i) \otimes K = (H^{i+j}(X/W) \otimes K)_{[i,i+1)},$$

solving one of Grothendieck's questions in this case.

In particular,

$$H^n(X, W\mathcal{O}) \otimes K = (H^n(X/W) \otimes K)_{[0,1)}.$$

Construction inspired by Artin-Mazur formal groups Φ^i, with Cartier modules $H^i(X, W\mathcal{O})$: $W\mathcal{O} = TC(\mathbf{G}_m)$, $C^i = TC(SK_{i+1})$ (TC = typical curves), SK_{i+1} = symbolic part of Quillen's K group.

1975: Deligne sketches differential geometric approach to Bloch's construction, with no K-groups, working without restrictions of dimension or characteristic.
Inspired by work of Lubkin (1970) on de Rham complexes of Witt vectors.
Carried out in [I, Complexe de de Rham-Witt et cohomologie cristalline, Ann. ENS, 4ème série, 12, 1979, 501-661].

What is the de Rham-Witt complex?

Let k as before, and X a k-scheme. The de Rham-Witt complex of X/k is a strictly commutative differential graded algebra on the Zariski site of X:

$$W\Omega^\bullet_X = (W\Omega^0_X \xrightarrow{d} W\Omega^1_X \xrightarrow{d} \cdots),$$

with $W\Omega^0_X = W\mathcal{O}_X$, and each component $W\Omega^i_X$ is equipped with additive operators F, V, extending the usual ones on $W\mathcal{O}_X$, satisfying the following relations

$$FV = VF = p,\ xVy = V(Fx.y),\ Fx.Fy = F(xy),\ Fd[a] = [a]^{p-1}d[a]$$

(for $[a] = (a, 0, \cdots, 0, \cdots)$ the Teichmüller representative of $a \in \mathcal{O}_X$), and

$$FdV = d.$$

$FV = VF = p$, $xVy = V(Fx.y)$, $Fx.Fy = F(xy)$, $Fd[a] = [a]^{p-1}d[a]$

(for $[a] = (a, 0, \cdots, 0, \cdots)$ the Teichmüller representative of $a \in \mathcal{O}_X$), and

$$FdV = d.$$

Implies: for $n \geq 1$, $V^n W\Omega^\bullet_X + dV^n W\Omega^\bullet_X$ is a differential graded ideal. Let

$$W_n\Omega^\bullet_X := W\Omega^\bullet_X/(V^n W\Omega^\bullet_X + dV^n W\Omega^\bullet_X) = (W_n\mathcal{O}_X \to \cdots)$$

be the quotient. The projective system

$$W_\bullet\Omega^\bullet_X = (\cdots \to W_{n+1}\Omega^\bullet_X \to W_n\Omega^\bullet_X \to \cdots \to W_1\Omega^\bullet_X),$$

together with the induced operators $F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$, $V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$, is characterized by a certain universal property: universal F-V-pro-complex over $W\mathcal{O}_X$, in Langer-Zink's terminology.

$$W_\bullet\Omega^\bullet_X = (\cdots \to W_{n+1}\Omega^\bullet_X \to W_n\Omega^\bullet_X \to \cdots \to W_1\Omega^\bullet_X),$$

Moreover, the sheaves $W_n\Omega^i_X$ are quasi-coherent on $W_n(X)$, the canonical map

$$\Omega^\bullet_{W_n\mathcal{O}_X/W_n} \to W_n\Omega^\bullet_X$$

is surjective, and an isomorphism for $n = 1$. For $X = \mathrm{Spec}(k)$, $W_n\Omega^\bullet_k = W_n(k)$.

Short definition for X/k smooth

$$W\Omega^\bullet_X = \widehat{\Omega}^\bullet_{W\mathcal{O}_X/W}/\overline{T},$$

where $\overline{T}$ = closure of p-torsion $T = \Omega^\bullet_{W\mathcal{O}_X/W}[p^\infty]$ for topology given by

$$\widehat{\Omega}^\bullet_{W\mathcal{O}_X/W} = \varprojlim \Omega^\bullet_{W_n\mathcal{O}_X/W_n},$$

i.e.,

$$W\Omega^\bullet_X = \varprojlim \Omega^\bullet_{W\mathcal{O}_X/W}/(T + K_n),$$

where $K_n := \mathrm{Ker}(\Omega^\bullet_{W\mathcal{O}_X} \to \Omega^\bullet_{W_n\mathcal{O}_X})$.

Remark. $T^0 = 0$, but $T^1 \neq 0$ if $\dim(X) > 0$, e.g., if $x_0 := [t]$, $x_1 = Vx_0$, then $x_1^p = p^p x_0 \Rightarrow$
if $y = x_1^{p-1}dx_1 - p^{p-1}dx_0 \in \Omega^1_{W(\mathbf{F}_p[t])}$, $y \neq 0$, $py = 0$.

Operators F, V, etc.: Use $Fa \equiv a^p \bmod p$ for $a \in W\mathcal{O}_X$, F induces φ on Ω^1, uniquely divisible by p on Ω^1/T, thus $\varphi = p^i F$ on Ω^i/T; V, relations follow.

Additional properties for X/k smooth.

- $W\Omega^i_X$ p-torsion free for all i, in particular, F, V injective
- $W\Omega^i_X = 0$ for $i > \dim(X)$
- (saturation) $d^{-1}(pW\Omega^{i+1}_X) = FW\Omega^i_X$ (in particular, F bijective on $W\Omega^{\dim(X)}_X$.

This property was the motivation for BLM (Bhatt-Lurie-Mathew) approach (see below).

- $W\Omega^\bullet_X/pW\Omega^\bullet_X \to W_1\Omega^\bullet_X = \Omega^\bullet_X$ is a quasi-isomorphism.

Main theorems

- Comparison with crystalline cohomology
- Structure of slope spectral sequence, and applications

Comparison with crystalline cohomology

Theorem 1. For X/k smooth, there is a canonical, functorial isomorphism of graded algebras

$$H^*(X/W) \xrightarrow{\sim} H^*(X, W\Omega^\bullet),$$

with Frobenius φ on $H^*(X/W)$ given by endomorphism $p^\bullet F$ of the complex $W\Omega^\bullet_X$.

Comes from refined, local theorem:

Theorem 1'. There exist a compatible system of (canonical, functorial) isomorphisms

$$Ru_*\mathcal{O}_{X/W_n} \xrightarrow{\sim} W_n\Omega^\bullet_X,$$

where

$$u : (X/W_n)_{crys} \to X_{\mathrm{zar}}$$

is Berthelot's canonical map ($u^{-1}(U) = (U/W_n)_{\mathrm{crys}}$).

In particular, if $R\Gamma(X/W) := R\varprojlim R\Gamma(X, Ru_*\mathcal{O}_{X/W_n})$,

Corollary. ForX/k proper and smooth,

$$R\Gamma(X/W) \xrightarrow{\sim} R\Gamma(X, W\Omega^\bullet_X)$$

is a perfect complex of W-modules, and, for X of pure dimension d, the σ^{-1}-linear endomorphism v of $R\Gamma(X/W)$ induced by the endomorphism of $W\Omega^\bullet_X$ given by $p^{d-1-i}V$ in degree i (and F^{-1} in degree d) satisfies

$$\varphi v = v\varphi = p^d.$$

Structure of slope spectral sequence and applications

Theorem 2. For X/k proper and smooth, the spectral sequence

$$E_1^{ij} = H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}(X, W\Omega^\bullet_X)(\xrightarrow{\sim} H^{i+j}(X/W))$$

(called slope spectral sequence) degenerates at E_1 modulo p-torsion, $H^j(X, W\Omega^i_X)/H^j(X, W\Omega^i_X)[p^\infty]$ is finitely generated over W, with V topologically nilpotent, $H^j(X, W\Omega^i_X)[p^\infty]$ is killed by a power of p, and the degeneration induces an isomorphism

$$H^j(X, W\Omega^i_X) \otimes K \xrightarrow{\sim} (H^{i+j}(X/W) \otimes K)_{[i,i+1)}.$$

In particular,

$$H^j(X, W\mathcal{O}_X) \otimes K \xrightarrow{\sim} (H^j(X/W) \otimes K)_{[0,1)}$$

When Artin-Mazur's functor Φ^j is representable by a smooth formal group, then $H^j(W\mathcal{O})$ is its Cartier module of typical curves, and $\dim_K(H^j(X, W\mathcal{O}_X) \otimes K)$ is the height of its largest p-divisible quotient.

Refinements and complements

by I., I.-Raynaud, Nygaard, Ekedahl, especially, theory of coherent graded modules over the Raynaud ring (see slide 49)

In particular:

- (Nygaard) New (simpler) proofs of Rudakov-Shafarevich theorem on K3's ($H^0(X, T_X) = 0$), and Ogus' th. ($\mathrm{Nwt}_n \geq \mathrm{Hdg}_n$) (via Nygaard filtration).

- (I.) (generalization of Igusa-Artin-Mazur inequality) For $k = \overline{k}$,

$$\mathrm{rk}(\mathrm{NS}(X)) = b_2 - 2\mathrm{dim}(H^2(W\mathcal{O}) \otimes K) - \mathrm{rk}\, T_p(\mathrm{Br}(X)).$$

- (answers to Grothendieck's question) (I., Ekedahl) $H^j(X, W\Omega^i)/V$-torsion = Cartier module of smooth formal group G^{ij}; dimensions of p-divisible quotient and unipotent part of G^{ij} are interesting numerical invariants of slope spectral sequence (Ekedahl's theory of Hodge-Witt numbers). If $H^j(X, W\Omega^i)$ is p-torsion free, G^{ij} is p-divisible, and $\dim(G^{ij}) = \dim_k H^j(X, W\Omega^i)/V$, $\mathrm{ht}(G^{ij}) = \dim_k H^j(X, W\Omega^i)/p$.

• (I.) Study of torsion of $H^2(X/W)$ (discovery of exotic torsion)

• (I. I.-Raynaud, Ekedahl, Bloch, Gabber, Kato, Kerz, Morrow) Logarithmic Hodge-Witt sheaves, relation with Milnor K-groups, and fixed points of F on $H^*(X, W\Omega^*)$

Further developments

• Hyodo-Kato's theory of log crystalline cohomology and log de Rham-Witt complexes (used in formulation - and proof - of Fontaine-Jannsen's comparison conjecture C_{st})

• Relative variants (Langer-Zink) (application to theory of displays), overconvergent variants (Davis-Langer-Zink), arithmetic variants (Hesselholt-Madsen), recent links with THH and cyclotomic spectra (BMS2), connections with integral p-adic Hodge theory and prismatic cohomology (BMS1, BS).

2.2. New approach to de Rham-Witt (BLM)

For X/k proper, but singular, $H^*(X/W)$ bad (Bhatt's examples with $\dim_K H^*(X/W) \otimes K = \infty$), but Berthelot's rigid cohomology $H^*_{\mathrm{rig}}(X/K)$ good (finite dimensional over K, and φ an isomorphism) (NB. vast generalizations by Kedlaya et al.). In particular, slope decomposition:

$$H^n_{\mathrm{rig}}(X/K) = \oplus_{i \in \mathbf{Z}} H^n_{\mathrm{rig}}(X/K)_{[i,i+1)},$$

BLM approach yield new dRW complexes $\mathcal{W}\Omega^\bullet_X$ with:

- $\mathcal{W}\Omega^\bullet_X = W\Omega^\bullet_X$ for X/k smooth,
- for certain singular X/k (conjecturally all proper X/k):

$H^*(X, \mathcal{W}\Omega^\bullet_X)$ finitely generated over W

$H^*(X, \mathcal{W}\Omega^\bullet_X) \otimes K \xrightarrow{\sim} H^*_{\mathrm{rig}}(X/K)$

Slope spectral sequence and cohomological interpretation of slope decomposition as in the proper smooth case.

Construction of BLM's de Rham-Witt complexes

A Dieudonné algebra is a strict cdga

$$A = (A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots)$$

with each A^i equipped with an additive endomorphism F satisfying:

$$(Fx)(Fy) = F(xy),\ dF = pFd,\ Fa \equiv a^p \bmod pA^0$$

for $a \in A^0$.

The Dieudonné algebra A is called saturated if, for all i, A^i is p-torsion free, $F : A^i \to A^i$ is injective, and $d^{-1}(pA^{i+1}) = FA^i$.

If A is saturated, there exists a unique additive $V : A^i \to A^i$ such that $FV = p$. It follows that:

$$VF = p,\ FdV = d,\ xVy = V(Fx.y).$$

Then, for all $n \geq 1$, $V^nA + dV^nA$ is a dg ideal in A. One sets

$$\mathcal{W}_nA := A/(V^nA + dV^nA).$$

Examples.

(1) $A = \Omega^\bullet_{\mathbf{Z}_p[t_1,\cdots,t_r]/\mathbf{Z}_p}$; lift $t_i \mapsto t_i^p$ of Frobenius gives endomorphism φ of $\Omega^\bullet$, with $\varphi = p^i F$ on Ω^i; (A, d, F) is a Dieudonné algebra; not saturated.

Variant: R/k smooth; B/W smooth formal lifting, with lifting F of Frobenius; then $A := \widehat{\Omega}^\bullet_{B/W}$, with $F = p^{-i}\varphi$ on $\widehat{\Omega}^i$ ($\varphi =$ endomorphism of A induced by $F : B \to B$) is a Dieudonné algebra; not saturated.

(2) R/k smooth; $(W\Omega^\bullet_R, d, F)$ is a saturated Dieudonné algebra (cf. p. 30, Additional properties).

Let A be a saturated Dieudonné algebra. Recall:

$$VF = p,\ FdV = d,\ xVy = V(Fx.y).$$

$$\mathcal{W}_n A := A/(V^n A + dV^n A).$$

Define

$$\mathcal{W}A := \varprojlim \mathcal{W}_n A.$$

It's again a saturated Dieudonné algebra.
One says that A is strict if the canonical map

$$A \to \mathcal{W}A$$

is an isomorphism. For example, $W\Omega^{\bullet}_R$ in Example (2) is strict.

Note: A strict $\Rightarrow A^0/VA^0$ reduced, and $A^0 \xrightarrow{\sim} W(A^0/VA^0)$.

$\mathbf{DA}$: Dieudonné algebras; $\mathbf{DA}_{\mathrm{str}}$: strict Dieudonné algebras

Theorem 1 (BLM). The functor

$$\mathbf{DA}_{\mathrm{str}} \to \mathbf{F}_p\mathrm{-alg},\ A \mapsto A^0/VA^0$$

has a left adjoint $R \mapsto \mathcal{W}\Omega^\bullet_R$:

$$\mathrm{Hom}_{\mathbf{F}_p-alg}(R, A^0/VA^0) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{DA}_{\mathrm{str}}}(\mathcal{W}\Omega^\bullet_R, A).$$

The strict Dieudonné algebra $\mathcal{W}\Omega^\bullet_R$ is called the saturated de Rham-Witt complex of R.

Functorial in R. For R perfect, $\mathcal{W}\Omega^\bullet_R = W(R)$.

Proof of Theorem 1. Easy. Uses Deligne-Ogus décalage functor η_p.

For a complex M of p-torsion free abelian groups, define the subcomplex

$$\eta_p M \subset M[1/p],\ (\eta_p M)^i = p^i M^i \cap d^{-1}(p^{i+1}M^{i+1}).$$

A p-torsion free Dieudonné algebra A is saturated if and only if

$$\alpha_F : A \to \eta_p A,\ a \in A^i \mapsto p^i F a$$

is an isomorphism. The saturation functor

$$\mathrm{Sat} : \mathbf{DA} \to \mathbf{DA}_{\mathrm{sat}}$$

defined by

$$\mathrm{Sat}(A) := \varinjlim_{\alpha_F} (\eta_p)^n (A/A[p^\infty])$$

is left adjoint to the inclusion. For $A = \Omega^\bullet_{\mathbf{Z}_p[t_1,\cdots,t_r]/\mathbf{Z}_p}$ as in Example (1), $\mathrm{Sat}(A)$ played crucial role in study of classical dRW in smooth case (Deligne's complex of integral forms).

Construction of saturated dRW

Put

$$\mathcal{W}\Omega_R^\bullet := \mathcal{W}\mathrm{Sat}(\Omega^\bullet_{W(R_{\mathrm{red}})}).$$

There is indeed a unique structure of Dieudonné algebra on $\Omega^\bullet_{W(R_{\mathrm{red}})}$ inducing F in degree 0.

Construction globalizes on schemes, yielding, for $X/\mathbf{F}_p$, the saturated de Rham-Witt complex

$$\mathcal{W}\Omega_X^\bullet = (\mathcal{W}\Omega_X^0 \xrightarrow{d} \mathcal{W}\Omega_X^1 \xrightarrow{d} \cdots)$$

a (strictly commutative) dga on X_{zar} (over $W(k)$ if X/k, k as before), with F, V satisfying same formulas as for the (classical) dRW complex, except that:

$$\mathcal{W}\Omega_X^0 = W(\mathcal{W}\Omega_X^0/V\mathcal{W}\Omega_X^0),$$

and adjunction map

$$\mathcal{O}_X \to \mathcal{W}\Omega_X^0/V\mathcal{W}\Omega_X^0$$

not necessarily an isomorphism.

Basic properties

In what follows, we assume X/k, k as before. First of all,

$$\mathcal{W}\Omega^\bullet_X = \varprojlim \mathcal{W}_n\Omega^\bullet_X,$$

with

$$\mathcal{W}_n\Omega^\bullet_X := \mathcal{W}\Omega^\bullet_X/(V^n + dV^n),$$

and $\mathcal{W}_n\Omega^i_X$ quasi-coherent over $W_n(X)$, and compatible with étale localization. For $U = \mathrm{Spec}(R)$ open in X,

$$\Gamma(U, \mathcal{W}_n\Omega^i) = \mathcal{W}_n\Omega^i_R.$$

The inverse system

$$\mathcal{W}_\bullet\Omega^\bullet_X = ((\mathcal{W}_n\Omega^\bullet_X)_{n\geq 1}, F, V)$$

is a Langer-Zink F-V-pro-complex over $W_\bullet\mathcal{O}_X$, hence a canonical map (of F-V-pro-complexes)

$$can : W_\bullet\Omega^\bullet_X \to \mathcal{W}_\bullet\Omega^\bullet_X.$$

Theorem 2. (BLM). For X/k smooth, the canonical map

$$can : W_\bullet\Omega^\bullet_X \to \mathcal{W}_\bullet\Omega^\bullet_X$$

is an isomorphism, hence so is the resulting map

$$can : W\Omega^\bullet_X \to \mathcal{W}\Omega^\bullet_X.$$

Proof. Formal consequence of known structure of $W\Omega^\bullet_R$ for $R = \mathbf{F}_p[t_1, \cdots, t_r]$.

Bonus.

• Independent, budget proofs for main results of I. on $W\Omega^\bullet_X$ in smooth case, including comparison with crystalline cohomology ("laborious calculations" of I. avoided), and Ogus' key lemma in proof of $\mathrm{Ntw}_n \geq \mathrm{Hdg}_n$, namely $\varphi : Ru_*\mathcal{O}_{X/W} \to Ru_*\mathcal{O}_{X/W}$ induces isomorphism $Ru_*\mathcal{O}_{X/W} \xrightarrow{\sim} L\eta_p Ru_*\mathcal{O}_{X/W}$.

Bonus (cont'd)

• Abstract formulation of I.-Katz-Raynaud's reconstruction of $W_\bullet\Omega^\bullet_X$ from $Ru_*\mathcal{O}_{X/W_\bullet}$ (in smooth case), in terms of a general fixed point theorem for $L\eta_p$.

New features

(a)

$$\mathcal{W}\Omega^\bullet_X \xrightarrow{\sim} \mathcal{W}\Omega^\bullet_{X_{\mathrm{red}}}.$$

More generally: if $X \to Y$ is a morphism of k-schemes which is a universal homeomorphism with trivial residue extensions,

$$\mathcal{W}\Omega^\bullet_Y \to \mathcal{W}\Omega^\bullet_X$$

is an isomorphism. In particular, if R^{sn} is the Swan seminormalization of an $\mathbf{F}_p$-algebra R, then

$$\mathcal{W}\Omega^\bullet_R \to \mathcal{W}\Omega^\bullet_{R^{\mathrm{sn}}}$$

is an isomorphism, which gives the following formula for R^{sn}:

$$R^{\mathrm{sn}} = \mathcal{W}\Omega^0_R / V\mathcal{W}\Omega^0_R.$$

(b) If $X \to Y$ is a universal homeomorphism (with possibly non-trivial residue extensions), then

$$\mathcal{W}\Omega^\bullet_Y \otimes K \to \mathcal{W}\Omega^\bullet_X \otimes K$$

is an isomorphism. In particular, the Frobenius endomorphism $\varphi : \mathcal{W}\Omega^\bullet_X \to \mathcal{W}\Omega^\bullet_X$ is an isogeny, i.e.

$$\varphi \otimes K : \mathcal{W}\Omega^\bullet_X \otimes K \to \mathcal{W}\Omega^\bullet_X \otimes K$$

is an isomorphism. More precisely, if X/k is of finite type, and there is an integer d such that $\Omega^1_{X/k}$ is generated by at most d elements, then

$$\mathcal{W}\Omega^i_X = 0$$

for $i > d$. As in the smooth case (but $d > \dim(X)$ if X is singular), this implies that $F : \mathcal{W}\Omega^d_X \to \mathcal{W}\Omega^d_X$ is bijective, hence $v : \mathcal{W}\Omega^\bullet \to \mathcal{W}\Omega^\bullet_X$ defined by $p^{d-1-i}V$ in degree i (and F^{-1} for $i = d$) satisfies

$$\varphi v = v\varphi = p^d.$$

Remark. In contrast with the simplicity of the proofs of the general properties of $\mathcal{W}\Omega^\bullet$, the results in (a) and (b) require delicate arguments of commutative (and homological) algebra.

In particular, the proof of (b) uses the theory of derived de Rham-Witt complex $LW\Omega^\bullet_X$ and its (curious) relation with the saturated one (which, roughly, says that $\mathcal{W}\Omega^\bullet_X$ is the derived p-completion of the saturation of $LW\Omega^\bullet_X$).

Questions and prospects

(1) The finiteness problem.

For the classical dRW complex, the canonical map $\Omega^\bullet_{W_n\mathcal{O}_X} \to W_n\Omega^\bullet_X$ ($n \geq 1$) is surjective, in particular, for X/k of finite type, $W_n\Omega^i_X$ is coherent over $W_n(X)$ for all i.
The similar map

$$\Omega^\bullet_{W_n\mathcal{O}_X} \to \mathcal{W}_n\Omega^\bullet_X$$

is not surjective in general, as (a) above shows (e.g. if $R = k[x,y]/(x^2 - y^3)$, $\mathcal{W}_1\Omega^0_R = k[t]$). That raises:

Question 1. For X/k of finite type, is $\mathcal{W}_1\Omega^i_X$ coherent on X?

Remarks.

(i) One can show that if $\mathcal{W}_1\Omega^i_X$ is coherent on X for all i, then $\mathcal{W}_n\Omega^i_X$ is coherent on $W_n(X)$ for all i.

(ii) Let

$$R = W(k)_\sigma[F, V; d]/(FV = VF = p, FdV = d, d^2 = 0) = R^0 \oplus R^1$$

be the Raynaud ring of k, a non-commutative graded ring, where $R^0 = W_\sigma[F, V]/(FV = VF = p)$ is the Cartier-Dieudonné ring, and d is placed in degree 1. (Left) graded modules M over R correspond to complexes M of W-modules, where each component M^i is equipped with (semi-linear) operators F, V, satisfying

$$FV = VF = p,\ FdV = d,$$

e.g., a saturated Dieudonné complex (over W) is a graded R-module.

Derived category $D(R)$ extensively studied by I.-Raynaud, Ekedahl, especially the full subcategory $D^b_c(R)$ of $D^b(R)$ consisting of complexes of (graded) R-modules M, with coherent cohomology complexes $H^i(M)$, characterized by the fact that

$$M \to R\varprojlim R/(V^nR + dV^nR) \otimes^L_R M$$

is an isomorphism and each $H^i(R/(VR + dVR) \otimes^L_R M)$ (a complex of k-vector spaces) has finite dimensional cohomology.

Can show (I.): if X/k is proper and Question 1 has a positive answer for X, then one has $R\Gamma(X, \mathcal{W}\Omega^\bullet) \in D^b_c(R)$, and, in particular (by I.-Raynaud, Ekedahl):

- $H^*(X, \mathcal{W}\Omega^\bullet)$ is finitely generated over W.

- As in the smooth case, the slope spectral sequence

$$E_1^{ij} = H^j(X, \mathcal{W}\Omega^i) \Rightarrow H^{i+j}(X, \mathcal{W}\Omega^\bullet)$$

degenerates at E_1 modulo torsion, $H^j(X, \mathcal{W}\Omega^i)/H^j(X, \mathcal{W}\Omega^i)[p^\infty]$ is finitely generated over W, with V topologically nilpotent, $H^j(X, \mathcal{W}\Omega^i)[p^\infty]$ is killed by a power of p, and the degeneration induces an isomorphism

$$H^j(X, \mathcal{W}\Omega^i) \otimes K \xrightarrow{\sim} (H^{i+j}(X, \mathcal{W}\Omega^\bullet) \otimes K)_{[i,i+1)}.$$

In particular, we have

$$H^j(X, \mathcal{W}\Omega^0) \otimes K \xrightarrow{\sim} (H^j(X, \mathcal{W}\Omega^\bullet) \otimes K)_{[0,1)}$$

Status of Question 1 (finiteness problem)

Positive answer known in the following cases:

(a) (I.) X has normal crossing singularities, i.e. is locally smooth over $Y = \mathrm{Spec}(k[t_1, \cdots, t_r]/(t_1 \cdots t_r))$.
If $D_i = V(t_i) \in \mathrm{Spec}(W[t_1, \cdots, t_r])$, then $\mathcal{W}\Omega^\bullet_Y$ is a du Bois type complex:

$$\mathcal{W}\Omega^\bullet_Y = \mathrm{Ker}(\oplus_i W\Omega^\bullet_{D_i/W} \to \oplus_{i<j} W\Omega^\bullet_{D_i \cap D_j)/W})$$

(and similarly for $\mathcal{W}_n\Omega^\bullet_Y$). In particular,

$$\mathcal{W}_1\Omega^\bullet_Y = \mathrm{Ker}(\oplus_i \Omega^\bullet_{D_i \otimes k/k} \to \oplus_{i<j} \Omega^\bullet_{D_i \cap D_j) \otimes k/k}).$$

(b) (I.) X/k is a curve. Follows from invariance of $\mathcal{W}\Omega^\bullet$ by passing to the seminormalization ($\mathcal{W}\Omega^\bullet_R \xrightarrow{\sim} \mathcal{W}\Omega^\bullet_{R^{\mathrm{sn}}}$) and local calculation in seminormal case, using canonical factorization

$$X^n \to X^{sn} \to X$$

where X^n (resp. X^{sn}) is the normalization (resp. seminormalization) of X.

(c) (Ogus, work in progress) X/k is an affine, toric scheme, i.e., $X = \mathrm{Spec}(k[P])$ where P is a fine, saturated, torsion free monoid.

Proof uses lifting of Frobenius on $W[P]$, $F = a \mapsto pa$ on P.

Hope: affine, toric can be relaxed to toroidal singularities.

(d) First unknown cases: conic singularities not of the above type, e. g. for $p > 2$, $\sum_{1 \leq i \leq 5} x_i^2 = 0$.

(2) Comparison with rigid cohomology

Let X/k be proper. Recall Berthelot's rigid cohomology $H^*_{\mathrm{rig}}(X/K)$ is finite dimensional over K, and has slope decomposition

$$H^n_{\mathrm{rig}}(X/K) = \oplus_{i \in \mathbf{Z}} H^n_{\mathrm{rig}}(X/K)_{[i,i+1)}.$$

Question 2. Can we construct a (functorial, φ-compatible) isomorphism

$$H^*_{\mathrm{rig}}(X/K) \xrightarrow{\sim} H^*(X, \mathcal{W}\Omega^\bullet) \otimes K?$$

Remarks. (i) A positive answer would imply that

$$\varphi : R\Gamma(X, \mathcal{W}\Omega^\bullet) \to R\Gamma(X, \mathcal{W}\Omega^\bullet)$$

is an isogeny. This is a theorem by BLM: in fact,

$$\varphi \otimes K : \mathcal{W}\Omega^\bullet_X \otimes K \to \mathcal{W}\Omega^\bullet_X \otimes K$$

is already an isomorphism (see slide 46).

(ii) By a result of Berthelot-Bloch-Esnault, there is a (φ-compatible) canonical isomorphism

$$H^*_{\mathrm{rig}}(X/K)_{[0,1)} \to H^*(X, W\mathcal{O}) \otimes K.$$

By BLM, $\mathcal{W}\Omega^0_X = W(\mathcal{O}^{sn}_X)$, and, as $\mathcal{O}_X \to \mathcal{O}^{sn}_X$ is a universal homeomorphism,

$$H^*(X, W\mathcal{O}) \otimes K \to H^*(X, W(\mathcal{O}^{sn})) \otimes K$$

is an isomorphism (by elementary properties of $H^*(W\mathcal{O})$). Therefore, when Question 1 (finiteness) has a positive answer, hence (see slide 46)

$$H^j(X, \mathcal{W}\Omega^0) \otimes K \xrightarrow{\sim} (H^j(X, \mathcal{W}\Omega^\bullet) \otimes K)_{[0,1)},$$

a positive answer to Question 2 yields Berthelot-Bloch-Esnault's result.

Status of Question 2.

Positive answer known only in cases (a) (snc singularities) and (b) (curve) above, thanks to du Bois type property of $\mathcal{W}\Omega^\bullet$, and Tsuzuki's proper cohomological descent for $H^*_{\mathrm{rig}}(X/K)$.

Remark. $H^*(X, \mathcal{W}\Omega^\bullet)$ does not satisfy proper cohomological descent (as Frobenius shows), and does not satisfy cdh descent either (as an example of Bhatt-Lurie shows [1]).

[1] (Added on Sept. 10, 2021) This example doesn't quite show that, see ([L. Illusie, A new approach to de Rham-Witt complexes, after Bhatt-Lurie-Mathew, to appear in Rend. Sem. Math. Univ. Padova.], 6.1.3).

Further developments and problems

- (Zijian Yao) Log variants of BLM constructed.
- Relative variants (comparison with Langer-Zink)? Variants with F-crystal coefficients? Relation with Ekedahl's theory of F-gauges?
- Relation with prismatic cohomology?

3. Liftings mod p^2, dRW and derived dR complexes

1. Review of Deligne-I. and statement of main result

k: perfect field, $\mathrm{char}(k) = p > 0$, $W = W(k)$, $W_n = W_n(k)$
For X/k, relative Frobenius F, sitting in

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l] \arrow[d] & X \arrow[l, "F"'] \arrow[dl] \\ \mathrm{Spec}(k) & \mathrm{Spec}(k) \arrow[l, "F_k"'] & \end{tikzcd}$$

$$F_*\Omega^\bullet_{X/k} \in D^b(X', \mathcal{O}_{X'}).$$

For X/k smooth, Cartier isomorphism

$$C^{-1} : \Omega^i_{X'/k} \xrightarrow{\sim} \mathcal{H}^i F_*\Omega^\bullet_{X/k}.$$

Theorem 1 (Deligne-I.). Assume X/k smooth and $\dim(X) < p$. With any smooth lifting of X to W_2 is associated a decomposition in $D(X')$ $(= D(X', \mathcal{O}_{X'}))$:

$$\oplus \Omega^i_{X'}[-i] \xrightarrow{\sim} F_* \Omega^\bullet_X$$

($/k$ omitted for brevity), inducing C^{-1} on $\mathcal{H}^i$.

Recall: implies various Hodge degeneration and Kodaira vanishing theorems.

Thanks to the multiplicative structures on both sides, Th. 1 follows from:

Theorem 1' (Deligne-I.). Assume X/k smooth. With any smooth lifting of X to W_2 is associated a decomposition in $D(X')$:

$$\mathcal{O}_{X'} \oplus \Omega^1_{X'}[-1] \xrightarrow{\sim} \tau_{\leq 1} F_* \Omega^\bullet_X,$$

where $\tau_{\leq i}$ denotes a canonical truncation.

Goal of this talk: sketch a proof of the following stronger result:

Theorem 2 (I., 2019). For any smooth X/k there is a canonical, functorial isomorphism in $D(X')$:

$$(\tau_{\geq -1} L_{X'/W_2})[-1] \xrightarrow{\sim} \tau_{\leq 1} F_* \Omega^\bullet_X,$$

where $\tau_{\geq i}$ denotes a canonical truncation, and L_{X'/W_2} is the cotangent complex of X'/W_2.

Proof of Th. 2 $\Rightarrow$ Th. 1'.

Let $\widetilde{X}$ be a smooth lifting of X to W_2. Then

$$L_{X/W_2} = L_{X/k} \oplus L_{X/\widetilde{X}}.$$

But

$$L_{X/k} = \Omega^1_X, \ \tau_{\geq -1} L_{X/\widetilde{X}} = \mathcal{O}_X[1].$$

Remarks. (a) A generalization of Th. 2 to the prismatic set-up has recently been obtained (independently) by Bhatt, private communication. See comments at the end.

(b) If $L = (L^0 \to L^1)$ is a complex of $\mathcal{O}$-modules over a locally ringed space, with $\mathcal{H}^1(L)$ locally free of finite type, a splitting of L is a section of $L^1 \twoheadrightarrow \mathcal{H}^1(L)$. Splittings exist locally, and two splittings s_1, s_2 are locally related by an $h : \mathcal{H}^1(L) \to L^0$ such that $s_2 - s_1 = dh$. The sheaf of automorphisms of a splitting s is isomorphic to $\mathcal{H}^0(L)$.

Local splittings of L form a gerbe

$$\mathrm{Split}(L).$$

Similarly, smooth liftings of X to W_2 exist locally, and two such are locally isomorphic, the sheaf of automorphisms of a fixed lifting is the tangent sheaf T_X. Local liftings form a gerbe

$$\mathrm{Lift}(X/W_2).$$

Th. 2 implies the $S = \mathrm{Spec}(k)$ special case of a theorem of Deligne-I:

Theorem 1''. There is a natural equivalence of gerbes:

$$\mathrm{Lift}(X'/W_2) \xrightarrow{\sim} \mathrm{Split}(\tau_{\leq 1} F_* \Omega^\bullet_X).$$

(Note: here $\mathrm{Lift}(X'/W_2) \xrightarrow{\sim} \mathrm{Lift}(X/W_2)$.)

Indeed, elementary theory of cotangent complex and deformations yield:

$$\mathrm{Lift}(X/W_2) \xrightarrow{\sim} \mathrm{Split}(\tau_{\geq -1} L_{X/W_2})$$

$$\widetilde{U} \mapsto (\tau_{\geq -1} L_{U/W_2} \xrightarrow{\sim} L_{U/k} \oplus \tau_{\geq -1} L_{U/\widetilde{U}} = \Omega^1_U \oplus \mathcal{O}_U[1])$$

2. Sketch of proof of Th. 2

Preliminary observation: $L_{X/W}$ is a perfect complex, of perfect amplitude in $[-1, 0]$, and

$$L_{X/W} \xrightarrow{\sim} \tau_{\geq -1} L_{X/W_2}.$$

If $X \hookrightarrow Y$ is a closed embedding in Y/W smooth, with ideal J, then

$$L_{X/W} \xrightarrow{\sim} (J/J^2 \to \mathcal{O}_X \otimes \Omega^1_Y)$$

(placed in degrees -1, 0).

Main ingredient of proof: the de Rham-Witt complex $W\Omega^\bullet_X$ and its Nygaard filtration.

A priori, no dRW complex in sight. How can it come on the scene?

It comes as a by-product of the construction of the comparison map from crystalline cohomology to de Rham-Witt.

For simplicity, assume there exists a closed embedding $X \hookrightarrow Y$, with Y/W smooth, and endowed with a lifting F of Frobenius. By Cartier, there exists a unique F-compatible section

$$s_F : \mathcal{O}_Y \to W(\mathcal{O}_Y)$$

of the canonical projection. Composing with $W(\mathcal{O}_Y) \to W(\mathcal{O}_X)$, get a map $\Omega^\bullet_Y \to \Omega^\bullet_{W(\mathcal{O}_X)/W}$, hence (composing with $\Omega^\bullet_{W(\mathcal{O}_X)/W} \to W\Omega^\bullet_X$) a map (of dga)

$$c : \Omega^\bullet_Y \to W\Omega^\bullet_X,$$

which, in degree 0, is $\mathcal{O}_Y \to W\mathcal{O}_X$, sending $J \subset \mathcal{O}_Y$ into $VW\mathcal{O}_X \subset W\Omega^\bullet_X$.

Therefore, for $r \geq 0$,

$$c(J^r\Omega^\bullet_Y) \subset \mathcal{N}^r W\Omega^\bullet_X;$$

Here:

$$\Omega^\bullet_Y = J^0\Omega^\bullet_Y \supset J\Omega^\bullet_Y \supset \cdots \supset J^r\Omega^\bullet_Y \supset \cdots$$

is the J-adic filtration of $\Omega^\bullet_Y$, defined by

$$J^r\Omega^\bullet_Y = (J^r \xrightarrow{d} J^{r-1}\Omega^1_Y \to \cdots \xrightarrow{d} \Omega^r_Y \xrightarrow{d} \Omega^{r+1}_Y \xrightarrow{d} \cdots),$$

and

$$W\Omega^\bullet_X = \mathcal{N}^0 W\Omega^\bullet_X \supset \mathcal{N}^1 W\Omega^\bullet_X \supset \cdots \supset \mathcal{N}^r W\Omega^\bullet_X \supset \cdots$$

is the Nygaard filtration defined, for $r \geq 1$, by

$$\mathcal{N}^r W\Omega_X^\bullet = (p^{r-1}VW\mathcal{O}_X \xrightarrow{d} p^{r-2}VW\Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} VW\Omega_X^{r-1} \xrightarrow{d} W\Omega_X^r$$
$$\xrightarrow{d} W\Omega_X^{r+1} \xrightarrow{d} \cdots).$$

Thus, c induces a map

$$\mathrm{gr}_J^r \Omega_Y^\bullet \to \mathrm{gr}_{\mathcal{N}}^r W\Omega_X^\bullet.$$

For $r = 1$,

$$\mathrm{gr}_J^1 \Omega_Y^\bullet = (J/J^2 \xrightarrow{d} \mathcal{O}_X \otimes \Omega_Y^1),$$

with J/J^2 placed in degree 0. Composing with the canonical isomorphism $L_{X/W} \xrightarrow{\sim} (J/J^2 \to \mathcal{O}_X \otimes \Omega_Y^1)$ recalled above, we get a map (in $D(X)$)

(1) $$L_{X/W}[-1] \to \mathrm{gr}_{\mathcal{N}}^1 W\Omega_X^\bullet,$$

which a diagonal argument shows to be independent of the choice of the embedding in (Y, F). What is the RHS?

For $r \geq 1$, consider the map

$$\mathcal{N}^r W\Omega^i_{X'} \to F_* \tau_{\leq r} \Omega^\bullet_X$$

sending $p^{r-1-i} Vx$ to x for $i \leq r-1$, Fx for $i = r$ (and 0 for $i > r$). It induces a map (of complexes of $\mathcal{O}_{X'}$-modules)

$$\mathrm{gr}^r_{\mathcal{N}} W\Omega^\bullet_{X'} \to \tau_{\leq r} F_* \Omega^\bullet_X. \tag{2}$$

A basic result is:

Lemma (Nygaard, 1981). The map (2) is a quasi-isomorphism.

Composing (2) with (1) (for X')

$$L_{X'/W}[-1] \to \mathrm{gr}^1_{\mathcal{N}} W\Omega^\bullet_{X'}, \tag{1}$$

and recalling the isomorphism $L_{X'/W} \xrightarrow{\sim} \tau_{\geq -1} L_{X'/W_2}$, we get the map announced in Th. 2:

$$(\tau_{\geq -1} L_{X'/W_2})[-1] \to \tau_{\leq 1} F_* \Omega^\bullet_X. \tag{3}$$

To show that

$$(\tau_{\geq -1} L_{X'/W_2})[-1] \to \tau_{\leq 1} F_* \Omega^\bullet_X. \tag{3}$$

is an isomorphism, we may assume that X has a formal smooth lifting (Y, F) over W. Then (3) boils down to the map

$$\mathcal{O}_{X'} \oplus \Omega^1_{X'}[-1] \to F_*(\mathcal{O}_X \xrightarrow{d} Z\Omega^1_X)$$

induced by $F : Y \to Y$, which is a quasi-isomorphism, inducing the Cartier isomorphism C^{-1} on $\mathcal{H}^*$.

This disposes of the case where there exists a closed embedding $X \hookrightarrow Y$, with Y/W smooth, and endowed with a lifting F of Frobenius. The general case is reduced to this one by cohomological descent for an open Zariski cover of X.

Remarks. (a) Let M be a saturated Dieudonné complex in the sense of BLM: M is a complex of abelian groups, endowed with $F : M^i \to M^i$ satisfying $dF = pFd$, and such that M is p-torsion free, and

$$p^\bullet F : M \to \eta_p M$$

is an isomorphism (saturation condition). Then M^i is endowed with V such that $VF = FV = p$ and $FdV = d$. The Nygaard filtration

$$M \supset \cdots \supset \mathcal{N}^r M \supset \mathcal{N}^{r+1} M \supset \cdots$$

is defined, for $r \in \mathbf{Z}$, similarly to the case of $W\Omega^\bullet_X$, by $\mathcal{N}^r M^i = p^{r-1-i} V M^i$ for $i < r$ and $\mathcal{N}^r M^i = M^i$ for $i \geq r$. We have

$$\mathcal{N}^r M = (p^\bullet F)^{-1}(p^r M \cap \eta_p M)$$

Then, as $W\Omega^\bullet_X / pW\Omega^\bullet_X \to \Omega^\bullet_X$ is a quasi-isomorphism, the following (easy lemma) generalizes Nygaard's lemma:

Lemma (BLM). The map $p^\bullet F$ induces an isomorphism

$$\mathrm{gr}^r_{\mathcal{N}} M \xrightarrow{\sim} \tau_{\leq r}(M/pM).$$

(b) The Nygaard filtration has deep relations (see BMS2, BS3) with

- topological Hochschild homology,
- integral p-adic Hodge theory,
- prismatic cohomology.

3. Variants and generalizations

1. Let S be a k-scheme, $\widetilde{S}$ a flat lifting of S to W_2, X/S smooth, $X' = X \times_{(S,F_S)} S$, $F : X \to X'$ the relative Frobenius. The following generalization of Th. 1'' is proved in Deligne-I.:

Theorem 3.1. There is a natural equivalence of gerbes:

$$\mathrm{Lift}(X'/\widetilde{S}) \xrightarrow{\sim} \mathrm{Split}(\tau_{\leq 1} F_* \Omega^\bullet_X).$$

(Note: here $\mathrm{Lift}(X/\widetilde{S})$ and $\mathrm{Lift}(X'/\widetilde{S})$ are in general not equivalent.)

Implies a canonical isomorphism

$$(\tau_{\geq -1} L_{X'/\widetilde{S}})[-1] \xrightarrow{\sim} \tau_{\leq 1} F_* \Omega^\bullet_{X/S}.$$

An independent direct proof can be given, though no dRW complex is available.

Log variants of the above for log smooth morphisms of Cartier type (Kato, 1989).

2. Prismatic variant (Bhatt)

Let $(T = \mathrm{Spec}(A), S : V(I) \subset T, \delta)$ be a prism. By definition:

• (A, δ) is a δ-ring, with associated Frobenius lift $\varphi : a \mapsto a^p + p\delta(a)$

• I is an ideal in A defining a Cartier divisor S in T

• A is derived (I, p)-complete (e.g. p-complete and f-complete if $I = (f)$)

• $S \cap \varphi^{-1}(S) \subset V(p)$.

Assume in addition (T, I, δ) bounded, i.e., A/I has bounded p^∞-torsion.

Let X/S formally smooth. Let $(X/T)^{\Delta}$ be the prismatic site, and

$$\nu : (X/T)^{\Delta} \to X_{\mathrm{zar}}$$

be the canonical projection (similar to the Berthelot map $u : (X/W)_{\mathrm{crys}} \to X_{zar}$). Then:

Theorem 3.2 (Bhatt). There is a canonical isomorphism (in $D(X_{\mathrm{zar}}, \mathcal{O}_X)$

$$L_{X/T}[-1] \xrightarrow{\sim} \tau_{\leq 1}(R\nu_*(\mathcal{O}^{\Delta}_{X/T}) \otimes^L_A A/I) \otimes (I/I^2)$$

Recall 3.1. Th. 2 (I., 2019). For any smooth X/k there is a canonical, functorial isomorphism in $D(X')$:

$$(\tau_{\geq -1} L_{X'/W_2})[-1] \xrightarrow{\sim} \tau_{\leq 1} F_* \Omega^{\bullet}_X,$$

where $\tau_{\geq i}$ denotes a canonical truncation, and L_{X'/W_2} is the cotangent complex of X'/W_2.

Remark. Th. 3.2 (Bhatt) $\Rightarrow$ Th. 2:

Take $A = W(k)$, $I = (p)$, $\varphi(a) = \sigma^*(a)$. Then $A/I = k$. Use crystalline comparison theorem (BS):

$$\sigma^* R\nu_* \mathcal{O}^{\Delta}_{X/T} \xrightarrow{\sim} Ru_* \mathcal{O}_{X/W},$$

hence

$$R\nu_* \mathcal{O}^{\Delta}_{X'/T} \otimes^L_A A/I \xrightarrow{\sim} F_* \Omega^\bullet_{X/k}.$$

and

$$L_{X/T}[-1] \xrightarrow{\sim} \tau_{\leq 1}(R\nu_*(\mathcal{O}^{\Delta}_{X/T}) \otimes^L_A A/I) \otimes (I/I^2)$$

gives

$$(\tau_{\geq -1} L_{X'/W_2})[-1] = L_{X'/W}[-1] \xrightarrow{\sim} \tau_{\leq 1} F_* \Omega^\bullet_{X/k}.$$

Question: Common generalization of Th. 3.1 and Th. 3.2 ?

4. Inputs from derived de Rham complexes

Review of $L\Omega^\bullet$.

For an A-algebra R,

$$L\Omega^\bullet_{R/A} := \mathrm{Tot}(\Omega^\bullet_{P_\bullet(R)/A}),$$

where $P_\bullet(R)$ = standard simplicial resolution of R/A by polynomial algebras.

Comes with Hodge filtration $\mathrm{Fil}^i_{\mathrm{Hdg}} L\Omega^\bullet_{R/A}$ deduced from $\Omega^{\geq i}$, with

$$\mathrm{gr}^i L\Omega^\bullet_{R/A} \xrightarrow{\sim} L\Omega^i_{R/A}[-i] (:= L\Lambda^i L_{R/A}[-i]).$$

Globalizes on schemes: $L\Omega^\bullet_{X/S}$, $\mathrm{Fil}^i_{\mathrm{Hdg}}$, $\mathrm{gr}^i = L\Omega^i[-i]$.

Back to the Nygaard filtration

Theorem 4.1 (I., 2019). Let X/k be smooth. There exists a canonical filtered isomorphism:

$$c : L\Omega^{\bullet}_{X/W}/\mathrm{Fil}^{p}_{\mathrm{Hdg}} \xrightarrow{\sim} W\Omega^{\bullet}_{X}/\mathcal{N}^{p},$$

where $\mathcal{N}^{i} = \mathcal{N}^{i}W\Omega^{\bullet}_{X}$ is the Nygaard filtration, with filtrations induced by the Hodge and the Nygaard filtration.

Corollary 4.2. For $i < p$,

$$\mathrm{gr}^{i} L\Omega^{\bullet}_{X'/W} \xrightarrow{\sim} \tau_{\leq i} F_{*}\Omega^{\bullet}_{X}.$$

($i = 1$: Cor 4.2 = Th. 2 (I.))

Corollary 4.3. A lifting $\widetilde{X}$ of X to W_2 gives a DI-decomposition

$$\oplus_i \Omega^i_{X'}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_X.$$

Apply Cor. 4.2 for $i = p - 1$, using the decomposition

$$L_{X/W} = \mathcal{O}_X[1] \oplus \Omega^1_X$$

given by $\widetilde{X}$.

Remarks. (a) The isomorphism

$$c : L\Omega^\bullet_{X/W}/\mathrm{Fil}^p_{\mathrm{Hdg}} \xrightarrow{\sim} W\Omega^\bullet_X/\mathcal{N}^p,$$

of 4.1 does not extend to an isomorphism

$$L\Omega^\bullet_{X/W} \xrightarrow{\sim} W\Omega^\bullet_X.$$

Example (Bhatt). Take $X = \mathrm{Spec}(k)$. Then

$$R\varprojlim_n (L\Omega^\bullet_{k/W_n}) \xrightarrow{\sim} (\varprojlim_n W_n\langle x\rangle)/(x-p),$$

where $W_n\langle x\rangle$ means the PD-algebra on x. RHS has p-torsion (e.g. $(x-p)^{[p]}$).

(b) Generalization to prisms.

Theorem (Bhatt) Notation as in Th. 3.2:
$(T = \mathrm{Spec}(A), S = \mathrm{Spec}(A/I), \varphi : A \to A)$ a prism, X/S formal smooth, $\nu : (X/T)^{\Delta} \to X_{\mathrm{zar}}$ the canonical projection.

There exists a canonical filtered isomorphism

$$L\Omega^{\bullet}_{X/T}/\mathrm{Fil}^{p}_{\mathrm{Hdg}} \xrightarrow{\sim} (\varphi^* R\nu_* \mathcal{O}^{\Delta}_{X/T})/\mathcal{N}^p$$

Here $\mathcal{N}^i$ is the Nygaard filtration defined in such a way that the basic isomorphism of prismatic cohomology, factorizing φ:

$$\widetilde{\varphi} : \varphi^* R\nu_* \mathcal{O}^{\Delta}_{X/T} \xrightarrow{\sim} L\eta_I R\nu_* \mathcal{O}^{\Delta}_{X/T}$$

be a filtered isomorphism, where the RHS is equipped with the filtration induced by the I-adic filtration $(I^r \cap L\eta_I)$.

Remark. In the case $(A, I) = (W, p)$ one recovers the isomorphism c of 4.1. Indeed, in this case the basic isomorphism $\widetilde{\varphi}$ boils down to the isomorphism

$$p^{\bullet}F : W\Omega^{\bullet}_X \to \eta_p W\Omega^{\bullet}_X,$$

expressing the saturation of dRW, and (cf. Remark p. 64) $\mathcal{N}^r$ becomes the Nygaard filtration previously defined.

Lci variants

For X/k, replace smooth by locally complete intersection.
Partial results by Bhatt:
decompositions (in presence of liftings), and partial degeneration of Hodge to de Rham spectral sequences, both in char. $p > 0$ and in char. 0.
Work in progress.