Around the Thom-Sebastiani theorem

Luc Illusie[1]

with an appendix by Weizhe Zheng[2]

Abstract. For germs of holomorphic functions $f : (\mathbf{C}^{m+1}, 0) \to (\mathbf{C}, 0)$, $g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ having an isolated critical point at 0 with value 0, the classical Thom-Sebastiani theorem describes the vanishing cycles group $\Phi^{m+n+1}(f \oplus g)$ (and its monodromy) as a tensor product $\Phi^m(f) \otimes \Phi^n(g)$, where $(f \oplus g)(x, y) = f(x) + g(y), x = (x_0, ..., x_m), y = (y_0, ..., y_n)$. We prove algebraic variants and generalizations of this result in étale cohomology over fields of any characteristic, where the tensor product is replaced by a certain local convolution product, as suggested by Deligne. They generalize [7]. The main ingredient is a Künneth formula for $R\Psi$ in the framework of Deligne's theory of nearby cycles over general bases. In the last section, we study the tame case, and the relations between tensor and convolution products, in both global and local situations.

Key words and phrases: Étale cohomology, nearby cycles, vanishing cycles, isolated singularity, Milnor fiber, monodromy, variation, Künneth formula, additive convolution, tame ramification, Jacobi sum.

Mathematics Subject Classification (2010): Primary: 14F20. Secondary: 11T23, 18F10, 32S30, 32S40.

Contents

0. Introduction

If $f : (\mathbf{C}^{m+1}, 0) \to (\mathbf{C}, 0)$, $g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are germs of holomorphic functions having 0 as an isolated critical point with value 0, the germ $f \oplus g : (\mathbf{C}^{m+n+2}, 0) \to (\mathbf{C}, 0)$ defined by $(f \oplus g)(x, y) = f(x) + g(y)$ has also 0 as an isolated critical point, and the classical Thom-Sebastiani theorem [30] expresses its group of vanishing cycles at 0 as a tensor product:

$$\Phi^m(f) \otimes \Phi^n(g) \xrightarrow{\sim} \Phi^{m+n+1}(f \oplus g). \tag{0.1}$$

Here, if $h : (\mathbf{C}^r, 0) \to (\mathbf{C}, 0)$ is a germ of holomorphic function having an isolated critical point at 0, $\Phi^q(h) := R^q\Phi_h(\mathbf{Z})_0$ is the stalk at $0 \in \mathbf{C}^r$ of $H^q R\Phi_h(\mathbf{Z})$, where $R\Phi_h$ is the vanishing cycles functor of ([35], XIV); this is also $\widetilde{H}^q(M_h, \mathbf{Z})$, where M_h is a Milnor fiber of h at 0, and $\widetilde{H}^q = \operatorname{Coker} H^q(\mathrm{pt}) \to H^q$. The

[1]Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France; email: `Luc.Illusie@math.u-psud.fr`.

[2]Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; University of the Chinese Academy of Sciences, Beijing 100049, China; email: `wzheng@math.ac.cn`. Partially supported by National Natural Science Foundation of China Grant 11321101.

isomorphism (0.1) is compatible with the monodromy operators T_f, T_g, $T_{f\oplus g}$, i. e.

$$T_f \otimes T_g = T_{f\oplus g} \tag{0.2}$$

via (0.1). In ([1], [4]), Deligne gave a refinement of (0.2), with the monodromy operators replaced by the *variation* isomorphisms $V_f : R^m\Phi_f(\mathbf{Z})_0 \xrightarrow{\sim} H^m_{\{0\}}(R\Psi_f(\mathbf{Z}))$ (and similarly for g and $f \oplus g$), in the notation of ([35], XIV), where the group $H^m_{\{0\}}(R\Psi_f(\mathbf{Z}))$, which is dual to $R^m\Phi_f(\mathbf{Z})_0$, is also isomorphic to $H^m_c(M_f - \partial M_f, \mathbf{Z})$, ∂M_f denoting the boundary of M_f, namely,

$$V_f \otimes V_g = V_{f\oplus g} \tag{0.3}$$

via (0.1) and its dual. The above groups of vanishing cycles are of an algebraic nature, as it is known that they depend only on a suitable high order jet of the functions. However, the proofs of (0.2) and (0.3) are transcendental. They heavily rely on a description of the Milnor fiber $M_{f\oplus g}$ as homotopic to a join $M_f * M_g$.

It had been observed by Deligne long ago that, in positive characteristic, an ℓ-adic analogue of (0.1), compatible with Galois actions, could not hold, as could already be seen in the Picard-Lefschetz situation for quadratic singularities. He suggested that the tensor product on the left hand side should be replaced by a certain local convolution product. More precisely, consider the following set-up. Let k be an algebraically closed field of characteristic p. For $i = 1, 2$, let $f_i : X_i \to \mathbf{A}^1_k = \operatorname{Spec} k[t]$ a flat morphism of finite type with X_i smooth over k, of dimension $n_i + 1$. Let $n = n_1 + n_2$. Assume that f_i has an isolated critical point at a rational point x_i of the special fibre. Let $a : \mathbf{A}^1_k \times_k \mathbf{A}^1_k \to \mathbf{A}^1_k$ denote the sum map $(u, v) \mapsto u + v$. Let $f = f_1 \times_k f_2 : X_1 \times_k X_2 \to \mathbf{A}^1_k$. Then the composite morphism

$$af : X_1 \times_k X_2 \to \mathbf{A}^1_k$$

(i. e. $f_1 \oplus f_2$) has again an isolated critical point at the point $x = (x_1, x_2)$ of its special fiber. Let $A = \operatorname{Spec} k\{t\}$ be the henselization at the origin of $\mathbf{A}^1_k$, let $\overline{\eta}$ be a geometric point over the generic point η of A. Let ℓ be a prime number different from p. It is known that in this case the vanishing cycles group $R^q\Phi_{f_i}(\mathbf{Z}_\ell)_{x_i}$ (resp. $R^q\Phi_{af}(\mathbf{Z}_\ell)_x$) is zero for $q \neq n_i$ (resp. $q \neq n + 1$) and $R^{n_i}\Phi_{f_i}(\mathbf{Z}_\ell)_{x_i}$ (resp. $R^{n+1}\Phi_{sf}(\mathbf{Z}_\ell)_x$) is a free $\mathbf{Z}_\ell$-module of finite type (cf. ([35], I 4.6) for $p = 0$, ([10], 2.10) for the general case). Consider $R^{n_i}\Phi_{f_i}(\mathbf{Z}_\ell)_{x_i}$ as a sheaf on η, extended by zero on A, and the external tensor product on the henselization $A^2_{(0,0)}$ of $A \times_k A$ (or $\mathbf{A}^2_k$) at $(0, 0)$

$$M := R^{n_1}\Phi_{f_1}(\mathbf{Z}_\ell)_{x_1} \boxtimes R^{n_2}\Phi_{f_2}(\mathbf{Z}_\ell)_{x_2} = \mathrm{pr}_1^* R^{n_1}\Phi_{f_1}(\mathbf{Z}_\ell)_{x_1} \otimes \mathrm{pr}_2^* R^{n_2}\Phi_{f_2}(\mathbf{Z}_\ell)_{x_2}$$

Denote again by $a : A^2_{(0,0)} \to A$ the map induced by the sum map. We have $R^q\Phi_a(M)_{(0,0)} = 0$ for $q \neq 1$, and in his seminar [2], Deligne sketched a construction of an isomorphism

$$R^1\Phi_a(M)_{(0,0)} \xrightarrow{\sim} R^{n+1}\Phi_{af}(\mathbf{Z}_\ell)_x, \tag{0.4}$$

such that when k is an algebraic closure of a field k_0, (0.4) is compatible with the action of $\operatorname{Gal}(k/k_0)$. This is an analogue of (0.1), where the vanishing cycles group on the left hand side replaces the tensor product. His construction used a suitable compactification of a and f. It has not been written up. The functor associating to a pair (V_1, V_2) of $\overline{\mathbf{Q}}_\ell$-sheaves on η (i.e., $\mathrm{Gal}(\overline{\eta}/\eta)$-modules), considered as sheaves on A extended by zero, the sheaf

$$R^1\Phi_a(V_1 \boxtimes V_2)_{(0,0)} \tag{0.5}$$

(denoted $V_1 *_1 V_2$ (3.9.1) in our paper) was then extensively studied by Laumon [20] under the name of *local (additive) convolution*. Fu [7] revisited the question, and gave a proof of (0.4) (with $\mathbf{Z}_\ell$ replaced by $\overline{\mathbf{Q}}_\ell$) (and a slight generalization of it), using Laumon's local Fourier transform [20]. In [5] Deligne conjectured generalizations taking for coefficients objects of D^b_c and replacing the sum map a by other types of morphisms (see [7] for a precise statement), generalizations which seemed out of reach of the method of [7].

In this paper, we address Deligne's expectations. Here is the idea. By a basic result of Gabber (a key ingredient in [7]), $R\Psi$ commutes with external tensor products ([9], 4.7). However, the external products in question in *loc. cit.* are products $Y_1 \times_S Y_2$, for S a trait, and S-schemes Y_1, Y_2, while here we need external products of the form $f_1 \times_k f_2$, with target of dimension 2. In order to deal with such morphisms, we use Deligne's theory of nearby cycles over general bases, developed in [12] and [21].

For a morphism of finite type $f : X \to Y$ between noetherian schemes, and $K \in D^b_c(X, \Lambda)$, $\Lambda = \mathbf{Z}/n\mathbf{Z}$, $n \geq 1$ invertible on Y, the nearby cycles complex $R\Psi_f(K)$ is an object of $D^b(X \overleftarrow{\times}_Y Y, \Lambda)$, where $X \overleftarrow{\times}_Y Y$ is the *vanishing topos* of f, a topos, not a scheme, which, in the case where Y is a strictly local trait, is a slight enrichment of the topos of sheaves on the special fibre endowed with an action of the inertia group of Y. While, if Y is a henselian trait, the formation of $R\Psi_f(K)$ commutes with (surjective) base change of traits ([33], Th. finitude, 3.7), for $\dim Y > 1$, the formation of $R\Psi_f K$ in general does not commute with base change. We say that (f, K) is Ψ-*good* when the formation of $R\Psi_f K$ commutes with arbitrary base change $Y' \to Y$ (see 1.5 for a precise formulation). In this case, we have $R\Psi_f K \in D^b_c(X \overleftarrow{\times}_Y Y, \Lambda)$, i. e., for all q, $R^q\Psi_f K$ is a constructible sheaf of Λ-modules on $X \overleftarrow{\times}_Y Y$ (1.6.1), which vanishes for q sufficiently large. Now, a key result of this paper is the following (Theorem 2.3):

Theorem. *Let S be a noetherian scheme, and for $i = 1, 2$, let $f_i : X_i \to Y_i$ be a morphism of S-schemes of finite type. Let $X := X_1 \times_S X_2$, $Y := Y_1 \times_S Y_2$, and $f := f_1 \times_S f_2 : X \to Y$. Let $K_i \in D_{ctf}(X_i, \Lambda)$, with Λ as above (or more generally, a noetherian $\mathbf{Z}/n\mathbf{Z}$-algebra), with n invertible on S (ctf meaning that K_i is in D^b_c and its stalks are of finite tor-dimension). Assume that (f_i, K_i) is Ψ-good. Then the external product $(f, K = K_1 \boxtimes^L K_2)$ is again Ψ-good, and the natural map*

$$R\Psi_{f_1}K_1 \boxtimes^L R\Psi_{f_2}K_2 \to R\Psi_f(K_1 \boxtimes^L K_2) \tag{0.6}$$

is an isomorphism.

The proof given in this paper uses the following ingredients: (a) Gabber's theorem quoted above, on the commutation of $R\Psi$ with external products over a trait, (b) Orgogozo's theorem ([21], 2.1) to the effect that there exists a modification $g : Y' \to Y$ such that base change by g renders (f, K) Ψ-good, (c) Gabber's theorem of oriented cohomological descent ([23], 2.2.3). A new, simpler proof, under slightly relaxed hypotheses, has just been obtained by W. Zheng, see A.3. It needs none of the above ingredients, and relies only on general nonsense on oriented products and projection formulas. It is similar in spirit to that of the Künneth formula ([34], III (1.6.4)).

Isomorphisms of type (0.4) are then more or less formal applications of 2.3 and general transitivity properties of $R\Psi$.

In §1 we recall basic definitions and facts about nearby cycles on general bases. §2 is devoted to the statement and proof of 2.3. Applications of Thom-Sebastiani type are discussed in §4, after reviewing standard material on global and local additive convolutions in §3. Formula (0.4) is obtained in 4.7. Generalizations conjectured by Deligne in [5] are discussed in 4.12. In § 5 we study the convolution product $V_1 *_1 V_2$ (3.9.1) for V_1 and V_2 tamely ramified, and analyze its "arithmetic difference" with $V_1 \otimes V_2$. The results are due to Deligne ([2] and private communication). Their presentation owes much to discussions with Laumon. In the situation considered at the beginning of this introduction, we give formulas for monodromy and variation in the tame case (5.24), (5.33.10), recovering (0.1), (0.2) and (0.3).

Conventions: For a morphism $f : X \to Y$ and a sheaf (or complex) K on Y we will write $K|X$ for f^*K when no confusion can arise. We will sometimes, by abuse, say that a diagram is commutative when it is commutative up to a canonical isomorphism. Rings are assumed to be commutative and unital.

1 Review of nearby cycles over general bases

1.1

Recall the following construction, due to Deligne (see [12]). Given morphisms of topoi $f : X \to S$, $g : Y \to S$, there is defined a topos $X \overset{\leftarrow}{\times}_S Y$, called the *(left) oriented product* of X and Y over S, together with maps $p_1 : X \overset{\leftarrow}{\times}_S Y \to X$, $p_2 : X \overset{\leftarrow}{\times}_S Y \to Y$, and a 2-map $\tau : gp_2 \to fp_1$, which is universal for these data. If X, Y, S are the categories of sheaves on small sites C_1, C_2, D satisfying standard exactness properties, $X \overset{\leftarrow}{\times}_S Y$ is the category of sheaves on a site C, consisting of pairs of maps $(U \to V \leftarrow W)$ above $(X \xrightarrow{f} S \xleftarrow{g} Y)$, i.e., $U \to f^*V$, $W \to g^*V$, with a topology generated by covering families $(U_i \to V \leftarrow W)$ for $(U_i \to U)$ covering in C_1, $(U \to V \leftarrow W_i)$ for $(W_i \to W)$ covering in C_2, and

(1.1.1)
$$\begin{tikzcd} & V' \arrow[d] & W' \arrow[l] \arrow[d] \\ U \arrow[r] \arrow[ur] & V & W \arrow[l] \end{tikzcd} ,$$

where the square is cartesian. If $\mathcal{F}$ is a sheaf on C, i.e., an object of $X \overset{\leftarrow}{\times}_S Y$, the restriction map $\mathcal{F}(U \to V \leftarrow W) \to \mathcal{F}(U \to V' \leftarrow W')$ is an isomorphism for any diagram of type (1.1.1). If e_X, e_S, e_Y denotes the final objects of X, S, Y, the maps p_1, p_2 are given by $p_1^*(U) = (U \to e_S \leftarrow e_Y)$, $p_2^*(W) = (e_X \to e_S \leftarrow W)$.

By the universal property of oriented topoi, the datum of a diagram of topoi

$$
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "u"'] & S' \arrow[d, "v"] & Y' \arrow[l, "g'"'] \arrow[d, "w"] \\
X \arrow[r, "f"] & S & Y \arrow[l, "g"']
\end{tikzcd}
\tag{1.1.2}
$$

and 2-maps $a : gw \to vg'$, $b : vf' \to fu$ defines a morphism of topoi

$$(u, v, w; a, b) : X' \overleftarrow{\times}_{S'} Y' \to X \overleftarrow{\times}_S Y, \tag{1.1.3}$$

called the *functoriality morphism*, denoted sometimes $\overleftarrow{u \times_v w}$ (or even $\overleftarrow{u \times v}$) for brevity.

An important property, which follows from the behavior of sheaves with respect to coverings of type (1.1.1), is that, when u, v, w are localization morphisms in X, S, Y, $X' = X$, $u = \mathrm{Id}_X$, and the right hand square is *cartesian* (with $a : gw \xrightarrow{\sim} vg'$), then $(u, v, w; a, b)$ is an equivalence ([12], 1.11).

We will be mostly concerned with the case where X, Y, S are schemes, f, g morphisms of schemes, and we consider the corresponding morphisms of topoi of sheaves for the étale topology, still denoted f, g. The functoriality map (1.1.3) will be only used when (1.1.2) is a commutative diagram of schemes, except for a crucial construction in (1.11.3).

Points of the topos $X \overleftarrow{\times}_S Y$, i.e., morphisms from the punctual topos pt to $X \overleftarrow{\times}_S Y$ are triples (x, y, c), where $x : \mathrm{pt} \to X$, $y : \mathrm{pt} \to Y$ are points of X and Y, and c is a morphism $gy \to fx$. Recall ([32], VIII, 7.9) that, when T is a scheme, points of the étale topos T correspond to usual geometric points of T, i.e., morphisms t from the spectrum of a separably closed field to T, and morphisms $t \to s$ in T to morphisms of the corresponding strictly localized schemes $T_{(t)} \to T_{(s)}$.

1.2

Let $f : X \to S$, $g : Y \to S$ be morphisms of schemes, and let $\mathrm{pr}_1 : X \times_S Y \to X$, $\mathrm{pr}_2 : X \times_S Y \to Y$ be the projections. As $f\mathrm{pr}_1 = g\mathrm{pr}_2$, by the universal property of the oriented product, we get a diagram where the upper triangles commute:

$$
\begin{tikzcd}
 & X \times_S Y \arrow[dl, "\mathrm{pr}_1"'] \arrow[d, "\Psi_{f,g}"] \arrow[dr, "\mathrm{pr}_2"] & \\
X \arrow[dr, "f"'] & X \overleftarrow{\times}_S Y \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y \arrow[dl, "g"] \\
 & S &
\end{tikzcd}
\tag{1.2.1}
$$

The inverse image functor by $\Psi_{f,g}$ is given by $\Psi^*_{f,g}(U \to V \leftarrow W) = U \times_V W$, in the notation of 1.1.

The most interesting case for us is when $Y = S$, $g = \mathrm{Id}_S$. The oriented product $X \overleftarrow{\times}_S S$ is then called the *vanishing topos* of f. Diagram (1.2.1) reduces to

$$
\begin{tikzcd}
& X \arrow[dl, "\mathrm{Id}_X"'] \arrow[d, "\Psi_f"] \arrow[dr, "f"] & \\
X \arrow[dr, "f"'] & X \overset{\leftarrow}{\times}_S S \arrow[l, "p_1"'] \arrow[r, "p_2"] & S \arrow[dl, "\mathrm{Id}_S"] \\
& S &
\end{tikzcd}
\tag{1.2.2}
$$

where $\Psi_f := \Psi_{f,g}$.

Let Λ be a ring. The direct image functor by Ψ_f, denoted

$$R\Psi_f := R(\Psi_f)_* : D^+(X, \Lambda) \to D^+(X \overset{\leftarrow}{\times}_S S, \Lambda), \tag{1.2.3}$$

is called the *nearby cycles* functor (relative to f).

The canonical morphism $p_1^* \to (\Psi_f)_*$ gives rise to a functor, called the *vanishing cycles* functor

$$R\Phi_f : D^+(X, \Lambda) \to D^+(X \overset{\leftarrow}{\times}_S S, \Lambda), \tag{1.2.4}$$

with a functorial exact triangle in $K \in D^+(X, \Lambda)$,

$$p_1^* K \to R\Psi_f K \to R\Phi_f K \to . \tag{1.2.5}$$

Actually, by the method of ([35], XIII 1.4), one sees that (1.2.5) underlies an object

$$R\underline{\Psi}_f K \tag{1.2.6}$$

of the filtered derived category $DF^{[0,1]}(X, \Lambda)$, with $F^1 R\underline{\Psi}_f K = p_1^* K$ and $\mathrm{gr}^0_F R\underline{\Psi}_f K = R\Phi_f K$.

The relation between these objects and the classical nearby and vanishing cycles defined in ([35], XIII) is as follows. Assume that S is the spectrum of a henselian discrete valuation ring, with closed (resp. generic) point s (resp. η). The topos denoted $X_s \times_s S$ in (*loc. cit.* 1.2) is $X_s \overset{\leftarrow}{\times}_S S$, the union of the open subtopos $X_s \overset{\leftarrow}{\times}_S \eta$ and complementary closed subtopos $X_s = X_s \overset{\leftarrow}{\times}_S s$. Sheaves on $X_s \overset{\leftarrow}{\times}_S S$ are described by triples $F = (F_s, F_\eta, \varphi : F_{\overline{s}} \to F_{\overline{\eta}})$ ([35], XIIII 1.2.4). In particular, we have a natural equivalence (given by $p_2 : s \overset{\leftarrow}{\times}_S S \to S$)

$$s \overset{\leftarrow}{\times}_S S \xrightarrow{\sim} S, \tag{1.2.7}$$

by which we will generally identify those two topoi (cf. ([35], XIII 1.2.2 (b)). The functor $R\Psi_\eta : D^+(X_\eta, \Lambda) \to D^+(X_s \overset{\leftarrow}{\times}_S \eta, \Lambda)$ ([35], XIII 1.3.3, 2.1.2) is

$$R\Psi_\eta : K \mapsto \overleftarrow{i}^* R\Psi_{f,\eta} K = (R\Psi_{f,\eta} K) | X_s \overset{\leftarrow}{\times}_S \eta \tag{1.2.8}$$

where $\overleftarrow{i} : X_s \overset{\leftarrow}{\times}_S \eta \hookrightarrow X \overset{\leftarrow}{\times}_S \eta$ is the closed subtopos defined by the inclusion i of X_s in X. The functor $R\Psi : D^+(X, \Lambda) \to D^+(X_s \overset{\leftarrow}{\times}_S S, \Lambda)$ ([35], XIII, 1.3.3, 2.1.2) is given by

$$R\Psi : K \mapsto (R\Psi_f K) | X_s \overset{\leftarrow}{\times}_S S, \tag{1.2.9}$$

where $(R\Psi_f K)|X_s \overleftarrow{\times}_S S$ defines the triple $(K|X_s, R\Psi_\eta(K), \varphi)$. In particular, the functor $R\Phi_\eta : D^+(X, \Lambda) \to D^+(X_s \overleftarrow{\times}_S \eta, \Lambda)$ is given by

$$R\Phi_\eta : K \mapsto (R\Phi_f K)|X_s \overleftarrow{\times}_S \eta. \tag{1.2.10}$$

1.3

We now recall the description of stalks of $R\Psi$ at points of the oriented product. In the situation of (1.2.1), let $(x, y, c : g(y) \to f(x))$ be a point of $X \overleftarrow{\times}_S Y$ (1.1). Neighborhoods of (x, y, c) consisting of $(U \to V \leftarrow W)$ with U, V, W affine étale neighborhoods of x, $f(x)$, y respectively, form a cofinal system, of which $X_{(x)} \times_{S_{(f(x)))}} Y_{(y)}$ is the projective limit, where $Y_{(y)} \to S_{(f(x))}$ is the composition of $Y_{(y)} \to S_{(g(y))}$ and the specialization $c : S_{(g(y))} \to S_{(f(x))}$. Therefore, by ([32], VII, 5.8), we have, for $K \in D^+(X \times_S T, \Lambda)$,

$$R\Psi_{f,g}(K)_{(x,y,c)} = R\Gamma(X_{(x)} \times_{S_{(f(x)))}} Y_{(y)}, K). \tag{1.3.1}$$

In particular, for $Y = S$, $g = \mathrm{Id}_S$ in (1.2.1), and a point $(x, c : t \to s)$ of $X \overleftarrow{\times}_S S$, we have

$$R\Psi_f(K)_{(x,t\to s)} = R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, K), \tag{1.3.2}$$

and an exact triangle

$$K_x \to R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, K) \to R\Phi_f(K)_{(x,t\to s)} \to . \tag{1.3.3}$$

Thus, for $X = S$, $f = \mathrm{Id}_S$, $x = s$, $R\Phi_{\mathrm{Id}_S} K$ measures the defect of the specialization maps $K_s \to K_t$ to be isomorphisms.

The fiber product $X_{(x)} \times_{S_{(s)}} S_{(t)}$ is called the *Milnor tube* at $(x, c : t \to s)$, in contrast with the *Milnor fiber* $X_{(x)} \times_{S_{(s)}} t$, a closed subscheme of the Milnor tube.

It is sometimes useful to consider, instead of Milnor tubes and fibers, Orgogozo's *sliced* nearby (resp. vanishing cycles) $\Psi_{s,t}$ (resp. $\Phi_{s,t}$) ([21], proof of 6.1). Namely, for a specialization map $c : t \to s$ of geometric points of S, let $X_{(s)} := X \times_S S_{(s)}$, consider the inclusion $i_s : X_s \hookrightarrow X_{(s)}$, and the morphism $j_{(t)} : X_{(t)} := X_{(s)} \times_{S_{(s)}} S_{(t)} \to X_{(s)}$ (with $S_{(t)} \to S_{(s)}$ given by c). Define the sliced nearby cycles functor

$$R\Psi_{(f,c)} : D^+(X_{(t)}, \Lambda) \to D^+(X_s, \Lambda) \tag{1.3.4}$$

by

$$R\Psi_{(f,c)} K = i_s^* R j_{(t)*}(K),$$

and let us write $R\Psi_{(s,t)}$ for short when no confusion can arise. For $K \in D^+(X, \Lambda)$, we have

$$R\Psi_{(s,t)} K = \overleftarrow{i}^{\,*}_{(s,t)} R\Psi_f K,$$

where $\overleftarrow{i}_{(s,t)} : X_s = X_s \overleftarrow{\times}_s t \to X \overleftarrow{\times}_S S$ is the morphism given by $(i_s : X_s \to X, s \to S, t \to S, c : t \to s)$ (1.1.2). The stalk of $R\Psi_{(s,t)} K$ at a geometric point x above s is $(R\Psi_f K)_{(x,c)}$ (1.3.2). The sliced vanishing cycles functor $R\Phi_{s,t}$ is defined by the exact triangle

$$K|X_s \to R\Psi_{s,t} K \to R\Phi_{s,t} K \to, \tag{1.3.5}$$

where the first map is the adjunction map. We have $R\Phi_{(s,t)} K = \overleftarrow{i}^{\,*}_{(s,t)} R\Phi_f K$.

1.4

A commutative diagram of schemes

$$
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & S' \arrow[d, "h"] \\
X \arrow[r, "f"] & S
\end{tikzcd}
\tag{1.4.1}
$$

defines a morphism of oriented topoi (1.1.3)

$$h' \overleftarrow{\times}_h h : X' \overleftarrow{\times}_{S'} S' \to X \overleftarrow{\times}_S S,$$

hence a commutative diagram

$$
\begin{tikzcd}
X' \arrow[r, "h'"] \arrow[d, "\Psi_{f'}"'] & X \arrow[d, "\Psi_f"] \\
X' \overleftarrow{\times}_{S'} S' \arrow[r, "h' \overleftarrow{\times}_h h"] & X \overleftarrow{\times}_S S
\end{tikzcd}
\tag{1.4.2}
$$

For $K \in D^+(X, \Lambda)$, we have a base change morphism

$$(h' \overleftarrow{\times}_h h)^* R\Psi_f K \to R\Psi_{f'} h^* K. \tag{1.4.3}$$

Definition 1.5. We say that the formation of $R\Psi_f K$ commutes with base change, or that the pair (f, K) is Ψ-*good*, if, for any cartesian diagram (1.4.1), (1.4.3) is an isomorphism.

Examples will be discussed in 1.7, after we have recalled results of Orgogozo in [21].

1.6

In the situation of (1.2.1), assume that S is noetherian, and f and g are of finite type. It is shown in ([12], 2.5) that $X \overleftarrow{\times}_S Y$ is then a coherent topos and the projections p_1, p_2 are coherent morphisms ([32], VI, 2.3, 2.4.5, 3.1). Moreover, by ([21], 9) one has a good notion of constructibility for sheaves on $X \overleftarrow{\times}_S Y$. Let Λ be a noetherian ring. A sheaf of Λ-modules F on $X \overleftarrow{\times}_S Y$ is called *constructible* if it has a presentation $L \to M \to F \to 0$, where L and M are finite sums of Λ-modules of the form $\Lambda^{(U \to V \leftarrow W)}$, for $(U \to V \leftarrow W)$ objects of the defining site of $X \overleftarrow{\times}_S Y$, with $U \to X$, $V \to S$, $W \to Y$ separated, étale, and of finite presentation. It is equivalent to saying that there exist finite partitions of X and Y into locally closed subsets: $X = \cup_{\alpha \in A} X_\alpha$, $Y = \cup_{\beta \in B} Y_\beta$, such that, for all (α, β), the restriction of F to the subtopos $X_\alpha \overleftarrow{\times}_S Y_\beta$ is locally constant of finite type. As in the case of noetherian schemes, constructible sheaves of Λ-modules form a thick subcategory (in the strong sense, i.e., closed under subobjects, quotients and extensions) so that in particular the full subcategory $D_c^+(X \overleftarrow{\times}_S Y, \Lambda)$ of $D^+(X \overleftarrow{\times}_S Y, \Lambda)$ consisting of complexes K such that $\mathcal{H}^i K$ is constructible for all i is a triangulated subcategory.

The following theorem ([21], 2.1, 3.1, 8.1) is the main result of [21].

Theorem 1.6.1. *In the situation of (1.2.2), assume that S is noetherian, f of finite type, and Λ is a $\mathbf{Z}/n\mathbf{Z}$-algebra, with n invertible on S. Let $K \in D_c^b(X, \Lambda)$. Then there exists a modification $a : S' \to S$ such that if $X' = X \times_S S'$ and K' is the inverse image of K on X', the following conditions are satisfied :*

(i) $R\Psi_{f'}K'$ belongs to $D_c^b(X' \overleftarrow{\times}_{S'} S', \Lambda)$;

(ii) the formation of $R\Psi_{f'}K'$ commutes with any base change $S'' \to S'$, i.e., (f', K') is Ψ-good (1.5).

In (*loc. cit.*) $\Lambda = \mathbf{Z}/n\mathbf{Z}$, but the proof goes on without any change for Λ as above. Note that (ii) implies (i) by ([21] 8.1, 10.5).

Example 1.7. Let Λ be as in 1.6.1, except in (b) where it can be any ring.

(a) If S is regular of dimension ≤ 1, (f, K) is Ψ-good (as any modification $S' \to S$ has a section). This contains, in particular, the universal local acyclicity of pairs (X, K) for X of finite type over a field ([33], Th. finitude, 2.16), and the compatibility of classical nearby cycles with base change by surjective maps of traits ([33], Th. finitude, 3.7).

(b) A pair (f, K) is locally acyclic ([33], Th. finitude, 2.12) if and only if $R\Phi_f K = 0$ and $R\Phi_f K$ commutes with locally quasi-finite base change. In particular, (f, K) is universally locally acyclic if and only if (f, K) is Ψ-good and $R\Phi_f K = 0$, i. e. $R\Phi_{f'}(K') = 0$ for all $S' \to S$, where $K' = K|S'$ and $f' = f \times_S S'$.

Indeed, if (f, K) is locally acyclic, then, by ([33], Th. finitude, Appendice, 2.6), for any geometric point x of X with image s in S, if $f_{(x,s)} : X_{(x)} \to S_{(s)}$ is the morphism deduced by localization, the formation of $Rf_{(x,s)*}K$ (where K denotes again, by abuse, its restriction to $X_{(x)}$) commutes with any finite base change (hence $R\Psi_f K$ and $R\Phi_f K$ commute with any locally quasi-finite base change, cf. A.2 (2)). In particular, for any geometric point $t \to S_{(s)}$, the restriction map $R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, K) \to R\Gamma(X_{(x)_t}, K)$ from the Milnor tube to the Milnor fiber (cf. (1.3.1)) is an isomorphism, as it is the stalk at t of the base change map by the closure of the image of t in $S_{(s)}$. Therefore, as the composition $K_x \to R\Gamma(X_{(x)} \times_{S_{(s)}} S_{(t)}, K) \to R\Gamma(X_{(x)_t}, K)$ is an isomorphism by the definition of local acyclicity, the first map of this composition is an isomorphism as well, i. e. $(R\Phi_f K)_{(x,t)} = 0$. That proves the "only if" part of the assertion. The converse is immediate. This argument is copied from ([27], Prop. 1.7).

(c) By ([21], 6.1), if (f, K) is universally locally acyclic on an open subset U of X whose complement is quasi-finite over S, then (f, K) is Ψ-good.

(d) If $f : X \to S$ is the blow-up of the origin in $S = \mathbf{A}_k^2$, k an algebraically closed field, $\Lambda = \mathbf{Z}/n\mathbf{Z}$, then (f, Λ) is not Ψ-good ([21], 11).

We will see other examples of Ψ-goodness in the next section (2.3).

1.8

The results of the next two subsections will be used only in the proof of 2.3.

There are variants of the constructions in 1.2 for morphisms of simplicial schemes $f_\bullet : X_\bullet \to S_\bullet$, $g_\bullet : Y_\bullet \to S_\bullet$, see ([23], 2.2). We will use them freely, and sometimes, by abuse, write $X_\bullet \overleftarrow{\times}_{S_\bullet} Y_\bullet$ for the total topos of this simplicial topos.

Let Λ be as in 1.6.1. It follows from 1.6.1 that there exists a hypercovering for the h-topology on S ([23], 2.1.3) $a : T_\bullet \to S$ such that for all $n \in \mathbf{N}$, $R\Psi_{f_n}(K|X_{T_n})$ belongs to $D^b_c(X_{T_n} \overleftarrow{\times}_{T_n} T_n, \Lambda)$ and is of formation compatible with base change. Such a hypercovering will be called *admissible for* (f, K). Admissible hypercoverings form a cofinal system in the category of hypercoverings of S for the h-topology, up to homotopy.

Note that, for *any* hypercovering $a : T_\bullet \to S$ for the h-topology, $R\Psi_f(K)$ is recovered from a^*K by a canonical isomorphism

$$R\Psi_f(K) \xrightarrow{\sim} R\overleftarrow{a}_* R\Psi_{f_{T_\bullet}}(K|X_{T_\bullet}), \tag{1.8.1}$$

where $f_{T_\bullet} : X_{T_\bullet} \to T_\bullet$ is the morphism deduced by base change, and $\overleftarrow{a} : X_{T_\bullet} \overleftarrow{\times}_{T_\bullet} T_\bullet \to X \overleftarrow{\times}_S S$ is the corresponding augmentation[3]. Indeed, we have a commutative diagram (cf. (1.4.2))

$$\begin{tikzcd} X_{T_\bullet} \arrow[r, "a"] \arrow[d, "\Psi_{f_{T_\bullet}}"] & X \arrow[d, "\Psi_f"] \\ X_{T_\bullet} \overleftarrow{\times}_{T_\bullet} T_\bullet \arrow[r, "\overleftarrow{a}"] & X \overleftarrow{\times}_S S \end{tikzcd}, \tag{1.8.2}$$

and by cohomological descent the adjunction map $K \to Ra_*a^*K$ is an isomorphism. The isomorphism (1.8.1) is induced by this isomorphism via (1.8.2).

1.9

By ([23], 2.2.3), $\overleftarrow{a} : X_{T_\bullet} \overleftarrow{\times}_{T_\bullet} T_\bullet \to X \overleftarrow{\times}_S S$ is of 1-cohomological descent, which means that, for any $L \in D^b(X \overleftarrow{\times}_S S, \Lambda)$, the adjunction map

$$L \to R\overleftarrow{a}_* \overleftarrow{a}^* L \tag{1.9.1}$$

is an isomorphism. For $L = R\Psi_f K$, we have a commutative diagram

$$\begin{tikzcd} & R\overleftarrow{a}_* \overleftarrow{a}^* R\Psi_f K \arrow[dd] \\ R\Psi_f K \arrow[ur] \arrow[dr] & \\ & R\overleftarrow{a}_* R\Psi_{f_{T_\bullet}}(K|X_{T_\bullet}) \end{tikzcd}$$

in which the vertical map

$$R\overleftarrow{a}_* \overleftarrow{a}^* R\Psi_f(K) \to R\overleftarrow{a}_* R\Psi_{f_{T_\bullet}}(K|X_{T_\bullet}) \tag{1.9.2}$$

is $R\overleftarrow{a}_*$ applied to the base change map

$$\beta : \overleftarrow{a}^* R\Psi_f K \to R\Psi_{f_{T_\bullet}}(K|X_{T_\bullet}),$$

[3] I am indebted to Gabber for this observation.

the upper oblique map is the isomorphism (1.9.1) and the lower one the isomorphism (1.8.1). Therefore (1.9.2) is an isomorphism. However, in general β is not an isomorphism, as in general the formation of $R\Psi_f K$ is not compatible with base change.

In the rest of this section we collect a few general facts that will be used in sections 3 and 4.

1.10

For morphisms of schemes $f : X \to Y$, $g : Y \to Z$, the diagram (of type (1.1.2))

(1.10.1)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] & Y \arrow[l, "\mathrm{Id}"'] \arrow[d, "g"] \\ & Z & Z \arrow[l, "\mathrm{Id}"] \end{tikzcd}$$

induces a morphism

$$\overleftarrow{g} := \overleftarrow{\mathrm{Id_X} \times g} : X \overleftarrow{\times}_Y Y \to X \overleftarrow{\times}_Z Z,$$

hence a commutative diagram of type (1.4.2)

(1.10.2)
$$\begin{tikzcd} X \arrow[r, "\Psi_f"] \arrow[dr, "\Psi_{gf}"'] & X \overleftarrow{\times}_Y Y \arrow[r, "p_1"] \arrow[d, "\overleftarrow{g}"] & X \ , \\ & X \overleftarrow{\times}_Z Z \arrow[ur, "q_1"'] & \end{tikzcd}$$

where we denote by $q_1 : X \overleftarrow{\times}_Z Z \to X$, instead of p_1, the canonical projection.

Let Λ be a commutative ring. For $K \in D^+(X, \Lambda)$, (1.10.1) induces an isomorphism

(1.10.3)
$$R\overleftarrow{g}_* R\Psi_f K \xrightarrow{\sim} R\Psi_{gf} K.$$

With this identification the base change map $q_1^* K \to R\overleftarrow{g}_* p_1^* K$ associated with the right triangle of (1.10.2) gives a map

(1.10.4)
$$R\Phi_{gf} K \to R\overleftarrow{g}_* R\Phi_f K,$$

which is not an isomorphism in general, as the trivial case where $X = Y$, $f = \mathrm{Id}_X$, $K = \Lambda$ already shows ($R\Phi_{\mathrm{Id_X}} \Lambda = 0$). In 1.16, we give a formula for the cone of (1.10.4), under certain hypotheses.

1.11

Let $f : X \to Y$ be a morphism of schemes, and $K \in D^+(X, \Lambda)$. We have seen how to calculate $R\Psi_f(K)$ by *slices* (1.3.4). Another way to unravel $R\Psi_f K$ is to consider its restrictions to *local sections*.

Let $x : \operatorname{Spec} k \to X$ be a geometric point of X, with image the geometric point $y = f(x)$ of Y, and let $X_{(y)} := X \times_Y Y_{(y)}$. The definition of oriented products implies (cf. ([12], 1.4, 1.11) that the morphisms of topoi

$$\operatorname{Spec} k \overleftarrow{\times}_Y Y \to \operatorname{Spec} k \overleftarrow{\times}_{Y_{(y)}} Y_{(y)} \to Y_{(y)} \tag{1.11.1}$$

are isomorphisms, where the second map is given by p_2 and the first one is the inverse of the localization map ([12], 1.11). This generalizes the strictly local case of (1.2.7). We will write $x \overleftarrow{\times}_Y Y$ for $\operatorname{Spec} k \overleftarrow{\times}_Y Y$, and call it the *stalk of* $X \overleftarrow{\times}_Y Y$ *at* x. It is constant along X_y, of value $Y_{(y)}$.

Recall ([12] 2.2) that, as the topos $Y_{(y)}$ is local, the point x defines a canonical section

$$\sigma_{x,y} : Y_{(y)} \to X_{(y)} \overleftarrow{\times}_{Y_{(y)}} Y_{(y)} \tag{1.11.2}$$

of the projection $p_2 : X_{(y)} \overleftarrow{\times}_{Y_{(y)}} Y_{(y)} \to Y_{(y)}$.

This section sends a point $(t \to y)$ of $Y_{(y)}$ to the point $(x \to y \leftarrow t)$ of $X_{(y)} \overleftarrow{\times}_{Y_{(y)}} Y_{(y)}$. Formally, $\sigma_{x,y}$ is defined as follows. Let $\varepsilon : Y_{(y)} \to \mathrm{pt}$ be the projection on the punctual topos pt. Consider the 2-map

$$c_y : \mathrm{Id}_{Y_{(y)}} \to y\varepsilon$$

given by adjunction map $(y\varepsilon)^*K = (K_y)_{Y_{(y)}} = \Gamma(Y_{(y)}, K)_{Y_{(y)}} \to K$, for K a sheaf on $Y_{(y)}$. We have a diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "\mathrm{Id}"] \arrow[d, "x"'] & \mathrm{pt} \arrow[d, "y"] & Y_{(y)} \arrow[l, "\varepsilon"'] \arrow[d, "\mathrm{Id}"] \\ X_{(y)} \arrow[r, "f_{(y)}"] & Y_{(y)} & Y_{(y)} \arrow[l, "\mathrm{Id}"'] \end{tikzcd}, \tag{1.11.3}$$

in which the left hand square trivially commutes, and the right hand one is 2-commutative by the datum of the 2-map c_y. The canonical section $\sigma_{x,y}$ (1.11.2) is the functoriality map on the oriented products given by (1.11.3) (with the identification of $Y_{(y)}$ with $x \overleftarrow{\times}_Y Y \xrightarrow{\sim} x \overleftarrow{\times}_{Y_{(y)}} Y_{(y)}$). It sits in a commutative diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[d, "x"'] & \mathrm{pt} \overleftarrow{\times}_{\mathrm{pt}} Y_{(y)} \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\sigma_{x,y}"'] & Y_{(y)} \arrow[d, "\mathrm{Id}"] \\ X_{(y)} & X_{(y)} \overleftarrow{\times}_{Y_{(y)}} Y_{(y)} \arrow[l, "p_1"'] \arrow[r, "p_2"] & Y_{(y)} \end{tikzcd}, \tag{1.11.4}$$

where the upper p_2 is one of the isomorphisms (1.11.1). When no confusion can arise, we will also denote by $\sigma_{x,y}$ the composition

$$\sigma_{x,y} : Y_{(y)} \to X \overleftarrow{\times}_Y Y \tag{1.11.5}$$

of (1.11.2) and the canonical map $X_{(y)} \overleftarrow{\times}_{Y_{(y)}} Y_{(y)} \to X \overleftarrow{\times}_Y Y$, and we will write σ_x for $\sigma_{x,y}$. For $L \in D^+(X \overleftarrow{\times}_Y Y, \Lambda)$, we will sometimes write

$$L | x \overleftarrow{\times}_Y Y = L | Y_{(y)} := \sigma_{x,y}^* L \in D^+(Y_{(y)}, \Lambda) \tag{1.11.6}$$

for the restriction (via $\sigma_{x,y}$) of L to the stalk of $X\overleftarrow{\times}_Y Y$ at x.

By construction, as the point $x : \mathrm{pt} \to X_{(y)}$ factors through $X_{(x)}$, so does $\sigma_{x,y}$: we have a commutative diagram

(1.11.7)
$$\begin{tikzcd}
& X_{(x)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \arrow[d] \arrow[dr, "p_2"] & \\
Y_{(y)} \arrow[ur, "\widetilde{\sigma}_{x,y}"] \arrow[r, "\sigma_{x,y}"] & X_{(y)}\overleftarrow{\times}_{Y(y)}Y_{(y)} \arrow[r, "p_2"] & Y_{(y)}
\end{tikzcd},$$

where the composition of the horizontal maps is the identity. Recall the following result, due to Gabber (a special case of ([12], 2.3.1)):

Proposition 1.12. *The canonical map*

(1.12.1)
$$\gamma : p_{2*} \to \widetilde{\sigma}^*_{x,y},$$

defined by $p_2\widetilde{\sigma}_{x,y} = \mathrm{Id}$, *is an isomorphism.*

It follows that, if $f_{(x,y)} : X_{(x)} \to Y_{(y)}$ is the map deduced from f by localization at x and y, by the commutative diagram (upper right triangle of (1.2.2))

(1.12.2)
$$\begin{tikzcd}
X_{(x)} \arrow[d, "\Psi_{f_{(x,y)}}"'] \arrow[dr, "f_{(x,y)}"] & \\
X_{(x)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \arrow[r, "p_2"] & Y_{(y)}
\end{tikzcd},$$

γ induces an isomorphism:

(1.12.3)
$$\gamma_{x,y} : \widetilde{\sigma}^*_{x,y}R\Psi_{f_{(x,y)}}(L) \overset{\sim}{\to} p_{2*}R\Psi_{f_{(x,y)}}(L) \overset{\sim}{\to} Rf_{(x,y)*}(L)$$

for any $L \in D^+(X_{(x)}, \Lambda)$. On the other hand, for $K \in D^+(X, \Lambda)$, by (1.3.2) the base change map

(1.12.4)
$$(R\Psi_f K)|X_{(x)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \to R\Psi_{f_{(x,y)}}(K|X_{(x)})$$

defined by the commutative diagrams

(1.12.5)
$$\begin{tikzcd}
X_{(x)} \arrow[r] \arrow[d, "\Psi_{f_{(x,y)}}"'] & X_{(y)} \arrow[r] \arrow[d, "\Psi_{f_{(y)}}"'] & X \arrow[d, "\Psi_f"'] \\
X_{(x)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \arrow[r] & X_{(y)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \arrow[r] & X\overleftarrow{\times}_Y Y
\end{tikzcd},$$

is an isomorphism, in which the horizontal arrows are defined by the localization maps. By composing $\gamma_{x,y}$ (for $L = K|X_{(x)}$) and (1.12.4), we get an isomorphism

(1.12.6)
$$\gamma_{x,y} : \sigma^*_{x,y}R\Psi_f K \overset{\sim}{\to} Rf_{(x,y)*}(K|X_{(x)}),$$

where $\sigma_{x,y}$ is the composition of $\widetilde{\sigma}_{x,y}$ and the localization map $X_{(x)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \to X\overleftarrow{\times}_Y Y$.

At a geometric point t of $Y_{(y)}$ (1.12.6) induces an isomorphism

$$(\sigma_x^* R\Psi_f K)_t \xrightarrow{\sim} R\Gamma(X_{(x)} \times_{Y_{(y)}} Y_{(t)}, K) = (R\Psi_f K)_{(x,y\leftarrow t)}, \tag{1.12.7}$$

and in particular, an isomorphism

$$(\sigma_x^* R\Psi_f K)_y \xrightarrow{\sim} R\Gamma(X_{(x)}, K) = K_x. \tag{1.12.8}$$

The specialization morphism $K_x \to (\sigma_x^* R\Psi_f K)_t$ is identified with the stalk at t of $\sigma_x^*(p_1^* K \to R\Psi_f K)$, so that, taking into account the fact that $p_1\widetilde{\sigma}_{x,y} : Y_{(y)} \to X_{(x)}$ projects $Y_{(y)}$ onto the closed point of $X_{(x)}$, (1.2.5) induces a distinguished triangle of $D^+(Y_{(y)}, \Lambda)$

$$(K_x)_{Y_{(y)}} \to \sigma_x^* R\Psi_f K \to \sigma_x^* R\Phi_f K \to, \tag{1.12.9}$$

where $(K_x)_{Y_{(y)}}$ denotes the *constant complex* on $Y_{(y)}$ of value K_x. Note that taking the stalk at y of (1.12.9) and using (1.12.8), we get

$$(\sigma_x^* R\Phi_f K)_y = 0. \tag{1.12.10}$$

Thus, locally at y, $R\Psi_f K$ defines a family of complexes

$$\sigma_x^* R\Psi_f K \xrightarrow{\sim} Rf_{(x,y)*}(K|X_{(x)})$$

of $D^+(Y_{(y)}, \Lambda)$, parametrized by the geometric points x of X above y, generalizing the classical $(R\Psi_f K)_x$ when Y is a trait, with geometric closed point y ([35], XIII, 2.1.1).

Proposition 1.13. *Let $f : X \to Y$, $g : Y \to Z$, and $\overleftarrow{g} : X \overleftarrow{\times}_Y Y \to X \overleftarrow{\times}_Z Z$ be as in 1.10. Let x be a geometric point of X with images $y = f(x)$ in Y and $z = g(y)$ in Z. For $L \in D^+(X \overleftarrow{\times}_Y Y, \Lambda)$, the commutative diagram*

$$\begin{tikzcd} x \overleftarrow{\times}_Y Y = Y_{(y)} \arrow[r, "\sigma_{x,y}"] \arrow[d, "g_{(y,z)}"'] & X \overleftarrow{\times}_Y Y \arrow[d, "\overleftarrow{g}"] \\ x \overleftarrow{\times}_Z Z = Z_{(z)} \arrow[r, "\sigma_{x,z}"] & X \overleftarrow{\times}_Z Z \end{tikzcd} \tag{1.13.1}$$

gives an isomorphism

$$(R\overleftarrow{g}_* L)|x \overleftarrow{\times}_Z Z \xrightarrow{\sim} Rg_{(y,z)*}(L|x \overleftarrow{\times}_Y Y) \tag{1.13.2}$$

Proof. The map (1.13.2) is the base change map associated with the square (1.13.1). This square is the composite of the following squares

$$\begin{tikzcd} Y_{(y)} \arrow[r, "\widetilde{\sigma}_{x,y}"] \arrow[d] & X_{(x)} \overleftarrow{\times}_{Y_{(y)}} Y_{(y)} \arrow[r] \arrow[d] & X_{(x)} \overleftarrow{\times}_Y Y \arrow[r] \arrow[d] & X \overleftarrow{\times}_Y Y \arrow[d] \\ Z_{(z)} \arrow[r, "\widetilde{\sigma}_{x,z}"] & X_{(x)} \overleftarrow{\times}_{Z_{(z)}} Z_{(z)} \arrow[r] & X_{(x)} \overleftarrow{\times}_Z Z \arrow[r] & X \overleftarrow{\times}_Z Z \end{tikzcd} . \tag{1.13.3}$$

Base change in the right square is clear, as $X_{(x)} \to X$ is a limit of étale neighborhoods of x. Base change in the left square follows from Gabber's formula $p_{2*} = \widetilde{\sigma_x}^*$ (1.12) applied to $\widetilde{\sigma}_{x,y}$ and $\widetilde{\sigma}_{x,z}$. It remains to show base change in the middle square. Define $Y_{(z)}$ by the cartesian square

$$\begin{tikzcd} Y_{(z)} \arrow[r] \arrow[d] & Y \arrow[d] \\ Z_{(z)} \arrow[r] & Z \end{tikzcd} .$$

The middle square is thus decomposed into

$$\begin{tikzcd} X_{(x)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \arrow[r, "u"] \arrow[dr] & X_{(x)}\overleftarrow{\times}_{Y_{(z)}}Y_{(z)} \arrow[r] \arrow[d] & X_{(x)}\overleftarrow{\times}_{Y}Y \arrow[d] \\ & X_{(x)}\overleftarrow{\times}_{Z_{(z)}}Z_{(z)} \arrow[r] & X_{(x)}\overleftarrow{\times}_{Z}Z \end{tikzcd} ,$$

where u is the localization map defined by $Y_{(y)} \to Y_{(z)}$. Base change in the square is clear, as $Y_{(z)} \to Z_{(z)}$ is a limit of pull-backs of $Y \to Z$ by étale neighborhoods of z in Z. It then suffices to show base change in the triangle. Let $Y_1 := Y_{(z)}$. We have $Y_{(y)} = (Y_1)_{(y)}$. By ([13], XI 1.11) applied to the case the map g of *loc. cit.* is the identity (and Y of *loc. cit.* our Y_1), and passing to the limit on the étale neighborhoods of y in Y_1, u is an equivalence, hence base change in the triangle is trivial, which finishes the proof. $\square$

In particular, via the isomorphisms γ (1.12.6) and the commutative square

$$\begin{tikzcd} X_{(x)}\overleftarrow{\times}_{Y_{(y)}}Y_{(y)} \arrow[r, "p_2"] \arrow[d, "\overleftarrow{g}"'] & Y_{(y)} \arrow[d, "g_{(y,z)}"] \\ X_{(x)}\overleftarrow{\times}_{Z_{(z)}}Z_{(z)} \arrow[r, "p_2"] & Z_{(z)} \end{tikzcd} ,$$

the restriction to the stalk of $X\overleftarrow{\times}_Z Z$ at x of the transitivity isomorphism (1.10.3),

$$\sigma_{x,z}^* R\Psi_{gf}K \xrightarrow{\sim} \sigma_{x,z}^*(R\overleftarrow{g}_* R\Psi_f K),$$

translates into the isomorphism

$$R(gf)_{(x,z)*}K \xrightarrow{\sim} Rg_{(y,z)*}Rf_{(x,y)*}K, \tag{1.13.4}$$

coming from $g_{(y,z)}f_{(x,y)} = (gf)_{(x,z)}$.

Recall the following result of Gabber ([21], 3.1):

Theorem 1.14. *Assume that Λ is a $\mathbf{Z}/n\mathbf{Z}$-algebra for some integer $n \geq 1$. Let $f : X \to S$ be a morphism locally of finite type such that the dimension of its fibers is bounded by an integer N. Then, for any sheaf $\mathcal{F}$ of Λ-modules on X, we have*

$$R^q\Psi_f(\mathcal{F}) = 0$$

for all $q > 2N$.

Corollary 1.15. *Under the assumptions of 1.14, for any geometric point x of X, with image s in S, and any sheaf of Λ-modules $\mathcal{G}$ on $X_{(x)}$, we have*

$$R^q f_{(x,s)*}\mathcal{G} = 0$$

for all $q > 2N$.

Proof. By standard limit arguments ([32] VII 5.11, IX 2.7.2, 2.7.4) we may assume that $G = \mathcal{F}|X_{(x)}$ for some sheaf of Λ-modules $\mathcal{F}$ on X. Then the conclusion follows from 1.14 by (1.12.6). □

Proposition 1.16. *Let Λ be as in 1.14, and let $f : X \to Y$, $g : Y \to Z$ be morphisms of schemes. Assume that g is locally of finite type. Let $K \in D^+(X, \Lambda)$. Then, with the notation of (1.10.2), the canonical map (1.10.4) fits in a distinguished triangle*

(1.16.1) $$R\Phi_{gf}K \to R\overleftarrow{g}_* R\Phi_f K \to q_1^* K \otimes^L \overleftarrow{(f \times \mathrm{Id}_Z)}^* R\Phi_g(\Lambda)[1] \to .$$

Proof. Consider the composition given by (1.10.2):

$$q_1^* K \to R\overleftarrow{g}_* p_1^* K \to R\overleftarrow{g}_* R\Psi_f K.$$

By the usual method, we can view it as making $R\overleftarrow{g}_* R\Psi_f K$ (which, by (1.10.3) is identified with $R\Psi_{gf}K$) into an object of $DF^{[0,1]}(X \overleftarrow{\times}_Z Z, \Lambda)$, the filtered derived category of complexes with filtration of length one. By ([10], 4.1), it gives rise to a cross (i.e., a refined special 9-diagram):

(1.16.2)
$$\begin{tikzcd}
 & R\overleftarrow{g}_* p_1^* K \arrow[r] \arrow[d] & C \arrow[d] \\
q_1^* K \arrow[ur] \arrow[r] & R\Psi_{gf}K \arrow[r] \arrow[d] & R\Phi_{gf}K \arrow[dl] \\
 & R\overleftarrow{g}_* R\Phi_f K &
\end{tikzcd} ,$$

where

(1.16.3) $$C \simeq \mathrm{Cone}(q_1^* K \to R\overleftarrow{g}_* p_1^* K) \simeq \mathrm{Cone}(R\Phi_{gf}K \to R\overleftarrow{g}_* R\Phi_f K)[-1].$$

By 1.15, g satisfies the condition (*) of A.1. Therefore, by A.7 for $K' = \Lambda$, the projection formula map

$$R\overleftarrow{g}_* \Lambda \otimes^L q_1^* K \to R\overleftarrow{g}_* p_1^* K$$

is an isomorphism, and the upper oblique map in (1.16.2) is identified with $q_1^* K \otimes^L \alpha$, where $\alpha : \Lambda \to R\overleftarrow{g}_* \Lambda$ is the adjunction map. Therefore, we get

(1.16.4) $$C \simeq q_1^* K \otimes^L \mathrm{Cone}(\alpha : \Lambda \to R\overleftarrow{g}_* \Lambda).$$

Consider the commutative diagram (cf. ([28], proof of 2.1):

$$\begin{tikzcd}
 & Y \\
X \overleftarrow{\times}_Y Y \arrow[ur, "p_2"] \arrow[r, "\overleftarrow{f \times \mathrm{Id}_Y}"] \arrow[d, "\overleftarrow{\mathrm{Id}_X \times g}"'] & Y \overleftarrow{\times}_Y Y \arrow[u, "p_2"'] \arrow[d, "\overleftarrow{\mathrm{Id}_Y \times g}"'] \\
X \overleftarrow{\times}_Z Z \arrow[r, "\overleftarrow{f \times \mathrm{Id}_Z}"] & Y \overleftarrow{\times}_Z Z
\end{tikzcd}.$$

For $M \in D^+(Y, \Lambda)$, the induced base change map

$$(\overleftarrow{f \times \mathrm{Id}_Z})^* R(\overleftarrow{\mathrm{Id}_Y \times g})_* p_2^* M \to R(\overleftarrow{\mathrm{Id}_X \times g})_* p_2^* M$$

is an isomorphism, as one sees by applying $\sigma_{x,z}$ and using 1.13, both sides being identified with $Rg_{(y,z)*}(M|Y_{(y)})$. Note that

$$R(\overleftarrow{\mathrm{Id}_Y \times g})_* p_2^* M \simeq R\Psi_g(M)$$

by (1.10.3) and $R\Psi_{\mathrm{Id}_Y} = p_2^*$. Finally, for $M = \Lambda$, we get

$$R\overleftarrow{g}_* \Lambda \simeq (\overleftarrow{f \times \mathrm{Id}_Z})^* R\Psi_g \Lambda,$$

and $\alpha : \Lambda \to R\overleftarrow{g}_*\Lambda$ is identified with the canonical map $(\overleftarrow{f \times \mathrm{Id}_Z})^*(\Lambda \to R\Psi_g\Lambda)$, so we get (1.16.1) from (1.16.3) and (1.16.4). $\square$

Corollary 1.17. *Under the assumptions of 1.16, let x be a geometric point of X, with images y in Y and z in Z. The following conditions are equivalent :*

(i) The morphism

$$\sigma_{x,z}^* R\Phi_{gf} K \to \sigma_{x,z}^* (R\overleftarrow{g})_* R\Phi_f K \tag{1.17.1}$$

induced by (1.10.4) is an isomorphism.

(ii) The cone of $K_x \otimes^L (\alpha : \Lambda_{Z_{(z)}} \to Rg_{(y,z)*}\Lambda_{Y_{(y)}})$, where α is the adjunction map, is zero, K_x denoting the constant complex of value K_x on $Z_{(z)}$.

In particular, if g is locally acyclic at y ([33], Th. finitude, 2.12), e. g. smooth at y, these conditions are satisfied.

Proof. It suffices to apply $\sigma_{(x,z)}^*$ to (1.16.1), as

$$\sigma_{(x,z)}^*(q_1^* K \otimes^L (\overleftarrow{f \times \mathrm{Id}_Z})^* R\Phi_g(\Lambda)) \xrightarrow{\sim} \mathrm{Cone}(K_x \otimes^L (\alpha : \Lambda_{Z_{(z)}} \to Rg_{(y,z)*}\Lambda_{Y_{(y)}})),$$

as we have seen in the proof of 1.16. The last assertion follows from (ii) by the definition of local acyclicity (or from (1.16.1) by (1.7 (b))). $\square$

By (1.12.6) (applied to g and $\sigma_{y,z}$), for $L = \sigma_{x,y}^* R\Phi_f K = R\Phi_f(K)|Y_{(y)}$, we have an exact triangle

$$(L_y)_{Z_{(z)}} \to Rg_{(y,z)*} L \to \sigma_{x,z}^*(R\Phi_{g_{(y,z)}} L) \to, \tag{1.17.2}$$

where as before $(L_y)_{Z_{(z)}}$ denotes the constant complex of value $L_y = (Rg_{(y,z)*}L)_z = R\Gamma(Y_{(y)}, L)$, and the first map is the specialization map. Taking into account that $(\sigma^*_{x,y} R\Phi_f K)_y = 0$ (1.12.10), it gives an isomorphism:

$$Rg_{(y,z)*}\sigma^*_{x,y} R\Phi_f K \xrightarrow{\sim} R\Phi_{g_{(y,z)}} \sigma^*_{x,y} R\Phi_f K,$$

Thus, we get:

Corollary 1.18. *If, in 1.17, g is locally acyclic at y, then (i) yields an isomorphism*

$$\sigma^*_{x,z} R\Phi_{gf} K \xrightarrow{\sim} R\Phi_{g_{(y,z)}} \sigma^*_{x,y} R\Phi_f K. \tag{1.18.1}$$

Remark 1.19. In particular, in view of (1.7 (b)), if (f, K) is locally acyclic at x and g locally acyclic at y, then (gf, K) is locally acyclic at x : this is ([33], Th. finitude, Appendice, 2.7).

Remark 1.20. Let $f : X \to Y$, $g : Y \to Z$ be as 1.10, $h = gf$, and let Λ be as in 1.14. Let $K \in D^+(Y, \Lambda)$. Consider the corresponding map (1.4.3)

$$\overleftarrow{f}^* R\Psi_g K \to R\Psi_h(f^* K), \tag{1.20.1}$$

where $\overleftarrow{f} = X \overleftarrow{\times}_Z Z \to Y \overleftarrow{\times}_Z Z$ is the functoriality map. Assume that f is locally of finite type, and locally acyclic, i.e., (f, Λ) is locally acyclic (cf. (1.7, (b)). Then (1.20.1) is an isomorphism (cf. ([18], 3.2.3), where this result is stated without proof).

Indeed, by (1.12.6), taking a geometric point $x \to X$, and replacing X, Y, Z by their strict localizations at x and its images y, z in Y and Z respectively, and f, g, h by the corresponding localized morphisms, we are reduced to showing that the canonical map

$$Rg_{(y,z)*}K \to Rh_{(x,z)*}(f^*_{(x,y)} K) \tag{1.20.2}$$

is an isomorphism, where K denotes $K|Y_{(y)}$ by abuse. As f is locally of finite type, by 1.15, $f_{(x,y)*}$ is of finite cohomological dimension. As f is locally acyclic, the formation of $Rf_{(x,y)*}$ commutes with finite base change (1.7, (b)). Therefore, by ([33], Th. finitude, Appendice, 1.2 (a)) the projection formula map

$$Rf_{(x,y)*}\Lambda \otimes^L K \to Rf_{(x,y)*} f^*_{(x,y)} K \tag{1.20.3}$$

is an isomorphism[4]. By (1.7, (b)), we have $\Lambda \xrightarrow{\sim} Rf_{(x,y)*}\Lambda$. As $Rh_{(x,z)*}(f^*_{(x,y)}K) = Rg_{(y,z)*} Rf_{(x,y)*} f^*_{(x,y)} K$, (1.20.3) implies that (1.20.2) is an isomorphism, which finishes the proof[5].

[4] For $K \in D^b(Y, \Lambda)$. For $K \in D^+(Y, \Lambda)$, see A.5 (1).

[5] As W. Zheng observes, instead of using the projection formula of ([33], Th. finitude, Appendice, 1.2 (a)), one can apply Artin's results in ([32], XV). This argument does not use the assumption that f is locally of finite type. However, as Artin works with abelian torsion sheaves rather than with sheaves of Λ-modules, to deduce from ([32], XV 1.17) that $K \to Rf_{(x,y)*} f^*_{(x,y)} K$ is an isomorphism requires some technical preliminary reductions.

2 External tensor products

2.1

Fix a noetherian scheme S, and a noetherian $\mathbf{Z}/n\mathbf{Z}$-algebra Λ, with n invertible on S. For $i = 1, 2$, let $f_i : X_i \to Y_i$ be a morphism of finite type between noetherian S-schemes, and let $f = f_1 \times_S f_2 : X = X_1 \times_S X_2 \to Y = Y_1 \times_S Y_2$. Let $K_i \in D_{tf}(X_i, \Lambda)$, where the subscript *tf* means *finite tor-dimension*, and

$$K = K_1 \boxtimes_S^L K_2 := \mathrm{pr}_1^* K_1 \otimes^L \mathrm{pr}_2^* K_2 \in D_{tf}(X, \Lambda)$$

their external tensor product ($\otimes^L$ being taken over Λ). Consider the oriented topoi $X_i \overleftarrow{\times}_{Y_i} Y_i$, $X \overleftarrow{\times}_Y Y$. The morphisms $\mathrm{pr}_i : X \to X_i$, $Y \to Y_i$ define morphisms $\overleftarrow{\mathrm{pr}_i} : X \overleftarrow{\times}_Y Y \to X_i \overleftarrow{\times}_{Y_i} Y_i$. Let
(2.1.1)

$$R\Psi_{f_1} K_1 \boxtimes_S^L R\Psi_{f_2} K_2 := \overleftarrow{\mathrm{pr}_1}^* R\Psi_{f_1} K_1 \otimes^L \overleftarrow{\mathrm{pr}_2}^* R\Psi_{f_2} K_2 \in D_{tf}(X \overleftarrow{\times}_Y Y, \Lambda),$$

(note that $R\Psi_{f_i} K_i$ is in $D_{tf}(X_i \overleftarrow{\times}_{Y_i} Y_i, \Lambda)$ by ([21], Proposition 3.1)). We define a natural map

$$c : R\Psi_{f_1} K_1 \boxtimes_S^L R\Psi_{f_2} K_2 \to R\Psi_f K \tag{2.1.2}$$

as follows. Consider the diagram with commutative squares

$$\begin{tikzcd} X_1 \arrow[d, "\Psi_{f_1}"] & X \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\Psi_f"] & X_2 \arrow[d, "\Psi_{f_2}"] \\ X_1 \overleftarrow{\times}_{Y_1} Y_1 & X \overleftarrow{\times}_Y Y \arrow[l, "\overleftarrow{\mathrm{pr}_1}"'] \arrow[r, "\overleftarrow{\mathrm{pr}_2}"] & X_2 \overleftarrow{\times}_{Y_2} Y_2 \end{tikzcd} . \tag{2.1.3}$$

It produces base change maps (1.4.3)

$$\overleftarrow{\mathrm{pr}_i}^* R\Psi_{f_i} K_i \to R\Psi_f(\mathrm{pr}_i^* K_i), \tag{2.1.4}$$

hence a tensor product map

$$\overleftarrow{\mathrm{pr}_1}^* R\Psi_{f_1} K_1 \otimes^L \overleftarrow{\mathrm{pr}_2}^* R\Psi_{f_2} K_2 \to R\Psi_f(\mathrm{pr}_1^* K_1) \otimes^L R\Psi_f(\mathrm{pr}_2^* K_2),$$

which, composed with the canonical map

$$R\Psi_f(\mathrm{pr}_1^* K_1) \otimes^L R\Psi_f(\mathrm{pr}_2^* K_2) \to R\Psi_f K,$$

yields (2.1.2).

Remark 2.2. In the situation of 2.1, take $X_2 = Y_2$, $f_2 = \mathrm{Id}_{Y_2}$, so that $f : X \to Y$ is base changed from $f_1 : X_1 \to Y_1$ by $\mathrm{pr}_1 : Y = Y_1 \times_S Y_2 \to Y_1$. The base change map (2.1.4) for $i = 1$

$$\overleftarrow{\mathrm{pr}}_1^* R\Psi_{f_1} K_1 \to R\Psi_f(\mathrm{pr}_1^* K_1)$$

is a particular case of the Künneth map. Indeed, take $K_2 = \Lambda_{Y_2}$. then, by ([12], 4.7) applied to $\Psi = \Psi_{\mathrm{Id}_{Y_2}} : Y_2 \to Y_2 \overleftarrow{\times}_{Y_2} Y_2$, $R\Psi_\Lambda = \Lambda$, hence $\overleftarrow{\mathrm{pr}}_2^* R\Psi_{f_2} \Lambda = \Lambda$, the base change map $\overleftarrow{\mathrm{pr}}_2^* R\Psi_{f_2} \Lambda \to R\Psi_f(\mathrm{pr}_2^* \Lambda)$ is the adjunction map $\Lambda \to R\Psi_f \Lambda$, and the composition given above is just (2.1.4) for $i = 1$.

In particular, one cannot expect (2.1.2) to be an isomorphism in general (cf. (1.7, (d))).

Theorem 2.3. *With the notation of 2.1, assume that, for $i = 1, 2$, Y_i is of finite type over S, K_i is in $D_{ctf}(X_i, \Lambda)$, and (f_i, K_i) is Ψ-good (1.5). Then c (2.1.2) is an isomorphism.*

As mentioned in the introduction, a more general statement, with a much simpler proof, is given in A.3.

Proof. We proceed in several steps.

Step 1. We may assume $Y_1 = Y_2 = S$.

Consider the commutative diagram with cartesian squares

(2.3.1)
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \times_{Y_2} X_2 \arrow[r, "p_2"] \arrow[d] & X_2 \arrow[d, "f_2"'] \\
X_1 \times_{Y_1} Y \arrow[r] \arrow[d, "p_1"'] & Y \arrow[r] \arrow[d] & Y_2 \arrow[d] \\
X_1 \arrow[r, "f_1"] & Y_1 \arrow[r] & S
\end{tikzcd} .$$

By the assumptions on (f_i, K_i) the base change maps

$$\overleftarrow{p_i}^* R\Psi_{f_i} K_i \to R\Psi_{X_i \times_{Y_i} Y/Y}(p_i^* K_i),$$

are isomorphisms, hence we have an identification

$$R\Psi_{f_1} K_1 \boxtimes^L_S R\Psi_{f_2} K_2 = R\Psi_{(X_1 \times_{Y_1} Y)/Y}(p_1^* K_1) \boxtimes^L_Y R\Psi_{(Y \times_{Y_2} X_2)/Y}(p_i^* K_2),$$

which reduces the proof of 2.3 to the case where $Y_1 = Y_2 = S$.

Step 2. (2.1.2) is an isomorphism if we assume moreover that $Y_1 = Y_2 = S$ and (f, K) is Ψ-good.

We check that (2.1.2) is an isomorphism on the slices (1.3.4). Let $t \to s$ be a specialization of geometric points of S. With the notations of (1.3.4), we have morphisms $(i_\alpha)_s : (X_\alpha)_s \to (X_\alpha)_{(s)}$, $i_s : X_s \to X_{(s)}$, $(j_\alpha)_{(t)} : (X_\alpha)_{(t)} \to (X_\alpha)_{(s)}$, $j_{(t)} : X_{(t)} \to X_{(s)}$, ($\alpha = 1, 2$), where the subscript (s) (resp. (t)) means base change by the strict localization $S_{(s)} \to S$ (resp. $S_{(t)} \to S$). We need to show that the morphism

(2.3.2)
$$(i_1)_s^* R(j_1)_{(t)*} K_1 \boxtimes^L (i_2)_s^* R(j_2)_{(t)*} K_2 \to i_s^* R j_{(t)*} K$$

induced by (2.1.2) is an isomorphism, where K_α, K still denote the inverse images of K_α, K on $(X_\alpha)_{(t)}$, $X_{(t)}$. Choose a strictly local trait S' with closed point s' and generic geometric point t', and a morphism $S' \to S$ sending s' to s together with a morphism $t' \to t$, compatible with the specialization map $t \to s$. As (f_1, K_1), (f_2, K_2), (f, K) are Ψ-good, the morphism deduced from (2.2.2) by base change by $S' \to S$ is the morphism

(2.3.3)
$$(i_1)_{s'}^* R(j_1)_{(t')*} K'_1 \boxtimes^L (i_2)_{s'}^* R(j_2)_{(t')*} K'_2 \to i_s^* R j_{(t')*} K'$$

similar to (2.3.2), with $X \to S$ replaced by the base changed $X' \to S'$, and K'_α (resp. K') induced by K_α (resp. K). Therefore it suffices to check that (2.3.3) is an isomorphism, in other words, we may assume that S is a strictly local trait, with closed point s and generic geometric point t. By the comparison recalled in (1.2.8), (1.2.9), the conclusion then follows from Gabber's theorem on the compatibility of classical nearby cycles with external products ([9], 4.7).

Step 3. End of proof : (2.1.2) is an isomorphism if $Y_1 = Y_2 = S$.

Let $a : T_\bullet \to S$ be a hypercovering for the h-topology which is admissible for $(f : X = X_1 \times_S X_2 \to S, K = K_1 \boxtimes^L_S K_2)$ (1.8). Denote by $(K_i)_\bullet$ (resp. $K_\bullet$) the inverse image of K_i (resp. K) on $(X_i)_\bullet = X_i \times_S T_\bullet$ (resp. $X_\bullet = X \times_S T_\bullet$). We denote by

$$\overleftarrow{a} : (X_i)_\bullet \overleftarrow{\times}_{T_\bullet} T_\bullet \to X_i \overleftarrow{\times}_S S$$

($i = 1, 2$) and

$$\overleftarrow{a} : X_\bullet \overleftarrow{\times}_{T_\bullet} T_\bullet \to X \overleftarrow{\times}_S S$$

the morphisms of topoi induced by a ([23], 2.2.2), where by abuse the left hand sides denote the total topoi defined by the simplicial topoi. We have commutative squares

(2.3.4)
$$\begin{tikzcd} X_\bullet \overleftarrow{\times}_{T_\bullet} T_\bullet \arrow[r, "\overleftarrow{a}"] \arrow[d, "\overleftarrow{\mathrm{pr}}_i"'] & X \overleftarrow{\times}_S S \arrow[d, "\overleftarrow{\mathrm{pr}}_i"'] \\ (X_i)_\bullet \overleftarrow{\times}_{T_\bullet} T_\bullet \arrow[r, "\overleftarrow{a}"] & X_i \overleftarrow{\times}_S S \end{tikzcd} .$$

As (f_i, K_i) is Ψ-good, the base change maps

(2.3.5)
$$\overleftarrow{a}^* R\Psi_{f_i} K_i \to R\Psi_{(f_i)_\bullet} (K_i)_\bullet$$

are isomorphisms. Hence the same holds for their external tensor product, which, thanks to (2.3.4), can be re-written

(2.3.6)
$$\overleftarrow{a}^*(R\Psi_{f_1} K_1 \boxtimes^L_S R\Psi_{f_2} K_2) \to R\Psi_{(f_1)_\bullet}(K_1)_\bullet \boxtimes^L_S R\Psi_{(f_2)_\bullet}(K_2)_\bullet.$$

Consider the morphism of $D(X \overleftarrow{\times}_S S, \Lambda)$

(2.3.7)
$$c_\bullet : R\Psi_{(f_1)_\bullet}(K_1)_\bullet \boxtimes^L_S R\Psi_{(f_2)_\bullet}(K_2)_\bullet \to R\Psi_{f_\bullet} K_\bullet$$

defined similarly to (2.1.2), using the variant of diagram (2.1.3) with X_i (resp. X) replaced by $(X_i)_\bullet$ (resp. $X_\bullet$). (Note that because of (2.3.5) and (2.3.6) the left hand side is in D_{ctf}, while *a priori* the right hand side is only in D^+.) We will show that (2.3.7) is an isomorphism. As the family of restriction maps $D^+(X_\bullet \overleftarrow{\times}_{S_\bullet} S_\bullet, \Lambda) \to D^+(X_n \overleftarrow{\times}_{S_n} S_n, \Lambda)$ is conservative, it is enough to check that, for each $n \in \mathbf{N}$,

$$c_n : R\Psi_{(f_1)_n}(K_1)_n \boxtimes^L_S R\Psi_{(f_2)_n}(K_2)_n \to R\Psi_{f_n} K_n$$

is an isomorphism. This is indeed the case by Step 2, as (f_α, K_α) is Ψ-good and a is admissible for (f, K), hence (f_n, K_n) is Ψ-good. Now, consider the commutative square deduced from the functoriality of the construction of the Künneth maps,

(2.3.8)
$$\begin{tikzcd} R\Psi_{f_1} K_1 \boxtimes^L_S R\Psi_{f_2} K_2 \arrow[r, "c"] \arrow[d] & R\Psi_f K \arrow[d] \\ R\overleftarrow{a}_*(R\Psi_{(f_1)_\bullet}(K_1)_\bullet \boxtimes^L_S R\Psi_{(f_2)_\bullet}(K_2)_\bullet) \arrow[r] & R\overleftarrow{a}_* R\Psi_{f_\bullet} K_\bullet \end{tikzcd} ,$$

where the bottom horizontal map is $R\overleftarrow{a}_* c_\bullet$, and the vertical maps are the canonical maps deduced from the isomorphisms $a^* K_i = (K_i)_\bullet$, $a^* K = K_\bullet$. The bottom horizontal map is an isomorphism as we have just shown that $c_\bullet$ is an isomorphism. As (2.3.6) is an isomorphism, the left vertical map is an isomorphism by oriented descent (1.9.1). The right vertical arrow is the isomorphism (1.8.1). Therefore c is an isomorphism, which concludes the proof. $\square$

Corollary 2.4. *Under the assumptions of 2.3, (f, K) is Ψ-good.*

Proof. Indeed, as (f_i, K_i) is Ψ-good, the formation of $R\Psi_{f_1} K_1 \boxtimes_S^L R\Psi_{f_2} K_2$ commutes with arbitrary base change. More explicitly, for $g : S' \to S$, let $Y'_i := Y_i \times_S S'$, $X'_i := X_i \times_S S'$, $f'_i = f_i \times_S S' : X'_i \to Y'_i$, $X' := X \times_S S'$, $K'_i := K_i | X_i$. We then have a commutative diagram

$$\begin{tikzcd}
X'_1 \overset{\leftarrow}{\times}_{Y'_1} Y'_1 \arrow[d, "\overleftarrow{g}"'] & X' \overset{\leftarrow}{\times}_{Y'} Y' \arrow[l, "\overleftarrow{\mathrm{pr}_1}"'] \arrow[r, "\overleftarrow{\mathrm{pr}_2}"] \arrow[d, "\overleftarrow{g}"'] & X'_2 \overset{\leftarrow}{\times}_{Y'_2} Y'_2 \arrow[d, "\overleftarrow{g}"'] \\
X_1 \overset{\leftarrow}{\times}_{Y_1} Y_1 & X \overset{\leftarrow}{\times}_{Y} Y \arrow[l, "\overleftarrow{\mathrm{pr}_1}"'] \arrow[r, "\overleftarrow{\mathrm{pr}_2}"] & X_2 \overset{\leftarrow}{\times}_{Y_2} Y_2
\end{tikzcd}$$

Write Ψ_i (resp. Ψ'_i) for short for $R\Psi_{f_i} K_i$ (resp. $R\Psi_{f'_i} K'_i$). The above diagram gives a (trivial) isomorphism

$$\overleftarrow{g}^* (\overleftarrow{\mathrm{pr}_1}^* \Psi_1 \otimes^L \overleftarrow{\mathrm{pr}_2}^* \Psi_2) \xrightarrow{\sim} \overleftarrow{\mathrm{pr}_1}^* (\overleftarrow{g}^* \Psi_1) \otimes^L \overleftarrow{\mathrm{pr}_2}^* (\overleftarrow{g}^* \Psi_2). \tag{2.4.1}$$

By Ψ-goodness of (f_i, K_i), the base change maps $\overleftarrow{g}^* \Psi_i \to \Psi'_i$ are isomorphisms. By composing them with (2.4.1) we get an isomorphism

$$\overleftarrow{g}^* (\Psi_1 \boxtimes_S^L \Psi_2) \xrightarrow{\sim} \Psi'_1 \boxtimes_{S'}^L \Psi'_2, \tag{2.4.2}$$

with the notation of (2.1.1). This map fits into a commutative diagram

$$\begin{tikzcd}
\overleftarrow{g}^* (\Psi_1 \boxtimes_S^L \Psi_2) \arrow[r, "\overleftarrow{g}^* c"] \arrow[d, "(2.4.2)"'] & \overleftarrow{g}^* \Psi \arrow[d] \\
\Psi'_1 \boxtimes_{S'}^L \Psi'_2 \arrow[r, "c"] & \Psi'
\end{tikzcd}$$

where $\Psi := R\Psi_f K$, $\Psi' := R\Psi_{f'} K'$, and the right vertical map is the base change map, which is therefore an isomorphism, as c is an isomorphism. $\square$

In particular, combining with 2.3, we get:

Corollary 2.5. *In the situation of 2.3, assume that, for $i = 1, 2$, (f_i, K_i) is universally locally acyclic (1.7 (b)). Then (f, K) is universally locally acyclic.*

One can make 2.3 explicit on the stalks:

Corollary 2.6. *Under the assumptions of 2.3, let s be a geometric point of S. For $i = 1, 2$, let y_i be a geometric point of Y_i above s, x_i a geometric point of X_i above y_i, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ the corresponding geometric points* *of $X = X_1 \times_S X_2$ and $Y = Y_1 \times_X Y_2$. Then, with the notation of (1.11.2), the isomorphism c (2.1.2) and the commutative diagram*

(2.6.1)
$$\begin{tikzcd} X_1 \overleftarrow{\times}_{Y_1} Y_1 & X \overleftarrow{\times}_Y Y \arrow[l, "\overleftarrow{pr_1}"'] \arrow[r, "\overleftarrow{pr_2}"] & X_2 \overleftarrow{\times}_{Y_2} Y_2 \\ (Y_1)_{(y_1)} \arrow[u, "\sigma_{x_1,y_1}"'] & Y_{(y)} \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[u, "\sigma_{x,y}"'] & (Y_2)_{(y_2)} \arrow[u, "\sigma_{x_2,y_2}"'] \end{tikzcd},$$

induce an isomorphism in $D_{ctf}(Y_{(y)}, \Lambda)$:

(2.6.2)
$$\sigma^*_{x_1,y_1} R\Psi_{f_1} K_1 \boxtimes^L \sigma^*_{x_2,y_2} R\Psi_{f_2} K_2 \xrightarrow{\sim} \sigma^*_{x,y}(R\Psi_f K),$$

where $\boxtimes^L$ in the left hand side means $pr_1^* \otimes^L pr_2^*$, with pr_i as in (2.6.1).

Corollary 2.7. *In the situation of 2.1, assume that Y_i $(i = 1, 2)$ is regular of dimension ≤ 1. Then the map c (2.1.2) is an isomorphism, and (f, K) is Ψ-good.*

Proof. Indeed, for $(i = 1, 2)$, (f_i, K_i) is Ψ-good (1.7 (a)). □

2.8

Let's come back to the situation of 2.1. In order to analyze the behavior of $R\Phi$ under the Künneth map (2.1.2), we need to use the refined objects $R\underline{\Psi}_{f_i} K_i$ (1.2.6) of $DF^{[0,1]}_{tf}(X_i \overleftarrow{\times}_{Y_i} Y_i, \Lambda)$. Their external tensor product is a 2-step filtered object :

(2.8.1)
$$R\underline{\Psi}_{f_1} K_1 \boxtimes^L_S R\underline{\Psi}_{f_2} K_2 \in DF^{[0,2]}_{tf}(X \overleftarrow{\times}_Y Y, \Lambda),$$

with associated graded

(2.8.2)
$$\mathrm{gr}(R\underline{\Psi}_{f_1} K_1 \boxtimes^L_S R\underline{\Psi}_{f_2} K_2) = \mathrm{gr} R\underline{\Psi}_{f_1} K_1 \boxtimes^L_S \mathrm{gr} R\underline{\Psi}_{f_2} K_2,$$

i. e.

$$\begin{aligned} \mathrm{gr}^0(R\underline{\Psi}_{f_1} K_1 \boxtimes^L_S R\underline{\Psi}_{f_2} K_2) &= R\Phi_{f_1} K_1 \boxtimes^L_S R\Phi_{f_2} K_2 \\ \mathrm{gr}^1(R\underline{\Psi}_{f_1} K_1 \boxtimes^L_S R\underline{\Psi}_{f_2} K_2) &= (R\Phi_{f_1} K_1 \boxtimes^L_S p_1^* K_2) \oplus (p_1^* K_1 \boxtimes^L_S R\Phi_{f_2} K_2) \\ \mathrm{gr}^2(R\underline{\Psi}_{f_1} K_1 \boxtimes^L_S R\underline{\Psi}_{f_2} K_2) &= p_1^* K_1 \boxtimes^L_S p_2^* K_2. \end{aligned}$$

It follows that under the assumptions of 2.3, the isomorphism c (2.1.2) defines a filtered object $(R\Psi_f K, F_2) \in DF^{[0,2]}(X \overleftarrow{\times}_Y Y, \Lambda)$, with associated graded given by (2.8.2), where F_2 refines the filtration F of (1.2.6) :

$$\begin{aligned} F_2^2 = F^1 &= p_1^* K_1 \boxtimes^L_S p_2^* K_2 \\ R\Phi_f K &= \mathrm{gr}^0_F = F_2^0 / F_2^2. \end{aligned}$$

In particular, we have a distinguished triangle
(2.8.3)
$$(R\Phi_{f_1} K_1 \boxtimes^L_S p_1^* K_2) \oplus (p_1^* K_1 \boxtimes^L_S R\Phi_{f_2} K_2) \to R\Phi_f K \to R\Phi_{f_1} K_1 \boxtimes^L_S R\Phi_{f_2} K_2 \to .$$

Thus, in the situation of 2.6, as by (1.12.9) $\sigma^*_{x_i,y_i} p_1^* K_i$ is the constant complex on $Y_{(y_i)}$ of value $(K_i)_{x_i}$, we get a distinguished triangle

$$(2.8.4)\quad (\sigma^*_{x_1,y_1} R\Phi_{f_1} K_1 \boxtimes^L (K_2)_{x_2}) \oplus ((K_1)_{x_1} \boxtimes^L \sigma^*_{x_2,y_2} R\Phi_{f_2} K_2) \to \sigma^*_{x,y} R\Phi_f K$$
$$\to \sigma^*_{x_1,y_1} R\Phi_{f_1} K_1 \boxtimes^L \sigma^*_{x_2,y_2} R\Phi_{f_2} K_2 \to .$$

The following is a generalization of 2.5:

Corollary 2.9. *In the situation of 2.1, assume that, for $i = 1, 2$, there is an open subset U_i of X_i with $\Sigma_i = X_i - U_i$ quasi-finite over Y_i such that $(f_i, K_i)|U_i$ is universally locally acyclic ([33], Th. finitude, 2.12). Let $U := U_1 \times_S U_2$, $\Sigma := X - U = (\Sigma_1 \times_S X_2) \cup (X_2 \times_S \Sigma_2)$. Then (f_i, K_i) $(i = 1, 2)$ and (f, K) are Ψ-good, and $R\Phi_f(K)$ is concentrated on $\Sigma \overleftarrow{\times}_Y Y$.*

Proof. As (f_i, K_i) is Ψ-good (1.7 (c)), (f, K) is also Ψ-good (2.4). As $(f_i, K_i)|U_i$ is universally locally acyclic, $R\Phi_{(f_i|U_i)}(K_i|U_i) = 0$ (1.7 (b)). By (2.8.3) we therefore have $R\Phi_f K|U \overleftarrow{\times}_Y Y = 0$, which implies the last assertion, as the complement of the closed subtopos $\Sigma \overleftarrow{\times}_Y Y$ of $X \overleftarrow{\times}_Y Y$ is the open subtopos $U \overleftarrow{\times}_Y Y$. □

3 Interlude: additive convolution

3.1

We fix a perfect field k of characteristic exponent p, an algebraic closure $\overline{k}$ of k, and a *finite* ring Λ annihilated by an integer invertible in k. We denote by $\pi_1^t(\mathbf{G}_{m,k}, \{\overline{1}\})$ the tame quotient of the fundamental group of $\mathbf{G}_{m,k}$, which is an extension

$$(3.1.1)\qquad 1 \to I_t \to \pi_1^t(\mathbf{G}_{m,k}, \{\overline{1}\}) \to \mathrm{Gal}(\overline{k}/k) \to 1,$$

where

$$(3.1.2)\qquad I_t\ (\xrightarrow{\sim} \widehat{\mathbf{Z}}'(1) = \varinjlim_{(n,p)=1} \mu_n(\overline{k}))$$

is the tame quotient of the geometric fundamental group $I = \pi_1(\mathbf{G}_{m,\overline{k}}, \{\overline{1}\})$ of $\mathbf{G}_{m,k} = \mathbf{A}^1_k - \{0\}$ (here $\overline{1}$ means the unit of $\mathbf{G}_{m,\overline{k}}$). If $\overline{\eta}_0$ (resp. $\overline{\eta}_\infty$) is a geometric point above the generic point η_0 (resp. η_∞) of the henselization of $\mathbf{A}^1_k$ at $\{0\}$ (resp. $\{\infty\}$), we can consider the tame quotient $(I_t)_0$ (resp. $(I_t)_\infty$) of the inertia subgroup $I_0 \subset \mathrm{Gal}(\overline{\eta}_0/\eta_0)$ (resp. $(I_t)_\infty \subset \mathrm{Gal}(\overline{\eta}_\infty/\eta_\infty)$, which maps isomorphically to I_t (once a path is chosen from $\overline{\eta}_0$ (resp. $\overline{\eta}_\infty$) to $\{\overline{1}\}$).

Recall that a lisse sheaf L of Λ-modules on $\mathbf{G}_{m,k}$ is tamely ramified at 0 (resp. ∞) if the action of I_0 (resp. I_∞) on $L_{\overline{\eta}_0}$ (resp. $L_{\overline{\eta}_\infty}$) factors through $(I_t)_0$ (resp. $(I_t)_\infty$) (or, equivalently, through a finite quotient of it, as Λ has been assumed finite). The sheaf L is tamely ramified at 0 and ∞ if and only if the action of I on $L_{\{\overline{1}\}}$ factors through I_t.

More generally, given $L \in D^b_c(\mathbf{A}^1_k, \Lambda)$, we say that L is *tamely ramified* at 0 (resp. ∞) if, for all q, $(\mathcal{H}^q L)_{\eta_0}$ (resp. $\mathcal{H}^q L)_{\eta_\infty}$) is tame (i.e., the action of I_0 (resp. I_∞) on $\mathcal{H}^q L_{\overline{\eta}_0}$ (resp. $\mathcal{H}^q L_{\overline{\eta}_\infty}$) factors through $(I_t)_0$ (resp. $(I_t)_\infty$)).

We denote by

$$a : \mathbf{A}^1_k \times \mathbf{A}^1_k \to \mathbf{A}^1_k \tag{3.1.3}$$

the sum map.

A. Global additive convolution

Recall the following definition, due to Deligne.

Definition 3.2. The (additive) convolution functor

$$*^L : D_{ctf}(\mathbf{A}^1_k, \Lambda) \times D_{ctf}(\mathbf{A}^1_k, \Lambda) \to D_{ctf}(\mathbf{A}^1_k, \Lambda)$$

is defined by

$$K_1 *^L K_2 := Ra_*(K_1 \boxtimes^L_\Lambda K_2).$$

Remark 3.3. (a) There is a variant $*^L_!$ of $*^L$, with Ra_* replaced by $Ra_!$. These two constructions are exchanged by the dualizing functor on $\mathbf{A}^1_k$, $D := R\mathcal{H}om(-, \Lambda(1)[2])$:

$$D(K_1 *^L_! K_2) = DK_1 *^L DK_2. \tag{3.3.1}$$

The $*^L_!$ definition has the advantage of corresponding to the usual convolution of trace functions, when k is a finite field $\mathbf{F}_q$, i. e. if, for $K \in D_{ctf}(\mathbf{A}^1_k, \Lambda)$, and $n \geq 1$, f_K denotes the function on $\mathbf{A}^1_k(\mathbf{F}_{q^n})$ defined by $f_K(x) = \mathrm{Tr}(F^n, K_x)$, then

$$f_{K_1 *^L_! K_2} = f_{K_1} * f_{K_2}.$$

(b) Assume $p > 1$. If $\psi : \mathbf{F}_p \to \Lambda^*$ is a non trivial additive character, we can consider the Fourier transform $\mathcal{F} : \mathcal{F}_\psi : D_{ctf}(\mathbf{A}^1_k, \Lambda) \to D_{ctf}(\mathbf{A}^1_k, \Lambda)$, defined by the formula (1.2.1.1) of [20], with $\overline{\mathbf{Q}}_\ell$ replaced by Λ. By ([20], 1.2.2.7), we have

$$\mathcal{F}(K_1 *^L_! K_2) \xrightarrow{\sim} (\mathcal{F}K_1 \otimes^L \mathcal{F}K_2)[-1]. \tag{3.3.2}$$

By ([20], 1.3.2.2), $D\mathcal{F}_\psi = (\mathcal{F}_{\psi^{-1}} D)(1)$, so that applying (3.3.1), we get

$$\mathcal{F}(K_1 *^L K_2) \xrightarrow{\sim} (\mathcal{F}K_1 \otimes^{!L} \mathcal{F}K_2)(1)[1], \tag{3.3.3}$$

where $L_1 \otimes^{!L} L_2 := D(DL_1 \otimes^L DL_2)$. As $L_1 \otimes^{!L} L_2 = \Delta^!(L_1 \boxtimes^L L_2)$ where $\Delta : \mathbf{A}^1_k \to \mathbf{A}^2_k$ is the diagonal, we have $L_1 \otimes^{!L} L_2 = (L_1 \otimes^L L_2)[-2](-1)$, so, from (3.3.3) we get

$$\mathcal{F}(K_1 *^L K_2) \xrightarrow{\sim} (\mathcal{F}K_1 \otimes^L \mathcal{F}K_2)[-1], \tag{3.3.4}$$

which is actually the same formula as (3.3.2).

The following results are standard (cf. [2], and [19], [20] for the case of $*!$ and $\overline{\mathbf{Q}}_\ell$-coefficients):

Proposition 3.4. *Let $e : \mathbf{A}^1_k \to \operatorname{Spec} k$ be the projection, and $i : \{0\} \to \mathbf{A}^1_k$, $j : \mathbf{A}^1_k - \{0\} \to \mathbf{A}^1_k$ be the inclusions. Let $K_i \in D_{ctf}(\mathbf{A}^1_k, \Lambda)$ $(i = 1, 2)$.*

(1) For K_1 concentrated at $\{0\}$, i. e. $K_1 = i_* i^* K_1$, we have a canonical isomorphism

$$K_1 *^L K_2 \xrightarrow{\sim} e^*(i^* K_1) \otimes^L K_2.$$

(2) Assume that K_2 is geometrically constant, i. e. is of the form e^*L_2 for $L_2 \in D_{ctf}(\operatorname{Spec} k, \Lambda)$. Then we have a canonical isomorphism

$$K_1 *^L K_2 \xrightarrow{\sim} e^* Re_* K_1 \otimes^L K_2, \tag{3.4.1}$$

and

$$R\Phi_a(K_1 \boxtimes^L K_2) = 0. \tag{3.4.2}$$

If in addition K_1 is concentrated in degree 0, $(K_1)_0 = 0$, lisse on $\mathbf{A}^1_k - \{0\}$, and tamely ramified at $\{0\}$ and $\{\infty\}$, then

$$K_1 *^L K_2 = 0. \tag{3.4.3}$$

Similar statement with K_1 and K_2 interchanged.

(3) Assume that, for $i = 1, 2$, K_i is concentrated in degree 0, lisse on $\mathbf{A}^1_k - \{0\}$, tamely ramified at ∞, and $(K_i)_0 = 0$. Then

$$\mathcal{H}^q(K_1 *^L K_2) = 0 \tag{3.4.4}$$

for $q \neq 1$, and $\mathcal{H}^1(K_1 *^L K_2)$ is a constructible sheaf of finitely generated and projective Λ-modules, lisse on $\mathbf{A}^1_k - \{0\}$. Its generic rank r is

$$r = r_1 r_2 + r_1 s_2 + r_2 s_1, \tag{3.4.5}$$

where $r_i = \operatorname{rk}(K_i)$, $s_i = \operatorname{sw}_0(K_i)$, sw_0 denoting the Swan conductor at 0. Its rank at zero is

$$\operatorname{rk}(\mathcal{H}^1(K_1 *^L K_2)_0) = \operatorname{sw}_0(K_1 \otimes [-1]^* K_2), \tag{3.4.6}$$

where $[-1]$ is the automorphism of $\mathbf{A}^1_k = \operatorname{Spec} k[t]$ given by $t \mapsto -t$.

For any homomorphism $\Lambda \to \Lambda'$ of rings satisfying the assumptions of 3.1, if $K'_i := K_i \otimes_\Lambda \Lambda'$, the natural map

$$\mathcal{H}^1(K_1 *^L_\Lambda K_2) \otimes_\Lambda \Lambda' \to \mathcal{H}^1(K'_1 *^L_{\Lambda'} K'_2)$$

is an isomorphism.

For any geometric point t of $\mathbf{A}^1_k$, the canonical map

$$\mathcal{H}^1(K_1 *^L K_2)_t \to H^1(a^{-1}(t), K_1 \boxtimes^L K_2) \tag{3.4.7}$$

is an isomorphism.

(4) Under the assumptions of (3), let $a_{(0)} : \mathbf{A}^2_k \times_{\mathbf{A}^1_k} S \to S$ be the map deduced from a by base change to the henselization S at $\{0\}$ of $\mathbf{A}^1_k$. Then $R\Phi_{a_{(0)}}(K_1 \boxtimes^L K_2) \in D_{ctf}(a^{-1}(0) \overleftarrow{\times}_S \eta, \Lambda)$ is concentrated at $(0,0)$ and in degree 1, and we have

(3.4.8)

$$R\Psi_{a_{(0)}}(K_1 \boxtimes^L K_2)_{(0,0)}[1] = R\Phi_{a_{(0)}}(K_1 \boxtimes^L K_2)_{(0,0)}[1] = R^0\Phi_{\mathrm{Id}}(\mathcal{H}^1(K_1 *^L K_2))_0$$

where Id means the identity of $\mathbf{A}^1_{(0)}$, with

$$\operatorname{rk}(R^0\Phi_{\mathrm{Id}}(\mathcal{H}^1(K_1 *^L K_2)))_0 = r_1 r_2 + r_1 s_2 + r_2 s_1 - \operatorname{sw}_0(K_1 \otimes [-1]^* K_2). \tag{3.4.9}$$

Proof. We may (and will) assume k algebraically closed. We write A for $\mathbf{A}^1_k$.

(1) Put $L_1 := i^*K_1$, and let $i_1 := i \times \mathrm{Id} : \{0\} \times A \to A^2$. We have

$$K_1 \boxtimes^L K_2 = i_{1*}(e^*L_1 \otimes^L K_2),$$

and, as $ai_1 : \{0\} \times A \to A$ is the identity, we get

$$K_1 *^L K_2 = Ra_*(K_1 \boxtimes^L K_2) = R(ai_1)_*(e^*L_1 \otimes^L K_2) \xrightarrow{\sim} e^*L_1 \otimes^L K_2.$$

(2) As $\mathrm{pr}_2^*e^* = \mathrm{pr}_1^*e^*$, we have

$$K_1 *^L K_2 = Ra_*\mathrm{pr}_1^*(K_1 \otimes^L e^*L_2).$$

Let

$$\varphi : A^2 \xrightarrow{\sim} A^2$$

be the isomorphism given by $\varphi(x, y) = (x+y, x)$. We have $\mathrm{pr}_1\varphi = a$, $\mathrm{pr}_2\varphi = \mathrm{pr}_1$, hence

$$K_1 *^L K_2 = R\mathrm{pr}_{1*}\varphi_*\mathrm{pr}_1^*(K_1 \otimes^L e^*L_2),$$

but the base change map $\mathrm{pr}_2^* \to \varphi_*\mathrm{pr}_1^*$ is (trivially) an isomorphism, so we get

$$K_1 *^L K_2 \xrightarrow{\sim} R\mathrm{pr}_{1*}\mathrm{pr}_2^*(K_1 \otimes^L e^*L_2).$$

Finally, by smooth base change and the projection formula, we get

$$K_1 *^L K_2 \xrightarrow{\sim} e^*Re_*(K_1 \otimes^L e^*L_2) \xrightarrow{\sim} e^*Re_*K_1 \otimes^L K_2.$$

Similarly, we have

$$R\Phi_a(K_1 \boxtimes^L K_2) \xrightarrow{\sim} R\Phi_{\mathrm{pr}_1}\mathrm{pr}_2^*(K_1 \otimes^L e^*L_2),$$

and for any $M \in D_{ctf}(A, \Lambda)$, $R\Phi_{\mathrm{pr}_1}\mathrm{pr}_2^*M = 0$ by universal local acyclicity for schemes of finite type over a field ([33], Th. finitude, 2.16) (cf. 1.7 (b)). For the last assertion, it suffices to show $Re_*K_1 = 0$. If L_1 is the largest constant subsheaf of j^*K_1, we have $e_*K_1 = e_*(j_!L_1) = 0$. It remains to show $R^1e_*K_1 = 0$. We have

$$\chi(A, K_1) = \chi(A, j_!j^*K_1) = \chi(A, Rj_*j^*K_1) \tag{3.4.10}$$

as $\chi(\{0\}, i^*Rj_*j^*K_1) = 0$ by Laumon's theorem [16][6]. As K_1 is tamely ramified at $\{0\}$ and $\{\infty\}$, we have

$$\chi(A - \{0\}, j^*K_1) = \chi(A - \{0\}, \Lambda)\mathrm{rk}(j^*K_1) = 0,$$

by the (Ogg-Shafarevitch case of the) Grothendieck-Ogg-Shafarevitch formula, which finishes the proof of (3.4.3).

(3) Put $K = K_1 \boxtimes K_2$. We use the partial compactification of ([20], 2.7.1.1 (iii)),

$$A \times_k A \xrightarrow{(\mathrm{pr}_1, a)} A \times_k A \xrightarrow{\alpha \times \mathrm{Id}} D \times A,$$

[6] As the referee points out, in the case of a strictly local trait $(S, i : s \to S, j : \eta \to S)$, Laumon's theorem to the effect that for a sheaf $\mathcal{F}$ on η, $\chi(s, i^*Rj_*\mathcal{F}) = 0$ is an immediate consequence of the standard formulas giving $i^*R^qj_*\mathcal{F}$ (cf. [33], Dualité, proof of 1.3).

where $D = \mathbf{P}^1_k = A \cup \{\infty\}$, $\alpha : A \hookrightarrow D$ is the inclusion, and let $\overline{pr}_2 : D \times_k A \to A$ denote the projection. If t is a geometric point of A, we have

$$(Ra_*K)_t = R\Gamma(D \times \{t\}, (R(\alpha \times \mathrm{Id})_*(\mathrm{pr}_1, a)_*K)|D \times \{t\})$$

by properness of $\overline{\mathrm{pr}}_2$. As $(\mathrm{pr}_1, a)_*K$ is tamely ramified along $\{\infty\} \times_k A$, by ([33], Th. finitude, Appendice, 1.3.3 (i)) we have

$$(R(\alpha \times \mathrm{Id})_*(\mathrm{pr}_1, a)_*K)|D \times \{t\} \xrightarrow{\sim} R\alpha_{t*}((\mathrm{pr}_1, a)_*K|A \times \{t\}),$$

where $\alpha_t : A \times \{t\} \hookrightarrow D \times \{t\}$ is the fiber of $\alpha \times \mathrm{Id}$ at t. Finally, $(\mathrm{pr}_1, a)_*K|A \times \{t\} = a_{t*}(K|a^{-1}(t))$, where $a_t : a^{-1}(t) \xrightarrow{\sim} A \times \{t\}$ is induced by pr_1, and we get that the specialization map (3.4.7)

$$(Ra_*K)_t \to R\Gamma(a^{-1}(t), K).$$

is an isomorphism. This also implies (3.4.4), since $a^{-1}(t) \xrightarrow{\sim} A$ is affine, and $H^0(a^{-1}(t), K) = 0$ as $K|a^{-1}(t)$ is lisse outside $(t, 0)$ and $(0, t)$ and vanishes at these points. The compatibility of the formation of $\mathcal{H}^1(K_1 *^L K_2)$ with extension of scalars $\Lambda \to \Lambda'$ follows, as $K_1 *^L K_2$ is in $D_{ctf}(A, \Lambda)$. Let us show that $\mathcal{H}^1(K_1 *^L K_2)$ is lisse on $A - \{0\}$. Let $E = R(\alpha \times \mathrm{Id})_*(\mathrm{pr}_1, a)_*K$. For a geometric point t of $A - \{0\}$, let $a_{(t)}$ (resp. $\overline{pr}_{2(t)}$) denote the map deduced from a (resp. $\overline{pr}_2$) by henselization at t. As $(\mathrm{pr}_1, a)_*K$ is tamely ramified along $\{\infty\} \times_k A$, by a classical result (cf. ([26], 3.14)) $(\overline{pr}_2, E)$ is universally locally acyclic along $\{\infty\} \times_k A$, hence $R\Phi_{\overline{pr}_{2(t)}}E \in D^b_c(D \times_k \{t\}, \Lambda)$ is zero at (∞, t). On the other hand, at $(0, t)$ (resp. (t, t)), $R\Phi_{\overline{pr}_{2(t)}}E$, which is isomorphic to $R\Phi_{a_{(t)}}K$ at $(0, t)$ (resp. $(t, 0)$), is also zero. Indeed, on the strict localization of A^2 at (a geometric point above) $(0, t)$, $\mathrm{pr}_2^*K_2$ is constant, so we may assume $K_2 = \Lambda$. Then, as above, by the isomorphism φ in the proof of (2), $R\Phi_{a_{(t)}}K$ is identified with $R\Phi_{\mathrm{pr}_1}\mathrm{pr}_2^*K_1$ at $(t, 0)$, which is zero. The proof at $(t, 0)$ is similar. It follows that $(\overline{\mathrm{pr}}_2, E)$ is universally locally acyclic, and as $\overline{pr}_2$ is proper, this implies that the specialization and cospecialization maps at t are isomorphisms ([33], Th. finitude, Appendice, 2.4), in other words, that $\mathcal{H}^1(K_1 *^L K_2)$ is lisse on $\mathbf{G}_{m,k}$.

Let us prove (3.4.5). By (3.4.7),

$$r = \mathrm{rk}(H^1(a^{-1}(t), K))$$

for $t \in A - \{0\}$. As $K|a^{-1}(t)$ is tamely ramified at ∞, and of rank r_1r_2, and taking into account that $\chi(a^{-1}(t), K) = \chi_c(a^{-1}(t), K)$ by [16], the Grothendieck-Ogg-Shafarevich formula gives

$$\chi(a^{-1}(t), K) = r_1r_2 - (r_1r_2 + \mathrm{sw}_{(0,t)}(K)) - (r_1r_2 + \mathrm{sw}_{(t,0)}(K)).$$

As K_2 (resp. K_1) is lisse at $(0, t)$ (resp. $(t, 0)$), we have $\mathrm{sw}_{(0,t)}(K) = s_1r_2$ (resp. $\mathrm{sw}_{(t,0)}(K) = s_2r_1$), which gives

$$\chi(a^{-1}(t), K) = -r_1r_2 - s_1r_2 - s_2r_1,$$

hence (3.4.5) by (3.4.4). Finally, on $a^{-1}(0)$, K is isomorphic to $K_1 \otimes [-1]^*K_2$ on A, hence $\mathrm{sw}_0(K|a^{-1}(0)) = \mathrm{sw}_0(K_1 \otimes [-1]^*K_2)$. Using again (3.4.7) (at $t = 0$) and the Grothendieck-Ogg-Shafarevich formula, we get (3.4.6).

(4) This is similar to ([20], 2.7.1.1 (iii)), whose proof works for Λ as a coefficients ring. However, as here we consider $*^L$ and not $*^L_!$, a justification is needed. We have just seen that, for $E = R(\alpha \times \mathrm{Id})_*(\mathrm{pr}_1, a)_* K$, where $K = K_1 \boxtimes^L K_2$, $(\overline{\mathrm{pr}}_2, E)$ is universally locally acyclic outside $(0,0)$, i.e., $R\Phi_{\overline{pr}_2} E$ is concentrated at $(0,0)$. The triangle deduced from $E|\overline{pr}_2^{-1}(0) \to R\Psi_{\overline{pr}_2}(E) \to R\Phi_{\overline{pr}_2}(E) \to$ by applying $R\Gamma(\overline{pr}_2^{-1}(0), -)$ reads

$$Ra_*(K)_0 \to Ra_*(K)_{\overline{\eta}} \to R\Phi_{a(0)}(K)_{(0,0)} \to . \tag{3.4.11}$$

Indeed, $R\Gamma(\overline{pr}_2^{-1}(0), E|\overline{pr}_2^{-1}(0)) = (R\overline{pr}_{2*}E)_0$ by proper base change,

$$R\Gamma(\overline{pr}_2^{-1}(0), R\Psi_{\overline{pr}_2}(E)) = R\Gamma(\overline{pr}_2^{-1}(\overline{\eta}), E) = (R\overline{pr}_{2*}E)_{\overline{\eta}}$$

by properness of $\overline{pr}_2$, and $R\Gamma(\overline{pr}_2^{-1}(0), R\Phi_{\overline{pr}_2}(E)) = R\Phi_{\overline{pr}_2} E_{(0,0)} = \Phi_a(K)_{(0,0)}$ by concentration of $R\Phi_{\overline{pr}_2}(E)$ at $(0,0)$. As $R\overline{pr}_{2*}E = Ra_*K = K_1 *^L K_2$ by definition of E, and taking account (3.4.4), we get (3.4.8), and (3.4.9 by (3.4.11), (3.4.5), and (3.4.6). □

Remark 3.5. In [20], 2.7.1.1 (i)) it is proved that $K_1 *^L_! K_2$ is tamely ramified at $\{\infty\}$, using a Fourier transform for $\overline{\mathbf{Q}}_\ell$-sheaves. We give a proof of the similar results for $K_1 *^L K_2$ (and torsion coefficients) in 5.11.

If, for $i = 1, 2$, M_i is a lisse sheaf of finitely generated and projective Λ-modules on $\mathbf{G}_{m,k}$, tamely ramified at ∞, we will write

$$M_1 *_1 M_2 := j^*(j_! M_1 *_1 j_! M_2). \tag{3.5.1}$$

This is a sheaf on $\mathbf{G}_{m,k}$.

Corollary 3.6. *In the situation of 3.4, assume that K_1 is concentrated in degree 0, $(K_1)_0 = 0$, lisse on $\mathbf{G}_{m,k}$, and tamely ramified at $\{0\}$ and $\{\infty\}$, and that K_2 is geometrically constant (so that we have (3.4.3)). Then the triangle*

$$j_! j^* K_2 \to K_2 \to i_*(K_2)_0 \to, \tag{3.6.1}$$

where $i : \{0\} \to \mathbf{A}^1_k$ is the inclusion, induces an isomorphism

$$K_1 *^L j_! j^* K_2 \xrightarrow{\sim} K_1 \otimes^L e^*((K_2)_0)[-1]. \tag{3.6.2}$$

If M_i $(i = 1, 2)$ is a lisse sheaf of finitely generated and projective Λ-modules on $\mathbf{G}_{m,k}$, if M_1 is tamely ramified at $\{0\}$ and $\{\infty\}$ and M_2 is geometrically constant, the above isomorphism yields an isomorphism

$$M_1 *_1 M_2 \xrightarrow{\sim} M_1 \otimes M_2. \tag{3.6.3}$$

Similar statements with K_1 and K_2 (resp. M_1 and M_2) interchanged. In particular, we have natural isomorphisms

$$\Lambda *_1 M \xrightarrow{\sim} M *_1 \Lambda \xrightarrow{\sim} M, \tag{3.6.4}$$

for M satisfying the assumption of M_1 above.

Remark 3.7. Let $\mathcal{C}$ be the category of lisse, finitely generated and projective Λ-modules on $\mathbf{G}_{m,k}$, which are tamely ramified at ∞. We leave it to the reader to check that, in addition to (3.6.4), the functor

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C}, (M_1, M_2) \mapsto M_1 *_1 M_2$$

verifies associativity and commutativity isomorphisms (for the associativity, use the Künneth formula for $a_m \times_k a_n : \mathbf{A}_k^m \times_k \mathbf{A}_k^n \to \mathbf{A}_k^1 \times_k \mathbf{A}_k^1$ ([34], III, (1.6.4)), $a_r : \mathbf{A}_k^r \to \mathbf{A}_k^1$ denoting the sum map), satisfying the usual constraints.

B. Local additive convolution

The following construction is a slight variant of a construction of Laumon ([20], 2.7.2). Denote by $A_h = \operatorname{Spec} k\{t\}$ (resp. $A_{sh} = \operatorname{Spec} \overline{k}\{t\}$) the henselization at 0 (resp. the strict henselization at $\{\overline{0}\}$ of the affine line $A = \mathbf{A}_k^1 = \operatorname{Spec} k[t]$. Denote again by

$$a : (A^2)_h \to A_h \tag{3.7.1}$$

(resp. $a : (A^2)_{sh} \to A_{sh}$) the morphism induced by the sum map $a : A^2 \to A$. Let $\mathrm{pr}_i : (A^2)_h \to A_h$ (resp. $\mathrm{pr}_i : (A^2)_{sh} \to A_{sh}$) be the morphism induced by the i-th projection. We denote by $\overline{\eta}$ a geometric point of A_{sh} above the generic point η of A_h.

Definition 3.8. The *local (additive) convolution functor*

$$*^L : D_{ctf}(A_h, \Lambda) \times D_{ctf}(A_h, \Lambda) \to D_{ctf}(A_h, \Lambda)$$

is defined by

$$K_1 *^L K_2 := R\Psi_a(K_1 \boxtimes^L K_2)_{(0,0)}, \tag{3.8.1}$$

where $K_1 \boxtimes^L K_2 := \mathrm{pr}_1^* K_1 \otimes^L \mathrm{pr}_2^* K_2 \in D_{ctf}((A^2)_h, \Lambda)$, and the subscript $(0,0)$ denotes the restriction to the closed point $(0,0)$ of the usual nearby cycles complex for $a : (A^2)_h \to A_h$ [SGA 7 XIII 2.1.1]; in the notation of 1.2, this is $R\Psi_a(K_1 \boxtimes^L K_2)|(0,0)\overleftarrow{\times}_A A$ (cf. (1.2.9)), where $(0,0)\overleftarrow{\times}_A A$ is identified with $(0)\overleftarrow{\times}_A A \xrightarrow{\sim} A_h$ by (1.2.7).

For $K_i \in D_{ctf}(A_{sh}, \Lambda)$, we define $K_1 *^L K_2 \in D_{ctf}(A_{sh}, \Lambda)$ similarly.

For $K_i \in D_{ctf}(A_h, \Lambda)$, we have a distinguished triangle

$$(K_1)_0 \otimes^L (K_2)_0 \to K_1 *^L K_2 \to R\Phi_a(K_1 \boxtimes^L K_2)_{(0,0)} \to, \tag{3.8.2}$$

so, if $(K_1)_0$ or $(K_2)_0$ is zero,

$$K_1 *^L K_2 \xrightarrow{\sim} R\Phi_a(K_1 \boxtimes^L K_2)_{(0,0)}. \tag{3.8.3}$$

On the other hand, by (1.12.6), we have

$$(K_1 *^L K_2)|A_{sh} \xrightarrow{\sim} \sigma^*_{(\overline{0},\overline{0})} R\Psi_a(K_1 \boxtimes^L K_2) \xrightarrow{\sim} (K_1|A_{sh}) *^L (K_2|A_{sh}). \tag{3.8.4}$$

The following result is a variant of ([20], 2.7.1.3) :

Proposition 3.9. *For $V_i \in D_{ctf}(\eta, \Lambda)$ concentrated in degree zero (i. e. given by a representation of* $\mathrm{Gal}(\overline{\eta}/\eta)$ *in a finitely generated projective Λ-module), $j_! V_1 *^L j_! V_2$ is concentrated in degree 1, i. e.*

$$\mathcal{H}^i(j_! V_1 *^L j_! V_2) = 0$$

for $i \neq 1$, and $\mathcal{H}^1(j_! V_1 *^L j_! V_2)_{\overline{\eta}}$ is projective and of finite type over Λ.

Proof. The proof is the same as that of ([20], 2.7.1.3). By the Gabber-Katz extension theorem ([14], 1.4) (cf. ([20], 2.2.2.2)), we can choose a lisse sheaf K_i of finitely generated and projective Λ-modules on $\mathbf{G}_{m,k}$ extending V_i, i. e. such that $K_i|\eta = V_i$, and K_i is tamely ramified at $\{\infty\}$. Then the result follows from (3.4.8) and (3.8.3). □

A purely local proof of 3.9, in a more general setting, will be given in 4.12.

In the situation of 3.9, similarly to (3.5.1), we will put

(3.9.1) $$V_1 *_1 V_2 = j^* \mathcal{H}^1(j_! V_1 *^L j_! V_2)$$

(in ([20], 2.7.2) the notation $V_1 * V_2$ is used instead of $V_1 *_1 V_2$). If we denote by

(3.9.2) $$\mathcal{G} = \mathcal{G}(\eta, \Lambda)$$

the category of sheaves of finitely generated and projective Λ-modules over η, we thus have a functor

(3.9.3) $$*_1 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}.$$

Similarly to 3.7, one checks that it verifies associativity and commutativity constraints (for the associativity, the Künneth formula of ([34], III, (1.6.4)) has to be replaced by the Künneth formula of 2.3, in its punctual form (A.3.1). It also has a two-sided unit (see 3.11 below).

Proposition 3.10. *Let $K_i \in D_{ctf}(A_h, \Lambda)$ $(i = 1, 2)$.*

(1) If K_1 is concentrated at $\{0\}$, we have a canonical isomorphism

(3.10.1) $$i_*(K_1)_0 *^L K_2 \xrightarrow{\sim} e^*((K_1)_0) \otimes^L K_2$$

(resp. the similar one with K_1 and K_2 interchanged), where $e : A_h \to \operatorname{Spec} k$ is the projection and $i : \{0\} \to A_h$ the inclusion.

(2) Assume that K_1 (resp. K_2) has lisse cohomology sheaves and that $(K_2)_0 = 0$ (resp. $(K_1)_0 = 0$). Then

(3.10.2) $$K_1 *^L K_2 = 0.$$

Proof. (1) The proof is the same as that of (3.4 (1)).

(2) It suffices to check the resp. assertion. We may assume that A_h is strictly local, hence K_2 is constant, so we are reduced to the case where K_2 is the constant sheaf Λ. By ([33], Rapport, 4.6), we may assume that $K_1 = j_! V_1$, where V_1 is a bounded complex of sheaves of finitely generated and projective Λ-modules over η. By dévissage, we may assume that $K_1 = j_! V$ with V concentrated in degree zero. By the Gabber-Katz extension theorem, choose a lisse sheaf L on

$\mathbf{G}_{m,k}$, of finitely generated and projective Λ-modules, tamely ramified at $\{\infty\}$, such that $L_\eta \xrightarrow{\sim} V$. By (3.8.3),

$$j_! V *^L \Lambda \xrightarrow{\sim} R\Phi_a(j_! V \boxtimes \Lambda)_{(0,0)} \xrightarrow{\sim} R\Phi_a(j_! L \boxtimes \Lambda)_{(0,0)},$$

which is zero by (3.4.2).

Note that, for the resp. assertion of (2), it suffices to show $R\Phi_a(\mathrm{pr}_1^* K_1)_{(0,0)} = 0$. A purely local argument for this, in a more general setting, will be given in 4.12. $\square$

In view of (3.10 (1)), (3.10 (2)) implies the following local analogue of 3.6 :

Corollary 3.11. *Under the resp. assumption of 3.10 (2) the triangle*

$$j_! j^* K_2 \to K_2 \to i_*(K_2)_0 \to, \tag{3.11.1}$$

where $i : \{0\} \to A_h$ is the inclusion, induces an isomorphism

$$K_1 *^L j_! j^* K_2 \xrightarrow{\sim} K_1 \otimes^L e^*((K_2)_0)[-1]). \tag{3.11.2}$$

Similar statement with K_1 and K_2 interchanged. For $V_i \in \mathcal{G}(\eta, \Lambda)$, if V_1 or V_2 is unramified, i. e. extends to a lisse sheaf on A_h, the above isomorphism yields an isomorphism

$$V_1 *_1 V_2 \xrightarrow{\sim} V_1 \otimes V_2. \tag{3.11.3}$$

In particular, we have natural isomorphisms

$$\Lambda *_1 V \xrightarrow{\sim} V *_1 \Lambda \xrightarrow{\sim} V. \tag{3.11.4}$$

4 Applications to Thom-Sebastiani type theorems

In order to formulate an analogue of the Thom-Sebastiani for the sum map, it is convenient to introduce the following generalization of the local additive convolution product 3.8. We keep the notation of 3. In particular, A (resp. A_h, resp. A_{sh}) denotes the affine line $\mathbf{A}^1_k$ (resp. its henselization, resp. strict henselisation) at the origin, $a : A^2 \to A$ (resp. $a : (A^2)_h \to A_h$, resp. $a : (A^2)_{sh} \to A_{sh}$) denotes the sum map (resp. the map induced by it), and $\overline{\eta}$ a geometric point of A_{sh} above the generic point η of A_h.

Definition 4.1. For $i = 1, 2$, let $f_i : X_i \to A_h$ be a morphism of finite type. The *local (additive) convolution along* (X_1, X_2) is the functor

$$*^L : D_{tf}(X_1 \overset{\leftarrow}{\times}_{A_h} A_h, \Lambda) \times D_{tf}(X_2 \overset{\leftarrow}{\times}_{A_h} A_h, \Lambda) \to D^+(X_h \overset{\leftarrow}{\times}_{A_h} A_h, \Lambda) \tag{4.1.1}$$

defined by

$$K_1 *^L K_2 = R\overset{\leftarrow}{a}_* K, \tag{4.1.2}$$

where, in the right hand side of (4.1.1),

$$X_h := (X_1 \times_k X_2) \times_{A_h \times_k A_h} (A^2)_h$$

is viewed as a scheme over A_h by $a : (A^2)_h \to A_h$,

$$K = \overleftarrow{\mathrm{pr}_1}^* K_1 \otimes^L \overleftarrow{\mathrm{pr}_2}^* K_2,$$

$\overleftarrow{\mathrm{pr}_i} : X_h \overset{\leftarrow}{\times}_{(A^2)_h} (A^2)_h \to X_i \overset{\leftarrow}{\times}_{A_h} A_h$ is defined by the i-th projection, and

$$\overleftarrow{a} : X_h \overset{\leftarrow}{\times}_{(A^2)_h} (A^2)_h \to X_h \overset{\leftarrow}{\times}_{A_h} A_h$$

is induced by $a : (A^2)_h \to A_h$. The subscript tf means finite tor-dimension.

Remark 4.2. This definition generalizes 3.8. Indeed, take X_i to be the closed point $\{0\}$ of A_h. Then $\{0\} \overset{\leftarrow}{\times}_{A_h} A_h \xrightarrow{\sim} A_h$, $\{0,0\} \overset{\leftarrow}{\times}_{(A^2)_h} (A^2)_h \xrightarrow{\sim} (A^2)_h$, and for $K_i \in D_{ctf}(A_h, \Lambda)$, by (1.13.2) we have $R\overleftarrow{a}_*(K_1 \boxtimes^L K_2) \xrightarrow{\sim} K_1 *^L K_2$.

There is an obvious variant of 4.1 with A_h replaced by A_{sh}. In the situation of 4.1, let x_i be a geometric point of X_i above $\{0\}$, x the geometric point of X_h defined by (x_1, x_2). We have $x_i \overset{\leftarrow}{\times}_{A_h} A_h \xrightarrow{\sim} A_{sh}$, $x \overset{\leftarrow}{\times}_{A_h} A_h \xrightarrow{\sim} A_{sh}$. Assume that $K_i \in D_{tf}(X_i \overset{\leftarrow}{\times}_{A_h} A_h, \Lambda)$. Let $(K_i)_{|x_i} := K_i | x_i \overset{\leftarrow}{\times}_{A_h} A_h \in D_{tf}(A_{sh}, \Lambda)$. Then (by (1.13.2) again)

$$(K_1 *^L K_2) | x \overset{\leftarrow}{\times}_{A_h} A_h \xrightarrow{\sim} (K_1)_{|x_1} *^L (K_2)_{|x_2} \in D_{ctf}(A_{sh}, \Lambda). \tag{4.2.1}$$

Proposition 4.3. *For $K_i \in D_{ctf}(X_i \overset{\leftarrow}{\times}_{A_h} A_h, \Lambda)$, we have*

$$K_1 *^L K_2 \in D_{ctf}(X \overset{\leftarrow}{\times}_{A_h} A_h, \Lambda).$$

Proof. This is proved by a standard dévissage, based on the following facts, whose proof is straightforward from the definitions. Let Z be a scheme separated and of finite type over a trait $S = (s, \eta, \overline{s} \leftarrow \overline{\eta})$.

(1) The oriented product $Z \overset{\leftarrow}{\times}_S S$ is the union of the closed subtopos $Z_s \overset{\leftarrow}{\times}_S S$ and the complementary open subtopos $Z_\eta \times_S S$.

(a) The topos $Z_\eta \times_S S$ is equivalent to $Z_\eta \times_\eta \eta$, in turn equivalent to Z_η (as continuous $\mathrm{Gal}(\overline{\eta}/\eta)$-equivariant sheaves on $Z_{\overline{\eta}}$ are just sheaves on Z_η).

(b) As noted in 1.2, the topos $Z_s \overset{\leftarrow}{\times}_S S$ is the union of the closed subtopos $Z_s \overset{\leftarrow}{\times}_S s$ and the complementary open subtopos $Z_s \overset{\leftarrow}{\times}_S \eta$. The topos $Z_s \overset{\leftarrow}{\times}_S s$ is equivalent to $Z_s \overset{\leftarrow}{\times}_s s = Z_s$. The topos $Z_s \overset{\leftarrow}{\times}_S \eta$ is the topos of sheaves $F_{\overline{\eta}}$ on $X_{\overline{s}}$ with a continuous action of $\mathrm{Gal}(\overline{\eta}/\eta)$ compatible with the action of $\mathrm{Gal}(\overline{s}/s)$ on $X_{\overline{s}}$.

(2) A sheaf of Λ-modules F on $Z \overset{\leftarrow}{\times}_S S$ is constructible if and only if its inverse images on Z_η, Z_s, and $Z_s \overset{\leftarrow}{\times}_S \eta$ are.

(3) Let $\widetilde{S}$ be the strict localization of S at $\overline{s}$, and let F be as in (2). Then F is constructible if and only if its inverse image on $X \overset{\leftarrow}{\times}_{\widetilde{S}} \widetilde{S}$ is.

(4) Let K be an object of $D_{ctf}(Z \overset{\leftarrow}{\times}_S S, \Lambda)$. Then K is isomorphic, in $D(Z \overset{\leftarrow}{\times}_S S, \Lambda)$, to a bounded complex of constructible sheaves of Λ-modules, projective over Λ[7]. This follows from the argument of ([33], Rapport, 4.7, 4.8).

(5) Assume S strictly local, and let F be a constructible sheaf of Λ-modules on $Z_s \overset{\leftarrow}{\times}_S \eta$, projective over Λ. Then there exists a finite stratification $Z_s =$

[7]Flat and of finite type is equivalent to projective and of finite type.

$\coprod Z_\alpha$ into locally closed subschemas, and, for each α, a finite quotient G_α of $G = \mathrm{Gal}(\overline{\eta}/\eta)$, a Λ-G_α-module M_α projective and of finite type over Λ, such that the restriction of F to Z_α is a locally constant sheaf of $\Lambda[G_\alpha]$-modules of value M_α.

Let us now prove 4.3. By (3) we may assume k algebraically closed, i. e. replace A_h by A_{sh}, which we will denote by $S = (S, s, \eta, \overline{\eta})$. By (4) we may assume that K_i is a constructible sheaf of projective Λ-modules on $X_i \overset{\leftarrow}{\times}_S S$. By (1) and (2) we may assume that each K_i is of the form $u_* E_i$ or $v_! F_i$, for E_i (resp. F_i) constructible on $(X_i)_s \overset{\leftarrow}{\times}_S S$ (resp. $(X_i)_\eta$, where $u : (X_i)_s \overset{\leftarrow}{\times}_S S \hookrightarrow X_i \overset{\leftarrow}{\times}_S S$, $v : (X_i)_\eta \hookrightarrow X_i \overset{\leftarrow}{\times}_S S$ are the inclusions. As the external tensor product of $u_* E_1$ and $v_! F_2$ (resp. $v_! F_1$ and $u_* E_2$) is zero, we only have to treat the two cases: (i) $K_i = u_* E_i$ ($i = 1, 2$), (ii) $K_i = v_! F_i$ ($i = 1, 2$). Case (ii) follows from ([33], Th. finitude): we may assume that $a : A_h^2 \to A_h$ is induced by a map $b : V_1 \times_k V_2 \to W$, where V_1, V_2 and W are étale neighborhoods of $\{0\}$ in $\mathbf{A}_k^1$, K_i comes from a constructible sheaf L_i of projective Λ-modules on a scheme Z_i separated and of finite type over $V_i - \{0\}$; then, if $g_i : Z_i \to V_i - \{0\}$, $g = g_1 \times_k g_2 : Z_1 \times_k Z_2 \to (V_1 - \{0\}) \times_k (V_2 - \{0\})$, $K_1 *^L K_2$ is induced by $Rb_* Rg_* (L_1 \boxtimes L_2)$, which is in $D^b_{ctf}(W - \{0\}, \Lambda)$.

Let's treat case (i), i. e. $K_i = u_* E_i$. As u is a closed subtopos, $u_* E_1 *^L u_* E_2 = u_*(E_1 *^L E_2)$, with $E_1 *^L E_2 \in D(X_s \overset{\leftarrow}{\times}_S S, \Lambda)$ defined in a similar way to (4.1.2). By (1) (b), we need only treat the two cases (a) $E_i = u_* u^* E_i$ ($i = 1, 2$), (b) $E_i = v_! v^* E_i$ ($i = 1, 2$), where this time $u : (X_i)_s \hookrightarrow (X_i)_s \overset{\leftarrow}{\times}_S S$ and $v : (X_i)_s \overset{\leftarrow}{\times}_S \eta \hookrightarrow (X_i)_s \overset{\leftarrow}{\times}_S S$ denote respectively the closed and open subtopoi. We may again assume that E_1 and E_2 are both of type (a) or both of type (b). In case of type (a), the conclusion follows from ([33], Th. finitude). So let us assume we are in case (b), i. e. $E_i = v_! L_i$, with L_i a sheaf of constructible and projective Λ-modules on $(X_i)_s \overset{\leftarrow}{\times}_S \eta$. If Z_i is a (locally closed) subscheme of $(X_i)_s$ and $Z := Z_1 \times_k Z_2$, then (by (4.2.1))

$$(K_1 | Z_1 \overset{\leftarrow}{\times}_S S) *^L (K_2 | Z_2 \overset{\leftarrow}{\times}_S S) \to (K_1 *^L K_2) | Z \overset{\leftarrow}{\times}_S S \tag{4.3.1}$$

is an isomorphism. Therefore, by (5) we may assume that L_i is a locally constant, constructible sheaf of $\Lambda[G_i]$-modules of value M_i, projective over Λ, for G_i a finite quotient of $\mathrm{Gal}(\overline{\eta}/\eta)$. Let $t_i : T_i \to (X_i)_s$ be a finite étale cover such that $t_i^* L_i$ becomes constant. If $T := T_1 \times_k T_2$, then, by (4.2.1) again, the natural map

$$(v_! L_1 | T_1 \overset{\leftarrow}{\times}_S S) *^L (v_! L_2 | T_2 \overset{\leftarrow}{\times}_S S) \to (v_! L_1 *^L v_! L_2) | T \overset{\leftarrow}{\times}_S S \tag{4.3.2}$$

is an isomorphism, so we may assume L_i constant. As similarly the natural map

$$(v_! M_1 | (X_1)_s \overset{\leftarrow}{\times}_S S) *^L (v_! M_2 | (X_2)_s \overset{\leftarrow}{\times}_S S) \to v_! L_1 *^L v_! L_2$$

is an isomorphism, $v_! L_1 *^L v_! L_2$ is then constant, of value $v_! M_1 *^L v_! M_2$, which is in $D^b_{ctf}(s \overset{\leftarrow}{\times}_S S, \Lambda)$ $(= D^b_{ctf}(S, \Lambda))$ by 3.9. $\square$

Remark 4.4. I don't know if, more generally, for morphisms of the form $\overleftarrow{g} : X \overset{\leftarrow}{\times}_Y Y \to X \overset{\leftarrow}{\times}_S S$, for $g : Y \to S$ of finite type, with S regular of dimension 1, and $X \to Y$ of finite type, $R\overleftarrow{g}_*$ preserves D_{ctf}.

The following is an analogue of the classical Thom-Sebastiani theorem for the sum map.

Theorem 4.5. *With the notation of 4.1, consider the composite morphism*

$$af_h : X_h \xrightarrow{f_h} (A^2)_h \xrightarrow{a} A_h,$$

where f_h is deduced from $f = f_1 \times_k f_2$ by base change by $(A^2)_h \to A_h \times_k A_h$. Let $K_i \in D_{ctf}(X_i, \Lambda)$, $K = (K_1 \boxtimes K_2)|X_h$. Then (1.10.3), (1.10.4), (2.1.2) induce isomorphisms in $D_{ctf}(X_h \overleftarrow{\times}_{A_h} A_h, \Lambda)$

$$(R\Psi_{f_1} K_1) *^L (R\Psi_{f_2} K_2) \xrightarrow{\sim} R\Psi_{af_h}(K) \tag{4.5.1}$$

$$(R\Phi_{f_1} K_1) *^L (R\Phi_{f_2} K_2) \xrightarrow{\sim} R\Phi_{af_h}(K). \tag{4.5.2}$$

Proof. Up to replacing A_h by an étale neighborhood Y_i of $\{0\}$ in A and extending (f_i, K_i) over Y_i, the hypotheses of 2.3 are satisfied by (1.7 (a)). By passing to the limit under such neighborhoods, we deduce that the morphism

$$c : R\Psi_{f_1} K_1 \boxtimes^L R\Psi_{f_2} K_2 \xrightarrow{\sim} R\Psi_f(K_1 \boxtimes^L K_2). \tag{4.5.3}$$

of (2.1.2) is an isomorphism. By (1.3.2), the base change map

$$R\Psi_f(K_1 \boxtimes^L K_2)|X_h \overleftarrow{\times}_{(A^2)_h} (A^2)_h \to R\Psi_{f_h} K \tag{4.5.4}$$

is an isomorphism. By (1.10.3), applying $R\overleftarrow{a}_*$ to the restriction of (4.5.3) to $X_h \overleftarrow{\times}_{(A^2)_h} (A^2)_h$ yields (4.5.1).

As a is universally locally acyclic, by (1.17) the map (1.10.4)

$$R\Phi_{af_h} K \to R\overleftarrow{a}_* R\Phi_{f_h} K \tag{4.5.5}$$

is an isomorphism. By (2.8.3), to deduce (4.5.2) it thus suffices to show

$$R\overleftarrow{a}_*((R\Phi_{f_1} K_1 \boxtimes^L p_1^* K_2)|X_h \overleftarrow{\times}_{(A^2)_h} (A^2)_h)) = 0. \tag{4.5.6}$$

and similarly with $R\Phi_{f_1} K_1 \boxtimes^L p_1^* K_2$ replaced by $p_1^* K_1 \boxtimes^L R\Phi_{f_2} K_2$. We check (4.5.6) on local sections. Let $L := (R\Phi_{f_1} K_1 \boxtimes^L p_1^* K_2)|X_h \overleftarrow{\times}_{(A^2)_h} (A^2)_h$. Let x_i be a geometric point of X_i above $\{0\}$, x the corresponding point of X_h (above the closed point $(0,0)$ of $(A^2)_h$). By (4.2.1), we have

$$(R\overleftarrow{a}_* L)|x \overleftarrow{\times}_{A_h} A_h = (R\Phi_{f_1} K_1)_{|x_1} *^L (p_1^* K_2)_{|x_2}. \tag{4.5.7}$$

As $(p_1^* K_2)_{|x_2}$ is the *constant* complex on A_{sh} of value $(K_2)_{x_2}$ and $(R\Phi_{f_1} K_1)_{|x_1}$ vanishes at $\{0\}$, 3.10 implies $(R\overleftarrow{a}_* L)|x \overleftarrow{\times}_{A_h} A_h = 0$, hence (4.5.6). The proof for the similar one is the same.

□

4.6

With the notation of 4.1, let x_i be a rational point of $(X_i)_0$. Assume that f_i is smooth along $(X_i)_0$ outside $\{x_i\}$, flat, and locally of complete intersection of relative dimension n_i at x_i (this last condition is satisfied for example if X_i is regular at x_i of dimension $n_i + 1$). Recall ([10], 2.10) that under these assumptions $R\Phi_{f_i}(\Lambda)$ is concentrated at x_i, and we have

$$R^q\Phi_{f_i}(\Lambda)_{x_i} = 0 \tag{4.6.1}$$

for $q \neq n_i$, and that $R^{n_i}\Phi_{f_i}(\Lambda)_{x_i}$ is a finitely generated and projective Λ-module. From (4.5.2) we deduce:

Corollary 4.7. *Under the assumptions of 4.6, the restriction of $R\Phi_{af_h}(\Lambda)$ to $(X_1)_0 \times_k (X_2)_0$ is concentrated at the rational point $x = (x_1, x_2)$,*

$$R^q\Phi_{af_h}(\Lambda)_x = 0$$

for $q \neq n+1$, and, with the notation of (3.9.1), (4.5.2) induces an isomorphism of sheaves of Λ-modules over η, with finitely generated and projective stalks :

$$R^{n_1}\Phi_{f_1}(\Lambda)_{x_1} *_1 R^{n_2}\Phi_{f_2}(\Lambda)_{x_2} \xrightarrow{\sim} R^{n+1}\Phi_{af_h}(\Lambda)_x. \tag{4.7.1}$$

Remark 4.8. If in 4.7 we assume furthermore that X_i is essentially smooth over k and f_i is smooth outside x_i, then af_h is essentially smooth outside x, and $R\Phi_{af_h}(\Lambda)$ is concentrated at x.

Remark 4.9. Under the assumptions of 4.8, let $\mu(f_i)$ (resp. $\mu(af_h)$) be the Milnor number of f_i (resp. af_h) at x_i (resp. x) ([35], XVI 1.2). Assume Λ local. By Deligne's theorem ([35] XVI, 2.4) we have

$$\mu(f_i) = \mathrm{dimtot}\, R^{n_i}\Phi_{f_i}(\Lambda)_{x_i},\ \mu(af_h) = \mathrm{dimtot}\, R^{n+1}\Phi_{af_h}(\Lambda)_x,$$

Here, for $V \in \mathcal{G}(\eta, \Lambda)$ (3.9.3), $\mathrm{dimtot}\, V = \mathrm{rk}(V) + \mathrm{sw}(V)$, where $\mathrm{rk}(V)$ is the rank of V over Λ, and $\mathrm{sw}(V)$ the Swan conductor of V. By the definition of the Milnor number, in this situation, we have

$$\mu(af_h) = \mu(f_1)\mu(f_2).$$

For coefficients $\overline{\mathbf{Q}}_\ell$, this formula agrees with (4.7.1), as by a result of Laumon ([20], (2.7.2.1)) we have

$$\mathrm{dimtot}(V_1 *_1 V_2) = \mathrm{dimtot}(V_1)\mathrm{dimtot}(V_2)$$

for V_1, V_2 in $\mathcal{G}(\eta, \overline{\mathbf{Q}}_\ell)$. It is not clear that Laumon's arguments extend to finite coefficients. We give an independent proof in 5.12.

4.10

If r is an integer ≥ 1, one defines a multiple convolution product

$$*^L_{1\leq i\leq r} : \prod_{1\leq i\leq r} D_{ctf}(A_h, \Lambda) \to D_{ctf}(A_h, \Lambda)$$

by a formula similar to (3.8.1)

$$*^L_{1\leq i\leq r} M_i := R\overleftarrow{a}_*(\boxtimes^L_{1\leq i\leq r}M_i)_{(0,\cdots,0)}, \tag{4.10.1}$$

where $\overleftarrow{a} : \{0\}\overleftarrow{\times}_{(A^r)_h}(A^r)_h \to \{0\}\overleftarrow{\times}_{A_h}A_h$ is induced by the sum map $a : \mathbf{A}^r_k \to \mathbf{A}^1_k$, $(A^r)_h$ denoting the henselization of $\mathbf{A}^r_k$ at $(0,\cdots,0)$. We also have a variant of 4.1 for a family $(X_i)_{1\leq i\leq r}$ of schemes of finite type over A_h, and $M_i \in D_{ctf}(X_i\overleftarrow{\times}_{A_h}A_h,\Lambda)$:

$$*^L M_i := R\overleftarrow{a}_* M \in D_{ctf}(X_h\overleftarrow{\times}_{A_h}A_h,\Lambda) \tag{4.10.2}$$

where X is the pull-back to $(A^r)_h$ of $\prod_k X_i$, M the inverse image of $\boxtimes^L M_i$ on X_h and $\overleftarrow{a} : X_h\overleftarrow{\times}_{(A^r)_h}(A^r)_h \to X_h\overleftarrow{\times}_{A_h}A_h$ is induced by the sum map $A^r \to A$.

For $f_i : X_i \to A_h$ of finite type, $K_i \in D_{ctf}(X,\Lambda)$ $(1 \leq i \leq r)$, one gets isomorphisms in $D_{ctf}(X\overleftarrow{\times}_{A_h}A_h,\Lambda)$:

$$*^L_{1\leq i\leq r} (R\Psi_{f_i}K_i)|X\overleftarrow{\times}_{A_h}A_h \xrightarrow{\sim} R\Psi_{af}(K)|X\overleftarrow{\times}_{A_h}A_h \tag{4.10.3}$$

$$*^L_{1\leq i\leq r} (R\Phi_{f_i}K_i)|X\overleftarrow{\times}_{A_h}A_h \xrightarrow{\sim} R\Phi_{af_h}(K)|X\overleftarrow{\times}_{A_h}A_h \tag{4.10.4}$$

where f_h is deduced from $\prod f_i$ by base change to A^r_h.

Remark 4.11. (a) Let ℓ be a prime not equal to the characteristic of k. The results in 4.5 imply variants with Λ replaced by $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, a finite extension E_λ of $\mathbf{Q}_\ell$, $\mathcal{O}_{E_\lambda}$, and $\overline{\mathbf{Q}}_\ell$.

(b) If in 4.10, $X_i = A_h$ and f_i induced by $t \mapsto t^2$, $R\Phi_{af_h}(\Lambda)$ is concentrated at $\{0\} := \{0,\cdots,0\}$ and in degree $r-1$, and given by

$$R^{r-1}\Phi_{af_h}(\Lambda)_{\{0\}} = (*_1)_{1\leq i\leq r}R^0\Phi_{f_i}(\Lambda)_{\{0\}}$$

For $\mathrm{char}(k) \neq 2$, $R^{r-1}\Phi_{af_h}(\Lambda)_{\{0\}}$ is tame, and of rank 1, but differs from $\otimes_{1\leq i\leq r}R^0\Phi_{f_i}(\Lambda)_{\{0\}}$ by arithmetic characters (([35] XV, 2.2.5, D, E), ([7], 1.3)).

(c) If G is a smooth commutative group (or monoid) scheme over k, one can define a local convolution $*_G$ associated with the product map $m : G\times G \to G$, and we have results similar to 4.5, with A_h replaced by the henselization of G at $\{1\}$. The case $G = \mathbf{G}_m$ (resp. G an elliptic curve) seems to be of special interest, in view of [24] (resp. [29]). Note that in the case of the multiplicative convolution for $\mathbf{G}_m$ near $\{0\}$ or infinity, studied in [15], which is equivalent to the case of the local convolution for the multiplicative monoid A near $\{0\}$, while the analogue of (4.5.1) will hold, (4.5.2) will have to be replaced by formulas involving the nearby cycles of the product map near zero, through (1.16.1)) and the defect of vanishing of the analogue of (4.5.6).

4.12

In [5] Deligne proposes the following variant of these local convolutions. For $i = 1,2$, let C_i be a germ of smooth curve over k, by which we mean the henselization at a rational point of a smooth curve over k. Let D be another germ of smooth curve over k. Denote by o the closed points of C_i, D, and by

C the henselization of $C_1 \times_k C_2$ at the point (o, o). Let again denote by o the closed point of D. Consider a local k-morphism

$$a : C \to D \tag{4.12.1}$$

satisfying the condition

(D) *The restriction of a to the closed subscheme $C_1 \times \{o\}$ (resp. $\{o\} \times C_2$) of C is an isomorphism onto D.*

For $K_i \in D_{ctf}(C_i, \Lambda)$, let $K_1 \boxtimes^L K_2$ denote the object of $D_{ctf}(C, \Lambda)$ induced by $\mathrm{pr}_1^* K_1 \otimes^L \mathrm{pr}_2^* K_2$. As in 3.8 we define the a-local convolution

$$*_a^L : D_{ctf}(C_1, \Lambda) \times D_{ctf}(C_2, \Lambda) \to D_{ctf}(D, \Lambda)$$

by

$$K_1 *_a^L K_2 := R\Psi_a(K_1 \boxtimes^L K_2)_o. \tag{4.12.2}$$

If V_i is a sheaf of projective, finitely generated Λ-modules over the generic point η_i of C_i, and $j : \eta_i \to C_i$ denotes the inclusion, then $\mathcal{H}^q(j_! V_1 *_a j_! V_2) = 0$ for $q \neq 1$, and $H^1(j_! V_1 *_a^L j_! V_2)$ is projective, finitely generated over Λ. Indeed, condition (D) implies that a is essentially smooth at o, as the map it induces on the Zariski tangent spaces at o is of the form $(x, y) \mapsto \lambda x + \mu y$, with λ and μ in k^*. The local acyclicity of smooth maps implies that the Milnor fiber $F = a^{-1}(\overline{\eta})$ (where $\overline{\eta}$ is a geometric point of D over the generic point η) is irreducible. Therefore, if z_1 (resp. z_2) is the closed point of F cut out by $C_1 \times \{o\}$ (resp. $\{o\} \times C_2$, and $u : U := F - \{z_1\} - \{z_2\} \hookrightarrow F$ is the inclusion, then for any lisse sheaf $\mathcal{F}$ of Λ-modules on U, and in particular for $\mathcal{F} = (j_! V_1 \boxtimes j_! V_2)|U$, $H^0(F, u_! \mathcal{F}) = 0$. As a is of relative dimension 1, $H^q(j_! V_1 *_a j_! V_2) = 0$ for $q > 1$.

As in (3.9.1), we define

$$V_1 *_{a,1} V_2 := j^* \mathcal{H}^1(j_! V_1 *_a j_! V_2), \tag{4.12.3}$$

where $j : \eta \hookrightarrow D$ is the inclusion.

Furthermore, for any $K_i \in D_{ctf}(C_i, \Lambda)$, one has

$$R\Phi_a(p_i^* K_i)_o = 0, \tag{4.12.4}$$

where p_1 (resp. p_2) is induced by pr_1 (resp. pr_2), composed with $a(-, o)$ (resp. $a(o, -)$). This generalizes (3.10.2). One uses the isomorphism $\varphi : C \xrightarrow{\sim} C$ induced by $\varphi(x, y) = (a(x, y), x)$ (C_1, C_2 being identified with D by (D)), which verifies $p_1 \varphi = a$, $p_2 \varphi = p_1$. One finds that $\varphi_* R\Phi_a(p_1^* K_1)_o = R\Phi_{p_1}(p_2^* K_1)_o$, which is zero as follows (by a limit argument) from the universal local acyclicity for schemes of finite type over a field.

Now, if $f_i : X_i \to C_i$ is a morphism of finite type, and $X := (X_1 \times_k X_2) \times_{C_1 \times_k C_2} C$, one defines $*_a^L$ as in (4.1.2) :

$$K_1 *_a^L K_2 = R\overleftarrow{a}_* K, \tag{4.12.5}$$

where $K = (K_1 \boxtimes^L K_2)|X \overleftarrow{\times}_D D$. Then, for $f : X \to C$ induced by $f_1 \times_k f_2$, by the same argument as for Theorem 4.5, one gets formulas similar to (4.5.1) and (4.5.2), namely isomorphisms in $D_{ctf}(X_o \overleftarrow{\times}_D D, \Lambda)$:

$$(R\Psi_{f_1} K_1) *_a^L (R\Psi_{f_2} K_2) \xrightarrow{\sim} R\Psi_{af}(K) \tag{4.12.6}$$

$$(R\Phi_{f_1} K_1) *_a^L (R\Phi_{f_2} K_2) \xrightarrow{\sim} R\Phi_{af}(K). \tag{4.12.7}$$

5 The tame case

5.1

We keep the notation and hypotheses of 3.1. Let $n \geq 1$ be an integer invertible in k. As in ([19], 2.2) we denote by

$$\pi_n : \mathcal{K}_n \to \mathbf{G}_{m,k} \tag{5.1.1}$$

the μ_n-torsor on $\mathbf{G}_{m,k} = \mathbf{A}^1_k - \{0\}$ defined by the exact sequence

$$1 \to \mu_n \to \mathbf{G}_{m,k} \xrightarrow{\pi_n} \mathbf{G}_{m,k} \to 1.$$

(the *Kummer torsor*) ($\mathcal{K}_n = \mathbf{G}_{m,k}$ and π_n is the map $a \mapsto a^n$). A lisse sheaf L of Λ-modules on $\mathbf{G}_{m,k}$ is tamely ramified at $\{0\}$ and $\{\infty\}$ if and only if, for some $n \geq 1$, after a finite extension of k containing $\mu_n(\overline{k})$, L is trivialized by π_n.

Assume that k contains $\mu_n(\overline{k})$. If L is trivialized by π_n, L is recovered from the constant sheaf $\pi_n^* L$ on $\mathcal{K}_n$ of value $L_{\{\overline{1}\}}$ by the canonical isomorphism

$$L \xrightarrow{\sim} (\pi_{n*}\pi_n^* L)^{\mu_n} \xrightarrow{\sim} (\pi_{n*}\Lambda \otimes_\Lambda L_{\{\overline{1}\}})^{\mu_n}, \tag{5.1.2}$$

where μ_n ($= \mu_n(\overline{k})$) acts diagonally (on $\pi_{n*}\Lambda$ via its action on $\mathcal{K}_n$, and on $L_{\{\overline{1}\}}$ via $(\mu_n)_{\{\overline{1}\}}$, a quotient of $\pi_1^t(\mathbf{G}_{m,k}, \{\overline{1}\})$). Note that, as $\pi_{n*}\Lambda$ is locally free of rank one over $\Lambda[\mu_n]$, $\pi_{n*}\Lambda \otimes_\Lambda L_{\{\overline{1}\}}$ is a co-induced module, hence the norm map $N : x \mapsto \sum_{g \in \mu_n} gx$ on $\pi_{n*}\Lambda \otimes_\Lambda L_{\{\overline{1}\}}$ induces an isomorphism

$$N : \pi_{n*}\Lambda \otimes_{\Lambda[\mu_n]} L_{\{\overline{1}\}} \to (\pi_{n*}\Lambda \otimes_\Lambda L_{\{\overline{1}\}})^{\mu_n},$$

so that (5.1.2) yields an isomorphism

$$L \xrightarrow{\sim} \pi_{n*}\Lambda \otimes_{\Lambda[\mu_n]} L_{\{\overline{1}\}}, \tag{5.1.3}$$

whose stalk at $\overline{1}$ is the identity, as the stalk of $\pi_{n*}\Lambda$ at $\overline{1}$ is naturally identified with $\Lambda[\mu_n(\overline{k})]$. (In the identification of $\pi_{n*}\Lambda \otimes_{\Lambda[\mu_n]} L_{\{\overline{1}\}}$ with the co-invariants $(\pi_{n*}\Lambda \otimes_\Lambda L_{\{\overline{1}\}})_{\pi_n}$ we let μ_n act on the right on $\pi_{n*}\Lambda$ by $ag = g^{-1}a$.)

If L is only assumed to be *geometrically* trivialized by π_n, i.e., after extension to $\overline{k}$, we get an isomorphism (5.1.3) over $\mathbf{G}_{m,\overline{k}}$,

$$L_{\overline{k}} \xrightarrow{\sim} \pi_{n*}\Lambda_{\overline{k}} \otimes_{\Lambda[\mu_n]} L_{\{\overline{1}\}}, \tag{5.1.4}$$

where the subscript $\overline{k}$ means inverse image on $\mathbf{G}_{m,\overline{k}}$.

Remark 5.2. The above argument is a particular case of the following standard facts, that we recall for later use. Let T be a scheme, A a commutative ring, G a finite group.

(1) Let E be a lisse sheaf of finitely generated and projective A-modules on T, equipped with an action of G, and let E_0 denote the underlying sheaf of A-modules, with trivial action of G. Then the map

$$A[G] \otimes_A E_0 \to A[G] \otimes_A E$$

sending $g \otimes x$ to $g \otimes gx$, is a G-equivariant isomorphism, where the right hand side is equipped with the diagonal action of G.

(2) Let B be a locally free sheaf of left $A[G]$-modules of rank 1. For E as in (1), we have $\mathcal{H}^q(G, B \otimes_A E) = 0$ for $q > 0$, and the norm map $N : B \otimes_A E \to B \otimes_A E$, $b \otimes x \mapsto \sum g(b \otimes x)$ induces an isomorphism

$$(B \otimes_A E)_G \, (= B \otimes_{A[G]} E) \to \mathcal{H}^0(G, B \otimes_A E) = (B \otimes_A E)^G$$

(where in $B \otimes_{A[G]} E$, G acts on the right on B by $b.g = g^{-1}b$).

(3) Let t be a geometric point of T. Let $\pi : P \to T$ be a G-torsor on T. Then the natural map $E \to (\pi_* \pi^* E)^G$ is an isomorphism, and, if we assume moreover that E is trivialized by P, then, identifying $\pi^* E$ with the constant sheaf of value E_t on P (with its G-action), the projection formula and (2) (with $B = \pi_* A$) yield a G-equivariant isomorphism

$$\pi_* A \otimes_{A[G]} E_t \xrightarrow{\sim} E,$$

where G acts diagonally on the left hand side.

Remark 5.3. When k is algebraically closed, the functor $L \mapsto L_{\{1\}}$ is an equivalence from the category of constructible lisse sheaves of Λ-modules on $\mathbf{G}_{m,k}$ which are tamely ramified at $\{0\}$ and $\{\infty\}$, to the category of representations of I_t on finitely generated Λ-modules. As I_t is commutative, for such a sheaf L, the action of I_t on $L_{\{1\}}$ is I_t-equivariant, hence extends uniquely to an action on L, compatible with its action on the stalk.

Let $g \in I_t$, with image g_n in μ_n. Denote by g_L^* the corresponding automorphism of L. As (5.1.3) is functorial and compatible with passing to the stalk at 1, g_L^* corresponds to the automorphism of the right hand side of (5.1.3) given by $a \otimes b \mapsto a \otimes g_n b$. Equivalently, g_L^* is given by $a \otimes b \mapsto g_n^* a \otimes b$, where now g_n^* denotes the automorphism of $\pi_{n*}\Lambda$ induced by g, which is also that given by the action of $g_n \in \mu_n$. In other words, g_L^* is induced by the automorphism $g_{\pi_{n*}\Lambda}^*$ of the universal object $\pi_{n*}\Lambda$ via $I_t \twoheadrightarrow \mu_n$.

Remark 5.4. The preceding definitions and results have variants for Λ replaced by profinite rings R like $\mathbf{Z}_\ell$ ($\ell \neq p$), the ring of integers O_λ of a finite extension E_λ of $\mathbf{Q}_\ell$, E_λ, or $\overline{\mathbf{Q}}_\ell$, and $D_{ctf}(-, \Lambda)$ replaced by $D_c^b(-, R)$, taken in the sense of Deligne [3] when k satisfies the cohomological finiteness condition of (*loc. cit.*, 1.1.2 (d)), or in general in the sense of Ekedahl [6]. We will use them freely.

Theorem 5.5. *Let $n_i \geq 1$ ($i = 1, 2$) integers invertible in k, and $r = a_1 n_1 = a_2 n_2 = \mathrm{lcm}(n_1, n_2)$. Assume that k contains $\mu_r(\overline{k})$. With the notation of (3.5.1), consider*

$$\mathcal{B}_{\Lambda,k}(\underline{n}) = \mathcal{B}_{\Lambda,k}(n_1, n_2) = \pi_{n_1 *}\Lambda *_1 \pi_{n_2 *}\Lambda, \tag{5.5.1}$$

which is a lisse sheaf of finitely generated and free Λ-modules on $\mathbf{G}_{m,k}$, equipped with a natural action of $\mu_{\underline{n}} = \mu_{n_1} \times \mu_{n_2}$. Then :

(1) $\mathcal{B}_{\Lambda,k}(\underline{n})$ is geometrically trivialized by π_r (cf. 5.1), in particular, tamely ramified at $\{0\}$ and $\{\infty\}$. Moreover, we have

$$(j_! \pi_{n_1 *}\Lambda *^L j_! \pi_{n_2 *}\Lambda)_0 = 0, \tag{5.5.2}$$

where $j : \mathbf{G}_{m,k} \hookrightarrow \mathbf{A}^1_k$ is the inclusion.

(2) As a sheaf of $\Lambda[\mu_{\underline{n}}]$-modules, $\mathcal{B}_{\Lambda,k}(\underline{n})$ is locally free of finite type and of rank 1.

(3) The tame inertia group I_t acts on $\mathcal{B}_\Lambda(\underline{n})_{\{\bar{1}\}}$ through its quotient μ_r via $\mu_r \to \mu_{n_1} \times \mu_{n_2}$, $\varepsilon \mapsto (\varepsilon^{a_1}, \varepsilon^{a_2})$.

Proof. We may assume k algebraically closed.

(1) The following argument is borrowed from Deligne's seminar [2]. Write G for $\mathbf{G}_{m,k}$, A for $\mathbf{A}^1_k$, and drop the subscript k in the products for short. Let N be an integer ≥ 1. As k is algebraically closed, to prove that a sheaf L on G (for the étale topology) is trivialized by π_N, it suffices to show that it is equipped with an isomorphism $\alpha : m_N^* L \xrightarrow{\sim} \mathrm{pr}_2^* L$, where $m_N : G \times G \to G$ is the map $(\lambda, t) \mapsto \lambda^N t$. Indeed, restricting α to the subscheme $G \times \{1\}$ of G gives an isomorphism $\pi_N^* L \xrightarrow{\sim} e^*(L_{\{1\}})$, where $e : G \to \operatorname{Spec} k$ is the projection. We'll call such an α a weak action of G on L above m_N (no associativity or unity constraints are required on α).

We have

$$Ra_*((j_! \pi_{n_1 *} \Lambda \boxtimes j_! \pi_{n_2 *} \Lambda) | A^2 - a^{-1}(0)) = Ra_{\underline{n}*}(u_! \Lambda),$$

where $a_{\underline{n}} : Z \to G$ is the map $(x_1, x_2) \mapsto x_1^{n_1} + x_2^{n_2}$, with

$$X = A \times A - a_{\underline{n}}^{-1}(0) - \{0\} \times A - A \times \{0\}$$

and u the open inclusion

$$u : X \hookrightarrow Z := A \times A - a_{\underline{n}}^{-1}(0)$$

with complement $D = \{0\} \times A \cup A \times \{0\} - \{(0,0)\}$. Therefore it suffices to show that the pair (Z, D) is trivialized by π_r, which is equivalent to constructing a weak G-action on $a_{\underline{n}} : (Z, D) \to G$ above m_r. Such an action is given by

$$h : G \times Z \to Z,\ (\lambda, x_1, x_2) \mapsto (\lambda^{a_1} x_1, \lambda^{a_2} x_2).$$

Indeed, one readily checks that the diagram

$$\begin{tikzcd} G \times Z \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "a_{\underline{n}}"] \\ G \times G \arrow[r, "m_r"] & G \end{tikzcd}$$

is cartesian, where $f(\lambda, x_1, x_2) = (\lambda, x_1^{n_1} + x_2^{n_2})$. The other verifications are straightforward.

Let us prove (5.5.2). Recall that by (3.4 (3)), we have

$$(j_! \pi_{n_1 *} \Lambda *^L j_! \pi_{n_2 *} \Lambda)_0 \xrightarrow{\sim} R\Gamma(a^{-1}(0), j_! \pi_{n_1 *} \Lambda \boxtimes j_! \pi_{n_2 *} \Lambda | a^{-1}(0)).$$

Let K denote the complex on the right hand side. Let $D := a^{-1}(0)$, $v : D - \{0\} \hookrightarrow D$ the inclusion, $L := j_! \pi_{n_1 *} \Lambda \boxtimes j_! \pi_{n_2 *} \Lambda | D - \{0\}$, so that $K = R\Gamma(D, v_! L)$. We have $H^i K = 0$ for $i = 0$ and $i > 1$. So it suffices to show that $\chi(D, v_! L) = 0$. We have $\chi(D, v_! L) = \chi(D, Rv_* L) = \chi(D - \{0\}, L)$ (cf. end of proof of (3.4, (2))). By the isomorphism $G \xrightarrow{\sim} D - \{0\}$, $t \mapsto (t, -t)$, we have $L \xrightarrow{\sim} v_!(\pi_{n_1 *} \Lambda \otimes [-1]^* \pi_{n_2 *} \Lambda)$, where $[-1] : G \xrightarrow{\sim} G$, $t \mapsto -t$, so L is tamely ramified at $\{0\}$ and $\{\infty\}$. By the Ogg-Shafarevich formula, we have $\chi(D-\{0\},L) = \chi(D-\{0\})\mathrm{rk}(L) = 0$, which finishes the proof of (5.5.2).

(2) By (3.4 (3)) we know that $\mathcal{H}^q(Ra_*((j_!\pi_{n_1*}\Lambda \boxtimes j_!\pi_{n_2*}\Lambda)|A^2 - a^{-1}(0))$ is zero for $q \neq 1$, and $\mathcal{H}^1 = \pi_{n_1*}\Lambda *_1 \pi_{n_2*}\Lambda$ is a lisse sheaf of finitely generated and projective Λ-modules on G. Let m be its rank. We first show that $m = n_1 n_2$. By the second formula of (3.4 (3)), we have

$$m = \mathrm{rk}\, H^1(a^{-1}(1), M|a^{-1}(1))$$

where

$$M = (j_!\pi_{n_1*}\Lambda\boxtimes j_!\pi_{n_2*}\Lambda)|a^{-1}(1) = u_{1!}((\pi_{n_1*}\Lambda\boxtimes\pi_{n_2*}\Lambda)|a^{-1}(1)-\{(0,1)\}-\{(1,0)\}),$$

and $u_1 : a^{-1}(1) - \{(0,1)\} - \{(1,0)\} \hookrightarrow a^{-1}(1)$ is the open inclusion. Then

$$m = -\chi(a^{-1}(1), u_{1!}((\pi_{n_1*}\Lambda \boxtimes \pi_{n_2*}\Lambda)|a^{-1}(1) - \{(0,1)\} - \{(1,0)\})).$$

By (3.4.10),

$$m = -\chi(a^{-1}(1), Ru_{1*}(M|a^{-1}(1) - \{(0,1)\} - \{(1,0)\})),$$

so

$$m = -\chi(a^{-1}(1) - \{(0,1)\} - \{(1,0)\}, N) \tag{5.5.3}$$

where N is the (lisse) sheaf $M|a^{-1}(1) - \{(0,1)\} - \{(1,0)\}$. This sheaf is tamely ramified at ∞, $\{(0,1)\}$, and $\{(1,0)\}$. Indeed, if φ denotes the isomorphism $A \xrightarrow{\sim} a^{-1}(1)$, $t \mapsto (t, 1-t)$, we have

$$\varphi^* N \xrightarrow{\sim} \pi_{n_1*}\Lambda \otimes [t \mapsto 1-t]^*(\pi_{n_2*}\Lambda),$$

and each factor is tamely ramified at 0, 1, and ∞. As N is of rank $n_1 n_2$, by the Ogg-Shafarevitch formula, (5.5.3) gives

$$m = -n_1 n_2 \chi(a^{-1}(1) - \{(0,1)\} - \{(1,0)\}, \Lambda) = n_1 n_2.$$

As $K = j_!\pi_{n_1*}\Lambda\boxtimes j_!\pi_{n_2*}\Lambda$ is a constructible sheaf of $\Lambda[\mu_{\underline{n}}]$-modules on A^2 whose stalks are zero or free of rank 1, Ra_*K is in $D_{ctf}(A,\Lambda)$. As it is concentrated in one degree, it follows that $\mathcal{B}_{\Lambda,k}(\underline{n}) = R^1a_*(K|A^2 - a^{-1}(0))$ is finitely generated and projective over $\Lambda[\mu_{\underline{n}}]$. As its rank over Λ is $n_1 n_2$, its rank over $\Lambda[\mu_{\underline{n}}]$ is 1.

(3) With the notation of the proof of (1), as (Z,D) is trivialized by π_r, the action of I_t on $\mathcal{B}_{\Lambda,k}(\underline{n})$ via μ_r is given by transportation of structure via the action of μ_r on the second factor of $(Z_r, D_r) := (Z,D)\times_G (G,\pi_r)$, the pull-back of (Z,D) by $\pi_r : G \to G, t \mapsto t^r$. We have:

$$Z_r = \{(x_1,x_2,t)|x_1^{n_1} + x_2^{n_2} = t^r\}.$$

Consider the curve

$$C_{\underline{n}} = (\{(x_1,x_2)|x_1^{n_1} + x_2^{n_2} = 1\} - \{(\zeta_1,0)|\zeta_1^{n_1} = 1\} - \{(0,\zeta_2)|\zeta_1^{n_2} = 1\},$$

fiber (by $a_{\underline{n}}$) of $Z - D$ over 1. The isomorphism

$$\alpha_t : (x_1,x_2) \mapsto (t^{a_1}x_1, t^{a_2}x_2)$$

from $C_{\underline{n}}$ to $(Z_r - D_r)_t$ defines a trivialization

$$\alpha : C_{\underline{n}} \times G \xrightarrow{\sim} (Z_r - D_r)|G$$

by which the action of μ_r on the right hand side corresponds to that on the left hand side given on the first factor by $\varepsilon(x_1, x_2) \to (\varepsilon^{-a_1}x_1, \varepsilon^{-a_2}x_2)$. Thus the action of ε on the stalk at $\{1\}$ of $R^1 a_{n*}(u_! \Lambda)$, i.e., $H^1(C_{\underline{n}}, \Lambda)$, is the action deduced from $(x_1 \mapsto \varepsilon^{-a_1}x_1, x_2 \mapsto \varepsilon^{-a_2}x_2)$. As the action on the direct image sheaf corresponds to the inverse of the action on the space, the conclusion follows. □

Definition 5.6. We will call $\mathcal{B}_{\Lambda,k}(\underline{n})$ the *universal (global) convolution sheaf* (relative to $\underline{n} = (n_1, n_2)$). We will write it $\mathcal{B}_\Lambda(\underline{n})$ (or even $\mathcal{B}(\underline{n})$) when no confusion may arise.

Remark 5.7. (a) If k' is a perfect extension of k, $\mathcal{B}_{\Lambda,k'}(\underline{n})$ is deduced from $\mathcal{B}_{\Lambda,k}(\underline{n})$ by base change from $\mathbf{G}_{m,k}$ to $\mathbf{G}_{m,k'}$. Thus, $\mathcal{B}_{\Lambda,k}(\underline{n})$ is deduced by base change from $\mathcal{B}_{\Lambda,\mathbf{F}_p}(\underline{n})$.

If, for $i = 1, 2$, n_i divides m_i, and $s = \mathrm{lcm}(m_1, m_2)$, then $\mathcal{K}_{n_i}$ is deduced from $\mathcal{K}_{m_i}$ by $d_i : \mu_{m_i} \to \mu_{n_i}$ ($d_i = m_i/n_i$), hence $\mathcal{B}(\underline{n})$, as a sheaf of $\Lambda[\mu_{\underline{n}}]$-modules is deduced from $\mathcal{B}(\underline{m})$ by $\underline{d} = (d_1, d_2) : \mu_{\underline{m}} \to \mu_{\underline{n}}$ (and its action of μ_r from that of μ_s by $s/r : \mu_s \to \mu_r$).

(b) If Λ is a $\mathbf{Z}/\ell^\nu\mathbf{Z}$-algebra, with $\ell \neq p$, then, by (3.4, (3)), $\mathcal{B}_{\Lambda,k}(\underline{n}) = \mathcal{B}_{\mathbf{Z}/\ell^\nu\mathbf{Z},k}(\underline{n}) \otimes_{\mathbf{Z}/\ell^\nu\mathbf{Z}} \Lambda$. For $\nu \geq 1$ the $\mathcal{B}_{\mathbf{Z}/\ell^\nu\mathbf{Z},k}(\underline{n})$ define a $\mathbf{Z}_\ell$-sheaf, locally free of finite type and rank one over $\mathbf{Z}_\ell[\mu_{\underline{n}}]$,

(5.7.1) $$\mathcal{B}_{\mathbf{Z}_\ell,k}(\underline{n}) := \varprojlim_\nu \mathcal{B}_{\mathbf{Z}/\ell^\nu\mathbf{Z},k}(\underline{n}) = \pi_{n_1*}\mathbf{Z}_\ell *_1 \pi_{n_2*}\mathbf{Z}_\ell.$$

By extension of scalars to R, where R is O_λ, E_λ, or $\overline{\mathbf{Q}}_\ell$, as in 5.4, we get an R-sheaf, locally free of finite type and rank 1 over $R[\mu_{\underline{n}}]$,

(5.7.2) $$\mathcal{B}_{R,k}(\underline{n}) := R \otimes_{\mathbf{Z}_\ell} \mathcal{B}_{\mathbf{Z}_\ell,k}(\underline{n}) = \pi_{n_1*}R *_1 \pi_{n_2*}R.$$

Assume that $m \geq 1$ is invertible in k, k contains $\mu_m = \mu_m(\overline{k})$, and the finite extension E_λ of $\mathbf{Q}_\ell$ contains μ_m. We have a canonical decomposition (cf. ([19], (2.0.6)))

(5.7.3) $$\oplus_{\chi:\mu_m \to E_\lambda^*} \mathcal{K}_m(\chi^{-1}) \xrightarrow{\sim} \pi_{m*}E_\lambda,$$

where, on the left hand side, χ runs through the characters $\mu_m \to E_\lambda^*$, and $\mathcal{K}_m(\chi^{-1})$ is the lisse sheaf of rank one $\chi(\mathcal{K}_m) = \mathcal{H}om_{\mu_m}(V_\chi, \pi_{m*}E_\lambda)$ in the notation of ([19], (2.2.1)). This decomposition is in fact μ_m-equivariant, for the action of μ_m on the left hand side described at the end of 5.3. This can be reformulated in the following way. For each character $\chi : \mu_m \to E_\lambda^*$, let V_χ denote a 1-dimensional E_λ-vector space, endowed with the action of μ_m via χ. Then (5.7.3) can be rewritten as a μ_m-equivariant decomposition

(5.7.4) $$\oplus_{\chi:\mu_m \to E_\lambda^*} V_\chi \otimes \mathcal{K}_m(\chi^{-1}) \xrightarrow{\sim} \pi_{m*}E_\lambda,$$

given by composition, where μ_m acts on the left hand side by its action on the first factor. From this and (5.7.2) we deduce (for $\mu_r(\overline{k}) \subset k$) a canonical $\mu_{\underline{n}}$-equivariant decomposition

(5.7.5) $$\oplus_{(\chi_1,\chi_2)} V_{\chi_1^{-1}} \otimes V_{\chi_2^{-1}} \otimes \mathcal{J}_{E_\lambda,k}(\chi_1, \chi_2) \xrightarrow{\sim} \mathcal{B}_{E_\lambda,k},$$

where

$$\mathcal{J}_{E_\lambda,k}(\chi_1,\chi_2) := \mathcal{K}_{n_1}(\chi_1) *_1 \mathcal{K}_{n_2}(\chi_2), \tag{5.7.6}$$

and χ_i runs through the characters of μ_{n_i} with values in E_λ. In the sequel we will write $\mathcal{J}(\chi_1,\chi_2)$ for $\mathcal{J}_{E_\lambda,k}(\chi_1,\chi_2)$. This is a lisse E_λ-sheaf of rank 1 on $\mathbf{G}_{m,k}$, geometrically trivialized by π_r (5.5). The reason for the notation $\mathcal{J}$ is that, if k is a finite field, then this sheaf is related to a Jacobi sum. Indeed, we have the following result, due to Deligne (see ([19], 7.3.1, 7.3.4)):

Proposition 5.8. *Assume that $k = \mathbf{F}_q$ with n_i (hence r) dividing $q-1$, and that χ_1 and χ_2 are nontrivial. Then, we have a natural isomorphism (of lisse sheaves of rank 1) on $\mathbf{G}_{m,k}$:*

$$\mathcal{J}(\chi_1,\chi_2) \xrightarrow{\sim} \mathcal{K}_{n_1}(\chi_1) \otimes \mathcal{K}_{n_2}(\chi_2) \otimes \varepsilon(\chi_1,\chi_2) \tag{5.8.1}$$

where $\varepsilon(\chi_1,\chi_2)$ is the geometrically constant lisse sheaf inverse image on $\mathbf{G}_{m,k}$ of the sheaf on $\operatorname{Spec} k$ defined by the restriction of $\mathcal{J}(\chi_1,\chi_2)$ to the rational point $1 \in \mathbf{G}_m(k)$, corresponding to the representation of $\operatorname{Gal}(\overline{k}/k)$ sending the geometric Frobenius F relative to $\mathbf{F}_q$ to $q/J(\chi_1,\chi_2)$, where $J(\chi_1,\chi_2)$ is the Jacobi sum[8]

$$J(\chi_1,\chi_2) = -\sum_{t_i\in\mathbf{F}_q^*, t_1+t_2=1} \chi_1^{-1}(t_1^{(q-1)/n_1})\chi_2^{-1}(t_2^{(q-1)/n_2})$$

(a q-Weil number of weight 1 (resp. 2) if $\chi_1\otimes\chi_2$ is not trivial (resp. trivial)). In particular, if $\bar{t}$ is a geometric point over a rational point $t \in \mathbf{G}_m(k) = \mathbf{F}_q^*$, we have

$$\operatorname{Tr}(F^*, \mathcal{J}(\chi_1,\chi_2)_{\bar{t}}) = \chi_1(t)\chi_2(t)(q/J(\chi_1,\chi_2)).$$

Remark 5.9. (a) For R equal to O_λ or $O_\lambda/\mathfrak{m}^\nu$ no decomposition of $\mathcal{B}_{R,k}(\underline{n})$ analogous to (5.7.5) exists, as in general $\mathcal{B}_{R,k}(\underline{n})$ contains unipotent components.

(b) As the referee observes, instead of quoting [19], one can deduce 5.8 from [20], as follows. For simplicity, we'll treat only the case where $\chi_1\chi_2$ is nontrivial. We will write $\mathcal{K}$ for $\mathcal{K}_{n_i}$. Let $\mathcal{F}$ be the Fourier transform as in (3.3 (b)), with here $\Lambda = E_\lambda$. By (3.3.4) we have

$$\mathcal{F}(j_!\mathcal{K}(\chi_1) *^L j_!\mathcal{K}(\chi_2)) = \mathcal{F}(j_!\mathcal{K}(\chi_1)) \otimes \mathcal{F}(j_!\mathcal{K}(\chi_2))[-1],$$

hence, by ([20], 1.4.3.1, 1.4.3.2),

$$\mathcal{F}(j_!\mathcal{K}(\chi_1) *^L j_!\mathcal{K}(\chi_2)) = j_!\mathcal{K}((\chi_1\chi_2)^{-1}) \otimes G(\chi_1,\psi)G(\chi_2,\psi)[-1]. \tag{5.9.1}$$

On the other hand, as $\chi_1\chi_2$ is nontrivial, by ([20], 1.4.3.1, 1.4.3.2) again, we have

$$\mathcal{F}(j_!\mathcal{K}(\chi_1\chi_2)) = j_!\mathcal{K}((\chi_1\chi_2)^{-1}) \otimes G(\chi_1\chi_2,\psi). \tag{5.9.2}$$

As

$$j_!\mathcal{K}(\chi_1) *^L j_!\mathcal{K}(\chi_2) = j_!(\mathcal{K}(\chi_1) *_1 \mathcal{K}(\chi_2))[-1],$$

[8] Our $J(\chi_1,\chi_2)$ differs from Deligne's $J(\chi_1,\chi_2)$ in ([33], Sommes trigonométriques, (4.14.2)) by the sign $(\chi_1\chi_2)(-1)$.

comparing (5.9.1) and (5.9.2), we get

$$\mathcal{K}(\chi_1) *_1 \mathcal{K}(\chi_2) = \mathcal{K}(\chi_1\chi_2) \otimes \frac{G(\chi_1,\psi)G(\chi_2,\psi)}{G(\chi_1\chi_2,\psi)}. \tag{5.9.3}$$

This isomorphism is equivalent to (5.8.1). Indeed, by the classical relation between Jacobi sums and Gauss sums (cf. ([33], Sommes trigonométriques, 4.15.1)),

$$\begin{aligned} J(\chi_1,\chi_2) &= \frac{\tau(\chi_1,\psi)\tau(\chi_2,\psi)}{\tau(\chi_1\chi_2,\psi)} \\ &= \frac{g(\chi_1^{-1},\psi)g(\chi_2^{-1},\psi)}{g((\chi_1\chi_2)^{-1},\psi)} \\ &= \frac{qg(\chi_1\chi_2,\psi^{-1})}{g(\chi_1,\psi^{-1})g(\chi_2,\psi^{-1})}, \end{aligned}$$

hence

$$\frac{q}{J(\chi_1,\chi_2)} = \frac{g(\chi_1,\psi^{-1})g(\chi_2,\psi^{-1})}{g(\chi_1\chi_2,\psi^{-1})} = \frac{g(\chi_1,\psi)g(\chi_2,\psi)}{g(\chi_1\chi_2,\psi)}.$$

Theorem 5.5 has the following consequence:

Corollary 5.10. *Under the assumptions of 5.5, for $i = 1, 2$, let M_i be a lisse sheaf of finitely generated and projective Λ-modules on $\mathbf{G}_{m,k}$ geometrically trivialized by π_{n_i} (cf. 5.1). Let V_i be the stalk of M_i at $\{\overline{1}\}$ (with its action of μ_{n_i}), and $V = V_1 \otimes V_2$ (with its action of $\mu_{\underline{n}}$). Then the isomorphisms (5.1.4) for M_1 and M_2 (over $\mathbf{G}_{m,\overline{k}}$) induce an isomorphism*

$$\mathcal{B}(\underline{n})_{\overline{k}} \otimes_{\Lambda[\mu_{\underline{n}}]} V \xrightarrow{\sim} (M_1 *_1 M_2)_{\overline{k}} \tag{5.10.1}$$

(with the notation of (3.5.1)), where $\mu_{\underline{n}}$ acts on $\mathcal{B}(\underline{n})$ on the right by $bg = g^{-1}b$. In particular, $M_1 *_1 M_2$ is geometrically trivialized by π_r, and we have

$$\mathrm{rk}(M_1 *_1 M_2) = \mathrm{rk}(M_1)\mathrm{rk}(M_2).$$

Proof. We may assume k algebraically closed. As in the proof of (5.5 (1)), consider the open immersion

$$u : U = A \times A - a^{-1}(0) - \{0\} \times A - A \times \{0\} \hookrightarrow A \times A - a^{-1}(0).$$

Let $R_i := \pi_{n_i*}\Lambda$, and write B for $\mathcal{B}(\underline{n})$. By definition, $B = R^1a_*(u_!(R_1 \boxtimes R_2|U))$, so by (5.2 (2)) and the fact that B is locally free of rank 1 over $\Lambda[\mu_{\underline{n}}]$, the norm map gives an isomorphism

$$B \otimes_{\Lambda[\mu_{\underline{n}}]} V \xrightarrow{\sim} \mathcal{H}^0(\mu_{\underline{n}}, R^1a_*(u_!(R_1 \boxtimes R_2) \otimes_\Lambda V)|U).$$

By (5.2 (2)) and the fact that $R_1 \boxtimes R_2$ is locally free of rank 1 over $\Lambda[\mu_{\underline{n}}]$, we have

$$\mathcal{H}^0(\mu_{\underline{n}}, (R^1a_*(u_!(R_1 \boxtimes R_2) \otimes_\Lambda V)|U) \xrightarrow{\sim} R^1a_*u_!\mathcal{H}^0(\mu_{\underline{n}}, (R_1 \boxtimes R_2) \otimes_\Lambda V)|U)).$$

As the referee observes, this isomorphism could also be viewed as deduced from the canonical isomorphism

$$R\underline{\Gamma}^{\mu_{\underline{n}}}Ra_* \xrightarrow{\sim} Ra_*R\underline{\Gamma}^{\mu_{\underline{n}}}$$

applied to $(u_!(R_1 \boxtimes R_2) \otimes_\Lambda V)|U$, via the degeneration of the corresponding spectral sequences. By (5.2 (1)), as R_i is locally free of rank 1 over $\Lambda[\mu_{n_i}]$, the natural map

$$\mathcal{H}^0(\mu_{n_1}, R_1 \otimes V_1) \boxtimes \mathcal{H}^0(\mu_{n_2}, R_2 \otimes V_2) \to \mathcal{H}^0(\mu_{\underline{n}}, (R_1 \boxtimes R_2) \otimes_\Lambda V))$$

is an isomorphism, and by (5.1.3) (or (5.2 (3)), the right hand side is identified with $M_1 \boxtimes M_2$, so we finally get the isomorphism (5.10.1)

$$B \otimes_{\Lambda[\mu_{\underline{n}}]} V \xrightarrow{\sim} R^1 a_*(u_!(M_1 \boxtimes M_2|U)) = M_1 *_1 M_2.$$

□

Proposition 5.11. *For $i = 1, 2$, let $K_i \in D_{ctf}(\mathbf{A}^1_k, \Lambda)$. Assume that K_1 and K_2 are tamely ramified at $\{\infty\}$ (3.1). Then $K_1 *^L K_2$ (3.2) is also tamely ramified at $\{\infty\}$.*

As mentioned in the proof of (3.4 (3)), a similar result for $*^L_!$ and $\overline{\mathbf{Q}}_\ell$-coefficients was proved by Laumon ([20], 2.7.1.1 (i)). Note that here, unlike in (*loc. cit.*), we make no assumption on the ramification of K_1 and K_2 at finite distance. This generalization was suggested to me by T. Saito. I am indebted to him and to W. Zheng for the following proof, which is simpler than the one I had originally given.

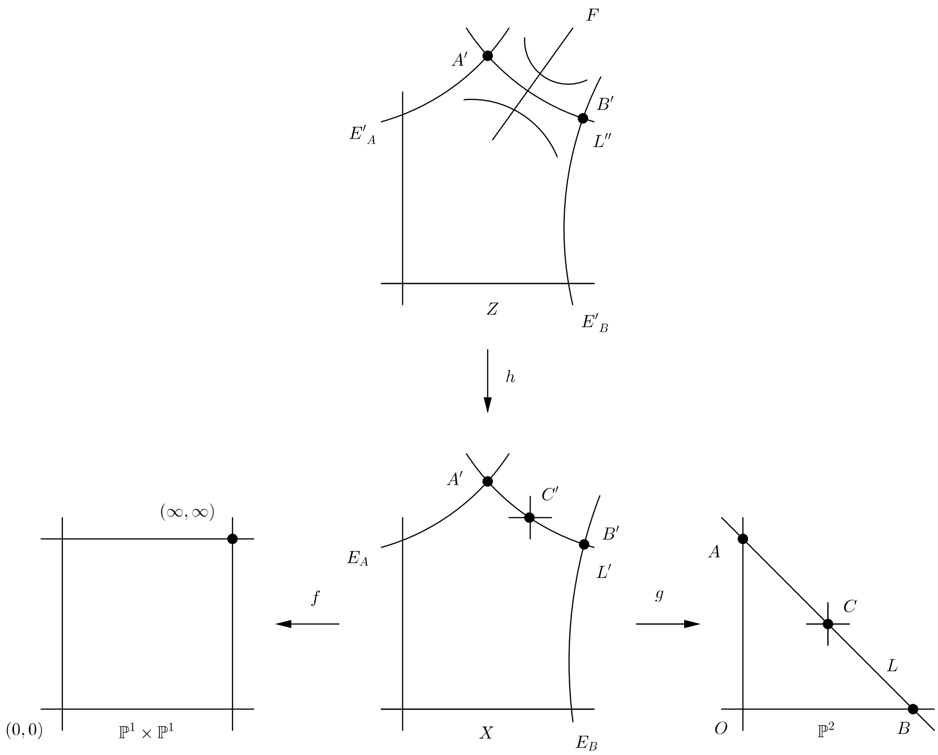
Proof. We may assume k algebraically closed, and we drop it from the notation. We need a compactification of $\mathbf{A}^2$ and of the sum map $a : \mathbf{A}^2 \to \mathbf{A}^1$. First, embed $\mathbf{A}^2$ in $\mathbf{P}^2$ in the usual way, $(x, y) \mapsto (x : y : 1)$, where $(x : y : z)$ are homogeneous coordinates in $\mathbf{P}^2$. Let L be the line $z = 0$. It joins the points $A = (0 : 1 : 0)$ and $B = (1 : 0 : 0)$. Let C be the point $(1 : -1 : 0)$ of L. Let $g : X \to \mathbf{P}^2$ be the blow-up of $\mathbf{P}^2$ at A and B. Denote by L' the strict transform of L, and by $E_A = g^{-1}(A)$, $E_B = g^{-1}(B)$ the exceptional divisors. Let A' (resp. B') be the point where L' meets E_A (resp. E_B). Let C' be the point of L' corresponding to C. The pencil of projective lines in $\mathbf{P}^2$ through A (resp. B) defines a projection $p_1 : X \to \mathbf{P}^1$ (resp. $p_2 : X \to \mathbf{P}^1$), which sends the strict transform of L to ∞. The map $f = (p_1, p_2) : X \to \mathbf{P}^1 \times \mathbf{P}^1$ contracts L' to the point (∞, ∞) of $\mathbf{P}^1 \times \mathbf{P}^1$. As $(L')^2 = -1$, f is the blow-up of (∞, ∞) in $\mathbf{P}^1 \times \mathbf{P}^1$ (with exceptional divisor L', E_A (resp. E_B) the strict transform of $\mathbf{P}^1 \times \infty$ (resp. $\infty \times \mathbf{P}^1$)). Let $h : Z \to X$ be the blow-up of C' in X, and $h' : Z' \to \mathbf{P}^2$ the blow-up of C in $\mathbf{P}^2$. Let $F = h^{-1}(C')$ and $F' = h'^{-1}(C)$ be the exceptional divisors. The pencil of projective lines

(5.11.1) $$\lambda(x + y) - \mu z = 0$$

through C induces the pencil of parallel lines $x + y - t = 0$ on $\mathbf{A}^2 = \mathbf{P}^2 - L$. It defines a $\mathbf{P}^1$-bundle $a' : Z' \to \mathbf{P}^1$, whose restriction to F' is an isomorphism (see Figure 1). We thus get a diagram with cartesian square:

$$\begin{tikzcd} & Z \arrow[r, "g'"] \arrow[d, "h"] & Z' \arrow[r, "a'"] \arrow[d, "h'"] & \mathbf{P}^1 . \\ \mathbf{P}^1 \times \mathbf{P}^1 & X \arrow[l, "f"'] \arrow[r, "g"] & \mathbf{P}^2 & \end{tikzcd}$$

Figure 1:



The map

$$\widetilde{a} := a'g' : Z \to \mathbf{P}^1$$

is a compactification of $a : \mathbf{A}^2 \to \mathbf{A}^1$. We have

$$\widetilde{a}^{-1}(\infty) = E'_A \cup L'' \cup E'_B,$$

where E'_A (resp. E'_B, resp. L'') denotes the strict transform of E_A (resp. E_B, resp. L') in Z, and ∞ is the point $(\lambda : \mu)$ with coordinate $\lambda = 0$, where λ and μ are defined in (5.11.1). The map $\widetilde{a}$ is proper, and is smooth outside A' and B', at which points it has semistable reduction. Indeed, $g' : Z \to Z'$ is the blow-up of Z' at the points A' and B' above A and B respectively, and $\widetilde{a}$ is the composition $a'g'$, where a' is smooth. The semistability at A' and B' in such a situation is standard and follows from an elementary calculation (see, e.g., ([25], Proof of Prop. 6') or ([11], 1.3)).

Let $j : \mathbf{A}^2 \hookrightarrow Z$ be the open immersion. Let $K := K_1 \boxtimes^L K_2$ on $\mathbf{A}^2$. We have

$$Ra_*K = R\widetilde{a}_* Rj_*K|(\mathbf{P}^1 - \infty).$$

We want to show that $R\widetilde{a}_* Rj_*K$ is tame at ∞. As $\widetilde{a}$ is proper, it suffices to show that $R\Psi_{\widetilde{a}}(Rj_*K)_\infty$ is tame. There are four cases:

(a) Tameness along $E'_A - A'$. The question is étale local at points of $E'_A - A'$, which we identify with $\mathbf{A}^1 \times \infty$ in $\mathbf{A}^1 \times \mathbf{P}^1$ (via fh). At such a point $(x_0, \infty) \in (\mathbf{A}^1 \times \mathbf{P}^1)(k)$, we have to show the tameness of $R\Psi_{\widetilde{a}} Rj_*(K)_{(x_0,\infty)}$. We may assume K_2 concentrated in degree zero and, by dévissage, using Abhyankar's lemma, we may further assume that $K_2 = \pi_{n*}\Lambda$, for an n invertible in k, where π_n is the Kummer torsor $y \mapsto y^n$. Let $T = \operatorname{Spec} k\{z\}$ be the strict localization of $\mathbf{P}^1$ at ∞, with parameter z induced by $1/y$, where (x, y) are coordinates on $\mathbf{A}^2$ as above. Let

$$W := (\mathbf{A}^1 \times \mathbf{P}^1)_{(x_0,\infty)} = T\{u\} = \operatorname{Spec} k\{u, z\}$$

be the strict localization of $\mathbf{A}^1 \times \mathbf{P}^1$ at (x_0, ∞), where u is the parameter induced by $x = x_0 + u$. The morphism $\widetilde{a} : W \to T$ is defined by $\widetilde{a}^*z = z/(1 + z(x_0 + u))$. Let α be the $k\{u\}$-automorphism of W defined by $\alpha^*(z) = z/(1+z(x_0+u))$, and let β be its inverse. By definition, we have $\mathrm{pr}_2\alpha = \widetilde{a}$, where $\mathrm{pr}_2 : W = T\{u\} \to T$ is the canonical projection, hence $\widetilde{a}\beta = \mathrm{pr}_2$, and as $\beta^* R\Psi_{\widetilde{a}} = R\Psi_{\mathrm{pr}_2}\beta^*$, it suffices to show the tameness of $R\Psi_{\mathrm{pr}_2}\beta^*(Rj_*K)$ at the closed point $(0, 0)$ of W. We have $\beta^*z = z/(1 - z(x_0 + u))$, so on $W[z^{-1}]$, $\mathrm{pr}_2\beta$ is induced by $b : \mathbf{A}^2 \to \mathbf{A}^1$, $b(x, y) = y - x$. Therefore, we have

$$R\Psi_{\mathrm{pr}_2}\beta^*(Rj_*K) = R\Psi_{\mathrm{pr}_2}(Rj_*(\mathrm{pr}_1^*K_1 \otimes^L b^*K_2)). \tag{5.11.2}$$

Let $\varepsilon : W' \to W$ be the finite étale cover of W of equation $v^n = 1-z(x_0+u)$. The pull-backs by ε of b^*K_2 and $\mathrm{pr}_2^*K_2$ are isomorphic. Therefore, by (5.11.2) we are reduced to showing the tameness at (x_0, ∞) of $R\Psi_{\mathrm{pr}_2}(Rj_*(\mathrm{pr}_1^*K_1 \otimes^L \mathrm{pr}_2^*K_2))$, where $j = (\mathrm{Id}, j_2) : \mathbf{A}^1 \times \mathbf{A}^1 \hookrightarrow \mathbf{A}^1 \times \mathbf{P}^1$. By Künneth, we have

$$R\Psi_{\mathrm{pr}_2}(Rj_*(\mathrm{pr}_1^*K_1 \otimes^L \mathrm{pr}_2^*K_2)) = R\Psi_{\mathrm{pr}_2}(\mathrm{pr}_1^*K_1 \otimes^L \mathrm{pr}_2^*Rj_{2*}K_2),$$

and by the projection formula,

$$R\Psi_{\mathrm{pr}_2}(\mathrm{pr}_1^*K_1 \otimes^L \mathrm{pr}_2^*Rj_{2*}K_2)_{(x_0,\infty)} = R\Psi_{\mathrm{pr}_2}(\mathrm{pr}_1^*K_1)_{(x_0,\infty)} \otimes^L (K_2)_{\overline{\eta}},$$

where $\overline{\eta}$ is a geometric generic point of T. By Deligne's generic universal local acyclicity theorem ([33], Th. finitude, 2.16), $R\Phi_{\mathrm{pr}_2}(\mathrm{pr}_1^*K_1)_{(x_0,\infty)} = 0$. Therefore, the inertia I acts tamely on $R\Psi_{\widetilde{a}}(Rj_*K)_\infty$ along $E'_A - A'$.

(b) Along $E'_B - B'$, the same calculation gives the desired tameness.

(c) Around A' or B', $\widetilde{a}$ is isomorphic to $m : \mathbf{A}^2 \to \mathbf{A}^1, m(x, y) = xy$ [9], and, by the description of X as the blow-up of (∞, ∞) in $\mathbf{P}^1 \times \mathbf{P}^1$, Rj_*K, restricted to $(xy \neq 0)$ is an external product of tamely ramified complexes, hence tamely ramified along $m^{-1}(0)$. For any lisse sheaf $\mathcal{F}$ on $(xy \neq 0)$ which is tamely ramified along $(xy = 0)$, $R\Psi_m(\mathcal{F})$ is tame, as follows by Abhyankar's lemma from the case where $\mathcal{F} = \pi_*\Lambda$, for a tame cover π of $\mathbf{A}^2$ of the form $x \mapsto x^n, y \mapsto y^n$. Therefore, $R\Psi_{\widetilde{a}}(Rj_*K)_\infty$ is tame around A' and B'.

(d) Along $L'' - A' - B'$, $\widetilde{a}$ is smooth, and Rj_*K is tamely ramified, hence by the last argument in (c), $R\Psi_{\widetilde{a}}(Rj_*K)_\infty$ is tame along $L'' - A' - B'$, which finishes the proof. □

[9] The local coordinates (x, y) here are not those used in (a).

The following corollary was suggested to me by T. Saito.

Corollary 5.12. *Let $(A_h, 0, \eta)$ be the henselization of $\mathbf{A}^1_k$ at 0. For $i = 1, 2$, let $V_i \in \mathcal{G}(\eta, \Lambda)$ (3.9.2). Then, with the notation of 4.9 and 3.9.1, we have:*

$$\mathrm{dimtot}(V_1 *_1 V_2) = \mathrm{dimtot}(V_1)\mathrm{dimtot}(V_2). \tag{5.12.1}$$

Let $r_i := \mathrm{rk}(V_i)$, $s_i := \mathrm{sw}(V_i)$, so that $\mathrm{dimtot}(V_i) = r_i + s_i$, and let $r := \mathrm{rk}((V_1 *_1 V_2)$, $s := \mathrm{sw}(V_1 *_1 V_2)$. We have

$$r = r_1r_2 + r_1s_2 + r_2s_1 - \mathrm{sw}(V_1 \otimes [-1]^* V_2), \tag{5.12.2}$$

$$s = s_1s_2 + \mathrm{sw}(V_1 \otimes [-1]^* V_2). \tag{5.12.3}$$

As mentioned in 4.9, the analogue of (5.12.1) for $\overline{\mathbf{Q}}_\ell$-coefficients was proved by Laumon in ([20], (2.7.2.1)). He also proved (5.12.2) and (5.12.3) in the same setting. The proof below is similar, except that we use $*$ instead of $*_!$ for the global convolution.

Proof. We may and will assume k algebraically closed. By the Gabber-Katz theorem [14], extend V_i to a lisse sheaf $\mathcal{F}_i$ on $(\mathbf{G}_m)_k$, tame at ∞. Put $A := \mathbf{A}^1_k$. Let $j : A - \{0\} \hookrightarrow A$ denote the inclusion. Recall (3.4.8) that

$$V_1 *_1 V_2 = R^0\Phi_{\mathrm{id}}(j_!\mathcal{F}_1 *^L j_!\mathcal{F}_2[1])_0, \tag{5.12.4}$$

where $j_!\mathcal{F}_1 *^L j_!\mathcal{F}_2 = Ra_*(j_!\mathcal{F}_1 \boxtimes j_!\mathcal{F}_2)$, $a : A^2 \to A$ being the global sum map. Formula (5.12.2) follows from (3.4.9). This does not use 5.11. We will use 5.11 to prove (5.12.3). For this, we compute the Euler-Poincaré characteristic $\chi(A, Ra_*(j_!\mathcal{F}_1 \boxtimes j_!\mathcal{F}_2))$ in two ways. First,

$$\chi(A, Ra_*(j_!\mathcal{F}_1 \boxtimes j_!\mathcal{F}_2)) = \chi(A^2, j_!\mathcal{F}_1 \boxtimes j_!\mathcal{F}_2),$$

and by Künneth ([34] III, (1.6.4)),

$$\chi(A^2, j_!\mathcal{F}_1 \boxtimes j_!\mathcal{F}_2) = \chi(A, j_!\mathcal{F}_1)\chi(A, j_!\mathcal{F}_2).$$

As $\mathcal{F}_i$ is tame at ∞, by the Grothendieck-Ogg-Shafarevich formula, and remembering that $\chi = \chi_c$ [16], we have

$$\chi(A, j_!\mathcal{F}_i) = -\mathrm{sw}_0(\mathcal{F}_i),$$

hence

$$\chi(A, j_!\mathcal{F}_1 *^L j_!\mathcal{F}_2[1]) = -s_1s_2.$$

By the Grothendieck-Ogg-Shafarevich formula again, and (3.4.6), using that $\mathcal{F}_1 *_1 \mathcal{F}_2$ is tame at ∞ by 5.11, we have, for $L := j_!\mathcal{F}_1 *^L j_!\mathcal{F}_2[1]$,

$$\chi(A, L) = \mathrm{rk}_0(L) + \chi_c(A - \{0\}, L) = \mathrm{sw}(V_1 \otimes [-1]^* V_2) - s$$

which gives (5.12.3). Combining (5.12.3) and (5.12.2), we get (5.12.1). $\square$

Remark 5.13. (a) The calculation made above shows that, conversely, (5.12.1) (or, equivalently, (5.12.3)) implies 5.11.

(b) Formula (5.12.1) is essentially ([17], Exemples 2.3.8 (a)), as the sum map a gives a good pencil in the situation of (*loc. cit.*).

(c) T. Saito gave an independent proof of (5.12.1) as a corollary of his theory of the characteristic cycle, more precisely, of the compatibility of the formation of the characteristic cycle with external products ([28], 2.6, 2.7).

Remark 5.14. Let $\mathcal{C} = \mathcal{C}(\mathbf{G}_{m,k}, \Lambda)$ be the full subcategory of sheaves of finitely generated and projective Λ-modules M on $\mathbf{G}_{m,k}$ which are tamely ramified at $\{0\}$ and $\{\infty\}$. Then, as in (3.9.3), we have a functor

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C}, \ (M_1, M_2) \mapsto M_1 *_1 M_2,$$

satisfying associativity and commutativity constraints, and, by (3.6.4), having the constant sheaf Λ as a two-sided unit.

Remark 5.15. The isomorphism (5.10.1) is compatible with the decomposition (5.7.5), in the following sense. Let $M_i = \mathcal{K}_n(\chi_i)$, for characters $\chi_i : \mu_{n_i} \to E_\lambda^*$, so that $V_i = V_{\chi_i^{-1}}$. Then, together with the decomposition (5.7.5), the isomorphism (5.10.1) reads

$$\oplus_{\chi_1', \chi_2'} \mathcal{J}_{E_\lambda, k}({\chi'}_1, {\chi'}_2) \otimes_{E_\lambda} (V_{{\chi'}_1^{-1}} \otimes V_{{\chi'}_2^{-1}}) \otimes_{E_\lambda[\mu_{\underline{n}}]} (V_{\chi_1^{-1}} \otimes V_{\chi_2^{-1}}) \xrightarrow{\sim} M_1 *_1 M_2,$$

and the left hand side boils down to $\mathcal{J}_{E_\lambda, k}(\chi_1, \chi_2)$.

Corollary 5.16. *Assume k algebraically closed. Let $g \in I_t$. Under the assumptions of 5.10, let g_i^* denotes the automorphism of M_i induced by the automorphism of the stalk of M_i at 1 given by g (cf. 5.3). Then the automorphism g^* of $M_1 *_1 M_2$ given by the action of g on $(M_1 *_1 M_2)_1$ is :*

$$g^* = g_1^* *_1 g_2^*. \tag{5.16.1}$$

Proof. This follows from (5.10.1) by the same argument as the one used in 5.3 to describe g_L^*. $\square$

5.17

The constructions and statements above have local counterparts, that we will now discuss. We keep the notation of the beginning of 4. We will now denote by $I \subset \mathrm{Gal}(\overline{\eta}/\eta)$ the inertia group, and I_t its tame quotient (which is identified with the group denoted I_t in (3.1.1) by the map $A_{sh} \to A$ and the choice of a path from $\overline{\eta}$ to $\{\overline{1}\}$). For $n \geq 1$ invertible in k, we denote by

$$\pi_n^{\mathrm{loc}} : \mathcal{K}_n^{\mathrm{loc}} \to A_h \tag{5.17.1}$$

the inverse image on $\eta \in A_h$ of the Kummer torsor $\mathcal{K}_n$ (5.1). For $L \in \mathcal{G}(\eta, \Lambda)$ (3.9.2), if $\mu_n(\overline{k}) \subset k$ and L is geometrically trivialized by π_n^{loc}, we have a canonical isomorphism similar to (5.10):

$$L_{\overline{k}} \xrightarrow{\sim} \pi_{n*}^{\mathrm{loc}} \Lambda_{\overline{k}} \otimes_{\Lambda[\mu_n]} L_{\overline{\eta}}. \tag{5.17.2}$$

For a pair of integers $\underline{n} = (n_1, n_2)$ as in 5.5, with $\mu_r(\overline{k}) \subset k$ ($r = \mathrm{lcm}(n_1, n_2)$), we denote by

$$\mathcal{B}^{\mathrm{loc}}_{\Lambda,k}(\underline{n}) \tag{5.17.3}$$

the inverse image of $\mathcal{B}_{\Lambda,k}(\underline{n})$ (5.5.1) on η. This is a tamely ramified representation of $\mathrm{Gal}(\overline{\eta}/\eta)$, geometrically trivialized by π_r, in the notation of 5.5. We will call $\mathcal{B}^{\mathrm{loc}}_{\Lambda,k}(\underline{n})$ the *universal (local) convolution sheaf* (relative to $\underline{n} = (n_1, n_2)$), and, as in 5.6, write it $\mathcal{B}^{\mathrm{loc}}_{\Lambda}(\underline{n})$ or $\mathcal{B}^{\mathrm{loc}}(\underline{n})$ when no confusion may arise.

Proposition 5.18. *(1) The isomorphism (3.4.8) for $K_i = j_!\pi_{n_i *}\Lambda$ induces an isomorphism*

$$\pi^{\mathrm{loc}}_{n_1 *}\Lambda *_1 \pi^{\mathrm{loc}}_{n_2 *}\Lambda \xrightarrow{\sim} \mathcal{B}^{\mathrm{loc}}_{\Lambda,k}(\underline{n}), \tag{5.18.1}$$

where the left hand site denotes a local convolution product (3.9.1).

(2) The stalk of $\mathcal{B}^{\mathrm{loc}}_{\Lambda,k}(\underline{n})$ at $\overline{\eta}$ is finitely generated and free of rank 1 over $\Lambda[\mu_{\underline{n}}]$, and the tame inertia group I_t acts on it through its quotient μ_r via $\mu_r \to \mu_{n_1} \times \mu_{n_2}$, $\varepsilon \mapsto (\varepsilon^{a_1}, \varepsilon^{a_2})$.

Proof. As K_i is tamely ramified at zero, we have, by (3.4.8) and (3.4.6),

$$\pi^{\mathrm{loc}}_{n_1 *}\Lambda *_1 \pi^{\mathrm{loc}}_{n_2 *}\Lambda \xrightarrow{\sim} \mathcal{H}^1(K_1 *^L K_2)_\eta,$$

which proves (1). (2) follows from (5.5 (2), (3)). □

Remark 5.19. By (5.12.3), for any tame representations $L_i \in \mathcal{G}(\eta, \Lambda)$ ($i = 1, 2$), $L_1 *_1 L_2$ is tame. However, as in 5.10, we have a more precise result:

Corollary 5.20. *For $i = 1, 2$, let n_i be as in 5.5, and let $L_i \in \mathcal{G}(\eta, \Lambda)$ be geometrically trivialized by $\pi^{\mathrm{loc}}_{n_i}$. Let $V_i = (L_i)_{\overline{\eta}}$ (with its action of μ_{n_i}), and $V = V_1 \otimes V_2$ (with its action of $\mu_{\underline{n}}$). Then the isomorphisms (5.17.2) for L_1 and L_2 induce an isomorphism*

$$\mathcal{B}^{\mathrm{loc}}(\underline{n})_{\overline{k}} \otimes_{\Lambda[\mu_{\underline{n}}]} V \xrightarrow{\sim} L_1 *_1 L_2 \tag{5.20.1}$$

(with the notation of (3.9.1)), where $\mu_{\underline{n}}$ acts on $\mathcal{B}(\underline{n})$ on the right by $bg = g^{-1}b$. In particular, $L_1 *_1 L_2$ is geometrically trivialized by π_r, and we have

$$\mathrm{rk}(L_1 *_1 L_2) = \mathrm{rk}(L_1)\mathrm{rk}(L_2).$$

Proof. One can imitate the proof of 5.10, or deduce the result from it, using the Gabber-Katz extension theorem. □

Remark 5.21. Let $\mathcal{G}_t$ be the full subcategory of $\mathcal{G} = \mathcal{G}(\eta, \Lambda)$ consisting of tame representations. It follows from 5.19 or 5.20 that $\mathcal{G}_t$ is stable under $*_1$.

Remark 5.22. Assume k is algebraically closed. Let $L \in \mathcal{G}(\eta, \Lambda)$ be a tame representation. As in 5.3, since I_t is commutative, the action of I_t on $L_{\{1\}}$ is I_t-equivariant, hence extends uniquely to an action on L, compatible with its action on $L_{\overline{\eta}}$. For $g \in I_t$, we will denote by g^* the automorphism of L induced by the automorphism g of $L_{\overline{\eta}}$.

Corollary 5.23. *Assume k algebraically closed. Let $g \in I_t$. Under the assumptions of 5.20, let g_i^* denotes the automorphism of L_i induced by the automorphism of $(L_i)_{\overline{\eta}}$ given by g (cf. 5.22). Then the automorphism g^* of $L_1 *_1 L_2$ given by the action of g on $(L_1 *_1 L_2)_{\overline{\eta}}$ is :*

$$g^* = g_1^* *_1 g_2^*. \tag{5.23.1}$$

Proof. The proof is the same as for 5.16. □

This implies the following "convolution" variant of the Thom-Sebastiani theorem for the monodromy operators:

Corollary 5.24. *Under the assumptions of 4.7, with k algebraically closed, assume that, for $i = 1, 2$, $R^{n_i}\Phi_{f_i}(\Lambda)_{x_i}$ is tame. Then $R^{n+1}\Phi_{af_h}(\Lambda)_x$ is tame, and if for $g \in I_t$, g_i^* denotes the monodromy automorphism of $R^{n_i}\Phi_{f_i}(\Lambda)_{x_i}$, and g^* that of $R^{n+1}\Phi_{af_h}(\Lambda)_x$, with the identification (4.7.1) we have*

$$g^* = g_1^* *_1 g_2^*. \tag{5.24.1}$$

Actually, one can recover the original formulation of the Thom-Sebastiani theorem ((0.1), (0.2)), involving a tensor product rather than a convolution. This is a consequence of the following corollaries of 5.5 and 5.18.

Corollary 5.25. *Assume k algebraically closed. There exists a projective system of $\mu_{\underline{n}}$-equivariant isomorphisms*

$$\beta_{\underline{n}} : \mathcal{B}_{\Lambda,k}(\underline{n}) \xrightarrow{\sim} \pi_{n_1*}\Lambda \otimes \pi_{n_2*}\Lambda \tag{5.25.1}$$

(resp.

$$\beta_{\underline{n}}^{\mathrm{loc}} : \mathcal{B}_{\Lambda,k}(\underline{n})^{\mathrm{loc}} \xrightarrow{\sim} \pi_{n_1*}^{\mathrm{loc}}\Lambda \otimes \pi_{n_2*}^{\mathrm{loc}}\Lambda), \tag{5.25.2}$$

for $\underline{n} = (n_1, n_2)$ running through the pairs of integers ≥ 1 invertible in k, ordered by divisibility.

Proof. By 5.18, it suffices to prove the assertion for the global convolution. First, fix n_i, let $r = \gcd(n_1, n_2)$, and let's prove the existence of an isomorphism (5.25.1). The functor $M \to M_{\{1\}}$ is an equivalence from the category of lisse $\Lambda[\mu_{\underline{n}}]$-modules on $\mathbf{G}_{m,k}$ tamely ramified at $\{0\}$ and $\{\infty\}$ to the category of (continuous) $\Lambda[\mu_{\underline{n}}][I_t]$-modules, finitely generated over Λ. The stalks at $\{1\}$ of $\mathcal{B}_{\Lambda,k}(\underline{n})$ and $\pi_{n_1*}\Lambda \otimes \pi_{n_2*}\Lambda$ are both free of rank 1 over $\Lambda[\mu_{\underline{n}}]$, and I_t acts on them through the diagonal map $\mu_r \to \mu_{\underline{n}}$ (5.5 (3)). They are therefore isomorphic, and this proves the existence of an isomorphism $\beta_{\underline{n}}$. To prove the existence of a projective system of isomorphisms $\beta_{\underline{n}}$'s, by (5.7 (a)), it suffices to do it for the diagonal system $\underline{n} = (n, n)$. Isomorphisms $\beta_{(n,n)}$ form a torsor under $(\Lambda[\mu_n \times \mu_n])^*$. As $(\Lambda[\mu_{n+1} \times \mu_{n+1}])^* \to (\Lambda[\mu_n \times \mu_n])^*$ is surjective, the conclusion follows. □

By 5.10 (resp. 5.20), this implies :

Corollary 5.26. *Under the assumptions of 5.25, the functors on $\mathcal{C} \times \mathcal{C}$ (5.14) (resp. $\mathcal{G}_t \times \mathcal{G}_t$ (5.21)), $(M_1, M_2) \mapsto M_1 *_1 M_2$ and $(M_1, M_2) \mapsto M_1 \otimes M_2$ are isomorphic.*

For $k = \mathbf{C}$, a transcendental proof of this was given by Deligne in [2].

Remark 5.27. Let k be as in 5.25, and let ℓ be a prime number invertible in k. For $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$, with $\nu \geq 1$ fixed, projective system of isomorphisms $\beta_\bullet = (\beta_{\underline{n}})$ form a torsor under $(\Lambda[[\mu\times\mu]])^* = \varprojlim(\Lambda[\mu_{n_1}\times\mu_{n_2}])^*$. As $(\mathbf{Z}/\ell^{\nu+1}[[\mu\times\mu]])^* \to (\mathbf{Z}/\ell^\nu[[\mu\times\mu]])^*$ is surjective, there exists an isomorphism of projective systems of $\mu_{n_1}\times\mu_{n_2}$-equivariant $\mathbf{Z}_\ell$-sheaves

$$(\mathcal{B}_{\mathbf{Z}_\ell,k}(\underline{n})) \xrightarrow{\sim} (\pi_{n_1*}\mathbf{Z}_\ell \otimes \pi_{n_2*}\mathbf{Z}_\ell) \tag{5.27.1}$$

(resp.

$$(\mathcal{B}^{\mathrm{loc}}_{\mathbf{Z}_\ell,k}(\underline{n})) \xrightarrow{\sim} (\pi^{\mathrm{loc}}_{n_1*}\mathbf{Z}_\ell \otimes \pi^{\mathrm{loc}}_{n_2*}\mathbf{Z}_\ell)). \tag{5.27.2}$$

This implies the analogue of 5.26 for $\mathbf{Z}_\ell$-sheaves, as well as O_λ-sheaves, E_λ-sheaves, and $\overline{\mathbf{Q}}_\ell$-sheaves (where E_λ is a finite extension of $\mathbf{Q}_\ell$, and O_λ its ring of integers).

5.28

No longer assume that k is algebraically closed. Let n_i be as in 5.5, with $\mu_r(\overline{k}) \subset k$. There is in general no isomorphism $\mathcal{B}_{\Lambda,\overline{k}}(\underline{n}) \xrightarrow{\sim} (\pi_{n_1*}\Lambda\otimes\pi_{n_2*}\Lambda)_{\overline{k}}$ compatible with the action of $\mathrm{Gal}(\overline{k}/k)$, as 5.8 shows. The obstruction is the class in $H^1(\operatorname{Spec} k, (\Lambda[\mu_{\underline{n}}])^*)$ of the geometrically constant, locally free $\Lambda[\mu_{\underline{n}}]$-module of rank 1, $B_{1,2}(\underline{n},\Lambda)\otimes(T_{1,2}(\underline{n},\Lambda))^{-1}$, where $B_{1,2}(\underline{n},\Lambda) = \mathcal{B}_{\Lambda,k}(\underline{n})$, $T_{1,2}(\underline{n},\Lambda) = \pi_{n_1*}\Lambda\otimes\pi_{n_2*}\Lambda$. Let ℓ be a prime number invertible in k, and let us now restrict to rings Λ killed by a power of ℓ. Then $B_{1,2}(\underline{n},\Lambda)\otimes(T_{1,2}(\underline{n},\Lambda))^{-1}$ comes by change of rings $\mathbf{Z}_\ell\to\Lambda$ and field extension $\mathbf{F}_p\to k$ from the geometrically constant, locally free $\mathbf{Z}_\ell[\mu_{\underline{n}}]$-module of rank 1 over $\mathbf{G}_{m,\mathbf{F}_p}$

$$J_{1,2}(\underline{n}) := B_{1,2}(\underline{n},\mathbf{Z}_\ell,\mathbf{F}_p)\otimes(T_{1,2}(\underline{n},\mathbf{Z}_\ell,\mathbf{F}_p))^{-1} \tag{5.28.1}$$

For variable $\underline{n}$, the $J_{1,2}(\underline{n})$'s form a projective system denoted $J_{1,2}$. The next result is due to Deligne :

Proposition 5.29. *Denote by A_i ($i = 1,2,3$) a copy of the projective system $(\pi_{n*}\mathbf{Z}_\ell)$ (n running through the integers prime to p) on $\mathbf{G}_{m,\mathbf{F}_p}$. Then the associativity property of $*_1$ (5.14) gives an isomorphism*

$$J_{12,3}\otimes J_{1,2} \xrightarrow{\sim} J_{1,23}\otimes J_{2,3} \tag{5.29.1}$$

with the notation of (5.28.1) and $J_{12,3} := ((A_1\otimes A_2)*_1 A_3)\otimes((A_1\otimes A_2)\otimes A_3)^{-1}$, $J_{1,23} := (A_1 *_1 (A_2\otimes A_3))\otimes(A_1\otimes(A_2\otimes A_3))^{-1}$.

Proof. The associativity isomorphism

$$(A_1 *_1 A_2) *_1 A_3 \xrightarrow{\sim} A_1 *_1 (A_2 *_1 A_3)$$

can be re-written

$$((A_1\otimes A_2)\otimes J_{1,2}) *_1 A_3 \xrightarrow{\sim} A_1 *_1 ((A_2\otimes A_3)\otimes J_{2,3}).$$

As $J_{1,2}$ and $J_{2,3}$ are geometrically constant, by (3.6.3) (and associativity and commutativity of $*_1$), the left hand side is isomorphic to

$$((A_1\otimes A_2) *_1 A_3)\otimes J_{1,2} \xrightarrow{\sim} ((A_1\otimes A_2)\otimes A_3)\otimes J_{12,3}\otimes J_{1,2},$$

and the right hand side to

$$((A_1 \otimes (A_2 \otimes A_3) \otimes J_{1,23})) *_1 J_{2,3} \xrightarrow{\sim} A_1 \otimes (A_2 \otimes A_3) \otimes J_{1,23} \otimes J_{2,3}.$$

As $A_1 \otimes A_2 \otimes A_3$ is locally free of rank 1 over $\mathbf{Z}_\ell[\mu \times \mu \times \mu]$ (where $\mu := (\mu_\bullet)$, (5.29.1) follows. $\square$

Remark 5.30. By 5.8, (5.29.1) implies the following identity between Jacobi sums :

$$J(\chi_1 \otimes \chi_2, \chi_3) J(\chi_1, \chi_2) = J(\chi_1, \chi_2 \otimes \chi_3) J(\chi_2, \chi_3), \tag{5.30.1}$$

for non-trivial characters $\chi_i : \mathbf{F}_q^* \to E_\lambda^*$ $(i = 1, 2, 3)$, with the notations of 5.8. ((5.30.1) follows from the classical identity between Gauss and Jacobi sums, see e. g. ([33], Sommes trigonométriques, (4.15.2)).)

Remark 5.31. For $i = 1, 2$ let M_i be a lisse $\overline{\mathbf{Q}}_\ell$-sheaf on $\mathbf{G}_{m,k}$, tamely ramified at $\{0\}$ and $\{\infty\}$, such that the image of the representation $\rho_i : \pi_1(\mathbf{G}_{m,k}, \overline{1}) \to \mathrm{GL}(V_i)$, where $V_i = (M_i)_{\overline{1}}$, is contained in a unipotent subgroup of $\mathrm{GL}(V_i)$. Using (5.29.1) Deligne has shown (unpublished) that under these assumptions there is an isomorphism $M_1 *_1 M_2 \xrightarrow{\sim} M_1 \otimes M_2$.

Remark 5.32. Constructions and results from 5.28 to 5.31 imply local variants (on A_h), with π_n replaced by π_n^{loc}, $\mathcal{B}$ by $\mathcal{B}^{\mathrm{loc}}$, etc.

5.33

Let us now return to the situation considered in 4.8, and assume k algebraically closed. Let us write (S, s) for $(A_h = A_{sh}, \{0\})$, (T, t) for $((A^2)_h, \{(0, 0)\})$, $f : X \to T$ for $f_h : X_h \to (A^2)_h$, and denote by $a : T \to S$ the morphism defined by the sum map.

Assume that $R^{n_i}\Psi_{f_i}(\Lambda)$ is tame at x_i (so that, by (4.5.1) and 5.20), $R\Psi_{af}(\Lambda)$ is tame at x). Then, by ([10], 3.5), if σ is a topological generator of I_t, the variation morphism (induced by $\sigma - 1$)

$$\mathrm{Var}(\sigma)_i : R^{n_i}\Phi_{f_i}(\Lambda)_{x_i} \to H^{n_i}_{\{x_i\}}((X_i)_s, R\Psi_{f_i}(\Lambda)) \tag{5.33.1}$$

is an isomorphism. Similarly,

$$\mathrm{Var}(\sigma) : R^{n+1}\Phi_{af}(\Lambda)_x \to H^{n+1}_{\{x\}}(X_s, R\Psi_{af}(\Lambda)), \tag{5.33.2}$$

where $X_s := (af)^{-1}(s)$, is an isomorphism. Note that $\mathrm{Var}(\sigma)$ commutes with I_t, hence defines an isomorphism $\mathrm{Var}(\sigma)_i^*$ between the sheaves (on S) $R^{n_i}\Phi_{f_i}(\Lambda)_{x_i}$ and $H^{n_i}_{\{x_i\}}((X_i)_s, R\Psi_{f_i}(\Lambda))$ (resp. an isomorphism $\mathrm{Var}(\sigma)^*$ between $R^{n+1}\Phi_{af}(\Lambda)_x$ and $H^{n+1}_{\{x\}}(X_s, R\Psi_{af}(\Lambda))$).

Recall that, by ([10], 3.7), we have a commutative diagram

$$\begin{tikzcd} R\Psi_{f_i}(\Lambda)[n_i] \arrow[r] \arrow[d, "(\sigma-1)_i"'] & i_{x_i*}R^{n_i}\Phi_{f_i}(\Lambda)_{x_i} \arrow[d, "\mathrm{Var}(\sigma)_i"] \\ R\Psi_{f_i}(\Lambda)[n_i] & i_{x_i*}H^{n_i}_{x_i}((X_i)_s, R\Psi_{f_i}(\Lambda)) \arrow[l] \end{tikzcd} , \tag{5.33.3}$$

where the horizontal maps are adjunction maps, which are dual to each other (up to a Tate twist). Moreover, by ([10], 3.8), $i_{x_i*}R^{n_i}\Phi_{f_i}(\Lambda)_{x_i} = R\Phi_{f_i}(\Lambda)[n_i]$, the entries of (5.33.3) are perverse sheaves, and $\mathrm{Var}(\sigma)_i$ is the canonical isomorphism from the co-image of the left vertical map to its image (in the category of perverse sheaves). In particular, $(\sigma-1)_i : R\Psi_{f_i}(\Lambda)[n_i] \to R\Psi_{f_i}(\Lambda)[n_i]$ factors uniquely through $R\Phi_{f_i}(\Lambda)[n_i]$:

(5.33.4)
$$\begin{tikzcd} R\Psi_{f_i}(\Lambda)[n_i] \arrow[r] \arrow[d, "(\sigma-1)_i"'] & R\Phi_{f_i}(\Lambda)[n_i] \arrow[dl, "\mathrm{Var}(\sigma)_i"] \\ R\Psi_{f_i}(\Lambda)[n_i] & \end{tikzcd}$$

The isomorphisms (4.5.1) and (4.5.2) define an isomorphism from the triangle

(5.33.5)
$$\begin{tikzcd} R\Psi_{f_1}(\Lambda)[n_1] *^L R\Psi_{f_2}(\Lambda)[n_2] \arrow[r] \arrow[d, "(\sigma-1)_1 *^L (\sigma-1)_2"'] & R\Phi_{f_1}(\Lambda)[n_1] *^L R\Phi_{f_2}(\Lambda)[n_2] \arrow[dl, "\mathrm{Var}(\sigma)_1 *^L \mathrm{Var}(\sigma)_2"] \\ R\Psi_{f_1}(\Lambda)[n_1] *^L R\Psi_{f_2}(\Lambda)[n_2] & \end{tikzcd}$$

to the triangle

(5.33.6)
$$\begin{tikzcd} R\Psi_{af}(\Lambda)[n+1] \arrow[r] \arrow[d, "\sigma-1"'] & R\Phi_{af}(\Lambda)[n+1] \arrow[dl, "\mathrm{Var}(\sigma)"] \\ R\Psi_{af}(\Lambda)[n+1] & \end{tikzcd}$$

Using now the factorizations given by the lower triangles of (5.33.3), we find that the isomorphisms (4.5.1) and (4.5.2) define an isomorphism from the triangle

(5.33.7)
$$\begin{tikzcd} & R\Phi_{f_1}(\Lambda)[n_1] *^L R\Phi_{f_2}(\Lambda)[n_2] \arrow[dl, "\mathrm{Var}(\sigma)_1 *^L \mathrm{Var}(\sigma)_2"'] \arrow[d, "\mathrm{Var}(\sigma)_1 *^L \mathrm{Var}(\sigma)_2"] \\ R\Psi_{f_1}(\Lambda)[n_1] *^L R\Psi_{f_2}(\Lambda)[n_2] & i_{x_1*}H^{n_i}_{x_1}((X_i)_s, R\Psi_{f_1}(\Lambda)) *^L i_{x_2*}H^{n_2}_{x_2}((X_2)_s, R\Psi_{f_2}(\Lambda)) \arrow[l] \end{tikzcd}$$

to the triangle

(5.33.8)
$$\begin{tikzcd} & R\Phi_f(\Lambda)[n+1] \arrow[dl, "\mathrm{Var}(\sigma)"'] \arrow[d, "\mathrm{Var}(\sigma)"] \\ R\Psi_{af}(\Lambda)[n+1] & i_{x*}H^{n+1}_x(X_s, R\Psi_{af}(\Lambda)) \arrow[l] \end{tikzcd} .$$

In particular, we have a commutative square of isomorphisms

(5.33.9)
$$\begin{tikzcd} R^{n_1}\Phi_{f_1}(\Lambda)_{x_1} *_1 R^{n_2}\Phi_{f_2}(\Lambda)_{x_2} \arrow[r] \arrow[d, "\mathrm{Var}(\sigma)^*_1 *_1 \mathrm{Var}(\sigma)^*_2"'] & R^{n+1}\Phi_{af}(\Lambda)_x \arrow[d, "\mathrm{Var}(\sigma)^*"'] \\ H^{n_1}_{\{x_1\}}((X_1)_s, R\Psi_{f_1}(\Lambda)) *_1 H^{n_2}_{\{x_2\}}((X_2)_s, R\Psi_{f_1}(\Lambda)) \arrow[r] & H^{n+1}_{\{x\}}(X_s, R\Psi_{af}(\Lambda)) \end{tikzcd} ,$$

where the upper horizontal map is the isomorphism (4.5.2), and the vertical maps are the isomorphisms defined by (5.33.1) and (5.33.2). In other words, with the above identifications, we have

$$\mathrm{Var}(\sigma)^* = \mathrm{Var}(\sigma)_1^* *_1 \mathrm{Var}(\sigma)_2^*. \tag{5.33.10}$$

Remark 5.34. (a) Presumably, the isomorphism

$$H^{n+1}_{\{x\}}(X_s, R\Psi_{af}(\Lambda)) \xrightarrow{\sim} H^{n_1}_{\{x_1\}}((X_1)_s, R\Psi_{f_1}(\Lambda)) *_1 H^{n_2}_{\{x_2\}}((X_2)_s, R\Psi_{f_1}(\Lambda)), \tag{5.34.1}$$

in (5.33.9) is the inverse of the isomorphism deduced by duality from the upper horizontal isomorphism, using the perfect pairings

$$R^{n_i}\Phi_{f_i}(\Lambda)_{x_i} \otimes H^{n_i}_{x_i}((X_i)_s, R\Psi_{f_i}(\Lambda)) \to H^{2n_i}_{\{x_i\}}((X_i)_s, K_{(X_i)_s}[-2n_i]) = \Lambda$$

(resp.

$$R^{n+1}\Phi_{af}(\Lambda)_x \otimes H^{n+1}_x((X_i)_s, R\Psi_{af}(\Lambda)) \to H^{2n+2}_{\{x\}}((X)_s, K_{(X)_s}[-2n-2]) = \Lambda),$$

where $n = n_1 + n_2$, and K denotes a dualizing complex ($= g^!\Lambda$, with g the projection to $\mathrm{Spec}\, k$). I haven't checked it.

(b) In view of 5.26, we could replace $*_1$ by $\otimes$ in (5.33.10). For $k = \mathbf{C}$, we recover (0.3).

Acknowledgements. I am grateful to Pierre Deligne, Ofer Gabber and Gérard Laumon for illuminating conversations, and Fabrice Orgogozo for a careful reading of a first draft of this paper and helpful comments. I thank T. Saito for helping me correct an argument in (1.7 (b)), suggesting 1.19, statements and proofs of 5.11, 5.12, and W. Sawin for a remark on (4.11 (c)) and the communication of references [24] and [29]. I thank Alexander Beilinson, Vladimir Drinfeld, Lei Fu, Takeshi Saito, and Weizhe Zheng for precious comments and criticism. I am especially grateful to Weizhe Zheng for sending me his note [31], and accepting to submit it as an appendix to this paper. I thank Tung-Yang Lee for the realization of the figure illustrating the proof of 5.11. Finally, I wish to heartily thank the referee for a meticulous examination of the manuscript and the request of many corrections and improvements. Parts of this paper were written or revised while I was visiting the Department of Mathematics of the University of Chicago, the Morningside Center of Mathematics of the AMSS in Beijing, the Graduate School of Mathematical Sciences at the University of Tokyo, and the Department of Mathematics at Academia Sinica, NCTS, Taipei. I warmly thank these institutions for their hospitality and support.

References

[1] P. Deligne, *Letter to N. A' Campo*, Nov. 14, 1972.

[2] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Séminaire à l'IHÉS, 1980.

[3] P. Deligne, *La conjecture de Weil : II*, Pub. Math. IHÉS **52** (1980), 137-252.

[4] P. Deligne, *Letter to L. Illusie*, May 5, 1999.

[5] P. Deligne, *Letter to L. Fu*, April 13, 2011.

[6] T. Ekedahl, *On the Adic Formalism*, The Grothendieck Festschrift, Vol. II, 197-218, Birkhäuser 1990.

[7] L. Fu, *A Thom-Sebastiani Theorem in Characteristic p*, Mathematical Research Letters 21 (2014), 101–119, *arXiv*:1105.5210.

[8] L. Illusie, *Théorie de Brauer et caractéristique d'Euler-Poincaré (d'après P. Deligne)*, in Caractéristique d'Euler-Poincaré, 161–172. Astérisque 82, Soc. Math. de France, 1981.

[9] L. Illusie, *I Autour du théorème de monodromie locale*, in *Périodes p-adiques*, Séminaire de Bures 1988, Astérisque 223, 9-57, SMF, 1994.

[10] L. Illusie, *Perversité et variation*, Manuscripta math. **112** (2003), 271-295.

[11] L. Illusie. On semistable reduction and the calculation of nearby cycles. In *Geometric aspects of Dwork theory. Vol. I, II*, pages 785–803. Walter de Gruyter, Berlin, 2004.

[12] L. Illusie, *XI Produits orientés*, in [13].

[13] L. Illusie, Y. Laszlo, F. Orgogozo (eds.) *Gabber's work on local uniformization and étale cohomology of quasi-excellent schemes. Seminar at École Polytechnique 2006–2008. (Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents. Séminaire à l'École Polytechnique 2006–2008.)* (French, English) Zbl 1297.14003 Astérisque 363-364. Paris: Société Mathématique de France (SMF) (2014).

[14] N. M. Katz. Local-to-global extensions of representations of fundamental groups. *Ann. Inst. Fourier (Grenoble)*, 36(4):69–106, 1986.

[15] N. Katz, *Gauss Sums, Kloosterman Sums, and Monodromy Groups*, Ann. of Math. Studies **116**, Princeton Univ. Press, 1988.

[16] G. Laumon, *Comparaison de caractéristiques d'Euler-Poincaré en cohomologie ℓ-adique*, Comptes Rendus de l'Acad. des Sc., Série I. Mathématique, **292**, no. 3 (1981), 209–12.

[17] G. Laumon. Caractéristique d'Euler-Poincaré des faisceaux constructibles sur une surface. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 193–207. Soc. Math. France, Paris, 1983.

[18] G. Laumon. Vanishing cycles over a base of dimension ≥ 1. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin, 1983.

[19] G. Laumon, *Majoration de sommes exponentielles attachées aux hypersurfaces diagonales*, Ann. scient. Éc. Norm. Sup., 4e série, t. 16 (1983), 1-58.

[20] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Pub. Math. IHÉS **65** (1987), 131-210.

[21] F. Orgogozo, *Modifications et cycles proches sur une base générale*, IMRN, vol. 2006, ID 25315, 38 p., 2006.

[22] F. Orgogozo, *II Topologies adaptées à l'uniformisation locale*, in [13].

[23] F. Orgogozo, *XII-A Descente cohomologique orientée*, in [13].

[24] A. Rojas-León. Local convolution of ℓ-adic sheaves on the torus. *Math. Z.*, 274(3-4):1211–1230, 2013.

[25] T. Saito. ϵ-factor of a tamely ramified sheaf on a variety. *Invent. Math.*, 113(2):389–417, 1993.

[26] T. Saito. *Wild ramification and the cotangent bundle. arXiv*:1301.4632, to appear in JAG.

[27] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf* (2015), *arXiv*:1510.03018.

[28] T. Saito, *Notes on the characteristic cycle of a constructible sheaf* (2016), *arXiv*:1603.07865.

[29] W. Sawin, *Local Convolution on Elliptic Curves and* $\mathbf{G}_m$, in preparation.

[30] M. Sebastiani and R. Thom, *Un résultat sur la monodromie*, Invent. Math. **13** (1971), 90-96.

[31] W. Zheng, *Künneth formula for nearby cycles*, preprint, 2016.

[32] [SGA 4] *Théorie des topos et cohomologie étale des schémas*, Séminaire de géométrie algébrique du Bois-Marie 1963-64, dirigé par M. Artin, A. Grothendieck, J.-L. Verdier, Lecture Notes in Math. 269, 270, 305, Springer-Verlag, 1972, 1973.

[33] SGA 4 1/2, *Cohomologie étale*, Séminaire de Géométrie Algébrique du Bois-Marie, par P. Deligne, Lecture Notes in Math. 569, Springer-Verlag 1977.

[34] SGA 5, *Cohomologie ℓ-adique et fonctions* L, Séminaire de géométrie algébrique du Bois-Marie 1965-66, dirigé par A. Grothendieck, Lecture Notes in Math. 589, Springer-Verlag, 1977.

[35] SGA 7, *Groupes de monodromie en géométrie algébrique*, Séminaire de Géométrie Algébrique du Bois-Marie 1967-1969, I dirigé par A. Grothendieck, II par P. Deligne et N. Katz, Lecture Notes in Math. 288, 340, Springer-Verlag 1972, 1973.