Étale cohomology and the Brauer group: exercises (II)

Unless explicitely specified, all schemes are equipped with their étale topology. Recall that if Γ is a profinite group, M is a discrete Γ-module, U is an open subgroup of Γ, and i is a stricly positive integer, then the kernel of the restriction map $H^i(\Gamma, M) \to H^i(U, M)$ is an n-torsion group, where $n := [\Gamma : U]$.

1. Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field κ. Set $X = \operatorname{Spec} \mathcal{O}_K$.

a) Let F be a finite and étale $\mathcal{O}_K$-group scheme. Show that $H^i(X, F) = 0$ for all $i \geq 2$. Does this still hold for $i = 1$?

b) Show that $\operatorname{Br} X = 0$.

c) Show that $\operatorname{Br} K$ is isomorphic to $H^1(\kappa, \mathbf{Q}/\mathbf{Z})$, then that the latter is isomorphic to $\mathbf{Q}/\mathbf{Z}$.

2. Let X be a projective, smooth, and geometrically integral variety over a field k of characteristic zero. Set $\overline{X} = X \times_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Assume that the group $\operatorname{Pic} \overline{X}$ is torsion-free (recall that this implies that it is also of finite type).

a) Show that the Galois cohomology group $H^1(k, \operatorname{Pic} \overline{X})$ is finite.

b) Set $\operatorname{Br}_1 X = \ker[\operatorname{Br} X \to \operatorname{Br} \overline{X}]$. Show that the cokernel of the map $\operatorname{Br} k \to \operatorname{Br}_1 X$ is finite.

3. Let X be a smooth variety over a field of characteristic zero k. Let $\bar{k}$ be an algebraic closure of k. Denote by $\mu_n \subset \bar{k}^*$ the Galois module of n-roots of unity and by $\mu = \bigcup_{n \geq 1} \mu_n$ the Galois module of all roots of unity in $\bar{k}^*$. The corresponding étale sheaves on X are still denoted respectively by μ_n and μ.

a) Let i be an integer with $i \geq 2$. Show that there is an exact sequence

$$0 \to H^{i-1}(X, \mathbf{G}_m)/n \to H^i(X, \mu_n) \to H^i(X, \mathbf{G}_m)[n] \to 0.$$

b) Show that there is an exact sequence

$$0 \to \operatorname{Pic} X \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} \to H^2(X, \mu) \to \operatorname{Br} X \to 0.$$

c) Assume k algebraically closed. Compute $H^2(X, \mu)$ when X is the affine space $\mathbf{A}^n_k$ and when X is the projective space $\mathbf{P}^n_k$.

d) Show that $H^3(X, \mu)$ is the torsion subgroup of $H^3(X, \mathbf{G}_m)$.

4. Let X be an integral, regular, and noetherian scheme with function field K.

a) Show that for every element $\alpha \in \operatorname{Br} K$, there exists a non empty Zariski open subset $U \subset X$ such that $\alpha \in \operatorname{Br} U$.

b) Assume further that X is a proper and smooth variety over a number field k. Recall that there exists a finite set of places S_0 of k (containing all archimedean places) such that there is a smooth and proper $\mathcal{O}_{k,S_0}$-scheme $\mathcal{X}$ with generic fibre X (where $\mathcal{O}_{k,S_0} \subset k$ is the ring of S_0-integers). Let $\beta \in \operatorname{Br} X \subset \operatorname{Br} K$. Show that there exists a finite set of places $S \supset S_0$ such that $\beta \in \operatorname{Br} \mathcal{X}_S$, where $\mathcal{X}_S$ is the inverse image of $\operatorname{Spec} \mathcal{O}_{k,S}$ by the structural map $\mathcal{X} \to \operatorname{Spec}(\mathcal{O}_{k,S_0})$.

c) Deduce from b) and from exercice 1.b) that $\beta(P_v) = 0$ for all places $v \notin S$ and all k_v-points $P_v \in X(k_v)$, where k_v is the completion of k at v.

5. Let X be a smooth and geometrically integral variety over a perfect field k. Let T be a k-*torus*, that is: a k-group scheme such that $T \times_k L$ is isomorphic to $\mathbf{G}_m^r$ for some finite (Galois) field extension L of k and some $r \geq 0$. Show that the group $H^2(X, T)$ is torsion.