A few hints and references for the exercices

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1. Chapters 1 and 2

1. See [1], Prop 1.12.

2. cf. [1], end of proof of Th. 2.6.

3. See [1], Prop 1.25.

4. See [1], Prop 1.31.

5. See [1], Lemma 1.38.
6. See [1], Th. 1.42.

9. For a), first show that there exists a normal open subgroup U of G such that U acts trivially on A and the canonical map $H^n(G/U, A) \to H^n(G, A)$ is onto. for b), write the cohomology long exact sequence associated to $0 \to A \to B \to A/B \to 0$.

12. cf. [3], §I.3, Propsition 13.

13. For a), consider a p-Sylow of G. For b), consider both cases: p divides the order of G or not.

14. See [3], §I.3, Corollary 4.

15. Use Prop 2.42 of the lecture notes.

17. See [3], §II.2, Proposition 4.

18. Use the identication of $\mathbf{Z}/n$ with μ_n given by the assumption that k contains a primitive n-th root of unity.

19. For a) and b), Use the fact that the extension $\bar{k}/k$ is normal, hence Galois with group $G = \mathbf{Z}/p$, so the cohomology of G is 2-periodic. For c), show that k^*/k^{*^p} and $k^*/N_{\bar{k}/k}(\bar{k}^*)$ both have same cardinality as $\mathrm{Br}\, k$. For d), see exercise 18. For e), use the fact that the minimal polynomial of α over k is X_a^p.

20. The case of an element whose (finite) order is not a power of 2 is settled by exercise 19. It is then sufficient to show that G has no element of order 4; for this, use the fact that an extension of degree 4 of k of the form $k(\sqrt{a}, \sqrt{-1})$ has Galois group $\mathbf{Z}/2 \times \mathbf{Z}/2$. For the last statement, use the fact that the absolute Galois group of k cannot be $(\mathbf{Z}/2)^2$ either because of the last statement of exercise 19.

2. Chapter 3

21. See [3], §I.5, Proposition 33.

22. cf. [3] , §I.5.4.

23. For a) and b), see [3], §1.5, Corollary 1 and Proposition 37. For c), use the class formulation for the action of a finite group on a finite set. For d) and e), cf. loc. cit., Proposition 39.

24. See [3], §I.5, Proposition 40 and Corollary 1.

25. Mimic the proof with cocycles of the abelian case.

27. Use Hilbert's 90.

28. Cf. [3], §III.1.3.

3. Chapters 4 and 5

29. For a) and b), use Artin-Schreier exact sequence and the vanishing of the cohomology of a coherent sheaf over an affine scheme (plus the identification of the étale cohomology groups of $\mathbf{G}_a$ with the Zariski cohomology groups of $\mathcal{O}_X$). For c), consider the inclusion $g : \eta \to Y$ of the generic point and show that the stalks of $R^r g_*(g^*\mathcal{F})$ are torsion (resp. zero if $\mathcal{F}$ is uniquely divisible); then use Leray spectral sequence when $\mathcal{F}$ is uniquely divisible, and reduce to the case $\mathcal{F}$ torsion-free in the general case.

30. For a), use Chevalley's result. For b) and c), use Leray spectral sequence and the fact that Galois cohomology in positive degree is torsion. For d), observe that multiplication by n is smooth and surjective on A for every positive integer n.

35. Start with the case $T = \mathbf{G}_m$. then use Hochschild-Serre spectral sequence.

36. Apply Brauer-Hasse-Noether Theorem.

38. Use the birational invariance of the Brauer group.

40. For a), use the finiteness theorem of [2], Th. 5.17. For b), use Corollary 8.15 of [1].

References

[1] D. Harari: *Galois cohomology and class field theory* (translated from the 2017 French original by Andrei Yafaev), Universitext, Springer, Cham 2020.

[2] D. Harari: Lectures notes 2023, https://www.imo.universite-paris-saclay.fr/~david.harari/enseignement/m2brauer23/

[3] J-P. Serre: *Cohomologie galoisienne*, fifth edition, Lecture Notes in Mathematics **5**, Springer-Verlag, Berlin, 1994.