Exercices M2 "The Brauer group"

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1. Chapters 1 and 2

1. Let G be a finite group. Let H be a subgroup of G. Let A be an H-module and B be a G-module. Check the formula $\mathrm{Hom}_H(B,A) = \mathrm{Hom}_G(B, I_G^H(A))$. How does this extend to a profinite group G ?

2. Let G be a finite group. Let A be an abelian group. Consider the induced module $I_G(A) \simeq \mathbf{Z}[G] \otimes A$. Show that the groups $\widehat{H}^0(G, I_G(A))$ and $\widehat{H}^{-1}(G, I_G(A))$ are zero.

3. Let G be a finite group. Let (A_j) be an inductive system of G-modules, set $A = \varinjlim(A_j)$. Show by induction on i and dimension shifting that

$$\varinjlim_j H^i(G, A_j) \simeq H^i(G, A)$$

for all $i \in \mathbf{N}$.

4. Let G be a finite group. Let $(A_n)_{n \in \mathbf{N}}$ be a projective system of finite G-modules, set $A = \varprojlim_n A_n$.

a) Show that for all $i \geq 0$, there is an isomorphism

$$H^i(G, A) \simeq \varprojlim_n H^i(G, A_n).$$

(embed each A_n into $I_G(A_n)$ and compare $\varprojlim I_G(A_n)$ with $I_G(A)$).

b) Take $G = \mathbf{Z}/p$ with p prime. Let M_n be the G-module $\mathbf{Z}$ with trivial action of G. Let ℓ be a prime with $\ell \neq p$, consider the projective system (M_n), the transition maps being multiplication by ℓ. Compare $\varprojlim_n H^2(G, M_n)$ and $H^2(G, M)$, where $M = \varprojlim_n M_n$.

c) Show that the analogue of a) is false for a profinite group G, even if A is assumed to be discrete (take $G = \mathbf{Z}_p$ and $A_n = \mathbf{Z}/p^n$ endowed with the trivial action of G).

5. Let G be a finite group. Let H be a subgroup of G. Show that if a G-module is injective, it is also injective as an H-module (reduce to the same statement for "induced" instead of "injective"). Deduce that an injective G-module is divisible as an abelian group.

6. Let G be a finite group. Let H be a subgroup of G. Let A be a G-module. Check directly (using cocycles) the exactness of

$$0 \to H^1(G/H, A^H) \overset{Inf}{\to} H^1(G, A) \overset{Res}{\to} H^1(H, A).$$

7. Let G be a profinite group. Let H be a closed subgroup of G. Let A be an abelian group.

a) Show that H is the projective limit of the H/V, where V runs over the open subgroups of H that are normal in G.

b) Let V be as above. Show that $I_G(A)^V \simeq I_{G/V}(A)$ is the inductive limit of a family of induced H/V-modules.

c) Deduce that for all $n > 0$, we have $H^n(H, I_G(A)) = 0$.

8. Let G be a profinite group. Let A be a finite G-module.

a) Show that for every $f \in I_G(A)$, there exists a normal open subgroup U of G such that for every $x \in G$, the value $f(x)$ depends only on the class of x in G/U.

b) Deduce that $I_G(A)$ is the direct limit of the $I_G^U(A)$, the limit being taken over all open normal subgroups U of G.

9. Let G be a profinite group. Let A be a finite G-module. Let $n > 0$.

a) Assume that $H^n(G, A)$ is finite. Show that there exists a finite G-module B and an injective morphism of G-modules $f : A \to B$ such that the map $H^n(G, A) \to H^n(G, B)$ induced by f is zero.

b) Give an example where the conclusion of a) is no longer valid if $H^n(G, A)$ is not assumed to be finite.

10. Let G be a profinite abelian group. Assume that for all positive integers $n > 0$, the group G/nG is finite.

a) Show that nG is open in G.

b) Let U be an open subgroup of G. Compare G/U and nG/nU, and deduce that nU is open in G.

c) Deduce that if A is a discrete finite G-module, then $H^1(G, A)$ is finite.

11. Let G be a profinite group. Let A be a discrete G-module. Assume that A is isomorphic to $\mathbf{Z}^r$ as an abelian group, for some $r \in \mathbf{N}$.

a) Show that if the action of G on A is trivial, then $H^1(G, A) = 0$.

b) Show that there exists an open normal subgroup U of G such that the inflation map $H^1(G/U, A^U) \to H^1(G, A)$ is an isomorphism.

c) Show that $H^1(G, A)$ is finite. Does this result extend to $H^r(G, A)$ for $r > 1$?

12. Let G be a profinite group. Let p be a prime number. Let M be a G-module, denote by $N = M[p]$ the p-torsion submodule of M and set $Q := M/pM$, $I := pM$. Let $n = \mathrm{cd}_p(G)$ (assumed to be finite). Let $q > n+1$.

a) Show that the map $H^q(G, M) \to H^q(G, I)$ induced by multiplication by p and the map $H^q(G, I) \to H^q(G, M)$ induced by the inclusion $I \to M$ are both injective.

b) Deduce that $H^q(G, M)[p] = 0$ and that $\mathrm{scd}_p(G) \leq n + 1$.

13. Let G be a profinite group. Let p be a prime number.

a) Show that if $\mathrm{cd}_p(G)$ is neither zero nor infinite, then the exponent of p in the order of G is infinite.

b) Show that the strict p-cohomological dimension of G cannot be 1.

14. Let G be a profnite group of cohomological dimension $n \in \mathbf{N}$.

a) Let M be finite type discrete G-module. Show that there exists an open normal subgroup U of G and an exact sequence of G-modules:

$$0 \to B \to \mathbf{Z}[G/U]^r \to M \to 0$$

for some $r \in \mathbf{N}$.

b) Show that if $H^{n+1}(U, \mathbf{Z}) = 0$, then $H^{n+1}(G, M) = 0$.

c) Deduce that $\mathrm{scd}(G) = n$ if and only if for every (normal) open subgroup U of G, we have $H^{n+1}(U, \mathbf{Z}) = 0$. How does this result extend to strict p-cohomological dimension ?

15. Let G be a profinite group of finite cohomological dimension. Show that every element $s \neq 1$ of G is of infinite order.

16. Let G be a profinite group of cohomological dimension n. Let A be a divisible discrete G-modules. Show that $H^q(G, A) = 0$ for all $q > n$.

17. Let p be a prime number. Let k be a field of characteristic $\neq p$ with separable closure $\bar{k}$. Let $n \in \mathbf{N}^*$. Prove the equivalence of the following:

a) $\mathrm{cd}_p(k) \leq n$;

b) For every algebraic separable extension $K \subset \bar{k}$ of k, we have

$$H^{n+1}(K, \bar{k}^*)[p] = 0$$

and the p-primary group $H^{n+1}(K, \bar{k}^*)\{p\} = 0$ is divisible;

c) Same as b), but restricted to extensions K/k that are finite and of degree prime to p.

18. Let k be a field. Let n be a positive integer, not divisible by the characteristic of k. Assume that k contains a primitive n-th root of unity. Show that every Galois extension of k with Galois group $\mathbf{Z}/n$ can be written $k(\sqrt[n]{a})$ with $a \in k^*$. Is there an analog for extension of Galois group $\mathbf{Z}/p$ in characteristic p ?

19. Let k be a field of characteristic zero. Assume that the algebraic closure $\bar{k}$ of k is a finite extension of k of prime degree p.

a) Show that $\mathrm{Br}\, k$ is a p-torsion group.

b) Show that $\mathrm{Br}\, k$ is isomorphic to $H^2(k, \mu_p)$ and to $H^3(k, \mu_p)$.

c) Deduce that $N_{\bar{k}/k}(\bar{k}^*) = k^{*p}$.

d) Show that k contains a primitive p-th root ζ of 1 and that $\bar{k} = k(\alpha)$ with $\alpha \notin k$ and $a := \alpha^p \in k$.

e) By computing the norm of α, deduce that $p = 2$ and $\bar{k} = k(\sqrt{-1})$.

20. Deduce from the previous exercise that if G is the absolute Galois group of a field of characteristic zero, then every non trivial element of finite order in G is of order 2. Deduce that every subgroup of finite order of G is trivial or of order 2.

2. Chapter 3

In all exercises, the symbol G denotes a profinite group.

21. Let A be a G-group. A *principal homogeneous space* of A is a non empty G-set P, equipped with a simply transitive right-action

$$(x, a) \mapsto x.a, \ \ x \in P, a \in A$$

of A which is compatible with the G-structures (that is: ${}^s(x.a) = ({}^sx).({}^sa)$ for all $s \in G, x \in P, a \in A$). An isomorphism between two principal homogeneous spaces P, P' is a bijective map $u : P \to P'$ compatible (in an obvious sense) with the left-action of G and the right action of A. Denote by $P(A)$ the set of isomorphism classes of principal homogeneous spaces of A.

a) Show that one can define a map $u : P(A) \to H^1(G, A)$ as follows: for $P \in P(A)$, choose $x \in A$; for each $s \in G$, denote by a_s the unique element of A such that ${}^sx = x.a_s$. Then take for $u(P)$ the class of the cocycle $s \mapsto a_s$.

b) Let $a \in Z^1(G, A)$ be a cocycle. Let P_a be the group A with the "twisted" action of G given by $s(x) = a_s.{}^sx$. Show that the operation of A on P_a by right translations yields a structure of principal homogenous space of A on P_a.

c) Show that u is bijective, with inverse map $v : H^1(G, A) \to P(A)$ induced by $a \mapsto P_a,\ a \in Z^1(G, A)$. Thus the pointed set $H^1(G, A)$ classifies principal homogeneous spaces of A.

22. Let B be a G-group. Let A be a G-subgroup of B. Give a definition of the coboundary $H^0(G, B/A) \to H^1(G, A)$ using the definition of $H^1(G, A)$ in terms of principal homogeneous spaces (see exercise 21).

23. Let B be a G-group. Let A be a G-subgroup of B.

a) Show that the kernel of the map $f : H^1(G, A) \to H^1(G, B)$ identifies with the quotient of $H^0(G, B/A)$ by the action of $H^0(G, B)$.

b) Let $b \in Z^1(G, B)$ be a cocycle with class $\beta \in H^1(G, B)$. Define the G-set ${}_b(B/A)$ as the set B/A with the twisted action of G given by $s(x) = b_s.{}^sx,\ s \in G, x \in B/A$. Show that $\beta \in \operatorname{Im} f$ if and only if $H^0(G,{}_b(B/A)) \neq \emptyset$.

c) Assume that G is a finite p-group (with p prime) and the index $[B : A]$ is finite, not divisible by p. Show that f is surjective. Does this extend to G profinite ?

Assume further that A is normal in B and set $C = B/A$.

d) Show that there is a right operation of C^G on $H^1(G, A)$ defined as follows: lift $c \in C^G$ to $b \in B$ and write ${}^sb = b.x_s$ with $x_s \in A$ for each $s \in G$. Then for every cocycle $a \in Z^1(G, A)$, define the class $[a].c$ as the class of the cocycle $s \mapsto b^{-1}a_s{}^sb$.

e) Show that two elements α, α' of $H^1(G, A)$ have the same image by f if and only if there exists $c \in C^G$ such that $\alpha' = \alpha.c$.

24. Let B be a G-group. Let A be an abelian and normal G-subgroup of B, set $C = B/A$. Define by $(c, \alpha) \mapsto c.\alpha$ the left action of C^G on $H^1(G, A)$ induced by the G-morphism $C^G \to \mathtt{Aut}_G(A)$ given by the action of C^G on A

(associated to the extension $1 \to A \to B \to C \to 1$). Let $\delta : C^G \to H^1(G, A)$ be the coboundary map.

a) Show that $\alpha^c = c^{-1}.\alpha + \delta(c)$ for all $c \in C^G, \alpha \in H^1(G, A)$, where α^c is defined by the right action of C^G on $H^1(G, A)$ defined on Exercice 23 d).

b) Show that $\delta(c'c) = \delta(c) + c^{-1}.\delta(c')$ for all $c, c' \in C^G$.

c) Deduce that if A is contained in the center of B, then δ is a morphism of groups.

25. Let A be a G-group. Let H be a closed normal subgroup of G. Show that there is an exact sequence of pointed sets

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A).$$

26. Let G be a finite group. Let A be a finite G-group whose order is prime to the order of G.

a) Assume that G is a p-group. Show that $H^1(G, A) = 0$ (use Exercise 23 c).

b) Deduce that if G is solvable, then $H^1(G, A) = 0$ (proceed by induction on $\#G$ and use Exercise 25).

c) Assume that A is a solvable group. Show by induction on $\#A$ that $H^1(G, A) = 0$.

d) Using Feit-Thomson Theorem (which says that every finite group of odd order is solvable), show that $H^1(G, A) = 0$ without additional assumption.

e) Does this extend to G profinite ?

27. Let K/k be a Galois extension of fields with group G. Compute $H^1(G, SL_n(K))$.

28. Let V be a quasi projective variety over a field k. Let K be a finite Galois extension of k with group G. Set $V_K = V \times_k K$. Let $A(K)$ be the group of K-automorphisms of V_K, which is a G-group for the action

$$(s.f)(x) = s.f(s^{-1}.x), \ f \in A(K), s \in G, x \in V.$$

a) Show that for every k-variety V' such that V'_K is isomorphic to V_K, the set P of K-isomorphisms between V'_K and V_K is a principal homogeneous space of $A(K)$.

b) Deduce from a) and Exercise 21 an injective map

$$\theta : E(K/k, V) \to H^1(G, A(K))$$

between the k-isomorphisms classes of V' as in a) and the cohomology set $H^1(G, A(K))$.

c) Let $s \mapsto c_s$ be a cocycle of $Z^1(G, A(K))$. Define a new operation of G on V_K by

$$s(x) = c_s({}^s x), \ \ s \in G, x \in V_K,$$

and denote by ${}_cV$ the quotient of V_K by this new action of G (the existence of this quotient as a k-variety is ensured by the assumption that V is quasi-projective). Show that the image of the class of ${}_cV$ by θ is the class of the cocycle c in $H^1(G, A(K))$.

d) Deduce from c) that k-forms of V are classified by the pointed set $H^1(k, \mathtt{Aut}\,(V_{\bar{k}})$.

e) Take for V the projective space $\mathbf{P}^n_k$. What does d) say about classification of its k-forms ? Describe the special case when k is a finite field.

3. Chapters 4 and 5

29. Let X be an affine $\mathbf{F}_p$-scheme of finite type.

a) Show that $H^i(X, \mathbf{Z}/p) = 0$ for every integer $i \geq 2$.

b) Assume that X is the affine space over $\mathbf{F}_p$. Show that $H^1(X, \mathbf{Z}/p) \neq 0$.

c) Let Y be a normal, connected and noetherian scheme. Let $\mathcal{F}$ be a constant sheaf on Y. Show that for any $r > 0$, the group $H^r(Y, \mathcal{F})$ is torsion, and that is is zero if $\mathcal{F}$ is uniquely divisible.

d) Deduce that $H^2(Y, \mathbf{Z})$ is isomorphic to $H^1(Y, \mathbf{Q}/\mathbf{Z})$, and that for every $n > 0$, there is an isomorphism $H^1(Y, \mathbf{Z}/n) \simeq_n H^1(Y, \mathbf{Q}/\mathbf{Z})$.

30. Let X be a smooth and integral variety over a field of characteristic zero k. Let A be an abelian variety (that is: a projective, smooth and connected algebraic group over k). Recall (Chevalley) that any k-rational map from X to A extends to a k-morphism $X \to A$. Let $j : \eta \to X$ be the inclusion of the generic point of X, set $A_\eta = A \times_k \eta$ and $A_X = A \times_k X$.

a) Show that $j_* A_\eta = A_X$ as étale sheaves on X.

b) Show that for all integers $q > 0$, the sheaves $R^q j_* A_\eta$ are torsion.

c) Deduce that the groups $H^i(X, A) := H^i(X, A_X)$ are torsion for all $i > 0$.

d) Let $i > 0$. Let $\alpha \in H^i(X, A)$. Show that there exists $n > 0$ such that α is in the image of the natural map $H^i(X, A[n]) \to H^i(X, A)$, where $A[n]$ is the n-torsion subgroup of A.

31. Let X be a noetherian scheme. Let $x \in X$ be a point of X; denote by $i : \mathrm{Spec}\,(k(x)) \to X$ the corresponding morphism. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathrm{Spec}\,(k(x))$.

a) Show that for every $q \geq 1$, the sheaves $(R^q i_*)(\mathcal{F})$ are torsion on $X_{\text{ét}}$.

b) Deduce that for all $p > 0$, the groups $H^p(X, i_*\mathcal{F})$ are torsion.

c) Assume further that X is integral and regular. Show that the groups $H^q(X, \mathbf{G}_m)$ are torsion for $q \geq 2$ (hint: use the sheaf of divisors D_X on X).

32. Let X be a projective, smooth, and geometrically integral variety over a field k of characteristic zero. Set $\overline{X} = X \times_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Assume that the group $\mathrm{Pic}\,\overline{X}$ is torsion-free (recall that this implies that it is also of finite type).

a) Show that the Galois cohomology group $H^1(k, \mathrm{Pic}\,\overline{X})$ is finite.

b) Set $\mathrm{Br}_1 X = \ker[\mathrm{Br}\,X \to \mathrm{Br}\,\overline{X}]$. Show that the cokernel of the map $\mathrm{Br}\,k \to \mathrm{Br}_1 X$ is finite.

33. Let X be a smooth variety over a field of characteristic zero k. Let $\bar{k}$ be an algebraic closure of k. Denote by $\mu_n \subset \bar{k}^*$ the Galois module of n-roots of unity and by $\mu = \bigcup_{n \geq 1} \mu_n$ the Galois module of all roots of unity in $\bar{k}^*$. The corresponding étale sheaves on X are still denoted respectively by μ_n and μ.

a) Let i be an integer with $i \geq 2$. Show that there is an exact sequence

$$0 \to H^{i-1}(X, \mathbf{G}_m)/n \to H^i(X, \mu_n) \to H^i(X, \mathbf{G}_m)[n] \to 0.$$

b) Show that there is an exact sequence

$$0 \to \mathrm{Pic}\,X \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} \to H^2(X, \mu) \to \mathrm{Br}\,X \to 0.$$

c) Assume k algebraically closed. Compute $H^2(X, \mu)$ when X is the affine space $\mathbf{A}^n_k$ and when X is the projective space $\mathbf{P}^n_k$.

d) Show that $H^3(X, \mu)$ is the torsion subgroup of $H^3(X, \mathbf{G}_m)$.

34. Let X be an integral, regular, and noetherian scheme with function field K. Show that for every element $\alpha \in \mathrm{Br}\,K$, there exists a non empty Zariski open subset $U \subset X$ such that $\alpha \in \mathrm{Br}\,U$.

35. Let X be a smooth and geometrically integral variety over a perfect field k. Let T be a k-*torus*, that is: a k-group scheme such that $T \times_k L$ is isomorphic to $\mathbf{G}_m^r$ for some finite (Galois) field extension L of k and some $r \geq 0$. Show that the group $H^2(X, T)$ is torsion.

36. Let X be a variety over a number field k. Assume that for every completion k_v of k, the set $X(k_v)$ of k_v-points of X is not empty. Show that the canonical map $\operatorname{Br} k \to \operatorname{Br} X$ is injective.

37. Let X be a projective conic over a field k, with $\operatorname{Char} k \neq 2$, given by the equation in $\mathbf{P}^2_k$:

$$x^2 - ay^2 - bz^2 = 0,$$

where $a, b \in k^*$.

a) Set $\overline{X} = X \times_k \bar{k}$. Show that $\operatorname{Br} \overline{X} = 0$.

b) Show that the degree map $\operatorname{Pic} \overline{X} \to \mathbf{Z}$ is an isomorphism.

c) Show that $\operatorname{Pic} X$ is generated by the class of a point $x \in X(k)$ if $X(k)$ is not empty, and by the class of a closed point of degree 2 if $X(k)$ is empty.

d) Deduce that there is an exact sequence

$$0 \to \mathbf{Z}/d \to \operatorname{Br} k \to \operatorname{Br} X \to 0,$$

where $d = 1$ (resp. $d = 2$) if $X(k) \neq \emptyset$ (resp. $X(k) = \emptyset$).

e) Show that the element $(a, b) \in \operatorname{Br} k$ generates the kernel of the map $\operatorname{Br} k \to \operatorname{Br} X$.

38. Let X be the projective $\mathbf{C}$-variety defined by the equation

$$a_0 x_0^2 + ... + a_n x_n^2 = 0$$

in the projective space $\mathbf{P}^n_{\mathbf{C}}$, where $a_0, ..., a_n$ are non-zero complex numbers and $n \geq 3$. Show that $\operatorname{Br} X = 0$.

39. Let k be a field with separable closure $\bar{k}$. Let X be a geometrically integral variety over k. Set $\overline{X} = X \times_k \bar{k}$ and $\bar{k}[X]^* = H^0(\overline{X}, \mathbf{G}_m)$. Define $\operatorname{Br}_1 X := \ker[\operatorname{Br} X \to \operatorname{Br} \overline{X}]$ and $U(X) = \bar{k}[X]^*/\bar{k}^*$.

a) Show that there is an exact sequence

$$0 \to H^1(k, \bar{k}[X]^*) \to \operatorname{Pic} X \to H^0(k, \operatorname{Pic} \overline{X}) \to H^2(k, \bar{k}[X]^*) \to \operatorname{Br}_1 X \to H^1(k, \operatorname{Pic} \overline{X}).$$

From now on, we assume that the set $X(k)$ of k-points of X is not empty.

b) Show that the inclusion $\bar{k}^* \hookrightarrow \bar{k}[X]^*$ induces an injective map $\operatorname{Br} k \to H^2(k, \bar{k}[X]^*)$.

c) Deduce that there is an isomorphism $H^1(k, \bar{k}[X]^*) \simeq H^1(k, U(X))$.

d) Assume further that $\operatorname{Pic} \overline{X} = 0$. Show that $\operatorname{Br}_1 X/\operatorname{Br} k$ is isomorphic to $H^2(k, U(X))$.

40. Let X be a smooth and geometrically integral variety over a field of characteristic zero k.

a) Assume k algebraically closed. Show that for every $n > 0$, the n-torsion subgroup ${}_n\mathrm{Br}\, X$ of $\mathrm{Br}\, X$ is finite.

b) Is it still true if $k = \mathbf{Q}$? If $k = \mathbf{R}$? If k is p-adic ?

c) Let $\alpha \in \mathrm{Br}\, X$. Assume that for every closed point $x \in X$, we have $\alpha(x) = 0$ in $\mathrm{Br}\,(k(x))$, where $k(x)$ is the residue field of x. Does this imply $\alpha = 0$?