Étale cohomology and the Brauer group: exercises (I)

Unless explicitely specified, all schemes are equipped with their étale topology. Recall that if Γ is a profinite group, M is a discrete Γ-module, U is an open subgroup of Γ, and i is a strictly positive integer, then the kernel of the restriction map $H^i(\Gamma, M) \to H^i(U, M)$ is an n-torsion group, where $n := [\Gamma : U]$.

1. a) Let $\pi : X''_{E''} \to X'_{E'}$ and $\pi' : X'_{E'} \to X_E$ be continuous morphism of sites. Show that $(\pi'\pi)_* = \pi'_*\pi_*$ and $(\pi'\pi)^* = \pi^*{\pi'}^*$.

b) Let $X' = \operatorname{Spec}(\mathbf{Z}/p)$ and $X = \operatorname{Spec}(\mathbf{Z}/p^2)$, equipped with their étale sites. Let $\pi : X' \to X$ be the corresponding closed immersion. Show that the canonical map of sheaves $\pi^*((\mathbf{G}_m)_X) \to (\mathbf{G}_m)_{X'}$ is not injective. Is it surjective ?

2. Let X be an affine $\mathbf{F}_p$-scheme of finite type.

a) Show that $H^i(X, \mathbf{Z}/p) = 0$ for every integer $i \geq 2$.

b) Assume that X is the affine space over $\mathbf{F}_p$. Show that $H^1(X, \mathbf{Z}/p) \neq 0$.

c) Let Y be a normal, connected and noetherian scheme. Let $\mathcal{F}$ be a constant sheaf on Y. Show that for any $r > 0$, the group $H^r(Y, \mathcal{F})$ is torsion, and that is is zero if $\mathcal{F}$ is uniquely divisible.

d) Deduce that $H^2(Y, \mathbf{Z})$ is isomorphic to $H^1(Y, \mathbf{Q}/\mathbf{Z})$, and that for every $n > 0$, there is an isomorphism $H^1(Y, \mathbf{Z}/n) \simeq_n H^1(Y, \mathbf{Q}/\mathbf{Z})$.

e) Give an example of a projective and smooth variety Y over an algebraically closed field of characteristic $p > 0$ such that $H^2(Y, \mathbf{Z}/p) \neq 0$.

3. Let X be a smooth and integral variety over a field of characteristic zero k. Let A be an abelian variety (that is: a projective, smooth and connected algebraic group over k). Recall (Chevalley) that any k-rational map from X to A extends to a k-morphism $X \to A$. Let $j : \eta \to X$ be the inclusion of the generic point of X, set $A_\eta = A \times_k \eta$ and $A_X = A \times_k X$.

a) Show that $j_* A_\eta = A_X$ as étale sheaves on X.

b) Show that for all integers $q > 0$, the sheaves $R^q j_* A_\eta$ are torsion.

c) Deduce that the groups $H^i(X, A) := H^i(X, A_X)$ are torsion for all $i > 0$.

d) Let $i > 0$. Let $\alpha \in H^i(X, A)$. Show that there exists $n > 0$ such that α is in the image of the natural map $H^i(X, A[n]) \to H^i(X, A)$, where $A[n]$ is the n-torsion subgroup of A.

4. Let X be a noetherian scheme. Let $x \in X$ be a point of X; denote by $i : \mathrm{Spec}\,(k(x)) \to X$ the corresponding morphism. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathrm{Spec}\,(k(x))$.

a) Show that for every $q \geq 1$, the sheaves $(R^q i_*)(\mathcal{F})$ are torsion on $X_{\text{ét}}$.

b) Deduce that for all $p > 0$, the groups $H^p(X, i_* \mathcal{F})$ are torsion.

c) Assume further that X is integral and regular. Show that the groups $H^q(X, \mathbf{G}_m)$ are torsion for $q \geq 2$ (hint: use the sheaf of divisors D_X on X).

5. A sheaf of abelian group $\mathcal{F}$ on $X_{\text{ét}}$ is *locally constant* if there is a covering $(U_i \to X)_{i \in I}$ of X for the étale topology such that the restriction $\mathcal{F}_{|U_i}$ is constant for all $i \in I$.

a) Let $f : Y \to X$ be a morphism of schemes. Show that if $\mathcal{F}$ is locally constant on X, then $f^* \mathcal{F}$ is locally constant on Y.

b) Show that the direct image $f_* \mathcal{F}$ of a locally constant sheaf $\mathcal{F}$ on Y is not necessarily locally constant.

c) Describe the locally constant sheaves on X when $X = \mathrm{Spec}\, k$ is the spectrum of a field.