Étale cohomology and Brauer group

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Contents

1. Preliminaries

1.1. Introduction: why is étale cohomology useful ?

Classical cohomology of sheaves on a topological space can be applied to a scheme X, which is equipped with Zariski topology. The behavior of the corresponding cohomology groups is satisfying when one restricts to quasi-coherent $\mathcal{O}_X$-modules. However, Zariski topology is often too coarse to deal with more general sheaves, including constant sheaves of finite groups. Here are a few examples:

- Galois cohomology of a field k cannot be described in terms of cohomology of the topological space $\operatorname{Spec} k$ (which consists of one single point).
- More generally, X-torsors under a group G (they are analogs of G-principal bundles in differential or analytic geometry) are not necessarily locally trivial for Zariski topology (unlike vector bundles). Whence the necessity of introducing a finer topology.
- Assume that X is a smooth complex variety. Cohomology groups like $H^i(X(\mathbf{C}), \mathbf{Z}/n\mathbf{Z})$ (taken with respect to the complex topology on $X(\mathbf{C})$) play an important role, but they do not coincide with the Zariski groups $H^i(X, \mathbf{Z}/n\mathbf{Z})$ because again, Zariski topology is much coarser than complex topology.

It is therefore necessary to extend the notion of topology, so that we get something non trivial even in the case $X = \operatorname{Spec} k$. It turns out that the good framework to do this is to consider *Grothendieck topologies* (see subsection 2.1.). An especially interesting case is *étale topology* and its associated cohomology, which will be the main topic of this course (although we will also briefly discuss other examples, like flat topology).

For applications in arithmetic and geometry, a very important example of étale cohomology group is the *Brauer group* of X, defined as $H^2(X, \mathbf{G}_m)$ (where $\mathbf{G}_m$ is the sheaf represented by the multiplicative group on X). It coincides with the classical Brauer group $\operatorname{Br} k$ when X is the spectrum of a field k, and has good functorial properties as well as good invariance properties. This object (which is still nowadays the topic of very active research) will be discussed in sections 6. and 7.

1.2. Flat and étale morphisms

Unless explicitely specified, all rings are assumed to be commutative. Notation like $f : Y \to X$ will always denote a morphism of schemes. The local ring of a scheme X at x is denoted by $\mathcal{O}_{X,x}$, its maximal ideal by $\mathcal{M}_{X,x}$, and its residue field $\mathcal{O}_{X,x}/\mathcal{M}_{X,x}$ by $k(x)$. By convention, a homomorphism of local rings $u : A \to B$ satisfies $u(\mathcal{M}_A) \subset \mathcal{M}_B$, where $\mathcal{M}_A$ and $\mathcal{M}_B$ are the respective maximal ideals of A and B. For every ring A and $f \in A$, the piece of notation A_f stands for the localization $A[1/f]$.

Recall that an algebra B over a ring A is *finitely generated* if B is isomorphic to a quotient $A[X_1, ..., X_r]/I$ where I is some ideal, *finitely presented* if there is such an isomorphism with I of finite type as an ideal (if B is finitely presented, then the kernel of any surjection $A[X_1, ..., X_r] \to B$ is a finite type ideal, see [19], Lemma 6.3; observe also that finitely generated coincides with finitely presented if A is noetherian). The algebra B is said to be *finite* over A if it is an A-module of finite type.

Similarly, an A-module M is *finitely presented* if M is isomorphic to a quotient A^r/I, where I is a finite type sub-A-module of A^r (actually in this case every surjective morphism $u : N \to M$ with N an A-module of finite type satisfies that $\ker u$ is of finite type as well by [19], Lemma 5.3). A finite type A-module is automatically finitely presented if A is noetherian.

In this paragraph, we recall the main properties of flat and étale morphisms. Most proofs can be found in paragraph I.2., I.3. and I.4 of [12] (caution: in this reference, it is implicitely assumed that all schemes are locally noetherian, so it is sometimes necessary to replace "finitely generated" or "of finite type" by "finitely presented" to deal with the general case).

Definition 1.1 A morphism $f : Y \to X$ is said to be *locally of finite type* (resp. *locally of finite presentation*) if for every pair of affine open subsets $V \subset Y$ and $U \subset X$ such that $f(V) \subset U$, the $\mathcal{O}_X(U)$-algebra $\mathcal{O}_Y(V)$ is finitely generated (resp. is finitely presented).

Both properties are *local on the base* (meaning that they hold if and only if X can be covered by affine subschemes X_i such that all induced morphisms $f^{-1}(X_i) \to X_i$ have the required property; see [24], §15 and §21), and they coincide if the scheme X is locally noetherian.

Recall also that f is *quasi-compact* if the inverse image of every open affine subset of X is quasi-compact (=can be covered by finitely many affine open subsets); again this property is local on the base, cf. [7], Prop. 2.12. The morphism f is *quasi-separated* if the diagonal morphism $\Delta : Y \to Y \times_X Y$ associated to f is quasi-compact (this is automatic if Y is locally noetherian, e.g. if f is locally of finite type and X locally noetherian), *separated* if Δ is a closed immersion. An important property of separated (resp. quasi-separated) scheme Y over an affine scheme S is that the intersection of two affine open subset U_1, U_2 still is affine (resp. is quasi-compact), since it is the inverse image of the affine subset $U_1 \times_S U_2$ by the diagonal morphism $Y \to Y \times_S Y$.

Definition 1.2 The morphism f is *of finite type* if it is locally of finite type and quasi-compact, *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

Again, these two properties are local on the base, and they are the same if we work with noetherian and separated S-schemes (where S is any scheme). Recall also that a *variety* over a field k is a separated k-scheme of finite type.

Definition 1.3 Let A be a ring. An A-module M is *flat* if the functor $.\otimes_A M$ (which is always right-exact) is exact on A-modules. A homomorphism $A \to B$ between two rings is flat if it makes B a flat A-module.

A morphism of schemes $f : Y \to X$ is *flat* at $y \in Y$ if the corresponding homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat, where $x := f(y)$. The morphism f is flat if it is flat at every $y \in Y$, *faithfully flat* if it is flat and surjective.

Let us recall a few properties of flat morphisms in the following two propositions:

Proposition 1.4 *a) Open immersions are flat.*

b) The composition of two flat morphisms is flat.

c) Flatness is stable by base change.

d) Let M be a finitely presented module over A. Then M is flat iff it is a projective A-module, iff the corresponding $\mathcal{O}_X$-module $\widetilde{M}$ is a locally free sheaf on $X := \operatorname{Spec} A$.

e) A flat morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ between non empty affine schemes is faithfully flat if and only if a sequence

$$M_1 \to M_2 \to M_3$$

of A-modules is exact whenever

$$M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B$$

is exact. This holds in particular if $A \to B$ is a flat homomorphism of local rings.

Reference : For a), b), and c), see [9], Proposition III.9.2. d) is [12], Theorem I.2.9. when A is noetherian, or [19], Lemma 83.1 in the general case. For e), see [12], Prop. 2.7.

□

Proposition 1.5 *a) Every finite and surjective morphism between regular schemes is flat.*

b) Let $f : Y \to X$ be a morphism of schemes with Y reduced. Assume that X is integral, regular and of dimension 1. Then f is flat if and only if every irreducible component of Y dominates X.

c) A flat and locally finitely presented morphism is an open. map.

d) Let $f : Y \to X$ be a flat morphism between schemes of finite type over a field k. Assume that X is irreducible and Y is pure. Then for every $x \in X$, the fiber $Y_x = Y \times_X \operatorname{Spec}(k(x))$ is empty or pure of dimension $\dim Y - \dim X$.

Reference : For a), see [11], chapter 6, Theorem 46. b) is [9], Proposition III.9.7. c) is Lemma 25.10 of [24]. d) follows from [9], Corollary III.9.6.

□

The following lemma is part of descent theory; it turns out to be quite useful:

Lemma 1.6 *Let $f : A \to B$ be a faithfully flat morphism of rings. Set $B^{\otimes r} = B \otimes_A B \otimes ... \otimes_A B$. Define $d^{r-1} : B^{\otimes r} \to B^{\otimes (r+1)}$ by the formula:*

$$d^{r-1} = \sum_{i=0}^{r} (-1)^i e_i,$$

where, for $0 \leq i \leq r$*:*

$$e_i(b_0 \otimes ... \otimes b_{r-1}) := b_0 \otimes ... \otimes b_{i-1} \otimes 1 \otimes b_i \otimes ... \otimes b_{r-1}.$$

Let M *be an* A*-module. Then the sequence*

$$0 \to M \stackrel{1\otimes f}{\to} M \otimes_A B \stackrel{1\otimes d_0}{\to} M \otimes_A B^{\otimes 2} \to ... \to M \otimes_A B^{\otimes r} \stackrel{1\otimes d^{r-1}}{\to} M \otimes_A B^{\otimes r+1} \quad (1)$$

is exact.

Proof (sketch of): [12], Proposition I.2.18 and Remark I.2.19. The fact that (1) is a complex is shown by the usual straightforward computation. Now the idea is that if A' is a faithfully flat A-algebra, it is sufficient (by Proposition 1.4, e)) to prove the required exactness after replacing M by $M' := M \otimes_A A'$, B by $B' := B \otimes_A A'$ and f by $f' = f \otimes \mathrm{Id} : A' = A \otimes_A A' \to B'$. Picking $A' = B$, we observe that $f' : B \to B \otimes_A B$ now has a retraction, namely $b \otimes b' \mapsto bb'$, so we reduce to the case when f has a retraction, i.e. there exists a homomorphism $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$. Now the map $k_r : B^{\otimes(r+2)} \to B^{\otimes(r+1)}$ defined (for $r \geq -1$) by

$$k_r(b_0 \otimes ... \otimes b_{r+1}) = g(b_0)b_1 \otimes b_2 \otimes ... \otimes b_{r+1}$$

is a homotopy, namely it satisfies $k_{r+1}d^{r+1} + d^r k_r = \mathrm{Id}$, which shows that the complex (1) is exact for $M = B$. The general case is similar.

□

Faithfully flat morphisms have good "descent" properties, which we summarize in the following statement:

Proposition 1.7 *a) Let* $f : Y \to X$ *be a faithfully flat and quasi-compact morphism. The morphism* f *is a strict epimorphism, that is: for every scheme* Z *and every morphism* $h : Y \to Z$ *such that* $h \circ p_1 = h \circ p_2$*, there exists a unique morphism* $g : X \to Z$ *such that* $g \circ f = h$*, where* p_1, p_2 *are the two projections* $Y \times_X Y \to Y$*. In other words the sequence of sets*

$$0 \to \mathrm{Hom}(X, Z) \to \mathrm{Hom}(Y, Z) \rightrightarrows \mathrm{Hom}(Y \times_X Y, Z)$$

is exact.

b) Let $f : Y \to X$ *be a morphism. Let* $X' \to X$ *be a faithfully flat and quasi-compact morphism. Consider the morphism* $f' : Y' = Y \times_X X' \to X'$ *obtained by base change. Then if* f' *is quasi-compact (resp. an isomorphism, separated, locally of finite type, of finite type, proper, affine, finite, flat, smooth...), so is* f*.*

Reference : [12], Th. I.2.17 for a) and [5], §2.6 and 2.7. for b). See also [25], §4.

□

Intuitively, a) corresponds to the fact that if h coincides on every pair of points having the same image by f, then h can be factorized through f. b) means that a lot of "good" properties can be checked after base change by a faithfully flat morphism.

The simplest case of a) is when $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $Z = \operatorname{Spec} C$ are affine. In this case it immediately follows from the exactness of

$$0 \to A \to B \overset{e_0 - e_1}{\to} B \otimes_A B,$$

which is a special case of Lemma 1.6. Indeed $e_0 : b \mapsto 1 \otimes b$ and $e_1 : b \mapsto b \otimes 1$ respectively correspond to the second and the first projection $Y \times_X Y \to Y$. Let us also remark that these good descent properties do not characterize faithfully flat morphisms, see [25], §4.

Definition 1.8 Let $f : Y \to X$ be locally of finite presentation[1]. Let $y \in Y$ and $x := f(y)$. The morphism f is *unramified* at y if $\mathcal{M}_{X,x}\mathcal{O}_{Y,y} = \mathcal{M}_{Y,y}$ and $k(y)$ is a finite separable field extension of $k(x)$. The morphism f is *unramified* if it is unramified at every $y \in Y$.

Equivalently (using the fact that the local ring at y of the fiber Y_x is $\mathcal{O}_{Y,y}/\mathcal{M}_{X,x}\mathcal{O}_{Y,y}$, which is proved in [7], Lemma 6.30), a locally of finite presentation morphism f is unramified if and only if for every $x \in X$, the fiber Y_x is a disjoint union $\coprod \operatorname{Spec} k_i$, where every k_i is a finite separable field extension of $k(x)$ (if f is of finite type, this is equivalent to saying that Y_x is the spectrum of a finite separable $k(x)$-algebra; in particular an étale and finite type morphism is quasi-finite).

Definition 1.9 Let $f : Y \to X$ be locally of finite presentation. Then f is said to be *étale at* $y \in Y$ if it is flat and unramified at y. The morphism f is *étale* if f is étale at every $y \in Y$ (that is: f is a flat and unramified morphism).

Equivalently, f is étale if and only if it is smooth of relative dimension 0.

Example 1.10 a) A field extension is unramified (hence étale) if and only if it is finite and separable.

[1] In [27], it is only required that an unramified morphism is locally of finite type, and the term G-unramified is used for what we call unramified.

b) Let L be a finite extension of $\mathbf{Q}_p$. Then $L/\mathbf{Q}_p$ is unramified (in the sense of number theory) if and only if the corresponding morphism $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathbf{Z}_p$ is unramified in the sense of Definition 1.9. Similarly a finite extension of number fields L/K is usually said to be unramified at a prime $\wp \in \mathcal{O}_K$ if the morphism $u : \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ is unramified at every prime of $\mathcal{O}_L$ lying above $\wp$. Observe that the flatness of u is automatic by Proposition 1.5, b).

c) A closed immersion between locally noetherian schemes is unramified (but not flat in general). An open immersion is étale.

d) Let k be a field of characteristic $\neq 2$. Let Y be the affine scheme $Y = \operatorname{Spec}(k[x, y]/(y^2 - x))$ and $X = \mathbf{A}_k^1$. The morphism $f : Y \to X, (x, y) \mapsto x$ is étale everywhere but at $(0, 0)$. The fiber at 0 is isomorphic to $\operatorname{Spec}(k[\varepsilon])$, where $k[\varepsilon] := k[T]/(T^2)$. For $a \in X(k) \setminus \{0\}$, the fiber at a is isomorphic to $\operatorname{Spec}(k(\sqrt{a})$ if a is not a square of k^*, to $(\operatorname{Spec} k) \coprod (\operatorname{Spec} k)$ otherwise.

Proposition 1.11 *a) The composition of two étale morphisms is étale.*

b) The property "étale" is local on the base, and stable by base change.

c) Let $f : Z \to Y$ and $g : Y \to X$ be morphisms. If g is unramified and $g \circ f$ is étale, then f is étale. In particular, if S is a scheme, any morphism between two étale S-schemes is automatically étale.

Reference : [12], Proposition I.3.3.

□

The notion of étale morphism is in some sense the analogue in algebraic geometry of a local diffeomorphism in differential or analytic geometry. More precisely, we have:

Proposition 1.12 *Let $f : Y \to X$ be an étale morphism between locally noetherian schemes. Let $y \in Y, x := f(y)$. Then:*

a) $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y}$.

b) The tangent map $T_{Y,y} \to T_{X,x} \otimes_{k(x)} k(y)$ is an isomorphism.

Reference : [7], Proposition 6.34.

□

Definition 1.13 Let R be a ring and $F, G \in R[T]$. The homomorphism $R \to R[T]_G/(F) = (R[T]/(F))_G$ is said to be *standard étale* if F is monic and F' is invertible in $R[T]_G/(F)$. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ between affine schemes is standard étale if the induced homomorphism of rings $A \to B$ is isomorphic to a standard étale homomorphism.

Such a morphism is indeed étale (cf. [12], Example I.3.4; flatness is easy because we deal with finite and free A-modules, and the "unramified" property follows from the characterization that uses the fibers).

Proposition 1.14 *Let $f : Y \to X$ be a morphism. Then f is étale if and only if for every $y \in Y$ and $x := f(y)$, it satisfies one of the following equivalent conditions:*

a) There exist affine open subsets $V \subset Y, U \subset X$ with the conditions $y \in V$ and $f(V) \subset U$, such that the induced morphism $V \to U$ is standard étale.

b) There exist affine open subsets $V = \operatorname{Spec} C$ of y and $U = \operatorname{Spec} A$ of $x := f(y)$ such that $C = A[T_1, ..., T_n]/(P_1, ..., P_n)$ and the determinant of the Jacobian matrix $(\partial P_i/\partial T_j)$ is invertible in C.

Reference : [12], Example I.3.4. and Theorem I.3.14. The latter uses Zariski's main Theorem (see [26], section 40): Let $f : Y \to X$ be a a separated and quasi-finite morphism, with X quasi-compact and quasi-separated; then f factors as the composition of an open immersion with a finite morphism. In particular, we have that for any finite type étale morphism $f : Y \to X$, there is a Zariski-dense open subset U of X such that the restriction $f^{-1}(U) \to U$ is finite and étale.

□

Observe that for a standard étale morphism $R \to C := R[T]_G/(F)$, the R-algebra C is also isomorphic to $R[T, X]/(F(T), GX - 1)$, with the determinant $F'(T)G$ of the corresponding Jacobian matrix invertible in C. Thus it is always possible to take $n = 2$ in the previous proposition.

Proposition 1.15 *Let $f : Y \to X$ be an étale morphism. If X is normal (resp. regular), then Y is normal (resp. regular).*

Reference : [12], Proposition I.3.17.

□

Recall that a scheme is *normal* if all its local rings are normal, that is: they are integrally closed domains (unlike in [7], we do not require a normal scheme to be connected in these notes).

Remark 1.16 If the scheme X is normal, then the standard étale morphism $V \to U$ of Proposition 1.14 can be chosen such that $U = \operatorname{Spec} A$, $V = \operatorname{Spec}(A[T]_G/(F))$, and F is irreducible over $\operatorname{Frac} A$ ([12], Proposition I.3.19).

Using the previous statements, one gets other characterizations of étale morphisms between algebraic varieties:

Proposition 1.17 *Let $f : Y \to X$ be a morphism of varieties over a field k.*

a) Assume that k is algebraically closed. Then f is étale at $y \in Y$ if and only if the induced map between completions $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{Y,y}}$ (where $x := f(y)$) is an isomorphism.

b) Assume that Y and X are smooth over k. Then f is étale at y if and only if it satisfies condition b) of Proposition 1.12.

For a proof, see [13], Prop 2.9.

1.3. Henselian rings, henselization

Definition 1.18 Let A be a local ring with residue field κ. The ring A is said to be *henselian* if the following condition is satisfied:

Let $F \in A[T]$ be a monic polynomial with image $\bar{F}$ in $\kappa[T]$. Then for every decomposition $\bar{F} = g_0 h_0$ in $\kappa[T]$ with g_0, h_0 monic and coprime, there exist monic liftings g, h of g_0, h_0 in $A[T]$ such that $F = gh$.

Actually the condition ensures that the liftings g, h are unique if they do exist ([12], Remark I.4.1).

Example 1.19 Let A be a noetherian local ring with maximal ideal $\mathcal{M}$. Assume that A is complete (for the topology associated to $\mathcal{M}$, which is Hausdorff because A is noetherian). Then A is henselian ([19], Lemma 153.9).

Theorem 1.20 *Let A be a local ring with residue field κ. Then the following conditions are equivalent:*

i) A is henselian.

ii) Let $F_1, ..., F_n \in A[T_1, ..., T_n]$. Then every non singular common zero $a_0 \in \kappa^n$ of the $\bar{F}_i$ ("non singular" meaning that the matrix $(\frac{\partial F_i}{\partial F_j}(a_0)) \in M_n(\kappa)$ is invertible) lifts to a common zero $a \in A^n$ of the F_i.

iii) Same as ii) with $n = 1$, that is: for every $F \in A[T]$ and every $a_0 \in \kappa$ such that $\bar{F}(a_0) = 0$ and $\bar{F}'(a_0) \neq 0$ in κ, there exists an $a \in A$ such that $\bar{a} = a_0$ and $F(a) = 0$.

iv) Every finite A-algebra is a product of local rings (hence it is a local ring if its spectrum is further assumed to be connected).

v) Let $X = \operatorname{Spec} A$. *Let* $f : Y \to X$ *be an étale morphism. If there is a point* $y \in Y$ *such that* y *and* $x := f(y)$ *have same residue field, then* f *has a section* $s : X \to Y$ *(that is: a morphism such that* $f \circ s = \mathrm{id}_X$*).*

These properties imply that for a smooth scheme $X \to A$ *over a local henselian ring* A *with residue field* k*, the reduction map* $X(A) \to X(k)$ *is surjective.*

Reference : [12], Theorem I.4.2 (and Proposition I.3.24 b) for the last statement). Observe that conversely, if a morphism $f : Y \to X$ has a section, then every point $y \in Y$ with image $x := f(y)$ satisfies $k(x) = k(y)$, because the inclusion of fields $k(x) \to k(y)$ induced by f has a retraction, which is therefore a surjective morphism of fields, hence an isomorphism.

□

Definition 1.21 Let A be a local ring with maximal ideal $\mathcal{M}$ and residue field κ. An *étale neighborhood* of A is a pair $(B, \wp)$, where B is an étale A-algebra and $\wp$ is a prime ideal of B lying over $\mathcal{M}$ such that the induced map $\kappa \to \kappa_B$ is an isomorphism, where $\kappa_\wp$ is the residue field of B at $\wp$.

Proposition 1.22 *Let* A *be a local ring with maximal ideal* $\mathcal{M}$*. Then there exists a (unique up to isomorphism) local ring* A^h*, equipped with a homomorphism* $A \to A^h$ *of local rings, such that: every homomorphism* $A \to B$ *of local rings with* B *henselian factors uniquely into* $A \to A^h \to B$*. Besides,* A^h *has same residue field as* A *and its maximal ideal is* $\mathcal{M}A^h$*.*

Definition 1.23 The ring A^h is the *henselization* of A.

Namely A^h is defined as the direct limit of B, where the limit is over all étale neighborhoods $(B, \wp)$ such that $\operatorname{Spec} B$ is connected (see for example [12], Lemma I.4.8). Besides, if A is noetherian, then so is A^h (see [1], III.4.2 or [19], section 155).

Example 1.24 a) If A is a noetherian local ring, it injects into its completion $\widehat{A}$, and by Example 1.19, the henselization A^h is a subring of $\widehat{A}$. If we assume further that A is a discrete valuation ring with quotient field K, then A^h is the subring of $\widehat{A}$ consisting of algebraic separable elements over K, and "separable" can be removed if A is *excellent*. This statement extends to any excellent normal local ring by Artin-Popescu approximation theorem (cf. [29], Theorem 2.4). Recall that any algebra finitely generated over an excellent ring (e.g. over **Z** or over a field) is excellent, as is a localization of an excellent ring (see [11], chapter 13 for more details on this notion).

b) The henselization of a quotient A/I is A^h/IA^h.

Definition 1.25 A local ring A is *strictly henselian* (or strictly local) if A is henselian and its residue field is separably closed.

Proposition 1.26 *Let A be a local ring with maximal ideal $\mathcal{M}$ and residue field κ. Then there is a (unique up to isomorphism) strictly henselian ring A^{sh}, equipped with homomorphisms of local rings $A \to A^h \to A^{\text{sh}}$ such that:*

a) A^{sh} has maximal ideal $\mathcal{M}A^{\text{sh}}$ and residue field $\bar{\kappa}$ (the separable closure of κ),

b) A^{sh} is a direct limit of étale A-algebras (or étale A^h-algebras).

The ring A^{sh} is called the strict henselization of A.

The ring A^{sh} has a universal property similar to A^h's: every homomorphism $A \to H$ with H strictly local extends to a homomorphism $A^{\text{sh}} \to H$, which is uniquely determined once the induced map between the residue fields of A^{sh} and H is given.

If we fix a separable closure $\bar{\kappa}$ of κ, we can construct A^{sh} as $\varinjlim B$, where B runs over all commutative diagrams:

$$\begin{array}{ccc} B & \longrightarrow & \bar{\kappa} \\ \uparrow & \nearrow & \\ A & & \end{array}$$

with $A \to B$ étale. For a normal ring A with henselization A^h, the strict henselization A^{sh} is also the maximal unramified extension of A^h in the usual sense (this is not ambiguous because an integral domain B which is finite over A^h is also a local ring by Theorem 1.20, iv).

Definition 1.27 Let X be a scheme. A *geometric point* of X is a morphism $u_x : \bar{x} \to X$, where $\bar{x}$ is the spectrum of a separably closed field. We denote by $x \in X$ the point $u_x(\bar{x})$.

An *étale neighborhood* of a geometric point $\bar{x}$ is a commutative diagram

$$\begin{array}{ccc} \bar{x} & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X \end{array}$$

with $U \to X$ étale.

Thus $\mathcal{O}_{X,\bar{x}} := \mathcal{O}_{X,x}^{\text{sh}}$ is the limit of $\mathcal{O}_U(U)$ over all étale neighborhoods of $\bar{x}$. Changing the geometric point $\bar{x}$ with image x is equivalent to changing the separable closure of the residue field $k(x)$ when constructing the strict henselization of $\mathcal{O}_{X,x}$. The ring $\mathcal{O}_{X,\bar{x}}$ will play the same role for the étale topology as the ring $\mathcal{O}_{X,x}$ for the Zariski topology.

2. Étale topology and sheaves

In this section (and the next one), some familiarity with the language of categories and functors is assumed. A good introduction is [30], Appendix; a summary of the main properties can be found in [8], Appendix A.

2.1. Grothendieck topologies

Let X be a scheme. Denote by Sch/X the category of X-schemes. Consider a full subcategory C_X of Sch/X (so the morphism between two objects of C_X are the morphisms of X-schemes).

Definition 2.1 A *Grothendieck topology* on C_X consists of the datum of a subclass E of morphisms in C_X (called the *open sets*) satisfying:

i) Every isomorphism is in E.

ii) A composition of morphism in E is in E.

iii) If $V \to U$ is in E and $W \to U$ is an *arbitrary* morphism in C_X, then the pull-back $V \times_U W \to W$ is in E.

A *covering* (for this Grothendieck topology) of an object $U \in C_X$ is a family of morphisms $f_i : U_i \to U$, where every f_i is in E and $\bigcup_i f_i(U_i) = U$. The pair consisting of C_X and the family of all coverings is called a *site*, and is denoted by X_E.

Remark 2.2 There is a more general definition of Grothendieck topologies and sites, associated to an arbitrary small category (cf. for instance [22]), but the above definition will be sufficient for our purposes.

Example 2.3 a) The *small Zariski site* X_{zar}: C_X is the category of open subschemes of X and E is the class of open immersions.

b) The *big Zariski site* X_{Zar}: C_X is the category of all X-schemes and E is the class of open immersions.

c) The *small étale site* $X_{\text{ét}}$: C_X is the category of all étale X-schemes and E is the class of étale maps (actually in this example, every morphism in C_X is in E thanks to Proposition 1.11, c).

d) The *big étale site* $X_{\text{Ét}}$: C_X is the category of all X-schemes and E is the class of étale maps.

e) The (big) *flat site* X_{fppf}: C_X is the category of all X-schemes and E is the class of flat and locally finitely presented morphisms.

In these lectures, "étale site" (resp. "Zariski site") will always refer to the small étale site (resp. small Zariski site), and "flat site" to the big flat site.

Definition 2.4 Let $\pi : X' \to X$ be a morphism of schemes. Let X_E (resp. $X'_{E'}$) be a site with underlying scheme X (resp. X'). The morphism π is said to induce a *continuous map of sites* $X'_{E'} \to X_E$ (often also denoted π) if it satisfies the two following properties:

i) For every $Y \in C_X$, the scheme $Y \times_X X'$ is in $C_{X'}$.

ii) For every covering $(U_i \to Y)$ in X_E, the family $(U_i \times_X X' \to Y \times_X X')$ is also a covering in $X'_{E'}$.

Observe that the axioms extend the classical property that a map of topological spaces is continous if and only if the inverse image of every open subset is an open subset.

Remark 2.5 a) Again, this definition can be extended to sites in the sense of [22], using a functor from one category to the other one.

b) For all sites in Example 2.3, condition ii) is equivalent to saying that for every open set $V \to U$ in X_E, the pull-back $V \times_X X' \to U \times_X X'$ still is an open set in $X'_{E'}$ (indeed in these examples the image of V is a Zarisi open subset of U by Proposition 1.5, c).

Example 2.6 a) Let X be a scheme. Then the identity map on X defines continuous maps of sites

$$X_{\text{fppf}} \to X_{\text{Ét}} \to X_{\text{ét}} \to X_{\text{zar}}.$$

From left to right, the topology gets coarser and coarser.

b) Every morphism of schemes $X' \to X$ induces a continuous map of sites $X'_E \to X_E$, where E is any one of the sites of Example 2.3, thanks to stability by base change of flatness, étaleness etc.

2.2. Presheaves and sheaves

Definition 2.7 Let X be a scheme. Let X_E be a site with underlying category C_X. A *presheaf* (of abelian groups) [2] on X_E is a contravariant functor $\mathcal{P}$ from C_X to the category of abelian groups. The *group of sections* of $\mathcal{P}$ over $Y \in C_X$ is $\Gamma(Y, \mathcal{P}) := \mathcal{P}(Y)$. To every morphism $u : Y' \to Y$ in C_X is associated a *restriction map* $\mathcal{F}(Y) \to \mathcal{F}(Y')$, which we usually denotes by $s \mapsto s_{|Y'}$ when the morphism u is understood.

[2] Similar definitions can be given for sets, rings etc.

Observe that the definition of a presheaf (unlike the definition of a sheaf below) only depends on C_X (not on E). As in the case of sheaves on a topological space, the kernel and cokernel of a morphism $\mathcal{F} \to \mathcal{G}$ of presheaves are just the presheaves

$$Y \mapsto \ker[\mathcal{F}(Y) \to \mathcal{G}(Y)];\ Y \mapsto \operatorname{coker}[\mathcal{F}(Y) \to \mathcal{G}(Y)].$$

In particular a sequence of presheaves in exact in $P(X_E)$ if and only if the corresponding sequences of sections over Y is exact for every $Y \in C_X$. This makes the category of presheaves on X_E an abelian category, denoted by $P(X_E)$ (or sometimes $P(X)$ if E is understood).

Definition 2.8 Let X be a scheme. Let G be an X-scheme. A *group scheme* structure on G consists of giving a group structure on $G(Y) := \operatorname{Hom}_X(Y, G)$ for every X-scheme Y, such that the maps $G(Y) \to G(Y')$ are group homomorphisms for every morphism $Y' \to Y$ of X-schemes.

Example 2.9 a) A commutative group scheme G over X defines a presheaf via $Y \mapsto G(Y)$. Examples are: the *additive group* $\mathbf{G}_{a,X} = X \times_{\mathbf{Z}} \mathbf{Z}[T]$, the multiplicative group $\mathbf{G}_{m,X} = X \times_{\mathbf{Z}} \mathbf{Z}[T^{\pm 1}]$, or the group of n-roots of unity $\mu_{n,X} = X \times_{\mathbf{Z}} (\mathbf{Z}[T]/(T^n - 1))$. For an affine $Y = \operatorname{Spec} A$, we have $\mathbf{G}_a(A) = A$, $\mathbf{G}_m(A) = A^*$, and $\mu_n(A)$ is the n-torsion subgroup of A^*. Observe that in the special case when G itself is in C_X and $\mathcal{F}$ is a presheaf on X_E, a section $s \in \mathcal{F}(G)$ induces a presheaf morphism $G \to \mathcal{F}$ (for $U \in C_X$, send $f \in G(U) = \operatorname{Hom}_X(U, G)$ to the restriction $s_{|U}$ induced by f), and vice-versa (if $\varphi : G \to \mathcal{F}$ is a presheaf morphism, take for s the image of Id_G by the map $G(G) \to \mathcal{F}(G)$ induced by φ).

b) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules (in the usual sense). Then we can define a presheaf $W(\mathcal{F})$ on each site of Example 2.3 by the formula

$$W(\mathcal{F})(Y) := \Gamma(Y, f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y) = H^0(Y, f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y).$$

for every $f : Y \to X$. In particular $W(\mathcal{O}_X) = \mathbf{G}_a$.

c) A presheaf on the Zariski site of X is just a presheaf on the topological space X, equipped with Zariski topology.

d) Let M be an abelian group. The *constant presheaf* $\mathcal{P}_M$ on X_E is defined by $\mathcal{P}_M(\emptyset) = 0$ and $\mathcal{P}_M(U) = M$ for every non-empty X-scheme U, with obvious restriction maps.

Definition 2.10 A presheaf $\mathcal{F}$ on a site X_E is a *sheaf* if for every scheme $Y \in C_X$ and every covering $(U_i \to Y)_{i \in I}$, the following properties hold:

i) Every section $s \in \mathcal{F}(Y)$ whose restriction to each U_i is zero satisfies $s = 0$.

ii) For every family $(s_i)_{i \in I}$ (where $s_i \in \mathcal{F}(U_i)$) such that the restriction of s_i and s_j to $U_i \times_Y U_j$ coincide for all pairs $i, j \in I$, there exists an $s \in \mathcal{F}(Y)$ whose restriction to each U_i is s_i.

A presheaf is said to be *separated* if it satisfies condition i). The two conditions can be summarized by the exactness, for every covering $(U_i \to U)$, of

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \times_Y U_j),$$

where the last map is defined by $(s_i)_{i \in I} \mapsto ((s_i - s_j)_{|U_i \times_Y U_j})_{i,j \in I}$. In other words, the sheaf condition says that the sequence

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_Y U_j)$$

is exact, which means[3] that the map $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$ induces a bijection between $\mathcal{F}(U)$ and the subset of $\prod_i \mathcal{F}(U_i)$ consisting of those elements whose images in $\prod_{i,j} \mathcal{F}(U_i \times_Y U_j)$ by the two twin arrows are the same. The advantage of this last formulation is that it extends to presheaves of sets or of non-abelian groups. Observe also that the property that the restrictions of s_i and s_j to $U_i \times_Y U_j$ coincide is in general non trivial even if $i = j$ (unlike the classical case of a sheaf on a topological space).

The category of sheaves is denoted by $S(X_E)$ (or sometimes $S(X)$ if E is understood). It is a full subcategory of $P(X_E)$, which will be shown later to be an abelian category (but not an abelian subcategory of $P(X_E)$, because the cokernels are not the same).

So far it is not obvious to check that a presheaf is a sheaf. Here is a useful criterion:

Proposition 2.11 *Let $\mathcal{F}$ be a presheaf for the étale or the flat site X_E on a scheme X. Then $\mathcal{F}$ is a sheaf if and only if it satisfies the two following conditions:*

i) For every $Y \in C_X$, the restriction of $\mathcal{F}$ to Y is a sheaf for the usual Zariski topology on Y.

ii) For every covering $U' \to U$ consisting of one single surjective map with both U' and U affine, the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U')$$

is exact.

[3]Sometimes the same piece of notation is used without puting a left-zero in the sequence.

Proof : Since open immersions are étale (hence flat and locally of finite presentation), condition i) is necessary; condition ii) obviously is also necessary, since it is the sheaf condition for the special case of a covering of an affine scheme consisting of one single affine scheme. We will prove the sufficiency in the case of the flat site (the proof for the étale site is similar).

Condition i) shows that $\mathcal{F}(V) = \prod \mathcal{F}(V_i)$ when $V = \coprod V_i$ is the disjoint union of the schemes V_i. It is therefore sufficient to prove that the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U') \qquad (2)$$

is exact when $f : U' \to U$ is such that $U' = \coprod_{j \in J} U'_j$ and $(U'_j \to U)$ is a covering for the flat site. By condition ii), (2) is indeed exact when J is finite and we assume further that U and all U'_j are affine, because U' is then affine.

Write $U = \bigcup_{i \in I} U_i$, where all U_i are affine open subsets of U. Each $f^{-1}(U_i)$ can also be written $f^{-1}(U_i) = \bigcup_{k \in E_i} U'_{ik}$, where U'_{ik} is an affine open subset of U'. Now $f(U'_{ik})$ is open (by Proposition 1.5, c) in the affine (hence quasi-compact) scheme U_i. Since U_i is covered by the $f(U'_{ik})$, we get a covering $(U'_{ik} \to U_i)_{k \in K_i}$, where each set K_i is finite. We can always assume that K_i contains a given $k \in E_i$; therefore, up to repeating (possibly infinitely many times) U_i in the decomposition $U = \bigcup U_i$, we can also asume that the family $(U'_{ik})_{i \in I, k \in K_i}$ is a covering of U'.

Now there is a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U') & \longrightarrow & \mathcal{F}(U' \times_U U') \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_{i \in I} \mathcal{F}(U_i) & \longrightarrow & \prod_{i \in I} \prod_{k \in K_i} \mathcal{F}(U'_{ik}) & \longrightarrow & \prod_{i \in I} \prod_{k,l \in K_i} \mathcal{F}(U'_{ik} \times_U U'_{il}) \\
 & & \downarrow & & \downarrow & & \\
 & & \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) & \longrightarrow & \prod_{i,j \in I} \prod_{k \in K_i, l \in K_j} \mathcal{F}(U'_{ik} \cap U'_{jl}) & &
\end{array}$$

The two columns are exact thanks to assumption i). The middle row is exact as well because it is a product (for $i \in I$) of sequences that were shown to be exact thanks to condition ii) (recall that K_i is finite and all U'_{ik} are affine). It follows that the map $\mathcal{F}(U) \to \mathcal{F}(U')$ is injective. Applying this result to the covering of $(U_i \cap U_j)$ by the $U'_{ik} \cap U'_{jl}$ for every $i, j \in I$, we get that the bottom arrow is injective as well. Now a diagram chase shows that the first line is exact, which proves the proposition.

□

Corollary 2.12 *a) Every presheaf defined by a commutative group scheme G is a sheaf for the flat, étale, and Zariski sites on X.*

b) Every presheaf $W(\mathcal{F})$ associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is a sheaf for the flat, étale, and Zariski sites on X.

Proof : a) Condition i) of Proposition 2.11 follows from the fact that if (U_i) is a Zariski open covering of a scheme X, then a family of X-morphisms $U_i \to G$ such that f_i and f_j have same restriction to $U_i \cap U_j$ (for every pair i, j) uniquely extends to an X-morphism $U \to G$ by glueing properties of morphisms of schemes. Condition ii) is a consequence of Proposition 1.7 a) for $Y = U'$, $X = U$ and $Z = G$ (the proposition follows rather easily from Lemma 1.6 in this case because U and U' are affine).

b) Since $\mathcal{F}$ is a sheaf for Zariski topology, condition i) is immediate. Condition ii) follows from Lemma 1.6 (actually only the exactness up to the first three non zero terms is required) applied to the faithfully flat morphism $\operatorname{Spec} B := U' \to \operatorname{Spec} A = U$ and $M := \mathcal{F}(U)$.

□

Example 2.13 a) Let M be an abelian group and let M_X (or simply M if X is understood) be the associated constant group scheme on X (as a scheme $M_X = \coprod_{m \in M} X$; in particular $M_X(Y) = M$ for every connected X-scheme Y). Then the constant presheaf $\mathcal{P}_M$ is not a sheaf, but we can define the *constant sheaf* $\mathcal{F}_M$ on X as the sheaf associated to the group scheme M_X. It is a special case of the sheaf associated to a presheaf (see Theorem 2.29 below). For instance the constant sheaf $\mathbf{Z}$ has the property that

$$\operatorname{Hom}_{S(X_E)}(\mathbf{Z}, \mathcal{F}) = \mathcal{F}(X)$$

for every sheaf $\mathcal{F}$ on X: indeed (decomposing X into the disjoint union of its connected component) one immediately reduces to the case when X is connected. Then a sheaf homomorphism $\mathbf{Z} \to \mathcal{F}$ induces a map on global sections $\mathbf{Z} \to \mathcal{F}(X)$, hence an element $s \in \mathcal{F}(X)$ (the image of 1). Conversely, such an element induces a sheaf homomorphism $\mathbf{Z} \to \mathcal{F}$: for every map $U \to X$ in E with U connected, one defines the corresponding map $\mathbf{Z} = \Gamma(U, \mathbf{Z}) \to \mathcal{F}(U)$ by sending $1 \in \mathbf{Z}$ to the restriction $s_{|U}$.

b) A product of sheaves on X_E obviously is a sheaf.

c) The intersection of a family $\mathcal{F}_i$ of subsheaves of a sheaf $\mathcal{F}$, defined by $(\bigcap \mathcal{F}_i)(U) := \bigcap \mathcal{F}_i(U)$, is again a sheaf.

d) If $\phi : \mathcal{F} \to \mathcal{F}'$ is a morphism of sheaves, then the inverse image of every subsheaf of $\mathcal{F}'$ is a sheaf; in particular the kernel $\ker \phi$ is a subsheaf of $\mathcal{F}$. This is in general not true for the cokernel.

Definition 2.14 Let X_E be a site. A *refinement* of a covering $(U_i)_{i\in I} \to U$ is a covering $(V_j)_{j\in J} \to U$ equipped with a map $\sigma : J \to I$ such that for every $j \in J$, the morphism $V_j \to U$ factorizes through $U_{\sigma j} \to U$.

Observe that the map σ is part of the datum of the refinement.

Remark 2.15 a) A subcovering $(U_i)_{i\in J}$ (where $J \subset I$) is a refinement of $\mathcal{U}$ as soon as the union of the images of U_i for $i \in J$ still is the whole U. Also if each U_i is a union of open subsets (V_{ij}), then (V_{ij}) refines (U_i).

b) If we change the class E to a class E' (on the same category C_X), such that every covering for E has a refinement consisting of a covering for E' and vice-versa, then the category $S(X_E)$ and $S(X_{E'})$ are the same ([20], Lemma 7.8.7). See also Example 3.39 for a statement about cohomology in this context. This motivates the next proposition.

Proposition 2.16 *Let X be a scheme. Let $\mathcal{U} = (U_i \xrightarrow{f_i} X)$ be a covering of X for the étale (resp. flat) topology.*

a) Assume that X is separated (over an affine scheme). Then $\mathcal{U}$ has a refinement consisting of affine étale (resp. flat) morphisms.

b) Assume that X is quasi-separated (e.g. locally noetherian). Then $\mathcal{U}$ has a refinement $\mathcal{V} = (V_i \to X)_{i\in I}$ consisting of finitely presented étale (resp. flat) morphisms. If X is further assumed quasi-compact, then I can be chosen finite.

Proof : We first write X as the union of open affine subset (X_j). Set $U_{ij} = f_i^{-1}(X_i)$, and decompose again each U_{ij} as the union of affine subset (V_{ijk}). We obtain a refinement $\mathcal{V} = (V_{ijk})$, where each $g_{ijk} : V_{ijk} \to X$ factorizes through the affine morphism $V_{ijk} \to X_j$. The assumption a) implies that the open immersion $X_j \to X$ is affine, hence the morphism g_{ijk} is affine as well. Assumption b) implies that this open immersion is quasi-compact, making g_{ijk} quasi-compact and separated, thus finitely presented (since it is locally finitely presented by definition of the flat and the étale site). Finally, if X is further assumed to be quasi-compact, the image of each g_{ijk} is an open subset of X (by Proposition 1.5, c), and it suffices to take a finite subfamily of (g_{ijk}) such that the union of the images of the corresponding g_{ijk} is the whole X.

□

The next example is very important. It describes the étale sheaves on $X = \operatorname{Spec} k$ when k is a field. Fix a separable closure $\bar{k}$ of k and define $\Gamma = \operatorname{Gal}(\bar{k}/k)$. If $\mathcal{F}$ is a sheaf on $X_{\text{ét}}$, set $\mathcal{F}(K) := \mathcal{F}(\operatorname{Spec} K)$ when K is a finite and separable field extension of k. We denote by C_Γ the category of *discrete* Γ*-modules*. Recall that objects of C_Γ are abelian groups equipped with an action of Γ such that all stablizers are open (hence of finite index) in the profinite group Γ (cf. [8], chapter 4).

Theorem 2.17 *For every sheaf* $\mathcal{F}$ *on* $X = \operatorname{Spec} k$*, define*

$$M_{\mathcal{F}} := \varinjlim_{K} \mathcal{F}(K),$$

where the limit is taken over all finite (Galois) field extensions $K \subset \bar{k}$*. Equip* $M_{\mathcal{F}}$ *with the action of* Γ *induced by its action on each* K*. Then* $M_{\mathcal{F}}$ *is a discrete* Γ*-module and the functor* $\mathcal{F} \mapsto M_{\mathcal{F}}$ *induces an equivalence of categories between* $S(X_{\text{ét}})$ *and* C_Γ*.*

Proof : The definition makes clear that $M_{\mathcal{F}}$ is the union of $M_{\mathcal{F}}^H$ over all open subgroups H of Γ, hence $M_{\mathcal{F}}$ is a discrete Γ-module. Now the goal is to associate to every $M \in C_\Gamma$ a presheaf $\mathcal{F}_M$ on $X_{\text{ét}}$, which will be shown to be a sheaf.

We first observe that (thanks to the sheaf condition) it is sufficient to define $\mathcal{F}_M(K)$ for a finite separable field extension K of k, because every étale X-scheme is of the form $U = \coprod \operatorname{Spec} K_i$ (where K_i is such an extension of k) and we then set

$$\mathcal{F}_M(U) := \prod_i \mathcal{F}(K_i). \tag{3}$$

Then $A(K) := \operatorname{Hom}_k(K, \bar{k})$ has a Γ-module structure induced by the action of Γ on $\bar{k}$, and we define

$$\mathcal{F}_M(K) := \operatorname{Hom}_\Gamma(A(K), M).$$

This makes $\mathcal{F}_M$ a presheaf on $X_{\text{ét}}$ (observe that if we start with a non necessarily discrete Γ-module M, we get the same result by replacing M with the associated discrete submodule $\bigcup_H M^H$, where H runs over all open subgroups of Γ). We now show:

Proposition 2.18 *The presheaf* $\mathcal{F}_M$ *is a sheaf.*

Proof : We observe that in the case of a subextension $K \subset \bar{k}$, we have $A(K) = \Gamma/H$, where $H := \operatorname{Gal}(\bar{k}/K)$, hence $\mathcal{F}_M(K) = M^H$ (a Γ-equivariant function from Γ to M is given by $\gamma \mapsto \gamma.a$, where a is a fixed element of M; and this function induces a function $\Gamma/H \to M$ if and only if $a \in M^H$).

Let us show that $\mathcal{F}_M$ is a sheaf, using Proposition 2.11. Condition i) is relation (3) in the definition of $\mathcal{F}_M$. To check condition ii), we can restrict to the case $U = \operatorname{Spec} L$, $U' = \operatorname{Spec} L'$, where $k \subset L \subset L'$ are finite and separable field extensions. Fix an embedding of L' into $\bar{k}$; then $\mathcal{F}_M(L)$ identifies to M^{Γ_L}, where $\Gamma_L := \operatorname{Gal}(\bar{k}/L)$, and similarly for L'. Therefore, if L'/L is Galois, we have $\mathcal{F}_M(L) = \mathcal{F}_M(L')^{\operatorname{Gal}(L'/L)}$. Now Galois theory identifies $L' \otimes_L L'$ with $\prod_{\sigma \in \operatorname{Gal}(L'/L)} L'_\sigma$, where $L'_\sigma = L'$ for every σ. Writing $L' = L[T]/F$ and $L' \otimes_L L' = L'[T]/F$ (where $F \in L[T]$ is a separable polynomial), we see that in this identification, the maps $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$ (from L' to $L' \otimes_L L'$) respectively coincide with the diagonal map and the map $x \mapsto \prod_{\sigma \in \operatorname{Gal}(L'/L)} \sigma.x$. This implies that the sequence

$$0 \to \mathcal{F}_M(L) \to \mathcal{F}_M(L') \rightrightarrows \mathcal{F}_M(L' \otimes_L L')$$

is exact, because the twin maps $\mathcal{F}_M(L') \rightrightarrows \mathcal{F}_M(L' \otimes_L L')$ of this sequence now can respectively be identified with the diagonal map and with the map $M^{\Gamma_{L'}} \to \prod_{\sigma \in \operatorname{Gal}(L'/L)} M^{\Gamma_{L'}}$ given by $a \mapsto \prod_{\sigma \in \operatorname{Gal}(L'/L)} \sigma.a$. In the general case where L'/L is not assumed to be Galois, we embed L' into a finite separable extension L_1 which is Galois over L. Consider the commutative diagram (where the right horizontal maps are defined as the difference of the twin maps)

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}_M(L) & \longrightarrow & \mathcal{F}_M(L') & \longrightarrow & \mathcal{F}_M(L' \otimes_L L') \\
 & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_M(L) & \longrightarrow & \mathcal{F}_M(L_1) & \longrightarrow & \mathcal{F}_M(L_1 \otimes_L L_1)
\end{array}$$

As seen before, the bottom row is exact. Since we have injections

$$\mathcal{F}_M(L) \simeq M^{\Gamma_L} \hookrightarrow \mathcal{F}_M(L') \hookrightarrow \mathcal{F}_M(L_1),$$

a diagram chase shows that the top row is exact is well. This shows that $\mathcal{F}_M$ is a sheaf.

□

To conclude the proof of Theorem 2.17, we note that a Γ-homomorphism $M \to M'$ clearly induces a morphism $\mathcal{F}_M \to \mathcal{F}_{M'}$. Conversely, every sheaf morphism $\phi : \mathcal{F} \to \mathcal{F}'$ induces a Γ-equivariant morphism $\mathcal{F}(K) \to \mathcal{F}'(K)$ for every finite extension $K \subset \bar{k}$ of k. Taking direct limits, we thus get a G-homomorphism $M_{\mathcal{F}} \to M_{\mathcal{F}'}$. It is now easy to check that the corresponding map $\operatorname{Hom}_\Gamma(M, M') \to \operatorname{Hom}(\mathcal{F}_M, \mathcal{F}'_M)$ is an isomorphism and that the canonical map $\mathcal{F} \to \mathcal{F}_{M_{\mathcal{F}}}$ is an isomorphism, making the categories $S(X_{\text{ét}})$ and C_Γ equivalent.

□

Unfortunately, property d) of Example 2.13 does not hold for cokernels instead of kernels (already for Zariski topology). It is therefore important to have a good notion of sheaf associated to a presheaf, which will be explained in the next paragraph.

2.3. Sheaf associated to a presheaf

We start by definitions of direct image and inverse image of a presheaf.

Definition 2.19 Let $\pi : X'_{E'} \to X_E$ be a continuous map of sites. For every presheaf $\mathcal{P}'$ on $X'_{E'}$, the *direct image presheaf* $\pi_p(\mathcal{P}')$ is defined by

$$(\pi_p(\mathcal{P}'))(U) = \mathcal{P}'(U \times_X X').$$

Definition 2.20 The *inverse image presheaf* π^p is defined as the left adjoint functor of π_p, that is:

$$\mathrm{Hom}_{P(X')}(\pi^p\mathcal{P}, \mathcal{P}') = \mathrm{Hom}_{P(X)}(\mathcal{P}, \pi_p\mathcal{P}')$$

for every presheaves $\mathcal{P}$ on X_E, $\mathcal{P}'$ on $X_{E'}$.

The existence of π_p is ensured by a general result in category theory (cf. [12], Prop. II.2.2). More explicitely, it can be constructed as follows for sites of Example 2.3, as the flat or the étale site:

Proposition 2.21 *For a fixed morphism $U' \to X'$ in $C'_{X'}$, consider commutative squares (where $U \to X$ is a morphism in C_X)*

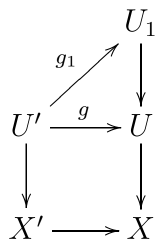
$$\begin{array}{ccc} U' & \xrightarrow{g} & U \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\pi} & X \end{array}$$

and define

$$(\pi^p\mathcal{P})(U') := \varinjlim \mathcal{P}(U),$$

where the limit runs over all these diagrams. Then π^p is the left adjoint functor of π_p.

It is of course understood that a morphism between two such diagrams (U_1, g_1) and (U, g) is a commutative diagram

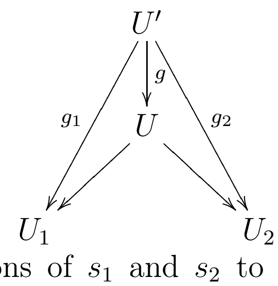


Proof : To give a morphism $\mathcal{P} \to \pi_p \mathcal{P}'$ is equivalent as giving compatible maps $f_U : \mathcal{P}(U) \to \mathcal{P}'(U \times_X X')$ for every $U \in C_X$; on the other hand, to give a morphism $\pi^p \mathcal{P} \to \mathcal{P}'$ is the same as defining maps $\mathcal{P}(U) \to \mathcal{P}'(U')$ for each diagram as above, which are compatible with the restriction maps. Such a diagram induces (by the universal property of the fibred product) a unique morphism $U' \to U \times_X X'$ that factorizes $U' \to U$, hence a morphism $\mathcal{P} \to \pi_p \mathcal{P}'$ induces a morphism $\pi^p \mathcal{P} \to \mathcal{P}'$, and conversely.

□

Remark 2.22 To be on the safe side, let us observe that the colimit in Proposition 2.21 is taken over a filtered category in all the relevant examples (Zariski, étale, or flat site), hence it is a direct limit in the usual sense (over an arbitrary site one has to use the colimit of abelian groups as defined in [19], Lemma 8.2). Indeed in these examples finite inverse limits exist in C_X: This a consequence of the existence of finite products and of difference kernels (=equalizer of two morphisms) in C_X, which follows easily from the first three statements of Proposition 1.4 in the case of the flat site, and from Proposition 1.11 in the case of the étale site; cf. [12], Remark II.1.13 and Appendix A.

Remark 2.23 Using the previous remark, it is possible (in the case of the Zariski, étale, or flat site) to represent an element of $(\pi^* \mathcal{P})(U')$ as a pair (s, g) where $U \to X$ is a morphism in C_X, $g : U' \to U$ is a morphism compatible with π, and $s \in \mathcal{P}(U)$, with the rule that we identify two such pairs (s_1, g_1) and (s_2, g_2) if there is a third pair (s, g) and a commutative diagram



such that the restrictions of s_1 and s_2 to U coincide. If a morphism $h : V' \to U'$ is given in $C_{X'}$, then the restriction of $(s, g) \in (\pi^p \mathcal{P})(U')$ to $(\pi^p \mathcal{P})(V')$ is just $(s, g \circ h)$.

Example 2.24 a) The inverse image of the constant presheaf $\mathcal{P}_M$ on X_E is the constant presheaf associated to the same group M on $X'_{E'}$.

b) Let $\pi : X' \to X$ be in C_X, take for $C'_{X'}$ the X'-schemes that are in C_X via π. Then $\pi^p(\mathcal{P})$ is just the restriction of $\mathcal{P}$ to the category $C'_{X'}$; in this case we can write $\mathcal{P}_{X'}$ for $\pi^p\mathcal{P}$.

Proposition 2.25 *The functor π_p is exact, and π^p is exact if X_E is one of the sites of Example 2.3, or in the case b) of Example 2.24.*

Proof : The exactness of π_p follows from the definition and the fact that a sequence of presheaves is exact if and only if the associated sequence of sections over U is exact for every $U \in C_X$. The exactness of π^p is obvious in the case b) of Example 2.24. In all sites of Example 2.3, if follows from the description of π^p using filtered direct limits, and the exactness of filtered direct limits in the category of abelian groups.

□

Proposition 2.26 *If $\mathcal{F}$ is a sheaf, its direct image $\pi_p\mathcal{F}$ is a sheaf.*

Proof : For every $U \in C_X$, set $U' := U \times_X X'$. Take a covering $(U_i \to U)$, then $(U'_i \to U')$ is also a covering. Since $\mathcal{F}$ is a sheaf, the sequence

$$0 \to \mathcal{F}(U') \to \prod_i \mathcal{F}(U'_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U'_i \times_{U'} U'_j)$$

is exact. Since $U'_i \times_{U'} U'_j = (U_i \times_U U_j)'$, we get that the sequence

$$0 \to \pi_p\mathcal{F}(U) \to \prod_i (\pi_p\mathcal{F})(U_i) \rightrightarrows \prod_{i,j} (\pi_p\mathcal{F})(U_i \times_U U_j)$$

is exact, which shows that $\pi_p\mathcal{F}$ is a sheaf.

□

Before constructing the sheaf associated to a presheaf on the étale site, we need the notion of stalk[4] of a presheaf. Since coverings for the étale topology are more general than Zariski open subsets, it will be associated not to a point of a scheme, but to a geometric point.

[4]In these notes, we will not consider the extension of the notion of stalk to more general sites. We refer to [16] for related results.

Definition 2.27 Let $\mathcal{P}$ be a presheaf on $X_{\text{ét}}$. Let $u_x : \bar{x} \to X$ be a geometric point of X. The *stalk* of $\mathcal{P}$ at $\bar{x}$ is

$$\mathcal{P}_{\bar{x}} := (u_x^p \mathcal{P})(\bar{x}).$$

In other words $\mathcal{P}_{\bar{x}} = \varinjlim \mathcal{P}(U)$, where the limit runs over all étale neighborhoods U (or all étale connected neighborhoods) of $\bar{x}$ in X.

Lemma 2.28 *Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. Let $s \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ over $U \in C_X$. If $s \neq 0$, then there exists $x \in X$ and a geometric point $\bar{x} \in U$ lying over x such that the restriction $s_{\bar{x}} \in \mathcal{F}_{\bar{x}}$ is not zero.*

Proof : Assume that $s_{\bar{x}} = 0$ for all maps $\mathcal{F}(U) \to \mathcal{F}_{\bar{x}}$ as in the lemma. For every $u \in U$, choose a geometric point $\bar{x}$ with image u. By definition of the stalk, there is an étale U-scheme V_u, whose image in U contains u, and such that the restriction of s to V_u is zero. Since U is covered by all $V_u \to U, u \in U$, the sheaf condition yields that $s = 0$.

□

Theorem 2.29 *Let X_E be a site with underlying scheme X. Let $\mathcal{P}$ be a presheaf on X_E. Then there is a sheaf $a\mathcal{P}$ on X_E and a morphism of presheaves $\phi : \mathcal{P} \to a\mathcal{P}$, such that any morphism ϕ' from $\mathcal{P}$ to a sheaf $\mathcal{F}$ factors uniquely through ϕ.*

Proof : We give the proof in the case of the étale site (the general case is more complicated, see [1], II.1.4 and also Example 3.12). Start with the case $X = \operatorname{Spec} K$, where K is a separably closed field. Then every étale X-scheme U is a disjoint union $U = \coprod X_i$ with $X_i = X$, and we define $a\mathcal{P}$ as

$$(a\mathcal{P})(\coprod X_i) = \prod \mathcal{P}(X_i),$$

and $\phi(U)$ as the map $\mathcal{P}(U) \to \prod \mathcal{P}(X_i)$ induced by the restriction maps.

We now deal with a general X. For each $x \in X$, choose a geometric point $u_x : \bar{x} \to X$ with image x. Then $u_x^p \mathcal{P}$ is a presheaf on $\bar{x}$, and we just defined the associated sheaf $P_{\bar{x}} = a(u_x^p \mathcal{P})$. According to Proposition 2.26, the direct image $(u_x)_p(P_{\bar{x}})$ is a sheaf on X, as well as

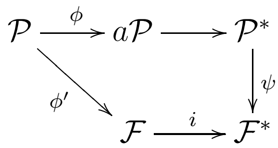
$$\mathcal{P}^* := \prod_{x \in X} (u_x)_p(P_{\bar{x}})$$

thanks to Example 2.13, b). For each $x \in X$, we have an adjunction morphism of presheaves $\mathcal{P} \to (u_x)_p(u_x^p \mathcal{P})$ and a sheafification map $u_x^p \mathcal{P} \to P_{\bar{x}}$, which induces a map $(u_x)_p(u_x^p \mathcal{P}) \to (u_x)_p(P_{\bar{x}})$, hence (by composition) maps

$$\mathcal{P} \to (u_x)_p(P_{\bar{x}}),$$

which induce a map $\phi : \mathcal{P} \to \mathcal{P}^*$. We define $a\mathcal{P}$ as the intersection of all subsheaves of $\mathcal{P}^*$ containing the image $\phi(P)$ (this is a sheaf by Example 2.13, c).

Let $\phi' : \mathcal{P} \to \mathcal{F}$ be a morphism from $\mathcal{P}$ to some sheaf $\mathcal{F}$. There is a commutative diagram



Here ψ is the map given by the maps $\phi'_{\bar{x}} : \mathcal{P}^*_{\bar{x}} \to \mathcal{F}^*_{\bar{x}}$ induced by ϕ'. The map i is injective by Lemma 2.28. We observe that by Example 2.13 d), $\psi^{-1}(\mathcal{F})$ is a subsheaf of $\mathcal{P}^*$ and it contains $\phi(\mathcal{P})$, hence also $a\mathcal{P}$. Therefore ψ induces a morphism $\psi_0 : a\mathcal{P} \to \mathcal{F}$, such that $\phi' = \psi_0 \circ \phi$ as required. Now if another $\psi_1 : a\mathcal{P} \to \mathcal{F}$ also satisfies $\phi' = \psi_1 \circ \phi$, then $\ker(\psi_0 - \psi_1)$ is a subsheaf of $\mathcal{P}^*$ containing $\phi(\mathcal{P})$, hence it contains $a\mathcal{P}$, which shows that $\psi_0 = \psi_1$. This concludes the proof.

□

Remark 2.30 Let $u_x : \bar{x} \to X$ be a geometric point of X. A presheaf on $\bar{x}$ (which is the spectrum of a separably closed field) is a sheaf if and only if it takes disjoint union of schemes to products of abelian groups; this easily implies that u_x^p takes sheaves to sheaves, which shows that $u_x^p \circ a \simeq a \circ u_x^p$. In particular, for a presheaf $\mathcal{P}$ on X, since

$$\mathcal{P}_{\bar{x}} = (u_x^p \mathcal{P})(\bar{x}) = (a u_x^p \mathcal{P})(\bar{x}),$$

we have

$$\mathcal{P}_{\bar{x}} \simeq (u_x^p a\mathcal{P})(\bar{x}) = (a\mathcal{P})_{\bar{x}}.$$

In other words, $\mathcal{P}$ and $a\mathcal{P}$ have the same stalks.

2.4. The category of sheaves

Theorem 2.29 (which constructs the sheafification of a presheaf) can be reformulated as follows: the inclusion functor $i : S(X_E) \to P(X_E)$ has the functor $a : P(X_E) \to S(X_E)$ as a left-adjoint. For a morphism of sheaves $u : \mathcal{F}_1 \to \mathcal{F}_2$, the cokernel (resp. image) of u in $S(X_E)$ is defined as $a(\operatorname{coker}^p u)$ (res. $a(\operatorname{Im}^p u)$), where $\operatorname{coker}^p u$ (resp. $\operatorname{Im}^p u$) is the cokernel (resp. image) of u in $P(X_E)$.

Theorem 2.31 *a) The inclusion functor i is left exact and the sheafification functor a is exact. In particular the image of a morphism $u : \mathcal{F}' \to \mathcal{F}$ in*

$S(X_E)$ identifies to a subsheaf of $\mathcal{F}$, the quotient $\mathcal{F}/\mathrm{Im}\, u$ in $S(X_E)$ is the sheaf cokernel coker u*, and a sequence of sheaves*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}" \tag{4}$$

is exact in $S(X_E)$ if and only if it is exact in $P(X_E)$. .

b) For $X_E = X_{\text{ét}}$, the sequence (4) is exact is exact if and only if for every geometric point $\bar{x}$ of X, the sequence

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \tag{5}$$

is exact.

c) A map of sheaves $\phi : \mathcal{F} \to \mathcal{F}'$ is surjective on $X_{\text{ét}}$ if and only if the map $\phi_{\bar{x}} : \mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}}$ is surjective for every geometric point $\bar{x}$ of X.

d) The category $S(X_E)$ is abelian.

Proof : a) The fact that i is left exact is obvious, because the kernel of a morphism of sheaves is the same in the categories $S(X_E)$ and $P(X_E)$. The functor a is right exact as adjoint of i; it remains to show that a is left exact, which we will show only in the case of the étale site (see [12], Theorem II.2.15 a) for the general case). Let $\mathcal{P}_1 \to \mathcal{P}_2$ be an injective morphism of presheaves. Using the notation of the proof of Theorem 2.29, the corresponding map $\mathcal{P}_1^* \to \mathcal{P}_2^*$ is injective because the functors u_x^p and $(u_x)_p$ are exact (Proposition 2.25), which implies (as $a\mathcal{P}_1$ and $a\mathcal{P}_2$ are respectively subpresheaves of $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$) that the morphism $a\mathcal{P}_1 \to a\mathcal{P}_2$ is injective in $P(X_E)$, hence also in $S(X_E)$.

b) Obviously, if (4) is exact, then (5) is also exact. Assume that (5) is exact for all geometric points $\bar{x}$ of X. Let $U \in C_X$ and $s' \in \mathcal{F}'(U)$ such that the image s of s' in $\mathcal{F}(U)$ is zero. Since the map $\mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ is assumed to be injective, this implies that $s'_{\bar{x}} = 0$ for all $\bar{x}$, hence $s' = 0$ by Lemma 2.28. To show the exactness "in the middle", take $s \in \mathcal{F}(U)$ such that the image of s in $\mathcal{F}''(U)$ is zero. This implies that $s_{\bar{x}} \in \mathcal{F}'_{\bar{x}} \subset \mathcal{F}_{\bar{x}}$ for all $\bar{x}$. Therefore, for every $u \in U$, there is an étale map $V_u \to U$ whose image contains u and such that the restriction $s_{|V_u} \in \mathcal{F}'(V_u) \subset \mathcal{F}(V_u)$. Since $(V_u \to U)$ is a covering of U and $\mathcal{F}'$ is a sheaf, this means that $s \in \mathcal{F}'(U)$ as required.

c) Let $u : \mathcal{F} \to \mathcal{F}'$ be a morphism of sheaves, denote by $\mathcal{P}$ its cokernel in $P(X_{\text{ét}})$, then its cokernel in $S(X_{\text{ét}})$ is $a\mathcal{P}$. Since

$$\mathcal{F} \to \mathcal{F}' \to \mathcal{P} \to 0$$

is exact in $P(X_{\text{ét}})$, the corresponding sequence of stalks

$$\mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}} \to \mathcal{P}_{\bar{x}} \to 0$$

obviously remains exact, which implies that the map $\mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}}$ is surjective if and only if $(a\mathcal{P})_{\bar{x}} = 0$ thanks to Remark 2.30. Since u is surjective in $S(X_{\text{ét}})$ if and only if $a\mathcal{P} = 0$, which (by Lemma 2.28) is equivalent to saying that $(a\mathcal{P})_{\bar{x}} = 0$ for all geometric points $\bar{x}$ of X, we are done.

d) It only remains to prove that every morphism $\phi : \mathcal{F} \to \mathcal{F}'$ in $S(X_E)$ induces an isomorphism between its coimage $\mathcal{F}/\ker\phi$ and its image $\operatorname{Im}\phi$. The image and coimage of ϕ in $S(X_E)$ are obtained by applying the functor a to the image and coimage in $P(X_E)$, which is an abelian category. Since a takes isomorphisms to isomorphisms, the map $\operatorname{Coim}\phi \to \operatorname{Im}\phi$ is an isomorphism in $S(X_E)$, since in the abelian category $P(X_E)$ the canonical map between the coimage and the image of ϕ is known to be an isomorphism.

□

Remark 2.32 On an arbitrary site, surjectivity of $\mathcal{F} \to \mathcal{F}'$ is equivalent to saying that for every $U \in C_X$ and every $s \in \mathcal{F}'(U)$, there exists a covering $(U_i \to U)$ and elements $s_i \in \mathcal{F}(U_i)$ such that $\phi(s_i) = s_{|U_i}$ for all i ("local surjectivity"). See [12], Theorem II.2.15 and proof of Theorem 2.11 (the latter defines the sheafification in the general case).

Example 2.33 a) The group schemes $\mathbf{G}_m$ and μ_n induce sheaves on the étale site or the flat site of a scheme X. The sequence

$$0 \to \mu_n \to \mathbf{G}_m \stackrel{.n}{\to} \mathbf{G}_m$$

is clearly exact in $P(X_E)$, hence also in $S(X_E)$. For every strictly local ring A with $n \in A^*$, the map $x \mapsto x^n$ is surjective from A^* to A^* by Hensel's lemma (that is: property iii) of Theorem 1.20), which proves (by Theorem 2.31, c) that the *Kummer sequence*

$$0 \to \mu_n \to \mathbf{G}_m \stackrel{.n}{\to} \mathbf{G}_m \to 0$$

is then exact on $X_{\text{ét}}$ as soon as the integer n is invertible on X.

b) The Kummer sequence is exact on X_{fppf} without any assumption on the residue characteristics of X. Indeed let U be an X-scheme and $u \in \mathcal{O}_U(U)^*$. Cover U by open affine subsets $U_i = \operatorname{Spec} A_i$ and let u_i be the restriction of u to U_i. Set $A'_i = A_i[T]/(T^n - u_i)$ and $U'_i = \operatorname{Spec} A'_i$. Then $(U'_i \to U)$ is a flat covering of U (indeed A'_i is free of rank n over A_i) such that the restriction of u to U'_i is an n-th power in $\mathbf{G}_m(U'_i)$ for every i. Remark 2.32 now yields the required exactness.

c) Let X be a scheme of characteristic $p > 0$ (that is an $\mathbf{F}_p$-scheme). Let F be the map $a \mapsto a^p$. Then there is an exact sequence (the *Artin-Schreier sequence*) of étale sheaves on X:

$$0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{G}_a \stackrel{F-1}{\to} \mathbf{G}_a \to 0.$$

The last map is surjective because if A is a strictly local ring, then the map $A \to A, x \mapsto x^p - x$ is onto (again by Hensel's lemma). The sequence of the first four terms is easily seen to be exact (even for the Zariski topology), the $\mathbf{F}_p$-group scheme $(\mathbf{Z}/p\mathbf{Z})$ being isomorphic to (Spec $(\mathbf{F}_p[T]/(T^p - T))$ (hence the same holds for their pullback to X).

d) Again, let X be a scheme of characteristic p. Let α_p be the subsheaf of $\mathbf{G}_a$ given by the kernel of $a \mapsto a^p$, that is the sheaf corresponding to the group scheme Spec $(\mathbf{F}_p[T]/T^p) \times_{\mathbf{F}_p} X$. Then there is an exact sequence of sheaves for the flat topology (but in general not for the étale topology, because over an imperfect separably closed field, the map $x \mapsto x^p$ is not surjective)

$$0 \to \alpha_p \to \mathbf{G}_a \stackrel{\cdot p}{\to} \mathbf{G}_a \to 0.$$

2.5. Direct and inverse images of sheaves

Definition 2.34 Let $\pi : X'_{E'} \to X_E$ be a continuous map of sites. The *direct image* $\pi_* \mathcal{F}'$ of a sheaf $\mathcal{F}'$ on $X'_{E'}$ is just $\pi_p \mathcal{F}'$. The *inverse image* of a sheaf $\mathcal{F}$ on X_E is $\pi^* \mathcal{F} := a(\pi^p \mathcal{F})$.

By definition, the functors π_* and π^* are adjoint in $S(X'_{E'}), S(X_E)$. If $\pi : X' \to X$ is in C_X, then π^* is just the restriction functor $\mathcal{F} \mapsto \mathcal{F}_{|X'}$.

Remark 2.35 For every presheaf $\mathcal{P}$ on X_E and every sheaf $\mathcal{F}$ on X'_E, we have

$$\mathrm{Hom}_{S_{E'}}(\pi^*(a\mathcal{P}), \mathcal{F}) = \mathrm{Hom}_{S_E}(a\mathcal{P}, \pi_p \mathcal{F}) = \mathrm{Hom}_{P_E}(\mathcal{P}, \pi_p \mathcal{F}) =$$

$$\mathrm{Hom}_{P_{E'}}(\pi^p \mathcal{P}, \mathcal{F}) = \mathrm{Hom}_{S(E')}(a(\pi^p \mathcal{P}), \mathcal{F}),$$

which shows that $\pi^*(a\mathcal{P}) = a(\pi^p \mathcal{P})$.

Proposition 2.36 *The functor π_* is left exact. The functor π^* is exact if X_E is one of the sites of Example 2.3, or if $\pi : X' \to X$ is in C_X.*

Proof : The first assertion follows from the exactness of π_p (Proposition 2.25), left-exactness of i, and last statement of Theorem 2.31, a). For the second assertion, let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence in $S(X_E)$. Then the sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{G}$$

is exact in $P(X_E)$, hence

$$0 \to \pi^*\mathcal{F}' \to \pi^*\mathcal{F} \to \pi^*\mathcal{G}$$

is exact in $S(X_E)$ because π^p (Proposition 2.25) and a (Theorem 2.31, a) are exact. It remains to prove the surjectivity of $\pi^*\mathcal{F} \to \pi^*\mathcal{G}$ in $S(X_E)$. Let $\mathcal{G}'$ be the cokernel of $\mathcal{F} \to \mathcal{G}$ in $P(X_E)$, we have $a\mathcal{G}' = 0$ because $\mathcal{F} \to \mathcal{G}$ has trivial cokernel in $S(X_E)$ and a is exact. Using again exactness of π^p and a, we see that

$$\pi^*\mathcal{F} \to \pi^*\mathcal{G} \to a(\pi^p\mathcal{G}')$$

is exact in $S(X_E)$. By Remark 2.35, we have $a(\pi^p\mathcal{G}') = \pi^*(a\mathcal{G}') = 0$, whence the result.

□

Note that although π_p is exact, the functor π_* is not right exact in general, because the cokernels in $S(X_E)$ and $P(X_E)$ are not the same (thus an exact sequence of sheaves does not necessarily remain exact as a sequence of presheaves).

Example 2.37 a) Let G_X be a commutative group scheme over X, it represents a sheaf (on one of the sites of Example 2.3). Let $G_{X'}$ be the X'-group scheme $G \times_X X'$.

There is a presheaf map $\pi^p G_X \to G_{X'}$, obtained by sending a pair (s, g) (where $g : U' \to U$ and $s \in G(U)$ are as in the explicit description after definition 2.20) to $sg \in G_X(U') = G_{X'}(U')$. This induces a canonical map of sheaves $\phi_G : \pi^* G_X \to G_{X'}$ (which can also be defined using the adjunction property of π^*). This map is an isomorphism if $\pi : X' \to X$ is in C_X, or if G itself is in C_X. The first case is easy (the inverse image $\pi^* G_X$ being just the restriction of G_X to X'). For the second one, we use Example 2.9, which yields for every sheaf $\mathcal{F}$ on $X'_{E'}$:

$$\mathrm{Hom}_{S(X')}(G_{X'}, \mathcal{F}) = \mathcal{F}(G_{X'}) = (\pi_*\mathcal{F})(G) = \mathrm{Hom}_{S(X)}(G_X, \pi_*\mathcal{F}).$$

Therefore $G_{X'} \simeq \pi^* G_X$ by uniqueness of adjoints.

b) In general ϕ_G is not an isomorphism: for instance let X be the spectrum of a field k of characteristic $p > 0$, let A be a k-algebra and $X' = \operatorname{Spec} A$. Take $G = \alpha_p$ and work on the small étale sites. Then $G_X = 0$ because every étale X-scheme U is reduced, hence $\alpha_p(U) = 0$. This implies that $\pi^* G_X = 0$ but $G_{X'}$ does not represent the zero sheaf as soon as A has a non-zero element a with $a^p = 0$. Thus ϕ_G is not surjective in this case.

c) It is also possible that ϕ_G is not injective. Set $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$, $X' = \operatorname{Spec} \mathbf{F}_p$ and $X = \operatorname{Spec}(\mathbf{Z}/p^2\mathbf{Z}) = \operatorname{Spec}(\mathbf{F}_p[\varepsilon])$, where $\varepsilon^2 = 0$. Consider the closed immersion $\pi : X' \to X$ and $G_X = \mathbf{G}_{m,X}$. For every finite field extension K of $\mathbf{F}_p$, there is an exact sequence

$$0 \to K \to K[\varepsilon]^* \to K^* \to 0$$

(the first map being $a \mapsto 1+a\varepsilon$), which yields an exact sequence of presheaves

$$0 \to \mathbf{G}_a \to \pi^p \mathbf{G}_m \to \mathbf{G}_{m,X'} \to 0.$$

Applying the exact functor a, we get an exact sequence of sheaves

$$0 \to \mathbf{G}_a \to \pi^* \mathbf{G}_m \overset{\Phi_G}{\to} \mathbf{G}_{m,X'} \to 0.$$

In particular the kernel of Φ_G is not 0.

c) Let $k \subset K$ be an inclusion of field, which induces a morphism $\pi : \operatorname{Spec} K \to \operatorname{Spec} k$. We choose compatible separable closures $\bar{k}$ and $\overline{K}$ of k and K. This induces a map of absolute Galois group $\psi : \Gamma_K \to \Gamma_k$ (which is injective if K is an algebraic separable extension of k). Consider the étale sites associated to $\operatorname{Spec} k$ and $\operatorname{Spec} K$. By Theorem 2.17, we can identify $S(\operatorname{Spec} k)$ with the category C_{Γ_k} of discrete Γ_k-modules, and similarly for K. Then taking the inverse image of a sheaf on $\operatorname{Spec} k$ corresponds to view a Γ_k-module M as a Γ_K-module via ψ. Since π_* and π^* are adjoint functors, we deduce that the direct image of a Γ_K-module N consists of the induced module (cf. [8], Remark I.1.14) $I_{\Gamma_k}^{\psi(\Gamma_K)}(N^{\ker \psi})$, which is just $I_{\Gamma_k}^{\Gamma_K}(N)$ if K/k is algebraic separable.

We now study the stalks of the inverse and the direct image of a sheaf for the étale topology.

Proposition 2.38 *Let $\pi : X' \to X$ be a morphism. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. Let $x' \in X'$, set $x = \pi(x')$ and choose compatible geometric points $\bar{x}'$, $\bar{x}$ respectively associated to x, x'. Then $(\pi^* \mathcal{F})_{\bar{x}'} \simeq \mathcal{F}_{\bar{x}}$. In particular, if we take for π the canonical morphism* $\operatorname{Spec}(\mathcal{O}_{X,\bar{x}}) = \operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}}) \to X$, *then* $\mathcal{F}_{\bar{x}} = (\pi^* \mathcal{F})_{\bar{x}} = (\pi^* \mathcal{F})(\operatorname{Spec}(\mathcal{O}_{X,\bar{x}}))$.

Proof : First observe that for a geometric point $u_x : \bar{x} \to X$, the stalk $\mathcal{F}_{\bar{x}}$ is isomorphic to $(u_x^* \mathcal{F})(\bar{x})$ (and similarly for the geometric point $\bar{x}' \to X'$ and the sheaf $\pi^* F$) because $u_x^* \mathcal{F} = (a u_x^p)(\mathcal{F})$ and $u_x^p \mathcal{F}$ have same group of sections over $\bar{x}$ by definition of sheafification over the spectrum of a separably closed field.

Now we may take $\bar{x} = \bar{x}'$, whence a commutative diagram

$$\begin{array}{ccc} X' & \xleftarrow{u_{x'}} & \bar{x}' \\ {\scriptstyle\pi}\downarrow & \swarrow {\scriptstyle u_x} & \\ X & & \end{array}$$

Then

$$(\pi^* \mathcal{F})_{\bar{x}'} = (u_{x'}^* \pi^* \mathcal{F})(\bar{x}') = (u_x^* \mathcal{F})(\bar{x}) = \mathcal{F}_{\bar{x}}.$$

□

The situation with direct image is more complicated. We need the following lemma, which will be extended later (Theorem 3.26, b) to étale cohomology:

Lemma 2.39 *Let I be a filtered category (e.g. a filtered ordered set) and $i \to X_i$ a contravariant functor from I to X-schemes. Asssume that all morphisms $X_j \to X_i$ are affine and that all schemes X_i (as well as X) are quasi-compact and quasi-separated. Set $X_\infty := \varprojlim X_i$. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$, with respective inverse image $\mathcal{F}_i$ and $\mathcal{F}_\infty$ on X_i, X_∞. Then*

$$\varinjlim_i \Gamma(X_i, \mathcal{F}_i) \xrightarrow{\simeq} \Gamma(X_\infty, \mathcal{F}_\infty).$$

Recall that the scheme $\varprojlim X_i$ is well defined because all transition morphisms $X_j \to X_i$ are assumed to be affine.

Proof (sketch of): Here are the main steps of the proof:

i) The first important observation is that if $f : U \to X_\infty$ is an étale, quasi-compact, and quasi-separated map, then it is obtained by base change from an étale map $f_i : U_i \to X_i$ for some $i \in I$. This follows from [19], Lemma 143.3, (9) (which is the case where all schemes considered are affine).

ii) We can work over the site $(X_\infty)_{\text{ét},qcqs}$ (whose definition is the same as the small étale site, except that étale morphisms are replaced by quasi-compact and quasi-separated étale morphisms; cf. Proposition 2.16). Although f_i (defined as above) is not unique, two such maps give rise to the same map $U_j \to X_j$ for some $j \geq i$, which makes the definition

$$\mathcal{G}(U) = \varinjlim_i \mathcal{F}_i(U_i)$$

unambiguous. This defines a presheaf on the site $(X_\infty)_{\text{ét},qcqs}$, which is a sheaf thanks to the sheaf condition on each $\mathcal{F}_i$ and the (left) exactness of the direct limit functor.

iii) One show (using the adjunction property of the inverse limit of a sheaf) that the obvious map $\mathcal{G} \to \mathcal{F}_\infty$ on $S(X_\infty)_{\text{ét},qcqs})$ is an isomorphism whence the result by taking global sections over X_∞.

□

Theorem 2.40 *Let $\pi : X' \to X$ be a quasi-compact and quasi-separated morphism. Let $x \in X$, $\bar{x} := \mathrm{Spec}\,(\overline{k(x)})$. Set $\widetilde{X} := \mathrm{Spec}\,(\mathcal{O}^{\mathrm{sh}}_{X,x})$ and $\widetilde{X}' := X' \times_X \widetilde{X}$. Let $\mathcal{F}$ be a sheaf on X' with inverse image $\widetilde{\mathcal{F}}$ on $\widetilde{X}'$. Then*

$$(\pi_* \mathcal{F})_{\bar{x}} = \widetilde{\mathcal{F}}(\widetilde{X}').$$

Proof : By definition we have

$$(\pi_* \mathcal{F})_{\bar{x}} = \varinjlim \mathcal{F}(U \times_X X'),$$

where the limit is over all étale neighborhoods U of $\bar{x}$ in X. We can restrict to affine U such that $\widetilde{X} = \varprojlim U$, whence $\widetilde{X}' = \varprojlim (U \times_X X')$ (products commute with inverse limits). The transition morphisms in the last equality are affine morphisms between quasi-compact and quasi-separated schemes; it remains to apply Lemma 2.39.

□

The situation is better with finite morphisms:

Theorem 2.41 *Let $\pi : X' \to X$ be a finite morphism. Let $\mathcal{F}$ be a sheaf on $X'_{\text{ét}}$. Let $\bar{x}$ be a geometric point of X associated to $x \in X$. Then $(\pi_* \mathcal{F})_{\bar{x}} = \prod \mathcal{F}_{\bar{x}'}^{d(x')}$, where the product is over all $x' \in \pi^{-1}(x)$ and $d(x')$ is the separable degree of $k(x')$ over $k(x)$ (which is a constant d if π is étale of constant degree d). In particular, for a closed immersion $i : Z \to X$ and a sheaf $\mathcal{F}$ on $Z_{\text{ét}}$, we have $(i_* \mathcal{F})_{\bar{x}} = 0$ if $\bar{x} \notin Z$ and $(i_* \mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}}$ if $\bar{x} \in Z$.*

Proof : With the notation of Theorem 2.40, we know that $\widetilde{X}'$ is finite over $\widetilde{X}$, which is the spectrum of the stricly local ring $\mathcal{O}_{X,\bar{x}}$. Therefore

$$\widetilde{X}' = \coprod_{x' \in \pi^{-1}(x)} \mathrm{Spec}\, \mathcal{O}_{X',\bar{x}'}^{d(x')}$$

by Theorem 1.20, which implies the result via Theorem 2.40.

□

Corollary 2.42 *If $\pi : X' \to X$ is a finite morphism (e.g. a closed immersion), then the functor π_* is exact.*

Proof : This follows from Theorem 2.41 and Theorem 2.31, c).

□

2.6. Extension by zero of a sheaf; functor $i^!$

Let $j : U \to X$ be an object of C_X for some site X_E. Then the functor $j^p : P(X) \to P(U)$ consists of restricting a presheaf on X to the open set (for the E-topology) U. We will now describe its left-adjoint, denoted $j_! : P(U) \to P(X)$. Its existence is ensured by the same general result ([12], Prop. II.2.2.) as the existence of the inverse image presheaf. Actually, one easily checks directly that that for $\mathcal{P} \in P(U)$ and $V \in C_X$, we have

$$(j_!\mathcal{P})(V) = \varinjlim \mathcal{P}(V'),$$

where the limit is over all commutative diagrams

$$\begin{array}{ccc} V' & \longleftarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

in C_X. Sorting these diagrams with respect to the corresponding homomorphism $\phi : V \to U$, we get

$$(j_!\mathcal{P})(V) = \bigoplus_{\phi \in \mathrm{Hom}_X(V,U)} \varinjlim_{S(\phi)} \mathcal{P}(V'),$$

where $S(\phi)$ is the set of diagrams as above such that the composite map $V \to V' \to U$ is ϕ. But $S(\phi)$ has a final object (the diagram with $V' = V$), so we obtain:

$$(j_!\mathcal{P})(V) = \bigoplus_{\phi \in \mathrm{Hom}_X(V,U)} \mathcal{P}(V_\phi),$$

where $V_\phi \in C_U$ is the object $V \xrightarrow{\phi} U$. For instance, if $j : U \to X$ is an open immersion, then $(j_!\mathcal{P})(V) = \mathcal{P}(V)$ if $V \to X$ factorizes through U, and it is zero otherwise.

Definition 2.43 The *extension by zero* functor $j_!$ associated to $j : U \to X$ is the composition of the functors

$$S(U) \xrightarrow{i} P(U) \xrightarrow{j_!} P(X) \xrightarrow{a} S(X).$$

It is left-adjoint to j^*.

Observe that $j_!$ is exact (it is right exact as left adjoint of j^*, and left exact as composite of left exact functors). In the case of an open immersion $j : U \to X$, the stalk $(j_!\mathcal{F})_{\bar{x}}$ is $\mathcal{F}_{\bar{x}}$ if $\bar{x} \in U$, and it is zero otherwise.

Let us now specialize to the case of a closed subscheme Z of X and set $U = X - Z$. Denote by $j : U \to X$ the corresponding open immersion and $i : Z \to X$ the corresponding closed immersion. Equip all schemes with the étale topology. For every sheaf $\mathcal{F}$ on $X_{\text{ét}}$, set

$$i^!\mathcal{F} := \ker[i^*\mathcal{F} \to i^*j_*j^*\mathcal{F}].$$

The idea is that $i_*i^!\mathcal{F}$ will appear as the largest subsheaf of $\mathcal{F}$ that is zero outside Z.

Lemma 2.44 *Let $\mathcal{F}$ be a sheaf on the étale site of a scheme X. There is an exact sequence of sheaves on $X_{\text{ét}}$:*

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0.$$

Proof : This can be checked on the geometric stalks. Let $\bar{x}$ be a geometric point of X. By Theorem 2.41, we have $(i_*i^*\mathcal{F})_{\bar{x}} = (i^*\mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}}$ if $\bar{x} \in Z$, and $(i_*i^*\mathcal{F})_{\bar{x}} = 0$ if $\bar{x} \in U = X - Z$. Similarly $(j_!j^*\mathcal{F})_{\bar{x}} = (j^*\mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}}$ if $\bar{x} \in U$, and $(j_!j^*\mathcal{F})_{\bar{x}} = 0$ if $\bar{x} \in Z = X - U$ (cf. Definition 2.43). Whence the result. □

Proposition 2.45 *a) The functor $i_* : S(Z) \to S(X)$ is fully faithful (i.e. it induces a bijection between homomorhism sets) and it is a left-adjoint of $i^! : S(X) \to S(Z)$.*

b) For every sheaf $\mathcal{F}$ on X, we have

$$(i^!\mathcal{F})(Z) = \ker[\mathcal{F}(X) \to \mathcal{F}(U)].$$

Proof : a) Let $\mathcal{F}$ be an étale sheaf on Z. The adjunction map $i^*i_*\mathcal{F} \to \mathcal{F}$ is an isomorphism because by Theorem 2.41, it induces an isomorphism on every geometric fiber. Therefore

$$\mathrm{Hom}_{S(X)}(i_*\mathcal{F}, i_*\mathcal{G}) = \mathrm{Hom}_{S(Z)}(i^*i_*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{S(Z)}(\mathcal{F}, \mathcal{G})$$

for every $\mathcal{F}, \mathcal{G} \in S(X)$, which means that i_* is fully faithful.

Now the square

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & j_*j^*\mathcal{G} \\ \downarrow & & \downarrow \\ i_*i^*\mathcal{G} & \longrightarrow & i_*i^*j_*j^*\mathcal{G} \end{array}$$

is cartesian (this is easily checked on the geometric stalks, dealing separately with the cases $\bar{x} \in U$ and $\bar{x} \in Z$). As $\mathrm{Hom}_{S(X)}(i_*\mathcal{F}, j_*j^*\mathcal{G}) = 0$ (this is checked again on the stalks), this implies:

$$\mathrm{Hom}_{S(X)}(i_*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{S(X)}(i_*\mathcal{F}, \ker[i_*i^*\mathcal{G} \to i_*i^*j_*j^*\mathcal{G}]).$$

The latter is also

$$\mathrm{Hom}_{S(X)}(i_*\mathcal{F}, i_*i^!\mathcal{G})$$

by definition of $i^!$ and left-exactness of i_*, and this last group identifies to $\mathrm{Hom}_{S(Z)}(\mathcal{F}, i^!\mathcal{G})$ because i_* is fully faithful. Finally

$$\mathrm{Hom}_{S(X)}(i_*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{S(Z)}(\mathcal{F}, i^!\mathcal{G}),$$

showing that i_* is the left-adjoint of $i^!$.

b) For any scheme S, denote by $\mathbf{Z}_S$ the constant sheaf $\mathbf{Z}$ on S. By a) and Example 2.13 a) , we have

$$(i^!\mathcal{F})(Z) = \mathrm{Hom}_{S(Z)}(\mathbf{Z}_Z, i^!\mathcal{F}) = \mathrm{Hom}_{S(X)}(i_*\mathbf{Z}_Z, \mathcal{F}).$$

We observe that $j^*\mathbf{Z}_X = \mathbf{Z}_U$ and $i^*\mathbf{Z}_X = \mathbf{Z}_Z$ by Example 2.37 a) (the group scheme $\mathbf{Z}$ being étale over X). Lemma 2.44 gives an exact sequence

$$0 \to j_!\mathbf{Z}_U \to \mathbf{Z}_X \to i_*\mathbf{Z}_Z \to 0,$$

and applying the left-exact contravariant functor $\mathrm{Hom}_{S(X)}(., \mathcal{F})$, yields an exact sequence

$$0 \to \mathrm{Hom}_{S(X)}(i_*\mathbf{Z}_Z, \mathcal{F}) \to \mathrm{Hom}_{S(X)}(\mathbf{Z}_X, \mathcal{F}) \to \mathrm{Hom}_{S(X)}(j_*\mathbf{Z}_U, \mathcal{F}).$$

To conclude, we observe that $\mathrm{Hom}_{S(X)}(\mathbf{Z}_X, \mathcal{F}) = \mathcal{F}(X)$ and

$$\mathrm{Hom}_{S(X)}(j_!\mathbf{Z}_U, \mathcal{F}) = \mathrm{Hom}_{S(U)}(\mathbf{Z}_U, \mathcal{F}_{|U}) = \mathcal{F}(U).$$

□

3. Cohomology : first properties

If not specified, the notation $S(X)$ denotes the category $S(X_{\text{ét}})$ of sheaves on the (small) étale site of X. We always implicitely assume that X_E is one of the sites of Example 2.3, and in particular that the inverse image functor π^* is exact. For a summary of properties of derived functors on abelian categories, see Appendix A of [8] or [30], chapters 2 and 5.

3.1. Some derived functors

Let X_E be a site. We want to define cohomology groups as (right) derived functors. Recall that an object A in an abelian category is *injective* if the contravariant functor $\mathrm{Hom}(., A)$ is exact (it is always left exact). An abelian category has *enough injectives* if every object can be embedded into an injective object.

Proposition 3.1 *The category $S(X_E)$ has enough injectives.*

Proof : We give the proof in the case of the étale site.[5] Let $u_x : \bar{x} \mapsto X$ be a geometric point of X. Then the category $S(\bar{x})$ is equivalent (Theorem 2.17) to the category **Ab** of abelian groups, hence it has enough injectives by [30], Exercise 2.3.2. Let $\mathcal{F} \in S(X)$. For each $x \in X$, we choose an embedding $u_x^*\mathcal{F} \hookrightarrow \mathcal{F}'_x$ into an injective sheaf. Set $\mathcal{F}_0 = \prod_{x\in X}(u_x)_*\mathcal{F}'_x$, it is injective in $S(X)$ (a product of injectives is injective; besides, direct image functors $(u_x)_*$ preserve injectives because its left adjoint $(u_x)^*$ is exact). Now $\mathcal{F}$ embeds into $\mathcal{F}_0$ via the embeddings $\mathcal{F} \hookrightarrow \prod_{x\in X}(u_x)_*u_x^*\mathcal{F}$ and $(u_x)_*(u_x^*\mathcal{F}) \hookrightarrow (u_x)_*\mathcal{F}'_x$. The injectivity of the first map can be checked on the stalks: indeed for a geometric point $\bar{x}$ with image $x \in X$, the canonical map $\mathcal{F}_{\bar{x}} \to ((u_x)_*u_x^*\mathcal{F})_{\bar{x}}$ is an isomorphism by Theorem 2.40 and equality $\mathcal{F}_{\bar{x}} = (u_x^*\mathcal{F})(\bar{x})$. The second map is injective because $(u_x)_*$ is left exact.

□

By definition, right derived functors $R^if(\mathcal{F})$ of a left-exact functor f on $S(X_E)$ are computed via an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to ...$$

of $\mathcal{F}$, then by taking the cohomology of the complex

$$f(I^0) \to f(I^1) \to ...$$

Now the functor $S(X_E) \to \mathbf{Ab}$ defined by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ is left exact, whence:

Definition 3.2 We denote by $H^r(X_E, .)$ ($r \in \mathbf{N}$) or $H^r(X, .)$ (if E is understood) the right derived functors of the functor $\Gamma(X, .) = H^0(X, .)$. For a sheaf $\mathcal{F}$ on X_E, the group $H^r(X_E, \mathcal{F})$ is called the *r-th cohomology group of X_E with values in $\mathcal{F}$.*

[5] Let $\mathcal{C}$ be an abelian category with products and coproducts, such that filtered colimts of exact sequences are exact, and with a family of generators. Then $\mathcal{C}$ has enough injectives. This yields the general case, using the sheaves $\mathbf{Z}_U := j_!\mathbf{Z}$ for various $j : U \to X$ in E as generators, cf. [12], Lemma III.1.3. and below.

In the case of the étale site, we shall denote $H^r(X_E, \mathcal{F})$ by $H^r_{\text{ét}}(X, \mathcal{F})$ or simply by $H^r(X, \mathcal{F})$. Notation $H^r_{\text{fppf}}(X, \mathcal{F})$ stands for cohomology on the flat site. As derived functors are *δ-functors* (see [30], Theorem 2.4.6. and § 2.5.1.), we have the following very important property:

Theorem 3.3 *For every short exact sequence of sheaves (on a site X_E)*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

there is a long exact sequence of the related cohomology groups

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to ...$$
$$... \to H^{r-1}(X, \mathcal{F}'') \to H^r(X, \mathcal{F}') \to H^r(X, \mathcal{F}) \to H^r(X, \mathcal{F}'') \to ...$$

Example 3.4 Other derived functors are interesting on sites like $X_{\text{ét}}$:

a) Let $\mathcal{F}_0$ be a sheaf on X_E. The covariant functor $\mathcal{F} \mapsto \text{Hom}_{S(X_E)}(\mathcal{F}_0, \mathcal{F})$ is left exact. Its right derived functors are denoted by $\mathcal{F} \mapsto \text{Ext}^r(\mathcal{F}_0, \mathcal{F})$. As in any abelian category, the groups $\text{Ext}^r(\mathcal{F}_0, \mathcal{F})$ can also be interpreted in terms of Yoneda extensions ([30], §3.4).

b) Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be sheaves on X_E. Then we define a sheaf (the sheaf condition is easy to check) $\underline{\text{Hom}}(\mathcal{F}_0, \mathcal{F}_1)$ on X_E by $U \mapsto \text{Hom}_{S(U)}((\mathcal{F}_0)_{|U}, (\mathcal{F}_1)_{|U})$. The right derived functors of $\underline{\text{Hom}}(\mathcal{F}_0, .)$ (from $S(X_E)$ to $S(X_E)$) are denoted $\underline{\text{Ext}}^r(\mathcal{F}_0, .)$.

c) Let $\pi : X'_{E'} \to X_E$ be a continuous map of sites. Then the functor $\pi_* : S(X'_{E'}) \to S(X_E)$ is left exact. Its derived functors are denoted $R^i\pi_*$. The sheaves $R^i\pi_*\mathcal{F}$ are the *higher direct images* of the sheaf $\mathcal{F}$.

d) Let $X = \text{Spec}\, k$, where k is a field with absolute Galois group $\Gamma = \text{Gal}\,(\bar{k}/k)$. Then the category of étale sheaves $S(X)$ is equivalent to the category of discrete Γ-modules C_Γ. Étale cohomology groups on X correspond to Galois cohomology groups $H^r(\Gamma, M)$ of a discrete Γ-module M, which are torsion groups if $r > 0$ ([8], Corollary 4.23). The derived functors Ext^r correspond to the Ext^r in the category of Γ-modules. Finally, $\underline{\text{Hom}}(\mathcal{F}_0, \mathcal{F}_1)$ translates into $\bigcup_H \text{Hom}_{C_H}(M_0, M_1)$, where H runs over all open subgroups of Γ. If M_0 is not assumed to be finitely generated, this is in general not the same as $\text{Hom}_{\mathbf{Ab}}(M_0, M_1)$, which might be non discrete when equipped with the Γ-action:

$$(\gamma.f)(x) = \gamma.f(\gamma^{-1}.x) \ , \gamma \in \Gamma, x \in M_0$$

(observe that $\text{Hom}_{C_H}(M_0, M_1) = \text{Hom}_{\mathbf{Ab}}(M_0, M_1)^H$) The associated derived functors are denoted $\underline{\text{Ext}}^r_\Gamma(M_0, M_1)$. See [8], §16.2.

e) Galois cohomology groups $H^i(k, M)$ can be described using cocycles ([8], Definition 4.17), which is a special case of Čech cohomology for the étale topology (see paragraph 3.2. below). For a constant sheaf (corresponding to a trivial action of Γ on the discrete G-module M), the first cohomology group $H^1(k, M)$ is just the group $\mathrm{Hom}_c(\Gamma, M)$ of continuous homomorphisms from the profinite group Γ to M. For instance $H^1(k, \mathbf{Z}) = 0$ (indeed $\mathbf{Z}$ has no non-trivial finite subgroup) and $H^2(k, \mathbf{Z}) \simeq H^1(k, \mathbf{Q}/\mathbf{Z})$ is isomorphic to $\mathrm{Hom}_c(\Gamma, \mathbf{Q}/\mathbf{Z})$ ([8], Example 1.52, which extends immediately to the cohomology of a profinite group; this comes from the fact that $H^i(k, \mathbf{Q}) = 0$ for all $i > 0$ because $\mathbf{Q}$ is uniquely divisible and Galois cohomology groups are torsion).

f) The inclusion functor $i : S(X_E) \to P(X_E)$ is left-exact. Its right derived functors are denoted $\underline{H}^r(X_E, .)$ (or simply $\underline{H}^r(.)$ if X_E is understood). Taking an injective resolution of a sheaf $\mathcal{F}$ in $S(X_E)$, one sees that $\underline{H}^r(X_E, \mathcal{F})$ is the presheaf $U \mapsto H^r(U, \mathcal{F})$ on X_E.

Remark 3.5 a) In some sense the functor $H^r(X_E, \mathcal{F})$ is contravariant on X_E. More precisely, if $\pi : X'_{E'} \to X_E$ is a continuous map, then the universal property of derived functors (as δ-functors) yields maps $H^r(X_E, \mathcal{F}) \to H^r(X'_{E'}, \pi^*\mathcal{F})$ induced by the obvious map $H^0(X_E, \mathcal{F}) \to H^0(X'_{E'}, \pi^*\mathcal{F})$. There is also a canonical map $H^r(X_E, \pi_*\mathcal{F}) \to H^r(X'_{E'}, \mathcal{F})$ (induced by the corresponding map for $r = 0$), which is an isomorphism if π_* is exact.

b) We can always identify (cf. Example 2.13, a) the functors $\Gamma(X, .)$ and $\mathrm{Hom}_{S(X_E)}(\mathbf{Z}, .)$, where $\mathbf{Z}$ is the constant sheaf on X_E; hence $H^r(X_E, .)$ and $\mathrm{Ext}^r_{S(X_E)}(\mathbf{Z}, .)$ also coincide.

c) Let $U \in C_X$. The functor $\mathcal{F} \mapsto \mathcal{F}_{|U}$ is exact from $S(X_E)$ to $S(U_E)$; besides, this functor takes injectives to injectives, because it actually has an exact left adjoint ("extension by zero") $j_!$ associated to the map $j : U \to X$ (see Definition 2.43). This implies that the notation $H^r(U, \mathcal{F})$ is not ambiguous: these groups can be obtained via derived functors of $\mathcal{F} \mapsto \Gamma(U, \mathcal{F})$ on $S(X_E)$, as well as the ones of $\mathcal{F}_U \mapsto \Gamma(U, \mathcal{F}_U)$ on $S(U_E)$.

It is a general fact ([30], Exercise 2.4.3) that to compute derived functors, the injective resolution can be replaced by any f*-acyclic* resolution, that is a resolution by objects J satisfying $R^i f(J) = 0$ for every $i > 0$. It is therefore important to determine a class of acyclic sheaves for the various functors we are interested in.

Definition 3.6 A sheaf $\mathcal{F}$ on X_E is sait to be *flabby* [6] if $H^r(U, \mathcal{F}) = 0$ for all $U \in C_X$ and all $r > 0$.

[6]Unfortunately, there is no uniform definition in the litterature for flabby or flasque.

In other words, a sheaf $\mathcal{F}$ is flabby if and only if $\underline{H}^r(\mathcal{F}) = 0$ for all $r > 0$. It immediately follows from Remark 3.5, c) that the restriction of a flabby sheaf $\mathcal{F} \in S(X_E)$ to any $U \to X$ in C_X still is a flabby sheaf.

Proposition 3.7 *a) Let $\pi : X'_{E'} \to X_E$ be a continuous map of sites. Let $\mathcal{F} \in S(X'_{E'})$. Then $R^i\pi_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^i(U', \mathcal{F}_{U'})$, where $U' := U \times_X X'$.*

b) With the notation as in a), a flabby sheaf on $X'_{E'}$ is acyclic for the functor π_*.

c) Assume that X_E is one of the sites of Example 2.3. Let $\mathcal{F}$ be an injective object of $S(X'_{E'})$. Then $\pi_*\mathcal{F}$ is injective in $S(X_E)$.

Proof : a) Let i be the inclusion $S(X') \to P(X')$. By definition $\pi_* = a\pi_p i$. Take an injective resolution $\mathcal{F} \to I^\bullet$ in $S(X')$. Now $R^i\pi_*\mathcal{F}$ is the i-th cohomology group of the complex $a\pi_p(iI^\bullet)$. Since a and π_p are exact (Proposition 2.25 and Theorem 2.31, a), they commute with cohomology and we get

$$R^i\pi_*\mathcal{F} = a\pi_p(\underline{H}^i(\mathcal{F})),$$

The conclusion follows because $\underline{H}^i(\mathcal{F})(U') = H^i(U', \mathcal{F})$ (Example 3.4, f).

b) This is an immediate consequence of a).

c) In these cases, the left adjoint π^* of π_* is exact (Proposition 2.25), whence the result.

□

3.2. Čech cohomology

For certain explicit computations (and also to give good criterions for a sheaf to be flabby), it is useful to define Čech cohomology on an arbitrary site X_E, which is modeled on the classical construction for topological spaces. We are now going to describe it in details.

Let $\mathcal{P}$ be a presheaf on X_E. Consider a covering $\mathcal{U} = (U_i \to X)_{i \in I}$ of X for the E-topology and set

$$U_{i_0 \dots i_r} := U_{i_0} \times_X \dots \times_X U_{i_r}$$

The definition that we give here is the same as the one of *flasque* in Artin [1] (p. 39), but it does agree neither with the classical definition of a flasque sheaf on a topological space, nor with the definition of flasque in [15] (the latter coincides with the notion of *totally acyclic* in [22]) for every $(r+1)$-uple $(i_0, ..., i_r)$, where each $i_j \in I$ (observe that unlike the classical case of a topological space, the i_j are not assumed to be pairwise distinct). There is an obvious restriction map

$$\mathrm{res}_j : \mathcal{P}(U_{i_0 \ldots \hat{i_j} \ldots i_r}) \to \mathcal{P}(U_{i_0 \ldots \ldots i_r}),$$

which is induced by the projection.

Definition 3.8 The *Čech complex* $C^\bullet(\mathcal{U}, \mathcal{P}) = (C^r(\mathcal{U}, \mathcal{P}))_{r \in \mathbf{N}}$ (associated to $\mathcal{U}$ and $\mathcal{P}$) is defined by

$$C^r(\mathcal{U}, \mathcal{P}) := \prod_{(i_0, \ldots, i_r) \in I^{r+1}} \mathcal{P}(U_{i_0 \ldots i_r})$$

with the differential maps

$$d^r : C^r(\mathcal{U}, \mathcal{P}) \to C^{r+1}(\mathcal{U}, \mathcal{P})$$

defined by

$$(d^r s)_{i_0 \ldots i_{r+1}} = \sum_{j=0}^{r+1} (-1)^j \mathrm{res}_j (s_{i_0 \ldots \hat{i_j} \ldots i_{r+1}}).$$

The verification that $d^{r+1} d^r = 0$ is straightforward (as in the classical case of a topological space).

Definition 3.9 The cohomology groups of the complex $(C^r(\mathcal{U}, \mathcal{P}))$ are denoted by $\check{H}^r(\mathcal{U}, \mathcal{P})$ and called the *Čech cohomology groups* of $\mathcal{P}$ with respect to $\mathcal{U}$.

For instance

$$\check{H}^0(\mathcal{U}, \mathcal{P}) = \ker[\prod_{i \in I} \mathcal{P}(U_i) \to \prod_{(i,j) \in I \times I} \mathcal{P}(U_{ij})],$$

whence a canonical map $\mathcal{P}(X) \to \check{H}^0(\mathcal{U}, \mathcal{P})$, which is injective is $\mathcal{P}$ is separated and an isomorphism if $\mathcal{P}$ is a sheaf.

Consider now a refinement $\mathcal{V} = ((V_j)_{j \in J}, \tau)$ of $\mathcal{U}$ (see Definition 2.14). Denote by $\phi_i : U_i \to X$ and $\psi_j : V_j \to X$ the corresponding maps. We can write $\psi_j = \phi_{\tau_j} \eta_j$ for some $\eta_j : V_j \to U_{\tau_j}$. This induces maps

$$\tau^r : C^r(\mathcal{U}, \mathcal{P}) \to C^r(\mathcal{V}, \mathcal{P})$$

defined by

$$(\tau^r s)_{j_0 \ldots j_r} = \mathrm{res}_{\eta_{j_0} \times \ldots \times \eta_{j_r}} (s_{\tau j_0 \ldots \tau j_r}),$$

which commute with the differentials. Whence a map

$$\rho : \check{H}^r(\mathcal{U}, \mathcal{P}) \to \check{H}^r(\mathcal{V}, \mathcal{P}).$$

Lemma 3.10 *The map ρ depends only on $\mathcal{P}$, $\mathcal{U}$, and $\mathcal{V}$ (not on τ or the η_j).*

Proof : Let $\tau', (\eta'_j)$ be another choice for $\tau, (\eta_j)$. Define a map $h^r : C^r(\mathcal{U},\mathcal{P}) \to C^{r-1}(\mathcal{V},\mathcal{P})$ by the formula

$$(h^r s)_{j_0\dots j_{r-1}} = \sum_{k=0}^{r-1}(-1)^k \mathrm{res}_{\eta_{j_0}\times\dots\times\dots\times(\eta_{j_k},\eta'_{j_k})\times\dots\times\eta'_{j_{r-1}}}(s_{\tau j_0\dots\tau j_k \tau' j_k\dots\tau' j_{r-1}}).$$

All h^r are homomorphisms and

$$d^{r-1}h^r + h^{r+1}d^r = (\tau')^r - \tau^r,$$

which proves that $(\tau')^r$ and τ^r induce the same map on $\check{H}^r(\mathcal{U},\mathcal{P})$, which is what we wanted to prove.

□

Lemma 3.10 imply that the groups $\check{H}^r(\mathcal{U},\mathcal{P})$ equipped with the maps $\rho : \check{H}^r(\mathcal{U},\mathcal{P}) \to \check{H}^r(\mathcal{V},\mathcal{P})$ form an inductive system[7]. Whence the following definition:

Definition 3.11 Set

$$\check{H}^r(X_E,\mathcal{P}) := \varinjlim_{\mathcal{U}} \check{H}^r(\mathcal{U},\mathcal{P}).$$

The groups $\check{H}^r(X_E,\mathcal{P})$ (or $\check{H}^r(X,\mathcal{P})$ if E is understood) are the *Čech cohomology groups* of $\mathcal{P}$ over X. We denote by $\underline{\check{H}}^r(X_E,\mathcal{P})$ (or simply $\underline{\check{H}}^r(\mathcal{P})$ if the site X_E is understood) the presheaf $U \mapsto \check{H}^r(U,\mathcal{P})$ on X_E (this is the Čech cohomology analog of the presheaf $\underline{H}^r(\mathcal{P})$ introduced in Example 3.4, f) in the context of derived functor cohomology.

Example 3.12 a) By definition, there is a canonical map of presheaves $\mathcal{P} \to \underline{\check{H}}^0(\mathcal{P})$, which is injective if and only if $\mathcal{P}$ is separated. In this case it turns out that $\underline{\check{H}}^0(\mathcal{P}) = a\mathcal{P}$ is the sheaf associated to $\mathcal{P}$ (see [12], proof of Theorem II.2.11 and [1], II.1.4. The difficulty is to check that $\underline{\check{H}}^0(\mathcal{P})$ is indeed a sheaf). In general the canonical map $\underline{\check{H}}^0(\mathcal{P}) \to a\mathcal{P}$ is only injective.

b) Assume that $X = \mathrm{Spec}\,A$ and $\mathcal{U}$ consists of one faithfully flat morphism $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$. Take $\mathcal{P} = \mathbf{G}_m$ and equip X with the flat topology. Then

[7] Strictly speaking, it is better to consider the coverings of X modulo the equivalence relation $\mathcal{U} \simeq \mathcal{V}$ if each is a refinement of the other. Then we get a filtered partially ordered set J_X because two coverings (U_i) and (V_j) have a common refinement $(U_i \times_X V_j)$. Then Lemma 3.10 shows that the functor $\mathcal{U} \mapsto \check{H}^r\mathcal{U},\mathcal{P})$ factors through J_X and we get an inductive system indexed by the filtered set J_X.

the groups $\check{H}^r(\mathcal{U}, \mathbf{G}_m)$ are the cohomology groups of the complex (where the differentials are defined as in the complex (1) of Lemma 1.6 for $M = A$)

$$0 \to B^* \to (B \otimes_A B)^* \to (B \otimes_A B \otimes_A B)^* \to ...$$

These groups were first considered by Amitsur.

Proposition 3.13 *Let*

$$0 \to \mathcal{P}' \to \mathcal{P} \to \mathcal{P}'' \to 0$$

be an exact sequence of presheaves. Then there is a long exact sequence

$$0 \to \check{H}^0(X, \mathcal{P}) \to ... \to \check{H}^r(X, \mathcal{P}) \to \check{H}^r(X, \mathcal{P}'') \to \check{H}^{r+1}(X, \mathcal{P}') \to ...$$

and similarly with X replaced by any covering $\mathcal{U}$ of X.

Proof : Let $\mathcal{U}$ be a covering of X for the E-topology. Then the sequence

$$0 \to C^r(\mathcal{U}, \mathcal{P}') \to C^r(\mathcal{U}, \mathcal{P}) \to C^r(\mathcal{U}, \mathcal{P}'') \to 0$$

is exact for all r as a product of exact sequences of abelian groups. Whence an exact sequence of complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{P}') \to C^\bullet(\mathcal{U}, \mathcal{P}) \to C^\bullet(\mathcal{U}, \mathcal{P}'') \to 0,$$

which (by [30], Th. 1.3.1) yields a long exact sequence

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{P}) \to ... \to H^r(\mathcal{U}, \mathcal{P}) \to \check{H}^r(\mathcal{U}, \mathcal{P}'') \to \check{H}^{r+1}(\mathcal{U}, \mathcal{P}') \to ...$$

Taking direct limit preserves exactness of a sequence of abelian groups, whence the required long exact sequence.

□

Remark 3.14 In general, an exact sequence of sheaves does not provide such an exact sequence because the corresponding sequence of presheaves does not always remain exact (but see the spectral sequences in Theorem 3.18 below that relate Čech cohomology to cohomology defined via derived functors). The point is that (as the next statement will show), Čech cohomology groups can be seen as derived functors from the category of presheaves (not of sheaves).

Proposition 3.15 *Let $U \in C_X$. Let $\mathcal{U}$ be a covering of U for the E-topology. Then the $\check{H}^r(\mathcal{U}, .)$ (resp. the $\check{H}^r(U, .)$) are the right derived functors of the functors $\check{H}^0(\mathcal{U}, .)$ (resp. $\check{H}^0(U, .)$) from $P(X_E)$ to $\mathcal{A}b$.*

Proof : The category $P(X_E)$ has enough injectives (this is a general fact, whose proof is similar to the same statement for abelian groups; see [21], Prop 6.1). Since we already know (Proposition 3.13 and its proof) that $\check{H}^*(\mathcal{U}, .)$ and $\check{H}^*(U, .)$ take short exact sequences to long exact sequences, it is sufficient to prove the following lemma (which immediately implies the similar statement for $\check{H}^r(U, \mathcal{P})$ by taking the inductive limit):

Lemma 3.16 *Let $\mathcal{P}$ be an injective object of $P(X_E)$. Then $\check{H}^r(\mathcal{U}, \mathcal{P}) = 0$ for all $r > 0$.*

Assume indeed that the lemma is proven. Then embed any presheaf $\mathcal{P}_0$ into an injective one $\mathcal{P}$, such that there is an exact sequence

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}' \to 0,$$

then $\check{H}^1(\mathcal{U}, \mathcal{P}_0) = \mathrm{coker}\,[\check{H}^0(\mathcal{U}, \mathcal{P}) \to \check{H}^0(\mathcal{U}, \mathcal{P}')]$, $\check{H}^i(\mathcal{U}, \mathcal{P}') \simeq \check{H}^{i+1}(\mathcal{U}, \mathcal{P}_0)$ for $i \geq 1$ (and similarly with $\check{H}^i$ replaced by the derived functors of $\check{H}^0$), whence the result by degree shifting.

Proof of the lemma: Set $\mathcal{U} = (U_i \to U)$. With the notation as above, the goal is to show that the complex

$$\prod \mathcal{P}(U_i) \to \prod \mathcal{P}(U_{i_0 i_1}) \to \prod \mathcal{P}(U_{i_0 i_1 i_2}) \to \dots$$

is exact. For every $j : W \to X$ in C_X, denote by $\mathbf{Z}_W$ the presheaf $j_!\mathbf{Z}$ on X associated to the constant presheaf $\mathbf{Z}$ on W as in paragraph 2.6. Then we have

$$\mathbf{Z}_W(V) = \bigoplus_{\mathrm{Hom}_X(V,W)} \mathbf{Z}; \quad \mathrm{Hom}(\mathbf{Z}_W, \mathcal{P}) = \mathrm{Hom}(\mathbf{Z}, j^*\mathcal{P}) = \mathcal{P}(W)$$

for every $V \to X$ in C_X. Therefore the above complex is also

$$\mathrm{Hom}(\bigoplus \mathbf{Z}_{U_i}, \mathcal{P}) \to \mathrm{Hom}(\bigoplus \mathbf{Z}_{U_{i_0 i_1}}, \mathcal{P}) \to \mathrm{Hom}(\bigoplus \mathbf{Z}_{U_{i_0 i_1 i_2}}, \mathcal{P}) \to \dots$$

and since $\mathcal{P}$ is injective, one reduces to show that the complex

$$\bigoplus \mathbf{Z}_{U_i} \leftarrow \bigoplus \mathbf{Z}_{U_{i_0 i_1}} \leftarrow \bigoplus \mathbf{Z}_{U_{i_0 i_1 i_2}} \leftarrow \dots \qquad (6)$$

is exact in $P(X)$.

Let $V \in C_X$. For every U-scheme W and $\phi \in \mathrm{Hom}_X(V, U)$, denote by $\mathrm{Hom}_\phi(V, W)$ the set of morphisms $V \to W$ that are made U-morphisms by ϕ. Then $\mathrm{Hom}_X(V, U_{i_0 i_1 \dots})$ is the disjoint union

$$\mathrm{Hom}_X(V, U_{i_0 i_1 \dots}) = \bigcup_{\phi \in \mathrm{Hom}_X(V,U)} \mathrm{Hom}_\phi(V, U_{i_0 i_1 \dots}) =$$

$$\bigcup_{\phi\in \mathrm{Hom}_X(V,U)} (\mathrm{Hom}_\phi(V,U_{i_0})\times(\mathrm{Hom}_\phi(V,U_{i_1})\times ...)$$

Now denote by $T(\phi)$ the disjoint union of the $\mathrm{Hom}_\phi(V,U_i)$ for $i\in I$. We get

$$\bigcup_{i_0,...,i_r} \mathrm{Hom}_X(V,U_{i_0 i_1 ... i_r}) = \bigcup_{\phi\in \mathrm{Hom}_X(V,U)} T(\phi)^r,$$

where $T(\phi)^r$ is the cartesian product of r copies of the set $T(\phi)$. This implies that $\bigoplus \mathbf{Z}_{U_{i_0 i_1 ... i_r}}(V)$ is actually the free abelian group on the set $\bigcup_{\phi\in \mathrm{Hom}_X(V,U)} T(\phi)^r$. Therefore the complex (6) evaluated at V can be rewritten

$$\bigoplus_{\phi\in \mathrm{Hom}_X(V,U)} [\bigoplus_{T(\phi)}\mathbf{Z} \leftarrow \bigoplus_{T(\phi)^2}\mathbf{Z} \leftarrow \bigoplus_{T(\phi)^3}\mathbf{Z} \leftarrow ...].$$

Fix $\phi\in \mathrm{Hom}_X(V,U)$ and define $S := T(\phi)$, we can assume $S\neq\emptyset$. Choose a distinguished element $a\in S$. Then the complex inside the brackets

$$\mathbf{Z}[S] \leftarrow \mathbf{Z}[S^2] \leftarrow \mathbf{Z}[S^3] \leftarrow ...,$$

where the differentials $\partial_r : \mathbf{Z}[S^{r+1}] \to \mathbf{Z}[S^r]$ are defined via

$$(s_0,...,s_r) \mapsto \sum_{j=0}^{r} (-1)^j (s_0,...,\hat{s}_j,...,s_r),\ (s_0,...,s_r)\in S^{r+1}$$

is known to be exact (see for instance [8], Lemma 1.26): indeed we have maps $u_r : \mathbf{Z}[S^{r+1}] \to \mathbf{Z}[S^{r+2}]$ defined by $u_r(s_0,...,s_r) = (a,s_0,...,s_r)$, which satisfy $u_{r-1}\circ d_r + d_{r+1}u_r = \mathrm{Id}$ for all $r\in\mathbf{N}^*$.

The lemma follows.

□

3.3. Čech cohomology compared to derived functor cohomology

We start with a general statement:

Proposition 3.17 *Čech cohomology of sheaves agrees with derived functor cohomology on X_E if and only if every short exact sequences of sheaves induces (functorially) a long exact sequence of Čech cohomology.*

Proof : The "only if" part is clear from the general properties of derived functors. For the "if" part, we first observe that $H^0(X,\mathcal{F}) = \check{H}^0(X,\mathcal{F})$ for every sheaf $\mathcal{F}$ on X_E. We now embed $\mathcal{F}$ into an injective sheaf I, which yields an exact sequence of sheaves $0 \to \mathcal{F} \to I \to Q \to 0$. Since I remains injective in $P(X_E)$ (because the left adjoint a of the inclusion functor $i : S(X_E) \to P(X_E)$ is exact), we have $\check{H}^r(X,I) = 0$ for all $r > 0$ by Lemma 3.16. The isomorphism $\check{H}^r(X,\mathcal{F}) \simeq H^r(X,\mathcal{F})$ follows by induction on r (dimension shifting).

□

We now relate Čech cohomology to derived functor cohomology via spectral sequences. Recall that when there is a spectral sequence E_2^{rs} converging to E^{r+s} (which is written $E_2^{rs} \Rightarrow E^{r+s}$), this implies that every term E^n has a filtration whose all successive quotients are isomorphic to a subquotient of E_2^{rs} for $r+s=n$. In particular if all of those E_2^{rs} are zero (resp. finite), then E^n is zero (resp. finite). Also, there is an *exact sequence of low degree terms* (cf. [12], Appendix B):

$$0 \to E_2^{1,0} \to E^1 \to E_2^{01} \to E_2^{20} \to [\ker E^2 \to E_2^{02}] \to E_2^{11} \to [\ker E_2^{30} \to E^3].$$

Theorem 3.18 *Let $\mathcal{F}$ be a sheaf on X_E. Let $\mathcal{U}$ be a covering of X. There are spectral sequences*

$$E_2^{rs} = \check{H}^r(\mathcal{U}, \underline{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X,\mathcal{F}).$$

$$\check{H}^r(X, \underline{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X,\mathcal{F}).$$

$$\underline{\check{H}}^r(X, \underline{H}^s(\mathcal{F})) \Rightarrow \underline{H}^{r+s}(X,\mathcal{F}).$$

Proof : The first two results are special cases of Grothendieck's spectral sequence of composed functors (cf. [30], §5.8), the first functor being $\mathcal{F} \mapsto \underline{H}^0(\mathcal{F})$ from $S(X_E)$ to $P(X_E)$, and the second one $\mathcal{P} \mapsto \check{H}^0(\mathcal{U},\mathcal{P})$ (resp. $\mathcal{P} \mapsto \check{H}^0(\mathcal{X},\mathcal{P})$) from $P(X_E)$ to $\mathcal{A}b$.

Indeed, since (as obviously $\underline{H}^0(\mathcal{F}) = \mathcal{F}$, and $\check{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}$ because $\mathcal{F}$ is a sheaf)

$$\check{H}^0(\mathcal{U}, \underline{H}^0(\mathcal{F})) = \check{H}^0(X, \underline{H}^0(\mathcal{F})) = H^0(X,\mathcal{F}),$$

the composition of the two functors is $\mathcal{F} \mapsto H^0(X,\mathcal{F})$, whose derived functors are the H^n.

It is therefore sufficient to check that for an injective object I of $S(X_E)$, its image by the first functor is acyclic for the second one, that is:

$$\check{H}^r(\mathcal{U}, \underline{H}^0(I)) = \check{H}^r(X, \underline{H}^0(I)) = 0$$

for all $r > 0$. But this follows from Lemma 3.16 because $\underline{H}^0$ is just the inclusion functor $i : S(X_E) \to P(X_E)$, which (as already seen in the proof of Proposition 3.17) preserves injectives.

Replacing X by an arbitrary open set $U \to X$ in the E-topology in the second spectral sequence provides the third one.

□

Corollary 3.19 *Let $\mathcal{F}$ be a sheaf on X_E. Then there are functorial isomorphisms*

$$\check{H}^0(X,\mathcal{F}) \simeq H^0(X,\mathcal{F});\ \check{H}^1(X,\mathcal{F}) \simeq H^1(X,\mathcal{F})$$

and an exact sequence

$$0 \to \check{H}^2(X,\mathcal{F}) \to H^2(X,\mathcal{F}) \to \check{H}^1(X,\underline{H}^1(\mathcal{F})) \to \check{H}^3(X,\mathcal{F}) \to H^3(X,\mathcal{F}).$$

Proof : Using the exact sequence of low degrees in the second spectral sequence of Theorem 3.18, it is sufficient to check that $\check{H}^0(X,\underline{H}^s(\mathcal{F})) = 0$ for every $s > 0$. Take an injective resolution $I^\bullet$ of $\mathcal{F}$ in $S(X_E)$. Then $a(\underline{H}^s(\mathcal{F})) = 0$ because since a (which is exact) commutes with cohomology, the sheaf $a(\underline{H}^s(\mathcal{F}))$ is obtained by taking cohomology of the complex $ai(I^\bullet) = I^\bullet$. Now, as already observed (cf. Example 3.12, a), the presheaf $\underline{\check{H}}^0(\underline{H}^s(\mathcal{F}))$ injects into $a(\underline{H}^s(\mathcal{F}))$, hence it is zero as well. In particular (taking global sections), we get $\check{H}^0(X,\underline{H}^s(\mathcal{F})) = 0$.

□

Remark 3.20 The vanishing of $\underline{\check{H}}^0(X,\underline{H}^s(\mathcal{F}))$ can be reformulated as: for every integer $s > 0$, every $U \in C_X$ and every $c \in H^s(U,\mathcal{F})$, there exists a covering $(U_i \to U)$ such that the restriction of c to $H^s(U_i,\mathcal{F})$ is zero for all i.

We now use Čech cohomology to give a useful characterization of flabby sheaves:

Proposition 3.21 *Let $\mathcal{F}$ be a sheaf on X_E. Then the following are equivalent:*

a) The sheaf $\mathcal{F}$ is flabby.

b) For every open set $U \to X$ (in the E-topology) and every covering $\mathcal{U}$ of U, we have $\check{H}^r(\mathcal{U},\mathcal{F}) = 0$ for all $r > 0$.

c) For every open set $U \to X$ (in the E-topology), we have $\check{H}^r(U,\mathcal{F}) = 0$ for all $r > 0$.

Proof : a) $\Rightarrow$ b): By definition of a flabby sheaf, we have $\underline{H}^s(\mathcal{F}) = 0$ for all $s > 0$. Hence the first spectral sequence of Theorem 3.18 yields

$$\check{H}^r(\mathcal{U}, \mathcal{F}) = H^r(U, \mathcal{F}) = 0$$

for all $r > 0$.

b) $\Rightarrow$ c) is obtained by taking the inductive limit.

c) $\Rightarrow$ a): The assumption is $\underline{\check{H}}^s(\mathcal{F}) = 0$ for all $s > 0$. Let us show by induction on $s > 0$ that $\underline{H}^s(\mathcal{F}) = 0$. The case $s = 1$ comes from Corollary 3.19. Assume that the required result holds for all positive integers $q < s$ and consider $\underline{\check{H}}^p(X, \underline{H}^q(\mathcal{F}))$ for $p + q = s$. By the induction hypothesis, it is zero for $0 < q < s$. Furthermore

$$\underline{\check{H}}^s(X, \underline{H}^0(\mathcal{F})) = \underline{\check{H}}^s(\mathcal{F}) = 0$$

by assumption, and $\underline{\check{H}}^0(X, \underline{H}^s(\mathcal{F})) = 0$ as seen in the proof of Corollary 3.19. Since the third spectral sequence of Theorem 3.18 shows that $\underline{H}^s(\mathcal{F})$ has a filtration whose each quotient is a subquotient of some $\underline{\check{H}}^p(X, \underline{H}^q(\mathcal{F}))$ for $p + q = s$, we are done.

□

Corollary 3.22 *If $\pi : X'_{E'} \to X_E$ is a continuous map of sites, the direct image $\pi_*\mathcal{F}'$ of every flabby sheaf $\mathcal{F}'$ on $X'_{E'}$ is flabby.*

Proof : Let $U \in C_X$. Let $\mathcal{U} = (U_i \to U)$ be a covering of U. Set $U'_i = U_i \times_X X'$ and $U' = U \times_X X'$, then $\mathcal{U}' := (U'_i \to U')$ is a covering of U' and by definition of π_* the Čech complexes $C^\bullet(\mathcal{U}', \mathcal{F}')$ and $C^\bullet(\mathcal{U}, \mathcal{F})$ are isomorphic. Now apply Proposition 3.21, b).

□

It is sometimes possible to extend corollary 3.19 to higher degrees:

Theorem 3.23 *Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules on a separated and noetherian scheme X. Then*

$$\check{H}^r(X_{\mathrm{Zar}}, \mathcal{F}) \simeq H^r(X_{\mathrm{Zar}}, \mathcal{F})$$

for every $r \in \mathbf{N}$.

Proof : This is classical; see [9], Th. III.4.5. or [7], Th. 9.6. The main step consists of proving that $H^r(U_{\text{Zar}}, \mathcal{F}) = 0$ for all $r > 0$ if U is an affine Zariski open subset of X. Then use Theorem 3.18.

□

The situation with étale topology is more complicated. We state the following theorem (due to Artin).

Theorem 3.24 *Let X be a quasi-projective scheme over a ring A. Let $\mathcal{F}$ be an étale sheaf on X. Then*

$$\check{H}^r(X_{\text{ét}}, \mathcal{F}) \simeq H^r(X_{\text{ét}}, \mathcal{F})$$

for every $r \in \mathbf{N}$.

Proof : By proposition 3.17, it is sufficient to show that an exact sequence of sheaves induces an exact sequence of the corresponding Čech complexes. Since the inclusion functor $i : S(X) \to P(X)$ is left-exact, we just have to show that a surjective map $\mathcal{F} \to \mathcal{F}'$ of sheaves induces a surjective map

$$\varinjlim_{\mathcal{U}}(\prod \mathcal{F}(U_{i_0 \dots i_r})) \to \varinjlim_{\mathcal{U}}(\prod \mathcal{F}'(U_{i_0 \dots i_r})),$$

where $\mathcal{U}$ runs over the set of all étale coverings (up to equivalence) of X and notation is as in paragraph 3.2.. Using Proposition 2.16 b), we can restrict to coverings $(U_i \to X)_{i \in I}$ with I finite and each U_i étale and of finite type over X. Replacing (U_i) by $\coprod_i U_i$, we can restrict to coverings consisting of one single morphism $f : U \to X$ with f étale and of finite type. Let $n \in \mathbf{N}^*$ and $s' \in \mathcal{F}'(U^n)$. As $\mathcal{F} \to \mathcal{F}'$ is surjective, there is an étale covering $W \to U^n$ such that $s'_{|W}$ lifs to some $s \in \mathcal{F}(W)$. Now by [12], Lemma 2.19, there is an étale covering $V \to U$ such that $V^n \to U^n$ factorizes through W, which implies that $s'_{|V^n}$ is the image of $s_{|V^n}$. Whence the result.

□

Remark 3.25 Not much is known about such a comparison result for flat cohomology, except in the case when $X = \operatorname{Spec} k$ is a field. Let k^{alg} be the algebraic closure of k. Then one can work with the Čech complex related to the sole covering $\operatorname{Spec}(k^{\text{alg}}) \to \operatorname{Spec} k$, whose cohomology gives flat cohomology for sheaves represented by k-group schemes G of finite type ([18], Th. 42 p. 208). Hence flat cohomology groups of G over $\operatorname{Spec} k$ coincide with étale (=Galois) cohomology groups if k is assumed to be perfect.

3.4. Cohomology and limits; stalks of the higher direct images

Recall that if $(\mathcal{F}_i)$ is an inductive system of sheaf, its direct limit is the sheaf associated to the presheaf $U \mapsto \varinjlim_i (\mathcal{F}_i(U))$ (it is indeed the direct limit in the category of sheaves). The next statement (whose detailed proof is rather complicated) is very useful to compute étale cohomology.

Theorem 3.26 *a) Assume that X is quasi-compact and quasi-separated. Then étale cohomology on X commutes with direct limit of sheaves.*

b) Let the notation and assumptions be as in Lemma 2.39. Then

$$\varinjlim H^r(X_i, \mathcal{F}_i) \stackrel{\simeq}{\to} H^r(X_\infty, \mathcal{F}_\infty).$$

Proof (sketch of): a) This uses the fact (cf. Proposition 2.16, b) that any étale covering of X has a refinement $(V_j \to X)_{j \in J}$ with J finite and all V_j (as well as the products $V_{j_1} \times_X ... \times_X V_{j_p}$ for $\{j_1, ..., j_p\} \subset J$) quasi-compact and quasi-separated. This easily implies the case $r = 0$. The general case is by induction on r, using the first spectral sequence of Theorem 3.18 and Remark 3.20. See [27], Lemma 51.4.

b) See [1], III.3. or [27], Theorem 51.3. The proof uses a) and the fact that the category of étale schemes of finite type over X_∞ is the direct limit of the categories of étale schemes of finite type over X_i. It is easier (via Lemma 2.39, which is the case $r = 0$) if one works with quasi-projective schemes over a ring, so that Čech cohomology can be used (thanks to Theorem 3.24).

□

Remark 3.27 The previous theorem actually also holds for the flat site. If $\mathcal{F}$ is represented by a group scheme G, the statement also holds if one takes for $\mathcal{F}_i$, $\mathcal{F}_\infty$ respectively the sheaves represented by $G \times_X X_i$, $G \times_X X_\infty$ (([15], VII.5).

Theorem 3.28 *Let $\pi : Y \to X$ be a quasi-compact and quasi-separated morphism. Let $\mathcal{F}$ be an étale sheaf on Y. Let $x \in X$ and $\bar{x}$ a geometric point with image x. Set $\widetilde{X} = \operatorname{Spec}(\mathcal{O}_{X,\bar{x}})$, $\widetilde{Y} = Y \times_X \widetilde{X}$ and define $\widetilde{\mathcal{F}}$ as the inverse image of $\mathcal{F}$ on $\widetilde{Y}$. Then*

$$(R^i \pi_*(\mathcal{F}))_{\bar{x}} \simeq H^i(\widetilde{Y}, \widetilde{\mathcal{F}}).$$

In particular, if $\mathcal{F}$ is defined by a group scheme G locally finitely presented over Y and we set $G_{\widetilde{Y}} = G \times_Y \widetilde{Y}$, we have an isomorphism $(R^i \pi_* G)_{\bar{x}} \simeq H^i(\widetilde{Y}, G_{\widetilde{Y}})$.

Proof : By Proposition 3.7 a), we have $R^i\pi_*\mathcal{F} = a\pi_p(\underline{H}^i(\mathcal{F}))$. Since a presheaf has the same stalks as its associated sheaf (Remark 2.30), we get

$$(R^i\pi_*\mathcal{F})_{\bar{x}} = \varinjlim H^i(U \times_X Y, \mathcal{F}_{|U\times_X Y}),$$

where the limit is over all étale neighborhoods U of $\bar{x}$ in X. Now the required isomorphism follows from Theorem 3.26 exactly as Theorem 2.40 follows from Lemma 2.39.

The case of a sheaf $\mathcal{F}$ represented by a group scheme G follows from the equality $\widetilde{\mathcal{F}} = G \times_Y \widetilde{Y}$, which comes from Example 2.37, a) (first case).

□

Remark 3.29 We will see later (proper base change theorem 5.7) that for a proper morphism π, the previous theorem can be considerably refined for a torsion sheaf $\mathcal{F}$ (that is: a sheaf such that $\mathcal{F}(U)$ is a torsion group for every $U \in C_X$ with U quasi-compact, see paragraph 5.1.). Indeed the stalk $(R^i\pi_*\mathcal{F})_{\bar{x}}$ is then isomorphic to $H^i(Y_{\bar{x}}, \mathcal{F}_{|Y_{\bar{x}}})$, where $Y_{\bar{x}} = Y \times_X \mathrm{Spec}\,(k(\bar{x}))$ is the geometric fiber of π at $\bar{x}$.

3.5. Some spectral sequences

Recall that we always assume that the sites are one of the sites of Example 2.3 (so that the inverse image functor is exact).

Theorem 3.30 *a) (Leray spectral sequence) Let $\pi : X'_{E'} \to X_E$ be continuous morphisms of sites. Let $\mathcal{F}$ be a sheaf on X'_E. Then there is a spectral sequence*

$$E_2^{rs} := H^r(X_E, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(X'_{E'}, \mathcal{F}).$$

b) Let $X''_{E''} \xrightarrow{\pi'} X'_{E"} \xrightarrow{\pi} X_E$ be a continuous morphism of sites. Then for every sheaf $\mathcal{F}$ on X''_E, there is a spectral sequence

$$(R^r\pi_*)(R^s\pi'_*)\mathcal{F} \Rightarrow R^{r+s}(\pi\pi')_*\mathcal{F}.$$

Proof : By Proposition 3.7 c), the direct image functors π, π' preserve injectives. Both results are therefore special cases of Grothendieck's spectral sequence of composed functors (cf. [30]).

□

Remark 3.31 There is also a spectral sequence associated to two sheaves $\mathcal{F}_1$, $\mathcal{F}_2$ on X_E, namely:

$$H^r(X_E, \underline{\mathrm{Ext}}^s(\mathcal{F}_1, \mathcal{F}_2)) \Rightarrow \mathrm{Ext}^{r+s}(\mathcal{F}_1, \mathcal{F}_2).$$

It is obtained the same way as the previous spectral sequences, once one knows that if $\mathcal{F}_2$ is injective, then $\underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2)$ is flabby, which is proven in [12], Cor III.2.13 (using Proposition 3.21). An example of this situation occurs in [8], Theorem 16.14.

Example 3.32 Let X be a quasi-compact scheme over a field k, set $\Gamma = \mathrm{Gal}\,(\bar{k}/k)$. We apply Theorem 3.30 to $\pi : X \to \mathrm{Spec}\, k$ on the étale sites. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$ with inverse image $\overline{\mathcal{F}}$ on $\overline{X} := X \times_k \bar{k}$. By Proposition 3.7, Theorem 2.17, and Theorem 3.26, we have that the sheaf $R^s\pi_*\mathcal{F}$ corresponds to the Γ-module

$$\varinjlim_K H^s(X_K, \mathcal{F}_{X_K}) \simeq H^s(\overline{X}, \overline{\mathcal{F}}),$$

where K runs over all finite extensions $K \subset \bar{k}$ of k and $\mathcal{F}_{X_K}$ is the inverse image of $\mathcal{F}$ on $X_K := X \times_k K$. We get the spectral sequence in étale cohomology

$$H^r(k, H^s(\overline{X}, \overline{\mathcal{F}})) \Rightarrow H^{r+s}(X, \mathcal{F}).$$

If $\mathcal{F}$ is represented by a commutative group scheme G over X, we can replace $\overline{\mathcal{F}}$, $\mathcal{F}$ respectively by $\overline{G} := G \times_k \bar{k}$ and G (cf. Remark 3.27). A consequence of the spectral sequence is that every $H^n(X, \mathcal{F})$ is filtered by groups such that each successive quotient of the filtration is a subquotient of some $H^r(k, H^s(\overline{X}, \overline{\mathcal{F}}))$ with $r + s = n$. This is quite useful to get finiteness or vanishing results. Another interesting consequence is the exact sequence of the first terms associated to the spectral sequence (cf. [12], Appendix B)

$$0 \to H^1(k, H^0(\overline{X}, \overline{\mathcal{F}})) \to H^1(X, \mathcal{F}) \to H^0(k, H^1(\overline{X}, \overline{\mathcal{F}})) \to H^2(k, H^0(\overline{X}, \overline{\mathcal{F}})) \to$$

$$\ker[H^2(X, \mathcal{F}) \to H^0(k, H^2(\overline{X}, \overline{\mathcal{F}}))] \to H^1(k, H^1(\overline{X}, \overline{\mathcal{F}})) \to H^3(k, H^0(\overline{X}, \overline{\mathcal{F}})).$$

Definition 3.33 Let X be a connected scheme, fix a geometric point $\bar{x}$ of X. A finite and étale morphism $\pi : Y \to X$ is a *finite Galois covering* if Y is connected and the right action of $\mathtt{Aut}_X(Y)$ on $F(Y) := \mathrm{Hom}_X(\bar{x}, Y)$ (which is free because Y is connected) is transitive (this does not depend on the choice of $\bar{x}$). The *Galois group* of such a covering is $\mathtt{Aut}_X(Y)$.

It is not difficult (but a bit tedious, see [12], Example III.2.6) to check that for a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ and a finite Galois covering Y of X, the Čech complex associated to the single covering $Y \to X$ is isomorphic to the standard complex of inhomogeneous cochains (cf. [8], Th. 1.27) of G with values in $\mathcal{F}(Y)$. This implies the following generalization of Example 3.32, called *Hochschild-Serre spectral sequence*:

Theorem 3.34 *Let $\pi : X' \to X$ be a finite Galois covering with group G. Let $\mathcal{F}$ a sheaf for the étale topology on X with restriction $\mathcal{F}_{X'}$ to X'. Then there is a spectral sequence in étale cohomology*

$$H^r(G, H^s(X', \mathcal{F}_{X'})) \Rightarrow H^{r+s}(X, \mathcal{F}).$$

Proof (sketch of): The left action of G on $\mathcal{F}(X')$ makes it a G-module. The section functor $\Gamma(X, .)$ is the composition of the functor $\mathcal{F} \mapsto \mathcal{F}(X')$ from $S(X)$ to C_G and the functor $M \mapsto M^G$ from C_G to **Ab** (the argument is roughly the same as in the proof of Theorem 2.17, cf. [12], Prop II.1.4). To apply Grothendieck's spectral sequence of composed functors, it remains to check that for an injective sheaf I on X, we have $H^r(G, I(X')) = 0$ for $r > 0$. Observing that I is a fortiori injective in the category of presheaves, this is a special case of Lemma 3.16 because $H^r(G, .)$ identifies with $\check{H}^r(X'/X, .)$.

□

Remark 3.35 Hochschild-Serre spectral sequence extends (when X is quasi-compact and quasi-separated) to an infinite Galois covering $X' \to X$ with group G (that is: X' is the projective limit of finite Galois covering $X_i \to X$ with group G_i and $G := \varprojlim G_i$). Indeed we can use Theorem 3.26 plus the fact that the cohomology of the group G is obtained as a direct limit of the cohomology of the G_i ([8], Proposition 4.18) coupled with the exactness of the direct limit functor.

3.6. Comparison of topologies

The first statement of this paragraph shows that in some sense, changing the category C_X does not change the cohomology if the class E of coverings remains the same.

Proposition 3.36 *Let C_X be a subcategory of a category of X-schemes C'_X, such that $(C_X)_E$ and $(C'_X)_E$ are sites associated to the same class of coverings E. Consider the continuous map of site $\pi : (C'_X)_E \to (C_X)_E$ induced by the identity map on X.*

a) The functor π_* is exact, and for every sheaf $\mathcal{F}$ on C_X there is an isomorphism $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$.

b) The functor $\pi^* : S((C_X)_E) \to S((C'_X)_E)$ is fully faithful.

c) The canonical maps

$$H^i(X, \pi_*\mathcal{F}') \to H^i(X, \mathcal{F}')$$

and

$$H^i(X, \mathcal{F}) \to H^i(X, \pi^*\mathcal{F})$$

are isomorphisms for every sheaves $\mathcal{F}'$ on $(C'_X)_E$ and $\mathcal{F}$ on $(C_X)_E$.

Proof : a) The exactness of π_* is clear. Let $U \in C_X$, then $\Gamma(U, \pi^p\mathcal{F}) = \Gamma(U, \mathcal{F})$ because the limit that defines $\Gamma(U, \pi^p\mathcal{F})$ can be taken over the single initial object (Id, U). Since $\mathcal{F}$ is a sheaf, this equality shows that $\pi^p\mathcal{F}$ and its associated sheaf $\pi^*\mathcal{F}$ have the same sections over $U \in C_X \subset C'_X$, hence:

$$\Gamma(U, \pi_*\pi^*\mathcal{F}) = \Gamma(U, \pi^*\mathcal{F}) = \Gamma(U, \pi^p\mathcal{F}) = \Gamma(U, \mathcal{F}).$$

b) comes from the second part of a), using the formula

$$\mathrm{Hom}(\pi^*\mathcal{F}, \pi^*\mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \pi_*\pi^*\mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$$

for every sheaves $\mathcal{F}$, $\mathcal{G}$ on X_E.

c) Since π_* is exact, the first map is an isomorphism. The composition of the second map with the isomorphisms $H^i(X, \pi^*\mathcal{F}) \simeq H^i(X, \pi_*\pi^*\mathcal{F}) \simeq H^i(X, \mathcal{F})$ is the identity, whence the result.

□

Example 3.37 a) For a sheaf $\mathcal{F}$ on the big étale site, corresponding to a sheaf $\pi_*\mathcal{F}$ on the small étale site, the cohomology groups $H^i_{\text{Ét}}(X, \mathcal{F})$ and $H^i_{\text{ét}}(X, \pi_*\mathcal{F})$ are the same. It does not always imply that the canonical map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is an isomorphism: take the sheaf represented by α_p on the big étale site of a regular scheme X; then $\pi_*\mathcal{F} = 0$ on the small étale site, but $\mathcal{F} \neq 0$ because on arbitrary (non reduced) X-scheme U, we don't necessarily have $\alpha_p(U) = 0$.

b) The statement does *not* imply that for a morphism $f : Y \to X$ of schemes and a group scheme G over X, the groups $H^i(Y_{\text{ét}}, G_Y)$ and $H^i(Y_{\text{ét}}, f^*G)$ coincide if we don't assume that f (or G) is étale, even though we know that $G_Y = f^*G$ on $Y_{\text{Ét}}$ by Example 2.37, a). Take for instance $G = \alpha_p$ on $X = \mathrm{Spec}\, k$ (where k is the spectrum of a field) and $Y = \mathrm{Spec}\,(k[T]/T^p)$. Then $G = 0$ on $X_{\text{ét}}$, hence we have $H^0(Y_{\text{ét}}, f^*G) = 0$; but $H^0(Y_{\text{ét}}, G_Y) = \alpha_p(Y) \neq 0$. Another example consists of taking k algebraically close, $G = \mathbf{G}_a$ and $Y = \mathbf{A}^1_k$. Then G is the constant sheaf k on $X_{\text{ét}}$, so f^*G is also the constant sheaf k, which implies $H^0(Y, f^*G) = k$. However

$$H^0(Y, G_Y) = H^0(Y, \mathbf{G}_a) = k[T].$$

There is also a statement when the class E of coverings is replaced by a smaller class, if one assumes that this smaller class contains a refinement of every covering in E.

Proposition 3.38 *Let $\pi : ((C_1)_X)_{E_1} \to ((C_2)_X)_{E_2}$ be a continuous morphism of sites induced by the identity on X, where $(C_2)_X$ is a subcategory of $(C_1)_X$ and $E_1 \supset E_2$. Assume that for every $U \in (C_2)_X$ and every covering of U for the E_1-topology, there is a covering of U for the E_2-topology that refines it. Then $\pi_* : S(X_{E_1}) \to S(X_{E_2})$ is exact and $H^i(X_{E_2}, \pi_*\mathcal{F}) \simeq H^i(X_{E_1}, \mathcal{F})$ for every sheaf $\mathcal{F}$ on X_{E_1}.*

Proof : The condition that π_* takes injectives to injectives is automatic (Proposition 3.7, c)). Since we obviously have $H^0(X_{E_2}, \pi_*\mathcal{F}) = H^0(X_{E_1}, \mathcal{F})$, it is sufficient to show that π_* is exact. We observe that π_* is simply the restriction of an E_1-sheaf to an E_2-sheaf, the E_2-topology being coarser. We already know that π_* is left exact. Let $\mathcal{F} \to \mathcal{F}'$ be a surjective map of sheaves on X_{E_1}, take $U \in (C_2)_X$ and $s \in \mathcal{F}'(U)$. By Remark 2.32, there exists a covering (U_i) of U for the E_1-topology such that every $s_{|U_i}$ is in the image of $\mathcal{F}(U_i)$. Let (V_j) be a covering of U for the E_2-topology, such that (V_j) is a refinement of (U_i). Then every $V_j \to U$ factors through some $U_i \to U$, which implies that $s_{|V_j}$ is in the image of $\mathcal{F}(V_j)$ for all j, hence $\mathcal{F} \to \mathcal{F}'$ remains surjective in $S(X_{E_2})$.

□

Example 3.39 Assume that X is locally noetherian. Thanks to Proposition 2.16 (which apply to every $U \in C_X$ in the cases below), we have the following examples of refinements as in Proposition 3.38:

a) Replace the class of all étale morphisms (in the definition of $X_{\text{ét}}$) by the class of étale morphisms of finite type, or even affine and of finite type étale morphisms morphisms if X is separated.

b) Similarly, we can take for C_X the class of finite type X-schemes and for E the class of flat and of finite type morphisms in the definition of the (big) flat site. If X is separated, it is also possible to compute cohomology on a "small"[8] flat site, taking for E the affine and flat morphisms of finite type and for C_X the affine flat X-schemes of finite type.

Theorem 3.40 *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and let $W(\mathcal{F})$ be the corresponding sheaf on X_{fppf}. Then $H^i(X_{\text{zar}}, \mathcal{F}) \simeq H^i(X_{\text{fppf}}, W(\mathcal{F}))$. A similar result holds if the flat site is replaced by the étale site.*

[8] Observe, however, that on this small flat site direct products do not exist in general: indeed if W is a flat X-scheme and U, V are W-schemes that are flat over X, then $U \times_W V$ might not be flat over X because it is not guaranteed that U or V is flat over W.

Proof : We give the proof for the flat site (it is similar for the étale site). Using Leray spectral sequence, one reduces to showing that $R^i\pi_*W(\mathcal{F}) = 0$ for all $i > 0$, where $\pi : X_{\mathrm{fppf}} \to X_{\mathrm{zar}}$ is the natural continuous map, or equivalently that $H^i(U_{\mathrm{fppf}}, W(\mathcal{F})) = 0$ for every open affine subset $U \subset X$. Using Example 3.39 b), one can use the small E-site on U, where E is the class of all affine flat morphisms of finite type. By Proposition 3.21 and the last statement of Proposition 2.16, it suffices to show that on this site, we have $\check{H}^i(\mathcal{U}, W(\mathcal{F})) = 0$ for every covering $\mathcal{U} = (U_i \to U)_{i \in I}$ of U with I finite. Replacing the family of U_i by $\coprod_i U_i$, we reduce to the case where $\mathcal{U}$ consists of one single morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ with B flat and finitely generated over A. Since $W(\mathcal{F}) = \widetilde{M}$ for some A-module M, the Čech complex is now the complex of Lemma 1.6, which we know is exact.

□

The following result is more difficult, see [12], Theorem III.3.9. Using again Leray spectral sequence, the method consists of proving that we have $H^i(X_{\mathrm{fppf}}, G) = 0$ for $i > 0$ when $X = \operatorname{Spec} A$ with A strictly local, which relies on computations in Čech cohomology.

Theorem 3.41 *Let G be a smooth, quasi-projective, and commutative group scheme over a scheme X. Then $H^i(X_{\text{ét}}, G) \simeq H^i(X_{\mathrm{fppf}}, G)$. Besides, if $X = \operatorname{Spec} A$ with A henselian, then $H^i(X, G) \simeq H^i(X_0, G_0)$ (for the flat or the étale topology), where X_0 is the closed point of X and $G_0 := G \times_X X_0$ is the closed fiber of G over X.*

Remark 3.42 A similar statement holds for étale cohomology compared to complex cohomology on a smooth $\mathbf{C}$-scheme X, provided one restricts to torsion coefficients ([12], Th. III.3.12). Theorem 3.41 is false without the smoothness assumption: take an imperfect field k of characteristic p and set $X = \operatorname{Spec} k$, $G = \mu_p$. Then by Kummer exact sequence and Hilbert's Theorem 90, we have $H^1_{\mathrm{fppf}}(k, \mu_p) = k^*/k^{*p}$, but $H^1_{\text{ét}}(k, \mu_p) = 0$ because the sheaf μ_p is zero on the étale site of X.

3.7. Cohomology with support

We consider the following situation: X is a scheme, U an open subscheme of X, and $Z = X - U$ is a closed subscheme of X. Denote by $i : Z \to X$ the corresponding closed immersion and $j : U \to X$ the open immersion.

Definition 3.43 The right derived functors of the left exact functor

$$\mathcal{F} \mapsto \ker[\mathcal{F}(X) \to \mathcal{F}(U)]$$

from $S(X_{\text{ét}})$ to **Ab** are denoted $H^i_Z(X, \mathcal{F})$ and are called the *cohomology groups with support on* Z.

Theorem 3.44 *Let* $\mathcal{F}$ *be a sheaf on* $X_{\text{ét}}$. *There is a long exact sequence*

$$0 \to H^0_Z(X, \mathcal{F}) \to \mathcal{F}(X) \to \mathcal{F}(U) \to H^1_Z(X, \mathcal{F}) ...$$

$$... \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \to H^{i+1}_Z(X, \mathcal{F}) \to ...$$

Proof : Apply Lemma 2.44 to the constant sheaf $\mathbf{Z}$ on $X_{\text{ét}}$, and define $\mathbf{Z}_U = j_! j^* \mathbf{Z}$, $\mathbf{Z}_Z = i_* i^* \mathbf{Z}$. Since $\mathbf{Z}$ is an étale group scheme over X, we have (by Example 2.37) that $i^*\mathbf{Z}$ is just the constant sheaf $\mathbf{Z}$ on Z. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. We use the long exact cohomology associated to the functor $\text{Hom}(., \mathcal{F})$ (which is contravariant and left-exact, with derived functors $\text{Ext}^i(., \mathcal{F})$). We get, for every sheaf $\mathcal{F}$ on $X_{\text{ét}}$, an exact sequence

$$... \to \text{Ext}^r(\mathbf{Z}, \mathcal{F}) \to \text{Ext}^r(\mathbf{Z}_U, \mathcal{F}) \to \text{Ext}^{r+1}(\mathbf{Z}_Z, \mathcal{F}) \to ...$$

As already explained (Remark 3.5, b), we have $\text{Ext}^r(\mathbf{Z}, \mathcal{F}) = H^r(X, \mathcal{F})$. Since

$$\text{Hom}_{S(X)}(\mathbf{Z}_U, \mathcal{F}) = \text{Hom}_{S(U)}(\mathbf{Z}, j^*\mathcal{F}) = \mathcal{F}(U),$$

the group $\text{Ext}^r(\mathbf{Z}_U, \mathcal{F})$ is obtained as a derived functor of $U \mapsto \Gamma(U, \mathcal{F})$, hence it is $H^r(U, \mathcal{F})$ (see Remark 3.5 c). Similarly, the groups $\text{Ext}^r(\mathbf{Z}_Z, \mathcal{F})$ are obtained as derived functors of

$$\text{Hom}_{S(X)}(\mathbf{Z}_Z, \mathcal{F}) = \text{Hom}_{S(Z)}(\mathbf{Z}, i^!\mathcal{F}) = H^0_Z(X, \mathcal{F})$$

(cf. Proposition 2.45), so they coincide with $H^r_Z(X, \mathcal{F})$.

□

Theorem 3.45 (Excision) *Let* $Z \subset X$ *and* $Z' \subset X'$ *be closed subschemes. Let* $\pi : X' \to X$ *be an étale morphism such that* $\pi(X' - Z') \subset X - Z$ *and* π *induces an isomorphism* $Z' \simeq Z$. *Then for every sheaf* $\mathcal{F}$ *on* $X_{\text{ét}}$, *the canonical map*

$$H^r_Z(X, \mathcal{F}) \to H^r_{Z'}(X', \pi^*\mathcal{F})$$

is an isomorphism for all $p \geq 0$.

Proof : Since π^* is exact and preserves injectives (its left adjoint $\pi_!$ being exact), it is sufficient to deal with the case $r = 0$. Set $U = X - Z$ and $U' = X' - Z'$, there is a commutative diagram with exact lines

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0_Z(X,\mathcal{F}) & \longrightarrow & H^0(X,\mathcal{F}) & \longrightarrow & H^0(U,\mathcal{F}) \\ & & \downarrow u & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0_{Z'}(X',\mathcal{F}_{|X'}) & \longrightarrow & H^0(X',\mathcal{F}) & \longrightarrow & H^0(U',\mathcal{F}) \end{array}$$

Let $\alpha \in H^0_Z(X,\mathcal{F})$ such that $u(\alpha) = 0$. Then the restriction of $\alpha \in H^0(X,\mathcal{F})$ to U and to X' are zero, which implies that $\alpha = 0$ because $(U \to X, X' \to X)$ is a covering of X. Now let $\beta \in H^0_{Z'}(X',\mathcal{F}_{|X'}) \subset H^0(X',\mathcal{F})$. The restriction of β and 0 to $X' \times_X U = \pi^{-1}(U) \subset U'$ agree, hence they glue to a section $\alpha \in H^0(X,\mathcal{F})$, which by construction is in $H^0_Z(X,\mathcal{F})$; thus $u(\alpha) = \beta$. Finally, u is bijective as required.

□

Corollary 3.46 *Let z be a closed point of X. Then for every sheaf $\mathcal{F}$ on $X_{\text{ét}}$, there is an isomorphism $H^r_z(X,\mathcal{F}) \to H^r_z(\operatorname{Spec}(\mathcal{O}^h_{X,z}),\mathcal{F})$.*

Proof : By theorem 3.45, we have $H^r_z(X,\mathcal{F}) \simeq H^r_y(Y,\mathcal{F})$ for every étale neighborhood (Y,y) of z that induces an isomorphism between the closed subschemes $\{y\}$ and $\{z\}$ (in particular y and z have same residue field). Taking the direct limit over such (Y,y) (cf. Theorem 3.26), we get the result.

□

4. Étale cohomology: more examples

Unless explicitely specified, all schemes are equipped with the étale topology. For every $\mathbf{Z}$-group scheme G and every scheme X (ex. $G = \mathbf{Z}, G = \mathbf{G}_m$), the piece of notation G_X (or simply G if X is understood) denotes the X-group scheme $G \times_{\operatorname{Spec}\mathbf{Z}} X$. When $X = \operatorname{Spec} A$ is affine, G_A stands for $G_{\operatorname{Spec} A}$.

4.1. Cohomology of Z

Lemma 4.1 *Let X be a scheme. Let K be a field equipped with a morphism $f : \operatorname{Spec} K \to X$ (i.e. a K-point of X is given). Then*

a) $H^1(X, f_*\mathbf{Z}_K) = 0$.
b) $H^1(X, f_*\mathbf{G}_{m,K}) = 0$.
c) $R^1 f_*\mathbf{Z}_K = 0$.
d) $R^1 f_*\mathbf{G}_{m,K} = 0$.

Proof : Leray spectral sequence (Theorem 3.30, a) yields for every étale sheaf $\mathcal{F}$ on $\operatorname{Spec} K$ (corresponding to a Galois module by Theorem 2.17) an injection $H^1(X, f_*\mathcal{F}) \to H^1(K, \mathcal{F})$. But $H^1(K, \mathcal{F}) = 0$ in both cases $\mathcal{F} = \mathbf{Z}$ (Example 3.4, e) and $\mathcal{F} = \mathbf{G}_m$ (this is Hilbert's Theorem 90 in Galois cohomology, cf. [8], Theorem 6.5).

By Proposition 3.7 a), the sheaf $R^1 f_* \mathbf{Z}_K$ is associated to the presheaf $U \mapsto H^1(U \times_X \operatorname{Spec} K, \mathbf{Z}_K)$. Since for every étale map $U \to X$, we know that the fiber $U \times_X \operatorname{Spec} K$ is empty or is the disjoint union of spectra of fields (see comment after Definition 1.8), the same argument as above shows that both $H^1(U \times_X \operatorname{Spec} K, \mathbf{Z}_K)$ and $H^1(U \times_X \operatorname{Spec} K, \mathbf{G}_{m,K})$ are zero.

□

Proposition 4.2 *Let X be an integral, noetherian and normal scheme. Then $H^1(X, \mathbf{Z}_X) = 0$.*

Proof : Using Proposition 3.38 and Example 3.39, we can make the calculation on the site X_E such that objects of C_X are étale X-schemes of finite type and morphisms in E are étale morphisms of finite type. Let $i : \eta \to X$ be the inclusion of the generic point of X. By Lemma 4.1, it is sufficient to show that the canonical map $\mathbf{Z}_X \to i_* \mathbf{Z}_\eta$ is an isomorphism. We can check this on sections over U, where U is a connected, étale and finite type X-scheme. Then U is normal by Proposition 1.15 and noetherian (it is of finite type over a noetherian scheme), hence it is integral by [23], Lemma 7.7. The generic fibre U_η of $U \to X$ is then integral, hence

$$\mathbf{Z}_X(U) = \mathbf{Z} = \mathbf{Z}_\eta(U_\eta) = (i_* \mathbf{Z}_\eta)(U).$$

□

Remark 4.3 The same argument shows that for an integral, noetherian and normal scheme and an arbitrary constant commutative group scheme M on X, we have $H^r(X, M) \simeq H^r(X, i_* M_\eta)$ for all $r \geq 0$. Using Leray spectral sequence, this provides an injective map $H^1(X, M) \to H^1(K, M)$, where K is the function field of X. Using the interpretation of $H^1(X, M)$ with torsors (see next paragraph), one sees that $H^1(X, M)$ corresponds to the group of continuous morphisms from the *étale fundamental group* (cf. [12], §I.5) of X (or its abelianized group) to M.

4.2. Torsors, descent, and twisted forms

In this paragraph only, we will most of the time work on the flat site, so we will use notation like $H^r(X, ...)$ for $H^r(X_{\text{fppf}}, ...)$. We denote by G be a locally finitely presented and flat group scheme over a scheme X.

We start with a classical statement in faithfully flat descent theory (see [2], §6.1 and 6.2):

Theorem 4.4 *Let $f : Y \to X$ be a faithfully flat and quasi-compact morphism. Denote by p_1, p_2 the projections $Y \times_X Y \to Y$ and by p_{ij} the projections $Y \times_X Y \times_X Y \to Y \times_X Y$ for $i > j$ (defined by $p_{ji}(y_1, y_2, y_3) = (y_j, y_i)$).*

Then, to give a quasi-coherent sheaf $\mathcal{F}$ over X is equivalent (via $\mathcal{F} \mapsto f^*\mathcal{F}$) as giving a quasi-coherent sheaf $\mathcal{F}'$ on Y equipped with a descent datum, that is: an isomorphism $\phi : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$ such that

$$p_{31}^*(\phi) = p_{32}^*(\phi) p_{21}^*(\phi). \tag{7}$$

A similar statement holds with schemes Z affine over X (where $Z' = Z \times_X Y$) instead of quasi-coherent sheaves.

Observe that if $\mathcal{F}'$ (resp. Z') comes from a quasi-coherent $\mathcal{O}_X$-module (resp. an affine X-scheme), we obviously can take for ϕ the identity map to obtain condition (7).

Definition 4.5 A (right) G-*torsor* (or *principal homogenous space* of G) over X is a faithfully flat, locally finitely presented scheme $Y \to X$, equipped with a (right) action of G such that the corresponding map

$$Y \times_X G \to Y \times_X Y, (y, g) \mapsto (y, y.g)$$

is an isomorphism. A torsor is *trivial* if it is isomorphic to G (acting on itself), which is the case as soon as $Y \to X$ has a section.

Example 4.6 A finite Galois covering of X with Galois group G (where G is an abstract finite group) is nothing but a (connected) X-torsor under the constant group scheme G.

Proposition 4.7 *Let $f : Y \to X$ be an X-scheme equipped with an action of the group scheme G. The following are equivalent:*

i) Y is a G-torsor over X.

ii) There a covering $(U_i \to X)$ for the flat topology such that every $Y_{U_i} := Y \times_X U_i$ is isomorphic as a G_{U_i}-scheme to G_{U_i} (equipped with the action by right-translation).

If G is smooth over X, flat topology can be replaced by étale topology in condition ii).

Condition ii) can be rephrased as : the torsor Y is *trivialized* by the covering $(U_i \to X)$.

Proof : i) $\Rightarrow$ ii): It suffices to take the covering given by the single morphism $Y \to X$ and the isomorphism given by the definition of a torsor.

ii) $\Rightarrow$ i): Set $U := \coprod U_i$. Then U is faithfully flat and locally of finite presentation over X and $Y_U \simeq G_U$. In particular Y_U is faithfully flat and locally of finite presentation over U, which implies (Proposition 1.7) the same for Y over X. By assumption $(Y \times_X G)_U \to (Y \times_X Y)_U$ is an isomorphism, hence so is $(Y \times_X G) \to (Y \times_X Y)$ by loc. cit.

Assume further that G is smooth over X. Then so is a G-torsor Y by loc. cit. Cover X by affine open subset X_i and set $Y_i = f^{-1}(X_i)$; then $f_i : Y_i \to X_i$ is a G-torsor. Since X_i is quasi-compact, the morphism f_i admits a *quasi-section*, that is: there is (by [12], Prop. I.3.26) an étale and surjective morphism $Y_i' \to X_i$ equipped with an X_i-morphism $Y_i' \to Y_i$. This implies that $(Y_i' \to X)$ is an étale covering of X, which satisfy condition ii) because each torsor f_i is trivialized by Y_i, hence by Y_i'.

□

It is actually useful to extend the definition of a torsor to (non necessarily representable) sheaves.

Definition 4.8 Let $\mathcal{F}$ be a sheaf of sets on X_{fppf} equipped with an action of G. It is said to be a *sheaf torsor* for G over X if it satisfies condition ii) of Proposition 4.7, where Y_{U_i} is replaced by the restriction of $\mathcal{F}$ to U_i, which must be isomorphic as a G_{U_i}-sheaf to the sheaf G_{U_i}.

Remark 4.9 In this definition, the group scheme G can be replaced by a sheaf of groups (not necessarily representable by a group scheme) on X, with the obvious modifications. Also, a sheaf torsor is trivial (isomorphic to G) if and only if it has a global section over X.

Lemma 4.10 *Let $\mathcal{F}$ be a sheaf torsor on a quasi-compact, quasi-separated scheme X. Then there exists a finitely presented and faithfully flat morphism $f : Y \to X$ such that the sheaves G_Y and $\mathcal{F}_{|Y}$ are isomorphic.*

Proof : This is a consequence of Proposition 2.16: indeed if $(f_i : U_i \to X)$ is a covering as in Definition 4.8, we can assume that I is finite and all f_i are flat and finitely presented. Then take $Y = \coprod_{i \in I} U_i$.

□

By Proposition 4.7, a scheme that is a G-torsor is a sheaf torsor, and two such schemes are isomorphic as G-torsors if and only if they are isomorphic as sheaf torsors (by Yoneda lemma). In general, a sheaf torsor is not always representable. However, we have:

Proposition 4.11 *Assume that the group scheme G is affine. Then a sheaf torsor $\mathcal{F}$ for G over X is representable by a G-torsor.*

Proof : Start with the case when X itself is affine. Let Y be as in Lemma 4.10 and denote by p_1, p_2 the two projections $Y \times_X Y \to Y$. We have $p_1^*(\mathcal{F}_{|Y}) = p_2^*(\mathcal{F}_{|Y})$ because both p_1 and p_2 are X-morphisms and $\mathcal{F}_{|Y}$ is the restriction of a sheaf defined over X. Since G_Y and $\mathcal{F}_{|Y}$ are isomorphic, this induces an isomorphism $p_1^*(G_Y) \to p_2^*(G_Y)$ which satisfies the conditions of Theorem 4.4, hence this defines an affine scheme $\mathcal{T}$ such that $\mathcal{T}_Y = G_Y$. Consider the sheaf $\underline{\mathrm{Hom}}(\mathcal{T}, \mathcal{F})$. By construction we have a section φ_Y of this sheaf over Y, which satisfies the glueing condition with respect to the covering $Y \to X$, hence comes from a unique morphism $\varphi : \mathcal{T} \to \mathcal{F}$ (and similarly for the converse isomorphism φ_Y^{-1}, hence φ is an isomorphism). Thus $\mathcal{F}$ is represented by $\mathcal{T}$.

In the general case, cover X by affine open subsets (X_i). For each i, let $\pi_i : \mathcal{T}_i \to X_i$ be an X_i-scheme that represents $\mathcal{F}_{|X_i}$. We can now glue the $\mathcal{T}_i$ along the $U_{ij} := \pi_i^{-1}(X_i \cap X_j)$ (the glueing conditions being ensured by the fact that $\mathcal{F}_{|(X_i \cap X_j)}$ is represented by both $\mathcal{T}_{ij} = \mathcal{T}_i \times_{X_i} (X_i \cap X_j)$ and $\mathcal{T}_{ji} = \mathcal{T}_j \times_{X_j} (X_i \cap X_j)$) to get an X-scheme that represents $\mathcal{F}$.

□

Our next goal is to relate (isomorphism classes) of sheaf torsors to étale cohomology. To include the case of a non commutative group scheme G, it is necessary to slightly extend the definition of the Čech H^1 as follows. Let $\mathcal{U} = (U_i \to X)_{i \in I}$ be a covering on a site X_E and let G be an X-group scheme, or more generally a sheaf of groups on X (with the composition law written multiplicatively). Set $U_{ij} := U_i \times_X U_j$ for $i, j \in I$ and similarly for U_{ijk}.

Definition 4.12 A 1-*cocycle* for $\mathcal{U}$ (with values in G) is a family $g = (g_{ij})_{i,j \in I}$ with $g_{ij} \in G(U_{ij})$, such that the restriction of g_{ik} and $g_{ij} g_{jk}$ to U_{ijk} coincide. Two cocycles g, g' are said to be *cohomologous* if there exists a family $(h_i)_{i \in I}$ with $h_i \in G(U_i)$ such that

$$g'_{ij} = ((h_i)_{|U_{ij}}) g_{ij} ((h_j)_{|U_{ij}})^{-1}, \quad \forall i, j \in I. \tag{8}$$

The quotient of the set of cocycles by the equivalence relation "being cohomologous" is denoted $\check{H}^1(\mathcal{U}, G)$ (recall that unless otherwise specified, the site E is the flat site in this paragraph). It coincides with the corresponding Čech cohomology group is G is commutative; in general it is only a pointed set, the distinguished element (denoted 0 or 1) being the class of the cocycle defined by $g_{ij} = 1$ for all i, j. Observe that thanks to the cocycle condition, a cocycle automatically restricts over each U_i to a trivial (:=cohomologous to 1) cocycle for the covering $(U_i \times_X U_j)_{j \in I}$.

Taking direct limit over all (classes of) coverings, we define

$$\check{H}^1(X, G) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, G).$$

There is still a (small) piece of long exact sequence for these non abelian cohomology sets:

Proposition 4.13 *Let*

$$1 \to G_1 \to G_2 \to G_3 \to 0$$

be an exact sequence of sheaves of groups. Then there is an exact sequence of pointed sets

$$1 \to G_1(X) \to G_2(X) \to G_3(X) \xrightarrow{\partial} \check{H}^1(X, G_1) \to \check{H}^1(X, G_2) \to \check{H}^1(X, G_3).$$

Here, exact sequence of sheaves of groups means that $1 \to G_1 \to G_2 \to G_3$ is exact as a sequence of presheaves, and $G_2 \to G_3$ is "locally surjective" in the sense of Remark 2.32.

Proof (sketch of): The only non obvious map is ∂, which we define as follows. Let $g \in G_3(X)$, take a covering $(U_i \to X)$ such that the restriction $g_{|U_i}$ lifts to some $g_i \in G_2(U_i)$. Then set

$$\partial(g)_{ij} = ((g_i)_{|U_{ij}})^{-1}(g_j)_{|U_{ij}}.$$

It is then straightforward to check that $\partial(g)$ is well-defined and the required exactness.

□

Theorem 4.14 *There is a bijection between the set $\mathcal{ST}(X, G)$ of isomorphism classes of sheaf torsors for G and $\check{H}^1(X, G)$. The image of the trivial torsor is* 0.

Proof : Start from a sheaf torsor $\mathcal{F}$ and take a covering $\mathcal{U} = (f_i : U_i \to X)$ that trivializes $\mathcal{F}$. In particular $\mathcal{F}(U_i) \neq \emptyset$; choose $s_i \in \mathcal{F}(U_i)$, there is a unique $g_{ij} \in G(U_{ij})$ such that $((s_i)_{|U_i}).g_{ij} = (s_j)_{|U_j}$. One immediately checks that the restriction to U_{ijk} of $s_i g_{ij} g_{jk}$ and $s_i g_{ik}$ coincide with $(s_k)_{|U_{ijk}}$, hence $g = (g_{ij})$ satisfies the cocycle condition. Replacing s_i by s'_i gives a cohomologous cocycle (write $s'_i = s_i.h_i$ with $h_i \in G(U_i)$, then the new cocycle $g'_{ij} = h_i^{-1} g_{ij} h_j$), and taking an isomorphic torsor clearly does not change the cohomology class of (g_{ij}) either. Whence a map $c : \mathcal{ST}(\mathcal{U}, G) \to \check{H}^1(\mathcal{U}, G)$ defined on the isomorphism classes of sheaf torsors trivialized by $\mathcal{U}$.

We now construct an inverse map. Consider the sheaves

$$\underline{C}^0(\mathcal{U}, G) = \prod_i (f_i)_*(G_{U_i});\ \underline{C}^1(\mathcal{U}, G) = \prod_{i,j} (f_{ij})_*(G_{U_{ij}}).$$

There is a morphism of sheaves $d : \underline{C}^0(\mathcal{U}, G) \to \underline{C}^1(\mathcal{U}, G)$, defined (for an arbitrary $V \to X$) by

$$(h_i) \mapsto (h_i^{-1})_{|U_{ij}} (h_j)_{|U_{ij}};\ \prod_i G(V \times_X U_i) \to \prod_{i,j} G(V \times_X U_{ij}).$$

Let g be a cocycle for G and $\mathcal{U}$, we associate a sheaf torsor $\mathcal{F} \subset \underline{C}^0(\mathcal{U}, G)$ to g as follows. For every $V \to X$, we take for $\mathcal{F}(V)$ the inverse image of $g_{|V}$ by d. The sheaf $\mathcal{F}$ is equipped wih the right action $((s_i, h)) \mapsto h^{-1} s_i$ of G. In the particular case of a trivial cocycle that can be written $g_{ij} = (g_i^{-1})_{|U_{ij}} (g_j)_{|U_{ij}}$ for some $g_i \in \prod G(U_i)$, we observe that G is isomorphic to $\mathcal{F}$ via

$$s \mapsto s^{-1}_{|V \times U_i}.(g_i)_{|V \times U_i};\ G(V) \to \prod_i G(V \times_X U_i).$$

. Indeed $(g_i)_{|V \times U_i}$ is in $\mathcal{F}(V)$ by definition and the kernel of the map $d(V) : \underline{C}^0(\mathcal{U}, G)(V) \to C^1(\mathcal{U}, G)(V)$ identifies to $G(V)$ because G is a sheaf, hence the inverse image of $g_{|V}$ by d is $G(V).(g_i)_{|V \times U_i}$. Since the cocycle g becomes trivial over each U_i and the definiion of $\mathcal{F}$ is compatible with the restriction maps, this proves that $\mathcal{F}_{|U_i}$ is isomorphic to G_{U_i} (compatibly with the action of G). Therefore $\mathcal{F}$ is a sheaf torsor for G.

It is now straightforward to check that the 1-cocycle associated to $\mathcal{F}$ by the map c is g, and conversely that if start from a sheaf torsor $\mathcal{F}_1$ and an associated cocycle g_1 (whose class is $c(\mathcal{F}_1)$), then the sheaf associated to the cocycle g_1 is $\mathcal{F}_1$. So c is a bijection. It remains to take the limit over all (classes of) coverings $\mathcal{U}$.

□

Remark 4.15 a) The bijection c is functorial in both X and G in the following sense. If $u : X' \to X$ is a morphism, then it induces a map $u^* : \check{H}^1(X, G) \to \check{H}^1(X', G_{X'})$ (defined by pulling-back coverings of X' and corresponding cocycles to X) such that $u^*(c(\mathcal{F})) = c(\mathcal{F}_{|X'})$ for every sheaf torsor $\mathcal{F}$. If $f : G \to G'$ is a morphism of X-group schemes, it induces an obvious map $f_* : \check{H}^1(X, G) \to \check{H}^1(X, G')$ by pushout of cocycles. The link to the bijection c is more complicated: starting from a sheaf torsor $\mathcal{F}$

for G, the sheaf torsor $\mathcal{F}'$ associated to $f_*(c(\mathcal{F}))$ is obtained by taking the *contracted product* $\mathcal{F} \times^G G'$; by definition this is the sheaf quotient of $\mathcal{F} \times G'$ by the action of G defined by $(s, g').g = (sg^{-1}, g'g)$, where G acts on G' via f. When G is commutative, the contracted product of two sheaf torsors also corresponds to the group law on the abelian group $\check{H}^1(X, G)$.

b) If the group scheme G is smooth over X, we can replace flat cohomology by étale cohomology in Theorem 4.14 thanks to Proposition 4.7. In particular Theorem 3.41 extends to the non-abelian H^1.

c) For an affine group scheme G, there is no need to distinguish between sheaf torsors and torsors (Proposition 4.11), hence we have a bijection between the set $\mathcal{T}(X, G)$ of isomorphism classes of torsors and $\check{H}^1(X, G)$.

Torsors and Čech cohomology sets are closely linked to the following notion:

Definition 4.16 Let Z be an "object" over X (example : a scheme, a sheaf of $\mathcal{O}_X$-modules or of $\mathcal{O}_X$-algebras...). A *twisted form* of Z for the flat (resp. étale, resp. Zariski) topology is an objet of the same type Z' with the property that there exists a covering $\mathcal{U} = (U_i \to X)$ for the flat (resp. étale, resp. Zariski) topology such that $Z \times_X U_i \simeq Z' \times_X U_i$ for all i.

Example 4.17 a) Let G be an X-group scheme. A G-torsor is a twisted form of G (as a right principal homogeneous space of G).

b) A locally free sheaf of rang n on X is a twisted form of $\mathcal{O}_X^n$ for Zariski topology; actually it is sufficient to demand that it is a twisted form of $\mathcal{O}_X^n$ for the étale or flat topology, by descent theory (see proof of lemma 4.19 below).

Keep the notation as above. Consider the presheaf $\underline{\mathtt{Aut}}(Z)$ defined by $U \mapsto \mathtt{Aut}_U(Z \times_X U)$, it is easily seen to be a sheaf when Z is a scheme as well as Z is a sheaf of $\mathcal{O}_X$-modules. Let Z' be a twisted form of Z trivialized by a covering $\mathcal{U} = (U_i \to X)$. We define an associated class $c(Z) \in \check{H}^1(\mathcal{U}, \underline{\mathtt{Aut}}(Z))$ as follows. Let $\varphi_i : Z_{U_i} \to Z'_{U_i}$ be an isomorphism, then $\alpha_{ij} = \varphi_i^{-1}\varphi_j$ is a cocycle, and we take for $c(Z)$ its class. It is easy to check that it is well-defined and that two twisted forms are X-isomorphic if and only if they have the same cohomology class.

Conversely, set $Y = \coprod_i U_i$ and as usual denote by p_1, p_2 the two projections $Y \times_X Y \to Y$. Start from a cocycle g with cohomology class

$c \in \check{H}^1(Y/X, \underline{\mathtt{Aut}}(Z))$, it can be viewed[9] as an isomorphism $p_1^* Z \to p_2^* Z$ that satisfies condition (7), that is as a descent datum on Z_Y. Summing up:

Proposition 4.18 *The map $Z \mapsto c(Z)$ induces an injection c from the set of isomorphism classes of twisted forms Z' of Z such that $Z'_Y \simeq Z_Y$ to the set $\check{H}^1(Y/X, \underline{\mathtt{Aut}}(Z))$. The map c is surjective iff every descent datum on Z_Y arises from a twisted form Z' defined over X.*

For example, surjectivity of c holds if the (flat, étale, or Zariski) covering $Y \to X$ is quasi-compact and Z is a quasi-coherent $\mathcal{O}_X$-module (resp. an affine X-scheme) thanks to Theorem 4.4.

4.3. Cohomology of $\mathbf{G}_m$, Hilbert's Theorem 90

The following lemma extends the classical Hilbert's Theorem 90 (cf. [8], Theorem 6.5) in Galois cohomology, which is the case $n = 1$ and A is a field. The group scheme GL_n is defined over $\mathbf{Z}$ by : for every $\mathbf{Z}$-algebra B, $\mathrm{GL}_n(B)$ is the group of invertible matrices $M_n(B)^*$; we also write GL_n for $\mathrm{GL}_n \times_{\mathbf{Z}} X$ if X is understood.

Lemma 4.19 *Let A be a local ring and $X = \operatorname{Spec} A$. Then $\check{H}^1(X, \mathrm{GL}_n) = 0$ (for the flat, étale, or Zariski topology).*

Proof : We give the proof for the flat site (the other cases are similar). Since X is affine, it is sufficient (using Proposition 2.16 and replacing a covering (U_i) by $\coprod U_i$) to prove that for every affine and faithfully flat morphism $Y = \operatorname{Spec} B \to X$, the set $\check{H}^1(Y/X, \mathrm{GL}_n)$ is trivial. We observe that $\mathrm{GL}_n = \underline{\mathtt{Aut}}(\mathcal{O}_X^n)$. Consider a descent datum on $\mathcal{O}_Y^n$. By Theorem 4.4, it comes from a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F} = \widetilde{M}$, where M is an A-module. As $\mathcal{F}_Y$ is isomorphic to $\mathcal{O}_Y^n$, the sheaf $\mathcal{F}$ is locally free of rank n, hence M is a flat and finitely presented A-module by [19] Lemma 10.83.2. Therefore M is a projective A-module by Proposition 1.4, d). This implies $M \simeq A^n$ by [19], Lemma 10.78.5, because A is local. Hence the class of $\mathcal{F}$ in $\check{H}^1(X, \mathrm{GL}_n)$ is trivial. We conclude with Proposition 4.18.

□

Definition 4.20 Let X be a scheme. The *Picard group* $\operatorname{Pic} X$ of X is the group $H^1_{\mathrm{zar}}(X, \mathcal{O}_X^*) = H^1_{\mathrm{zar}}(X, \mathbf{G}_m)$.

[9] To give an example of this correspondance, take $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Let Z be the sheaf $\mathcal{O}_X^n$ and N be the A-module B^n. Then a cocycle can be viewed either as an automorphism of $(B \otimes_A B)^n$ or as an isomorphism $N \otimes_A B \to B \otimes_A N$.

More concretely, recall that it is also the group of isomorphism classes of invertible sheaves of $\mathcal{O}_X$-modules (cf. [9], exercise III.4.5).

Theorem 4.21 *Let $\pi : X_{\mathrm{fppf}} \to X_{\mathrm{zar}}$ be the continuous map induced by the identity on X. Then $R^1\pi_*\mathbf{G}_m = 0$. The same holds if X_{fppf} is replaced by $X_{\text{ét}}$.*

Proof : We treat the case of the fppf site (the proof is similar for the étale site). By Proposition 3.7 a), we reduce to showing that for a Zariski open subset U of X, every class $\alpha \in H^1_{\mathrm{fppf}}(U, \mathbf{G}_m)$ is trivialized by restriction to all $H^1_{\mathrm{fppf}}(U_i, \mathbf{G}_m)$ for some Zariski covering $(U_i \to U)$. Using Theorem 3.26 and Remark 3.27, this is equivalent to requiring $H^1_{\mathrm{fppf}}(U, \mathbf{G}_m) = 0$ when $U = \operatorname{Spec} A$ with A local, which (thanks to Corollary 3.19, which says that the H^1 is the same for derived functor cohomology and Čech cohomology) is Lemma 4.19 for $n = 1$.

□

Remark 4.22 The proof of Lemma 4.19 shows in particular the exactness of

$$0 \to B^* \to (B \otimes_A B)^* \to (B \otimes_A B \otimes_A B)^*$$

when B is faithfully flat and finitely presented over A.

Corollary 4.23 *For every scheme X, there are isomorphisms*

$$\operatorname{Pic} X \simeq H^1_{\mathrm{zar}}(X, \mathbf{G}_m) \simeq H^1_{\text{ét}}(X, \mathbf{G}_m).$$

Proof : This follows from Leray spectral sequence (Theorem 3.30, a) and Theorem 4.21.

□

5. Some advanced theorems

In this section, we discuss a few fundamental results in étale cohomology. Unfortunately, detailed proofs are quite long (and would take a whole seminar by themselves), so we will only give an overview.

5.1. Torsion sheaves

Definition 5.1 Let X_E be a site. A sheaf of abelian groups $\mathcal{F}$ on X_E is a *torsion sheaf* if $\mathcal{F}$ is associated to a presheaf of abelian torsion groups.

Proposition 5.2 *a) A sheaf $\mathcal{F}$ is a torsion sheaf if and only if the canonical morphism $\varinjlim_{n\in\mathbf{N}^*} {}_n\mathcal{F} \to F$ is an isomorphism, where ${}_n\mathcal{F}$ is the n-torsion subsheaf of $\mathcal{F}$.*

b) If $\mathcal{F}$ is a torsion sheaf on one of the sites X_E of Example 2.3 and $U \in C_X$ is quasi-compact, then $\mathcal{F}(U)$ is torsion.

Proof : a) Assume that $\varinjlim_n {}_n\mathcal{F} = \mathcal{F}$. Then $\mathcal{F}$ is associated to the presheaf $U \mapsto \varinjlim({}_n\mathcal{F}(U))$, where $\varinjlim({}_n\mathcal{F}(U))$ is torsion (as a direct limit of abelian torsion groups). Hence $\mathcal{F}$ is a torsion sheaf.

Conversely assume that $\mathcal{F} = a\mathcal{P}$, where $\mathcal{P}$ is a presheaf of torsion abelian groups. The sequence

$$0 \to_n \mathcal{P} \to \mathcal{P} \stackrel{.n}{\to} \mathcal{P}$$

is exact in $P(X_E)$. Since the sheafification functor a is exact, this shows that ${}_n\mathcal{F} = a({}_n\mathcal{P})$. As a commutes with direct limits (as a left-adjoint functor), we obtain

$$\mathcal{F} = a\mathcal{P} = a(\varinjlim {}_n\mathcal{P}) = \varinjlim {}_n\mathcal{F}.$$

b) Let $\mathcal{P}$ be the presheaf $U' \mapsto \varinjlim {}_n\mathcal{F}(U')$ over U. It is separated as a subpresheaf of $\mathcal{F}$. Since U is quasi-compact and morphisms in E are open maps for all sites of Example 2.3, every covering $(U_i \to U)$ has a subcovering $(U_i \to U)_{i\in I}$ with I finite. As $\mathcal{F} = a\mathcal{P} = \underline{\check{H}}^0(\mathcal{P})$ (Example 3.12 a), we get

$$\mathcal{F}(U) = \check{H}^0(U, \mathcal{P}) = \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{P}),$$

where the limit can be taken over all coverings $\mathcal{U} = (U_i \to U)_{i\in I}$ with I finite. Since

$$\check{H}^0(\mathcal{U}, \mathcal{P}) = \ker[\prod_{i\in I} \mathcal{P}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{P}(U_i \times_U U_j)],$$

the groups $\check{H}^0(\mathcal{U}, \mathcal{P})$ are torsion (because each $\mathcal{P}(U_i)$ if torsion and I is finite). Hence, so is the direct limit $\mathcal{F}(U)$.

□

Proposition 5.3 *Let X be a scheme. A sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is torsion if and only if all its stalks $\mathcal{F}_{\bar{x}}$ are torsion groups.*

Proof : Let $\bar{x}$ be a geometric point of X. By Theorem 2.31, we have $({}_n\mathcal{F})_{\bar{x}} = {}_n(\mathcal{F}_{\bar{x}})$. The stalk functors clearly commute with direct limits of presheaves, hence also of sheaves (thanks to Remark 2.30). By loc. cit., the canonical map $\varinjlim({}_n\mathcal{F}) \to \mathcal{F}$ is an isomorphism if and only if $\varinjlim({}_n\mathcal{F})_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ is an isomorphism for all geometric points $\bar{x}$, which yields the result.

□

Example 5.4 a) The constant sheaf represented by a torsion group A is a torsion sheaf. Observe that $A(X)$ need not be torsion if X is not quasi-compact, e.g. $A = \mathbf{Q}/\mathbf{Z}$ and X is the disjoint union of infinitely many copies of $\operatorname{Spec} K$, where K is a field.

b) μ_n is a torsion sheaf.

c) Every torsion sheaf $\mathcal{F}$ can be written $\mathcal{F} = \bigoplus_\ell \mathcal{F}\{\ell\}$, where the sum is over all prime numbers ℓ and $\mathcal{F}\{\ell\} := \varinjlim_n({}_{\ell^n}\mathcal{F})$ is the *ℓ-primary torsion* of $\mathcal{F}$. One says that $\mathcal{F}$ is *of torsion prime to* p (where p is a prime number) if $\mathcal{F}\{p\} = 0$.

Proposition 5.5 *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a torsion sheaf on $X_{\text{ét}}$. Then the cohomology groups $H^r(X, \mathcal{F})$ are torsion for all $r \in \mathbf{N}$.*

Proof : Since X is quasi compact and quasi separated, we have (by Theorem 3.26, a) :

$$H^r(X, \mathcal{F}) = H^r(X, \varinjlim_n({}_n\mathcal{F})) = \varinjlim_n H^r(X,{}_n \mathcal{F}),$$

hence it is sufficient to prove that each $H^r(X,{}_n \mathcal{F})$ is torsion. Since multiplication by n is zero on ${}_n\mathcal{F}$, it is zero as well on $H^r(X,{}_n \mathcal{F})$, which shows that $H^r(X,{}_n \mathcal{F})$ is an n-torsion group.

□

Proposition 5.6 *a) Let $f : Y \to X$ be a morphism of schemes. If $\mathcal{F}$ is a torsion sheaf on $X_{\text{ét}}$, then the same holds for $f^*\mathcal{F}$.*

b) Assume further that f is quasi-compact and quasi-separated. Then $R^q f_*\mathcal{F}$ is a torsion sheaf for all $q \in \mathbf{N}$.

c) If $j : U \to X$ is an open immersion, then $j_!\mathcal{F}$ is a torsion sheaf for every torsion sheaf $\mathcal{F}$ on U.

d) If $i : Z \to X$ is a closed immersion, then $i^!\mathcal{F}$ is a torsion sheaf for every torsion sheaf $\mathcal{F}$ on X.

Proof : a) follows from Propositions 2.38 and 5.3. For b), we observe that by Theorem 2.40, the stalk of $R^q f_* \mathcal{F}$ at a geometric point $\bar{x}$ is $H^q(\widetilde{Y}, \widetilde{\mathcal{F}})$, where $\widetilde{Y}$ is quasi-compact and quasi separated (being quasi-compact and quasi separated over an affine scheme) and $\widetilde{\mathcal{F}}$ is a torsion sheaf by a). Therefore $H^q(\widetilde{Y}, \widetilde{\mathcal{F}})$ is torsion by Proposition 5.5, and we conclude with Proposition 5.3. c) and d) immediately follow from loc. cit. (recall that $i^! \mathcal{F}$ is a subsheaf of $i^* \mathcal{F}$).

□

5.2. Proper base change and smooth base change

In this paragraph, all schemes are equipped with the étale topology. For proofs of the main theorems of this paragraph, see [12], §VI.2 and V.4.

Theorem 5.7 (Proper base change) *Let $\pi : Y \to X$ be a proper morphism. Let $\mathcal{F}$ be a torsion sheaf on $Y_{\text{ét}}$.*

a) Let $f : X' \to X$ be a morphism, set $Y' := Y \times_X X'$. Let $f' : Y' \to Y$ and $\pi' : Y' \to X'$ be the corresponding projections.

$$\begin{array}{ccc} Y' & \xrightarrow{\pi'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\pi} & X \end{array}$$

Then

$$f^*(R^i \pi_* \mathcal{F}) \simeq R^i \pi'_*(f'^* \mathcal{F}).$$

b) Let $\bar{x} \to X$ be a geometric point, denote by $Y_{\bar{x}}$ the geometric fiber of Y over $\bar{x}$.

$$\begin{array}{ccc} Y_{\bar{x}} & \xrightarrow{\pi'} & \bar{x} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & X \end{array}$$

Then there is an isomorphism

$$(R^i \pi_* \mathcal{F})_{\bar{x}} \simeq H^i(Y_{\bar{x}}, \mathcal{F}_{|Y_{\bar{x}}}).$$

c) Let $k \subset K$ be separably closed fields. Assume that Y is a proper scheme over k and set $Y_K = Y \times_k K$. Then $H^i(Y, \mathcal{F}) \simeq H^i(Y_K, \mathcal{F}_{|Y_K})$, where we denote by $\mathcal{F}_{|Y_K}$ the pull-back of $\mathcal{F}$ to Y_K.

Observe that b) is a special case of a), taking for f the morphism $\bar{x} \to X$ and using Example 3.32. c) is obtained by showing that if $\pi : Y \to \operatorname{Spec} k$ is the structural morphism, then $R^i\pi_*\mathcal{F}$ is a so-called *locally constructible* sheaf, which implies that its sections over the two separably closed fields k and K are the same (compare with Theorem 3.41: the assumptions are of slightly different nature).

Corollary 5.8 *Let A be a henselian local ring. Let $S = \operatorname{Spec} A$, consider a proper morphism $\pi : X \to S$. Denote by s_0 the closed point of S and by $X_0 \to s_0$ the closed fiber of π. Let $\mathcal{F}$ be a torsion sheaf on $X_{\text{ét}}$ and $\mathcal{F}_0 := \mathcal{F}_{|X_0}$. Then $H^i(X, \mathcal{F}) \simeq H^i(X_0, \mathcal{F}_0)$.*

Proof : Let A^{sh} be the strict henselization of A, set $\overline{S} = \operatorname{Spec} A^{\text{sh}}$; denote by $\bar{s}_0$ the closed point of $\overline{S}$ and $\overline{X} = X \times_S \overline{S}$. By Theorem 5.7 b) and Theorem 3.28, we have isomorphisms

$$H^i(\overline{X}, \mathcal{F}_{|\overline{X}}) \simeq H^i(\overline{X}_0, (\mathcal{F}_0)_{|\overline{X}_0}).$$

Now define $G = \operatorname{Gal}(k(\bar{s}_0)/k(s_0)) = \operatorname{Gal}(A^{\text{sh}}/A)$. We have Hochschild-Serre spectral sequences (Remark 3.35)

$$H^i(G, H^j(\overline{X}, \mathcal{F}_{|\overline{X}})) \Rightarrow H^{i+j}(X, \mathcal{F}); \; H^i(G, H^j(\overline{X}_0, (\mathcal{F}_0)_{|\overline{X}}) \Rightarrow H^{i+j}(X, \mathcal{F}_0),$$

whence the result because the two left-hand terms are isomorphic.

□

There are also finiteness statements if $\mathcal{F}$ is further assumed to be *constructible* in the sense of [12], §V.1. We will restrict in these notes to the simpler case when $\mathcal{F}$ is represented by an étale group scheme.

Theorem 5.9 (Finiteness Theorem) *Let X be a proper scheme over a separably closed field k. Let G be an étale, finite type, commutative X-group scheme. Then the groups $H^i(X, G)$ are finite. In particular, with the notation and assumptions of Theorem 5.7, all $H^i(Y_{\bar{x}}, \mathcal{F}_{|Y_{\bar{x}}})$ are finite if we assume further that $\mathcal{F}$ is represented by an étale and finite type Y-group scheme.*

Again, the key is to show that the sheaf $R^i\pi_*\mathcal{F}$ is constructible, where $\pi : X \to \operatorname{Spec} k$ is the structural morphism.

Remark 5.10 Using Hochschild-Serre spectral sequence, we also see that the groups $H^i(X, G)$ are still finite if k is any field such that the Galois cohomology groups $H^i(k, M)$ are finite for every finite Galois module M, for instance if $k = \mathbf{R}$ or k is a p-adic field ([8], Corollary 8.15).

The conclusion of Theorem 5.7 a) still holds with the properness assumption on π strongly relaxed, provided a smoothness assumption is made on the base change f and a slight restriction is made on the torsion on the sheaf:

Theorem 5.11 (Smooth base change Theorem) *Let $\pi : Y \to X$ be a quasi-compact and quasi-separated morphism. Let $\mathcal{F}$ be a torsion sheaf on $Y_{\text{ét}}$ whose torsion is prime to all residue characteristics of X.*

a) Let $f : X' \to X$ be a smooth morphism, set $Y' := Y \times_X X'$. Let $f' : Y' \to Y$ and $\pi' : Y' \to X'$ be the corresponding projections. Then

$$f^*(R^i\pi_*\mathcal{F}) \simeq R^i\pi'_*(f'^*\mathcal{F}).$$

b) Assume that $\pi : Y \to X$ is a proper and smooth morphism, and that $\mathcal{F}$ is represented[10] *by an étale and finite type group scheme over Y. If X is connected, then the groups $(R^i\pi_*\mathcal{F})_{\bar{x}} \simeq H^i(Y_{\bar{x}}, \mathcal{F}_{|Y_{\bar{x}}})$ are isomorphic for all geometric points $\bar{x} \in X$.*

Corollary 5.12 *Let $k \subset K$ be algebraically closed fields. Let X be a quasi-compact and quasi-separated k-scheme. Let $\mathcal{F}$ be a torsion sheaf on X whose torsion is prime to* $\mathrm{Char}\, k$*. Then $H^i(X, \mathcal{F}) \simeq H^i(X_K, \mathcal{F}_{|X_K})$ for all $i \geq 0$.*

Proof : We have $K = \varinjlim A_j$, the limit being over all smooth k-algebras $A_j \subset K$: indeed every element a_j of $K - k$ is transcendental over k, hence $\mathrm{Spec}\,(k[a_j]) \simeq \mathbf{A}^1_k$, which is smooth over k. Now apply Theorem 5.11 a) to the morphisms $\mathrm{Spec}\, A_j \to \mathrm{Spec}\, k$. Taking the stalks at the geometric point $\bar{x}$ of $\mathrm{Spec}\, A_j$ corresponding to $\mathrm{Spec}\, K \to \mathrm{Spec}\, A_j$, we get isomorphisms $H^i(X, \mathcal{F}) \simeq (R^i\pi^j_*)(\mathcal{F}_{|X_{A_j}})_{\bar{x}}$, where π^j is the projection $X_{A_j} \to \mathrm{Spec}\, A_j$:

$$\begin{array}{ccc} X_{A_j} & \xrightarrow{\pi^j} & \mathrm{Spec}\,(A_j) \\ {\scriptstyle f^j}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\pi} & \mathrm{Spec}\, k \end{array}$$

Taking the limit (thanks to Theorem 3.26), we obtain an isomorphism

$$H^i(X, \mathcal{F}) \simeq (R^i\pi'_*\mathcal{F}_{|X_K})_{\bar{x}'} \simeq H^i(X_K, \mathcal{F}_{|X_K}),$$

where π' is the projection $X_K \to \mathrm{Spec}\, K$ and $\bar{x}'$ is the geometric point $\mathrm{Spec}\, K$.

□

Note that provided the slight additional assumption on the torsion on $\mathcal{F}$ is made, no properness assumption is necessary here. The statement still holds

[10] Again, it would be sufficient to assume $\mathcal{F}$ constructible and locally constant on $Y_{\text{ét}}$.

if k and K are only supposed to be separably closed, because for a universal homeomorphism $X_0 \to X$ (e.g. a closed immersion defined by a nilpotent sheaf of ideals, or a morphism $X_L \to X$ where the field extension L/k is radicial), the natural map $H^i(X, \mathcal{F}) \to H^i(X_0, \mathcal{F}_{|X_0})$ is an isomorphism. See [27], Proposition 45.4.

5.3. Purity and Gysin sequence

Let X be a scheme. We consider a closed subscheme $i : Z \hookrightarrow X$ and the open subscheme $U = X - Z$. Recall the functor $i^! : X_{\text{ét}} \to Z_{\text{ét}}$, which is left exact with left adjoint i_*. The latter is exact by Corollary 2.42, hence $i^!$ preserves injectives.

Proposition 5.13 *Denote by $\underline{H}^i_Z(X, .)$ the derived functors of $i^!$. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. There is a spectral sequence*

$$H^r(Z, \underline{H}^s_Z(X, \mathcal{F})) \Rightarrow H^{r+s}_Z(X, \mathcal{F}).$$

Proof : This is a special case of Grothendieck spectral sequence, since $i^!$ preserves injectives and $H^0_Z(X, \mathcal{F}) = H^0(Z, i^! \mathcal{F})$.

□

Definition 5.14 Let $n \in \mathbf{N}^*$ be invertible on X. Let $c \in \mathbf{Z}$. Define the following sheaves on X:

$$(\mathbf{Z}/n)(0) = (Z/n)_X; (\mathbf{Z}/n)(c) = (\mu_n^{\otimes c})_X, c > 0;$$
$$(\mathbf{Z}/n)(c) = \underline{\mathrm{Hom}}((\mathbf{Z}/n)(-c), \mathbf{Z}/n), c < 0.$$

For a sheaf $\mathcal{F}$ of $\mathbf{Z}/n$-modules, set $\mathcal{F}(c) := \mathcal{F} \otimes_{\mathbf{Z}/n} (\mathbf{Z}/n)(c)$.

The following recent theorem (called *absolute purity*) is due to Gabber (cf. [14]). The case of schemes of finite type over a perfect field was known before, see for example [12], Th. VI.5.1.

Theorem 5.15 (Gabber) *Assume that X and Z are regular and Z is of codimension c everywhere. Let $\mathcal{F}$ be a sheaf of n-torsion on $X_{\text{ét}}$, with n invertible on X. Let $m \in \mathbf{N}^*$. Then*

$$\underline{H}^m_Z(X, \mathcal{F})) = 0, m \neq 2c; \ \underline{H}^{2c}_Z(X, \mathcal{F}) \simeq (\mathcal{F})(-c)_Z.$$

Corollary 5.16 (Gysin sequence) *With the notation and assumptions as above, there is an exact sequence*

$$... \to H^{m-2c}(Z, \mathcal{F}(-c)) \to H^m(X, \mathcal{F}) \to H^m(U, \mathcal{F}) \to H^{m-2c+1}(Z, \mathcal{F}(-c)) \to ...$$

Proof : Apply exact sequence of Theorem 3.44; then identify $H^m_Z(X, \mathcal{F})$ with $H^{m-2c}(Z, \mathcal{F}(-c))$, using spectral sequence of Proposition 5.13 and Theorem 5.15.

□

An important consequence is the following extension of Theorem 5.9 (see [12], Corollary III.5.5 for a proof).

Theorem 5.17 *Let X be a smooth variety over a separably closed field k. Let G be an étale and commutative finite type X-group scheme whose torsion is prime to* $\operatorname{Char} k$. *Then the groups $H^i(X, G)$ are finite.*

Again, this holds more generally for a finite and locally constant sheaf whose torsion is prime to $\operatorname{Char} k$. Also, Theorem 5.17 is still valid for a singular variety if resolution of singularities is assumed.

6. The Brauer group

Unless explicitely specified, all schemes are equipped with the étale topology. A standout comprehensive reference for this topic is the book [4].

6.1. Definition and first examples

Definition 6.1 Let X be a scheme. The *Brauer group* of X is the étale cohomology group $H^2(X, \mathbf{G}_m)$.

Remark 6.2 a) What we call the Brauer group is sometimes called the *cohomological Brauer group*, to make the difference with a subgroup of $H^2(X, \mathbf{G}_m)$ defined in terms of Azumaya algebras. We denote the latter by $\operatorname{Br}_{\mathrm{Az}} X$. An *Azumaya algebra* is a twisted form of the sheaf $M_n(\mathcal{O}_X)$ for some n (for the étale topology), hence Azumaya algebras are classified (for a given n) by the cohomology set $\check{H}^1(X, \mathrm{PGL}_n)$. It is known in many cases (but not always true) that $\operatorname{Br}_{\mathrm{Az}} X = \operatorname{Br} X$, thanks to works by Gabber and Cesnavicius.

b) Since $\mathbf{G}_m$ is a smooth group scheme, the Brauer group $\operatorname{Br} X$ is also $H^2_{\mathrm{fppf}}(X, \mathbf{G}_m)$ by Theorem 3.40, b). It can also be computed as $H^2_{\mathrm{Ét}}(X, \mathbf{G}_m)$ on the big étale site thanks to Proposition 3.36.

c) Let $f : Y \to X$ be a morphism of schemes. Using the canonical map $\pi^*(\mathbf{G}_{m,X}) \to \mathbf{G}_{m,Y}$ (cf. Example 2.37 a), we get (via Remark 3.5, a) a morphism $\operatorname{Br} X \to \operatorname{Br} Y$ between Brauer groups. We can also use b) to define this map. In particular, if A is a ring, every A-point $x \in X(A) = \operatorname{Hom}(\operatorname{Spec} A, X)$ gives rise to a map $x^* : \operatorname{Br} X \to \operatorname{Br} A := \operatorname{Br}(\operatorname{Spec} A)$.

Example 6.3 a) Let $X = \operatorname{Spec} k$, where k is a field with absolute Galois group $\Gamma_k = \operatorname{Gal}(\bar{k}/k)$. Then $\operatorname{Br} X$ is the classical Brauer group $\operatorname{Br} k = H^2(\Gamma_k, \bar{k}^*)$ of the field k, thanks to Theorem 2.17 (cf. Example 3.4, d). As a Galois cohomology group of positive degree, it is a torsion group ([8], Corollary 4.23).

b) In particular, $\operatorname{Br} k$ is zero if k is separably closed, and more generally if k is a C_1 field like a finite field (see [8], Theorem 6.22). Other examples of C_1 fields include the function field of a curve over an algebraically closed field k_0 (Tsen's Theorem) and fraction fields k of excellent henselian discrete valuation rings with algebraically closed residue field (Lang's theorem). Without the "excellent" assumption, we still have $\operatorname{Br} k = 0$ (cf. [17], §X.7. : the proof easily extends from the complete case to the henselian case; it is also possible to use the approach of loc. cit., §XII.2). A refinement of Tsen's Theorem asserts that if k is the function field of a curve over a separably closed field whose characteristic is not ℓ, then the $\{\ell\}$-primary torsion of $\operatorname{Br} k$ is also zero.

c) Let A be a henselian local ring with residue field κ. Then the canonical map $\operatorname{Br} A \to \operatorname{Br} \kappa$ (associated to $\operatorname{Spec} \kappa \to \operatorname{Spec} A$) is an isomorphism by Theorem 3.40, b). This shows that $\operatorname{Br} A = 0$ if A is strictly local.

For any abelian group A and any positive integer n, we denote respectively by $A[n]$ and A/n the kernel and cokernel of multiplication by n on A. For a prime number ℓ, the piece of notation $A\{\ell\}$ stands for the ℓ-primary torsion subgroup of A.

Proposition 6.4 *Let X be a scheme. Let n be a positive integer, assume that n is invertible on X. Then there are exact sequences*

$$0 \to \operatorname{Pic} X/n \to H^2(X, \mu_n) \to (\operatorname{Br} X)[n] \to 0.$$

$$0 \to \operatorname{Br} X/n \to H^3(X, \mu_n) \to H^3(X, \mathbf{G}_m)[n] \to 0.$$

Proof : Apply the long exact sequence in étale cohomology to Kummer exact sequence of sheaves (Example 2.33, a)

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{.n} \mathbf{G}_m \to 0.$$

□

Observe that without assumption on n, the previous proposition still holds provided $H^2(X, \mu_n)$ and $H^3(X, \mu_n)$ are replaced by their fppf counterparts, thanks to Example 2.33 b) and Theorem 3.40 b).

6.2. Brauer groups and function fields

In this paragraph, we link the Brauer group of an integral scheme to the Brauer group of its function field. For a normal and integral scheme X, the piece of notation $X^{(1)}$ denotes the set of integral divisors (or, equivalently, of points of codimension 1) on X.

Proposition 6.5 *Let X be an integral, normal and noetherian scheme with function field F. Denote by $j : \eta = \operatorname{Spec} F \to X$ its generic point. For every integral divisor D of X, denote by $k(D)$ its function field and by i_D : $\operatorname{Spec}(k(D)) \to X$ the embedding of the generic point of D into X.*

a) There is an exact sequence of étale sheaves on X:

$$0 \to \mathbf{G}_{m,X} \to j_*\mathbf{G}_{m,F} \xrightarrow{u} D_X := \bigoplus_{D \in X^{(1)}} (i_D)_*\mathbf{Z}_{k(D)}.$$

b) Assume further that X is regular. Then the map u is surjective.

Proof : a) Let $U \to X$ be étale, connected and of finite type with generic fibre U_η. Using Example 3.39, it is sufficient to check that the corresponding sequence of sections over such a U is exact. The scheme U is integral by [23], Lemma 7.7., since it is noetherian and normal by Proposition 1.15; denote by $R(U)$ its function field. Then

$$\Gamma(U, j_*\mathbf{G}_{m,F}) = \Gamma(U_\eta, \mathbf{G}_m) = R(U)^*.$$

Define the morphism of sheaves u via the map

$$R(U)^* \to \bigoplus_{D \in X^{(1)}} ((i_D)_*\mathbf{Z}_{k(D)})(U) = \bigoplus_{E \in U^{(1)}} \mathbf{Z}$$

defined by the valuations associated to the integral divisors on U. Since $\mathbf{G}_{m,X}(U) = \mathcal{O}_U(U)^*$ is the group of invertible functions on U and the sequence

$$0 \to \mathcal{O}_U(U)^* \to R(U)^* \to \bigoplus_{E \in U^{(1)}} \mathbf{Z}$$

is exact by [10], Lemma 1.13 (the scheme U being integral and normal), we are done.

b) It is sufficient to check the surjectivity at the level of geometric stalks. Let $x \in X$, $A := \mathcal{O}^{\mathrm{sh}}_{X,x}$ and $K = \operatorname{Frac} A$. We have to check that the map $K^* \to \bigoplus_{\wp} \mathbf{Z}$, where the direct sum is over the prime ideals of height 1 and the map is given by the valuations, is surjective, or in other words that the ideal class group of A is trivial. Since A is local and regular (it is a direct limit of regular rings by Proposition 1.15), it is a UFD (cf. [11], pp. 139 and 142), whence the result.

□

Theorem 6.6 *Let X be a noetherian, integral and regular scheme with function field F. Then the canonical map $\operatorname{Br} X \to \operatorname{Br} F$ is injective. In particular $\operatorname{Br} X$ is a torsion group.*

Proof : Let $j : \operatorname{Spec} F \to X$ be the generic point of X. We have Leray spectral sequence

$$H^r(X, R^s j_* \mathbf{G}_m) \Rightarrow H^{r+s}(F, \mathbf{G}_m). \qquad (9)$$

We know that $R^1 j_* \mathbf{G}_m = 0$ by Lemma 4.1. The exact sequence of the first terms of the spectral sequence now yields an injection $H^2(X, j_* \mathbf{G}_m) \hookrightarrow \operatorname{Br} F$. On the other hand, we can apply cohomology to the exact sequence (cf. Proposition 6.5, b):

$$0 \to \mathbf{G}_{m,X} \to j_* \mathbf{G}_{m,F} \to \bigoplus_{D \in X^{(1)}} (i_D)_* \mathbf{Z}_{k(D)} \to 0,$$

which in turn gives an injective map $\operatorname{Br} X \to H^2(X, j_* \mathbf{G}_{m,F})$ because we have $H^1(X, \bigoplus_{D \in X^{(1)}} (i_D)_* \mathbf{Z}_{k(D)}) = 0$ by Lemma 4.1 and Theorem 3.26. Whence the result.

□

Without assumption, it is not always true that $\operatorname{Br} X$ is a torsion group (unlike the Azumaya Brauer group $\operatorname{Br}_{\mathrm{Az}} X$); see [4], §8.1.

6.3. Schemes of dimension 1

It is possible to say a lot more than Theorem 6.6 when X is of dimension 1 and some additional assumptions are made. For every profinite group G, denote by G^D the group of continuous homomorphism of G (or its abelianized group G^{ab}) to the discrete group $\mathbf{Q}/\mathbf{Z}$.

Proposition 6.7 *Let X be a noetherian, integral and regular scheme of dimension 1 with function field K. Assume that all residue fields $k(x)$ for $x \in X^{(1)}$ are perfect and denote by G_x the absolute Galois group of $k(x)$. Then there is an exact sequence*

$$0 \to \operatorname{Br} X \to \operatorname{Br} K \to \bigoplus_{x \in X^{(1)}} G_x^D \to H^3(X, \mathbf{G}_m) \to H^3(K, \mathbf{G}_m).$$

This statement will be partially extended to higher dimensional schemes in the next paragraph.

Proof : Let $j : \operatorname{Spec} K \to X$ be the generic point of X. Let $x \in X^{(1)}$, it is a closed point of X because X is of dimension 1. By Theorem 3.28, the stalk of $R^2 j_* \mathbf{G}_m$ at a geometric point $\bar{x}$ with image x is $H^2(K_x^{\mathrm{sh}}, \mathbf{G}_m) = \operatorname{Br} K_x^{\mathrm{sh}}$, where $K_x^{\mathrm{sh}} = \operatorname{Frac}(\mathcal{O}_{X,x}^{\mathrm{sh}})$. This group is known to be zero (Example 6.3, b) because the residue field of the henselian discrete valuation ring $\mathcal{O}_{X,x}^{\mathrm{sh}}$ is separably closed, hence algebraically closed (the residue field $k(x)$ of $\mathcal{O}_{X,x}^h$ being perfect). For a geometric point $\bar{\eta}$ with image the generic point of X, we still have $(R^2 j_* \mathbf{G}_m)_{\bar{\eta}} = 0$ (it is the Brauer group of a separably closed field). Finally $R^2 j_* \mathbf{G}_m = 0$. Using Leray spectral sequence (9), this yields

$$H^2(X, j_* \mathbf{G}_m) = \operatorname{Br} K; \;\; H^3(X, j_* \mathbf{G}_m) \hookrightarrow H^3(K, \mathbf{G}_m).$$

On the other hand, we also have Leray spectral sequence for the closed immersion $i_x : x \mapsto X$ with $x \in X^{(1)}$. Since $(i_x)_*$ is then exact, we have $R^q (i_x)_* = 0$ for all $q > 0$, which gives $H^r(X, (i_x)_* \mathbf{Z}) = H^r(k(x), \mathbf{Z})$ for all non negative integers r. Set $D_X = \bigoplus_{x \in X^{(1)}} (i_x)_* \mathbf{Z}$, we thus have (using Theorem 3.26) $H^r(X, D_X) = \bigoplus_{x \in X^{(1)}} H^r(k(x), \mathbf{Z})$. We observe that this group is zero for $r = 1$, and is $\bigoplus_{x \in X^{(1)}} G_x^D$ for $r = 2$ (cf. Example 3.4, e). Proposition 6.5 b) yields an exact sequence

$$0 \to \mathbf{G}_{m,X} \to j_* \mathbf{G}_{m,K} \to D_X \to 0. \tag{10}$$

Applying cohomology, we get an exact sequence

$$0 \to \operatorname{Br} X \to \operatorname{Br} K \to \bigoplus_{x \in X^{(1)}} G_x^D \to H^3(X, \mathbf{G}_m) \to H^3(X, j_* \mathbf{G}_m), \tag{11}$$

whence the results because $H^3(X, j_* \mathbf{G}_m) \hookrightarrow H^3(K, \mathbf{G}_m)$.

□

Example 6.8 a) Assume that A is a discrete valuation ring with perfect residue field κ and function field K. Then exact sequence (11) becomes

$$0 \to \operatorname{Br} A \to \operatorname{Br} K \to H^1(\kappa, \mathbf{Q}/\mathbf{Z}).$$

The map $\partial_A : \operatorname{Br} K \to H^1(\kappa, \mathbf{Q}/\mathbf{Z})$ is called the *residue map.* There are other definitions for this map (Serre residue, Witt residue; see also c) below), which coincide up to a sign (see [4], §1.4). If we assume further A henselian, then $\operatorname{Br} A \simeq \operatorname{Br} \kappa$, and this exact sequence has a section given by the composition of the inflation map $H^1(\kappa, \mathbf{Q}/\mathbf{Z}) \simeq H^2(\kappa, \mathbf{Z}) \to H^2(K, \mathbf{Z})$ with the map $H^2(K, \mathbf{Z}) \to H^2(K, \mathbf{G}_m)$ induced by $m \mapsto \pi^m, m \in \mathbf{Z}$, where π is a uniformizing parameter of A. In particular the residue map is surjective if A is henselian.

b) Assume further that K is of characteristic zero or X is an algebraic curve over a field (this works more generally if X is excellent, which is the case of any scheme of finite type over a field or over $\mathbf{Z}$). Then the fields K_x^{sh} are of cohomological dimension 1, because they are C_1 (cf. Example 6.3 b). This implies that $H^r(K_x^{\mathrm{sh}}, \mathbf{G}_m) = 0$ for every $r > 0$. In this case, we have $R^q j_* \mathbf{G}_m = 0$ for all $r > 0$, hence sequence (11) extends to a long exact sequence

$$... \to H^r(X, \mathbf{G}_m) \to H^r(K, \mathbf{G}_m) \to \bigoplus_{x \in X^{(1)}} H^{r-1}(k(x), \mathbf{Q}/\mathbf{Z}) \to H^{r+1}(X, \mathbf{G}_m) \to ...$$

Again, the special case of an excellent henselian discrete valuation ring as in a) yields short split exact sequences

$$0 \to H^r(A, \mathbf{G}_m) \to H^r(K, \mathbf{G}_m) \to H^{r-1}(\kappa, \mathbf{Q}/\mathbf{Z}) \to 0.$$

for all $r \geq 2$.

c) Let A and B be discrete valuation rings with respective residue fields κ_A, κ_B (assumed to be perfect) and fraction fields K, L. Assume that B is an extension of A with $A = K \cap B$ and denote by e its ramification index. Then there is a commutative diagram (where Res denote restriction maps in Galois cohomology)

$$\begin{array}{ccc} \mathrm{Br}\, L & \xrightarrow{\partial_B} & H^1(\kappa_B, \mathbf{Q}/\mathbf{Z}) \\ {\scriptstyle \mathrm{Res}}\uparrow & & \uparrow{\scriptstyle e.\mathrm{Res}} \\ \mathrm{Br}\, K & \xrightarrow{\partial_A} & H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}) \end{array}$$

To prove this, one can assume that A and B are henselian by going over to the henselizations. Let K^{nr} be the maximal unramified extension of K. Then $\mathrm{Br}\, K$ can be identified with $H^2(\mathrm{Gal}\,(K^{\mathrm{nr}}/K), K^{\mathrm{nr}^*})$ because $\mathrm{Br}\, K^{\mathrm{nr}} = 0$ (Example 6.3 b). Hence the valuation map $K^{\mathrm{nr}^*} \to \mathbf{Z}$ induces a map

$$\mathrm{Br}\, K \to H^2(\kappa_A, \mathbf{Z}) = H^1(\kappa_A, \mathbf{Q}/\mathbf{Z})$$

(observe that $\mathrm{Gal}\,(K^{\mathrm{nr}}/K)$ is the absolute Galois group of κ_A), which is one possible definition of ∂_A (and similarly for B). This easily provides the formula.

d) If X is a smooth curve over an algebraically closed field k, then its function field K is C_1 by Tsen's Theorem, so $H^r(K, \mathbf{G}_m) = 0$ for $r > 0$, and we also have $H^r(k(x), \mathbf{Q}/\mathbf{Z}) = 0$ for all closed points x of X, the field $k(x)$ being algebraically closed. Thus $H^q(X, \mathbf{G}_m) = 0$ for all $q \geq 2$.

6.4. Purity and residues

The next theorems identify more precisely the Brauer group of a regular integral scheme inside the Brauer group of its field of functions.

Theorem 6.9 *Let X be an integral, regular and notherian scheme. Let Z be a regular closed subscheme of X of pure codimension c, set $U = X - Z$. Let ℓ be a prime invertible on X.*

a) If $c \geq 2$, then the restriction map $(\mathrm{Br}\, X)\{\ell\} \to (\mathrm{Br}\, U)\{\ell\}$ is an isomorphism.

b) Assume $c = 1$; denote by $D_1, ..., D_m$ the connected components of Z and by $K_1, ..., K_m$ their respective function fields. There is an exact sequence

$$0 \to (\mathrm{Br}\, X)\{\ell\} \to (\mathrm{Br}\, U)\{\ell\} \to \bigoplus_{i=1}^{m} H^1(D_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$$

and likewise with $H^1(K_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ instead of $H^1(D_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$.

If all residual characteristics of X are zero, we can of course remove $\{\ell\}$ everywhere and replace $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ by $\mathbf{Q}/\mathbf{Z} = \bigoplus_\ell \mathbf{Q}_\ell/\mathbf{Z}_\ell$.

Proof : a) By Proposition 6.4, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}\, X/\ell^n & \longrightarrow & H^2(X, \mu_{\ell^n}) & \longrightarrow & (\mathrm{Br}\, X)[\ell^n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}\, U/\ell^n & \longrightarrow & H^2(U, \mu_{\ell^n}) & \longrightarrow & (\mathrm{Br}\, U)[\ell^n] & \longrightarrow & 0 \end{array}$$

Since X is regular, we can identify the Picard groups with the groups of Weil divisors, making the left hand vertical map surjective. The right hand vertical map is injective by Theorem 6.6. The snake lemma combined with Gysin exact sequence (Corollary 5.16, which comes from Gabber's purity theorem) now yields an exact sequence

$$0 \to (\mathrm{Br}\, X)[\ell^n] \to (\mathrm{Br}\, U)[\ell^n] \to H^{3-2c}(Z, \mu_{\ell^n}(-c)) \to H^3(X, \mu_{\ell^n}) \to H^3(U, \mu_{\ell^n})$$

for $c < 2$ and an isomorphism $(\mathrm{Br}\, X)[\ell^n] \simeq (\mathrm{Br}\, U)[\ell^n]$ for $c \geq 2$. For $c = 1$, the closed subscheme Z is the disjoint union of the D_i (which are its connected and irreducible components) and $H^{3-2c}(D_i, \mu_{\ell^n}(-c)) = H^1(D_i, \mathbf{Z}/\ell^n)$, whence the exact sequence

$$0 \to (\mathrm{Br}\, X)[\ell^n] \to (\mathrm{Br}\, U)[\ell^n] \to \bigoplus_{i=1}^{m} H^1(D_i, \mathbf{Z}/\ell^n) \to H^3(X, \mu_{\ell^n}) \to H^3(U, \mu_{\ell^n}).$$

Taking direct limit over n now yields the required result. We also observe that the restriction map $H^1(D_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell) \to H^1(K_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ is injective (Remark 4.3) because D_i is regular (hence normal), so it is possible to replace D_i by K_i in the exact sequence.

□

We now give a version of Theorem 6.9 where the assumptions on Z are relaxed:

Theorem 6.10 *Let X be an integral, regular, noetherian, excellent scheme. Let U be a non empty open subset of X, set $Z = X - U$ (with its reduced structure) and denote by c the codimension of Z. Let ℓ be a prime invertible on X.*

a) If $c \geq 2$, then the restriction map $(\mathrm{Br}\, X)\{\ell\} \to (\mathrm{Br}\, U)\{\ell\}$ is an isomorphism.

b) Assume $c = 1$; denote by $D_1, ..., D_m$ the irreducible components of Z of codimension 1 in X and by $K_1, ..., K_m$ their respective function fields. Then there is an exact sequence

$$0 \to (\mathrm{Br}\, X)\{\ell\} \to (\mathrm{Br}\, U)\{\ell\} \to \bigoplus_{i=1}^{m} H^1(K_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell).$$

Proof : As X is excellent, the singular locus of every closed subscheme Z of X is closed (and different from Z if Z is non empty and reduced, since it does not contain the generic point of an irreducible component of Z). Define a sequence of closed subsets (with their reduced structure)

$$Z = Z_0 \supset Z_1 \supset ... \supset ... \supset Z_m \supset Z_{m+1} = \emptyset$$

where Z_n is the union of the singular locus of Z_{n-1} and of its irreducible components of codimension $\geq n+1$ (the sequence stops because X is noetherian). Then Z is the disjoint union of the $W_n := Z_{n-1} - Z_n$, $1 \leq n \leq m+1$; each W_n is empty or is a regular closed subset of pure codimension n in $X - Z_n$, and the last one is $W_{m+1} = Z_m$.

The case where Z is regular of pure codimension 1 in X is settled by Theorem 6.9. Otherwise, repeated applications of Theorem 6.9 give isomorphisms

$$\mathrm{Br}\, X\{\ell\} \simeq \mathrm{Br}\,(X - Z_m)\{\ell\} \simeq \mathrm{Br}\,(X - Z_{m-1})\{\ell\} \simeq ... \simeq \mathrm{Br}\,(X - Z_1)\{\ell\}.$$

If $Z = Z_1$, the proof is over. If $Z \neq Z_1$, it remains to apply Theorem 6.9 to the codimension 1 regular closed subscheme $W_1 = Z - Z_1$ of $X - Z_1$.

□

Observe that without regularity assumption on Z, it is no longer possible to replace $H^1(K_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ by $H^1(D_i, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ in Theorem 6.10.

Corollary 6.11 *Let X be an integral, regular, noetherian, excellent scheme with function field K. Let ℓ be a prime invertible on X. Then there is an exact sequence*

$$0 \to (\mathrm{Br}\, X)\{\ell\} \to (\mathrm{Br}\, K)\{\ell\} \to \bigoplus_{D \in X^{(1)}} H^1(K_D, \mathbf{Q}_\ell/\mathbf{Z}_\ell),$$

where $X^{(1)}$ is the set of integral divisors (or points of codimension 1) of X and K_D is the function field (=residue field of the corresponding point of codimension 1) of D.

Proof : Take the direct limit over all non empty open subsets U of X in Theorem 6.10.

□

Remark 6.12 Let x be a point of codimension 1 of X with residue field $k(x)$. The residue map $(\mathrm{Br}\, K)\{\ell\} \to H^1(k(x), \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ appearing in the previous theorem coincides (up to a sign) with the map defined [11] in Example 6.8, a) with $A = \mathcal{O}_{X,x}$ (which can also be defined by going to the henselization A^h of A, cf Example 6.8 c). See [4], Theorem 3.7.3.

Corollary 6.13 *Let X be a regular, noetherian, integral, excellent scheme with function field K. Let ℓ be a prime number invertible on X. Then:*

a) The subgroup $(\mathrm{Br}\, X)\{\ell\}$ of $\mathrm{Br}\, K$ is given by

$$(\mathrm{Br}\, X)\{\ell\} = \bigcap_{x \in X^{(1)}} \mathrm{Br}\,(\mathcal{O}_{X,x})\{\ell\}.$$

b) Let $A_i \subset K$ for $i \in I$ be the discrete valuation rings with quotient field K which lie over X (that is: such that the map $\mathrm{Spec}\, K \to X$ factors through $\mathrm{Spec}\, A$). Then

$$(\mathrm{Br}\, X)\{\ell\} = \bigcap_{i \in I} (\mathrm{Br}\, A_i)\{\ell\} \subset (\mathrm{Br}\, K)\{\ell\}.$$

c) Assume further that X is proper over a scheme S. Let B_i be the the discrete valuation rings with quotient field K which lie over S. Then

$$(\mathrm{Br}\, X)\{\ell\} = \bigcap_{i \in I} (\mathrm{Br}\, B_i)\{\ell\} \subset (\mathrm{Br}\, K)\{\ell\}.$$

[11] It is not necessary to assume $k(x)$ perfect here, because ℓ is invertible on X.

Proof : a) follows from Corollary 6.11 and Remark 6.12. b) is an immediate consequence of a). c) is deduced from b) using the valuative criterion of properness.

□

Remark 6.14 K. Cesnavicius has proved recently that for every regular, noetherian and integral scheme X and every open subset U of X such that $\mathrm{codim}(X - U, X) \geq 2$, the restriction map $\mathrm{Br}\, X \to \mathrm{Br}\, U$ is an isomorphism (without restriction to $(\mathrm{Br}\, X)\{\ell\}$ with ℓ invertible on X). A consequence is that Corollary 6.13 actually still holds with the assumption on ℓ removed, see [4], Theorem 3.7.8. and Proposition 3.7.10.

6.5. Birational invariance of the Brauer group

We start with a definition due to D. Saltman.

Definition 6.15 Let $k \subset K$ be an extension of fields. The *unramified Brauer group of K over k* is the subgroup $\mathrm{Br}_{\mathrm{nr}}(K/k)$ of $\mathrm{Br}\, K$ consisting of those elements α such that for every discrete valuation ring A with quotient field K and such that $k \subset A$, the element α is in the image of the map $\mathrm{Br}\, A \hookrightarrow \mathrm{Br}\, K$

Observe that if we assume further that k is of characteristic zero, then the condition $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$ can be rephrased as: for every discrete valuation ring A with quotient field K and such that $k \subset A$, the residue $\partial_A(\alpha)$ is zero (since the residue field of A is then automatically perfect).

Proposition 6.16 *Let $k \subset K \subset L$ be field extensions. Then the image of* $\mathrm{Br}_{\mathrm{nr}}(K/k)$ *by the restriction map* $\mathrm{Br}\, K \to \mathrm{Br}\, L$ *is a subgroup of* $\mathrm{Br}_{\mathrm{nr}}(L/k)$.

Proof : Let B a discrete valuation ring containing k and with fraction field L. If its valuation v is trivial on K, then $K \subset B$ and the image of $\mathrm{Br}\, K$ in $\mathrm{Br}\, L$ is a subgroup of $\mathrm{Br}\, B$ (whose elements have trivial residue at B). Otherwise, $A := B \cap K$ is a discrete valuation ring with field of fractions K, and the restriction to $\mathrm{Br}\, L$ of an element of $\mathrm{Br}_{\mathrm{nr}}(K/k) \subset \mathrm{Br}\, A$ is in $\mathrm{Br}\, B \subset \mathrm{Br}\, L$.

□

Theorem 6.17 *Let X be a proper, integral, regular variety over a field k with function field $K = k(X)$. Then* $\mathrm{Br}_{\mathrm{nr}}(K/k)$ *is the subgroup* $\mathrm{Br}\, X$ *of* $\mathrm{Br}\, K$.

Proof : If k is of characteristic zero, this follows immediately of Corollary 6.13, c). In the general case, one has to use Cesnavicius's purity Theorem (see Remark 6.14).

□

Corollary 6.18 (Birational invariance) *Let X and Y be two k-birational proper, integral, regular varieties over a field k. Then* $\operatorname{Br} X \simeq \operatorname{Br} Y$.

Proof : The condition that X and Y are k-birational means that there are Zariski-dense open subsets $U \subset X$ and $V \subset Y$ such that U is k-isomorphic to V, which in turn is equivalent to saying that the function fields $k(X)$ and $k(Y)$ are k-isomorphic. Now apply Theorem 6.17.

□

Theorem 6.17 is especially useful to compute explicitely $\operatorname{Br} X$ (in particular when X is given as a smooth and projective model of a possibly singular variety, it is not necessary to explicitely write down equations for X to compute the unramified Brauer group of its function fields). See section 7. for examples.

7. Applications of the Brauer group

7.1. Birationality and stable birationality of varieties

In this paragraph, we denote by k a field with separable closure $\bar{k}$ and absolute Galois group $\Gamma = \operatorname{Gal}(\bar{k}/k)$. For a k-variety X, we set $\overline{X} := X \times_k \bar{k}$ and $\bar{k}[X]^* = H^0(\overline{X}, \mathbf{G}_m)$. We let $\operatorname{Br}_1 X := \ker[\operatorname{Br} X \to \operatorname{Br} \overline{X}]$ be the *algebraic Brauer group* of X.

Theorem 7.1 *Assume that $\bar{k}[X]^* = \bar{k}^*$ (e.g. X is proper and geometrically integral, or $X = \mathbf{A}^n_k$). Then there is an exact sequence*

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\Gamma} \to \operatorname{Br} k \to \operatorname{Br}_1 X \to H^1(k, \operatorname{Pic} \overline{X}) \to N \to 0,$$

where $N := \ker[H^3(k, \mathbf{G}_m) \to H^3(X, \mathbf{G}_m)]$. If $X(k) \neq \emptyset$, then the map $\operatorname{Br} k \to \operatorname{Br}_1 X$ is injective and $\operatorname{Br}_1 X / \operatorname{Br} k \simeq H^1(k, \operatorname{Pic} \overline{X})$.

Proof : By Hilbert's Theorem 90, we have $H^1(k, \bar{k}[X]^*) = H^1(k, \bar{k}^*) = 0$. Now the sequence just consists of the exact sequence of the first terms in Hochschild-Serre spectral sequence (Remark 3.35), given that $H^0(\overline{X}, \mathbf{G}_m) = \bar{k}^*$ by assumption. If we assume further that $X(k) \neq \emptyset$, then the structural morphism $X \to \operatorname{Spec} k$ has a section, hence the morphism $\operatorname{Br} k \to \operatorname{Br} X$ (as well as the morphism $H^3(k, \mathbf{G}_m) \to H^3(X, \mathbf{G}_m))$ has a retraction, hence is injective, which gives the second statement.

□

Proposition 7.2 *Let k be an algebraically closed field. Let C be a regular and integral curve over k (ex. $C = \mathbf{A}^1_k$). Then $\operatorname{Br} C = 0$. The same statement holds for $(\operatorname{Br} C)\{\ell\}$ if k is separably closed and ℓ is a prime distinct from $\operatorname{char} k$.*

Proof : Actually the Brauer group of the function field $k(C)$ is already trivial by Tsen's theorem (cf. Example 6.3) if k is algebraically closed, whence the result by Theorem 6.6. The version of Tsen's theorem for a separably closed ground field of characteristic $\neq \ell$ yields the second statement.

□

Theorem 7.3 *Let S be an integral, regular, noetherian scheme with function field K. Let ℓ be a prime distinct from $\operatorname{char} K$. Then the canonical map $(\operatorname{Br} S)\{\ell\} \to \operatorname{Br}(\mathbf{A}^n_S)\{\ell\}$ is an isomorphism.*

Proof : By induction on n, it is sufficient to deal with the case $n = 1$. Over a field K of characteristic $\neq \ell$, the map $(\operatorname{Br} K)\{\ell\} \to (\operatorname{Br}_1 \mathbf{A}^1_K)\{\ell\}$ is an isomorphism thanks to Theorem 7.1, because the Picard group of the affine space is zero. Besides, we have $(\operatorname{Br}_1 \mathbf{A}^1_K)\{\ell\} = (\operatorname{Br} \mathbf{A}^1_K)\{\ell\}$ by Proposition 7.2, whence the theorem when $S = \operatorname{Spec} K$.

In the general case, we observe that there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Br}(\mathbf{A}^1_S) & \longrightarrow & \operatorname{Br}(\mathbf{A}^1_K) \\ \uparrow & & \uparrow \\ \operatorname{Br} S & \longrightarrow & \operatorname{Br} K \end{array}$$

The horizontal maps are injective by Theorem 6.6 (both groups on the first line are subgroups of $\operatorname{Br}(K(T))$). Choose a section (which clearly exists, for example via the choice of a $\mathbf{Z}$-point of $\mathbf{A}^1_{\mathbf{Z}}$) of the structural morphism $\mathbf{A}^1_S \to S$, it induces a retraction s of the left vertical map and a retraction s_K of the right vertical map. By the case $S = \operatorname{Spec} K$, we already know that s_K is an isomorphism on the ℓ-primary torsion subgroups, hence so is s by diagram chasing.

□

Corollary 7.4 *Let k be a field of characteristic zero. Then the canonical map $\operatorname{Br} k \to \operatorname{Br} \mathbf{P}^n_k$ is an isomorphism for $n \geq 1$.*

Proof : Let $K \simeq k(T_1, ..., T_n)$ be the function field of $\mathbf{P}^n_k$ and $\mathbf{A}^n_k$. By Theorem 6.6, we have injective maps

$$\mathrm{Br}\, k \to \mathrm{Br}\, \mathbf{P}^n_k \to \mathrm{Br}\, \mathbf{A}^n_k \to \mathrm{Br}\, K,$$

and the corresponding map $\mathrm{Br}\, k \to \mathrm{Br}\, \mathbf{A}^n_k$ is surjective, whence the result.

□

Remark 7.5 Over an arbitrary perfect field k of characteristic p, we still have $(\mathrm{Br}\, k)\{p\} \simeq \mathrm{Br}\,(\mathbf{A}^1_k)\{p\}$ by Theorem 7.1 and Proposition 7.2, the separable closure of k being algebraically closed. This is no longer true over a non perfect field or for $\mathrm{Br}\,(\mathbf{A}^n_k)$ $(n \geq 2)$ over an algebraically closed field of characteristic p, see [4], Remark 6.1.2. Corollary 7.4 still holds in positive characteristic, but the proof is more complicated, see [4], Theorem 6.1.3.

Corollary 7.6 *Let X be a projective conic over a field k with* $\mathrm{Char}\, k \neq 2$*, given by the equation in $\mathbf{P}^2_k$:*

$$x^2 - ay^2 - bz^2 = 0,$$

where $a, b, c \in k^*$ are constants. Then there is an exact sequence

$$0 \to \mathbf{Z}/d \to \mathrm{Br}\, k \xrightarrow{u} \mathrm{Br}\, X \to 0,$$

where the index d of X is 1 *if $X(k) \neq \emptyset$, and $d = 2$ if $X(k) = \emptyset$.*

Recall that in general, the *index* of a k-variety X is the g.c.d. of the degrees of finite extension k' of k such that $X(k') \neq \emptyset$.

Proof : The degree map $\mathrm{Pic}\, \overline{X} \to \mathbf{Z}$ is a Galois-equivariant isomorphism and $\mathrm{Pic}\, X$ is generated by the class of a closed point of degree d. We have $\mathrm{Br}\, \overline{X} = 0$ thanks to the $\bar{k}$-isomorphism $\overline{X} \simeq \mathbf{P}^1$. Then apply Theorem 7.1.

□

We now go back to the unramified Brauer group. To make the statements simpler, we assume from now on that all fields are of characteristic zero.

Proposition 7.7 *Let $k \subset K \subset L$ be field extensions. Assume that $L = K(T_1, ..., T_n)$ is purely transcendental over K. Then the map* $\mathrm{Br}\, K \to \mathrm{Br}\, L$ *induces an isomorphism* $\mathrm{Br}_{\mathrm{nr}}\,(K/k) \to \mathrm{Br}_{\mathrm{nr}}\,(L/k)$.

Proof : Proposition 6.16 yields that the image of $\mathrm{Br}_{\mathrm{nr}}(K/k)$ in $\mathrm{Br}\,L$ is a subgroup of $\mathrm{Br}_{\mathrm{nr}}(L/k)$. Using induction on n, we can assume that $L = K(T)$. Let $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/k)$, then by definition $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/K)$, which is isomorphic to $\mathrm{Br}\,\mathbf{P}^1_K = \mathrm{Br}\,K$ by Theorem 6.17 and Corollary 7.4. Therefore β comes from a unique $\alpha \in \mathrm{Br}\,K$, and it remains to show that $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$.

Let A be a discrete valuation ring containing k with $\mathrm{Frac}\,A = K$, with uniformizing parameter π. Let B be the localization of $A[T]$ at the prime ideal spanned by π, it is a discrete valuation ring with uniformizing parameter π and fraction field L. The induced map $\kappa_A \to \kappa_B$ on residue fields corresponds to the canonical injection $\kappa_A \to \kappa_A(T)$, hence the corresponding restriction map $H^1(\kappa_A, \mathbf{Q}/\mathbf{Z}) \to H^1(\kappa_B, \mathbf{Q}/\mathbf{Z})$ is injective because κ_A is algebraically closed in κ_B. By Example 6.8, c), the equality $\partial_B(\beta) = 0$ implies $\partial_A(\alpha) = 0$ (indeed the ramification index is 1). Since this is true for all A as above, this exactly means that $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$.

□

Definition 7.8 Two k-varieties X and Y are *stably k-birationally equivalent* if there exists non negative integers m and n such that $X \times_k \mathbf{P}^k_m$ is k-birational to $Y \times_k \mathbf{P}^k_n$. A k-variety X is *stably k-rational* if $X \times \mathbf{P}^k_n$ is k-rational for some n.

Corollary 7.9 *Let k be a field. Let X and Y be integral k-varieties with respective function fields $k(X)$ and $k(Y)$. Then, if X and Y are stably k-birationally equivalent, we have* $\mathrm{Br}_{\mathrm{nr}}(k(X)/k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(Y)/k)$*. In particular, if X is stably k-rational, then* $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$*, is trivial, that is isomorphic to* $\mathrm{Br}\,k$.

This corollary is very important, because it can be used to proved that two varieties are *not* stably k-birationally equivalent, e.g. that a k-unirational variety is not stably k-rational. We will see an explicit example in the next paragraph.

7.2. Some explicit computations of residues

Let K be a field. Let n be a positive integer which is not divisble by $\mathrm{Char}\,K$. There is a cup-product pairing in Galois cohomology

$$H^1(K, \mathbf{Z}/n) \times H^1(K, \mu_n) \to H^2(K, \mu_n) = (\mathrm{Br}\,K)[n],\ (\alpha, \beta) \mapsto \alpha \cup \beta.$$

Assume further that K is the fraction field of a discrete valuation ring A with residue field κ whose characteristic does not divide n. There is a residue map $\partial_A : (\mathrm{Br}\,K)[n] \to H^1(\kappa, \mathbf{Z}/n)$. The following result relates the cup-product with this residue map:

Proposition 7.10 *Let* $\alpha \in H^1(A, \mathbf{Z}/n)$ *with image* $\alpha_0 \in H^1(\kappa, \mathbf{Z}/n)$. *Let* $b \in K^*$, *denote by* $v_A(b) \in \mathbf{Z}$ *its valuation and by* β *its image in* $H^1(K, \mu_n) = K^*/K^{*^n}$. *Then*

$$\partial_A(\alpha \cup \beta) = v_A(b)\alpha_0 \in H^1(\kappa, \mathbf{Z}/n).$$

Proof : See [4], §1.4.1., Formula (1.18).

□

Remark 7.11 The case $n = 2$ is especially interesting: in this case we can identify the K-group schemes $\mathbf{Z}/2$ and μ_2. For $(a, b) \in K^*$, the cup-product $(a, b) \in (\operatorname{Br} K)[2]$ of their classes in $H^1(K, \mu_2) = K^*/K^{*^2}$ is the classical *Hilbert symbol*, which is zero if and only if the projective conic

$$X : x^2 - ay^2 - bz^2 = 0$$

has a K-point (or, equivalently, b is a norm of the extension $K(\sqrt{a})/K$). Let $K(X)$ be the function field of X. The kernel of the map $u : \operatorname{Br} K \to \operatorname{Br} X \subset \operatorname{Br}(K(X))$ is generated by (a, b) thanks to Corollary 7.6: indeed (a, b) obviously belongs to $\ker u$ because X has a $K(X)$-point, and we know that $\ker u$ is non-trivial (isomorphic to $\mathbf{Z}/2$) if and only if $(a, b) \neq 0$.

Proposition 7.12 *Let* k *be a field of characteristic zero and* $a \in k^* - k^{*^2}$. *Let* $f_1(x)$ *and* $f_2(x)$ *be irreducible coprime polynomials of degree two. Consider the affine surface* V *defined in the affine space* $\mathbf{A}^3_k$ *by the equation*

$$y^2 - az^2 = f_1(x)f_2(x).$$

Let K *be the function field of* V. *Then:*

a) The element $\alpha := (a, f_1(x))$ *of* $\operatorname{Br} K$ *belongs to* $\operatorname{Br}_{\mathrm{nr}}(K/k)$.

b) Assume further that a *is a square neither in the field* $k[x]/(f_1(x))$ *nor in the field* $k[x]/(f_2(x))$. *Then* α *is not in the image of the canonical map* $\operatorname{Br} k \to \operatorname{Br} K$.

Proof (sketch of): a) follows from a computation of residues relying on Proposition 7.10; b) uses Remark 7.11 and the fact that K is also the function field of the conic

$$y_a^2 z^2 = f_1(x)f_2(x)$$

over the field $k(x)$.

□

Remark 7.13 A smooth projective model X of V (which exists by Hironoka's resolution of singularities Theorem) is an example of *Châtelet surface*. Thanks to Theorem 6.17, Proposition 7.12 exhibits a non-trivial element of $\mathrm{Br}\, X$, *without having to compute explicitely equations for* X.

Proposition 7.14 (D.H., 1994) *Let a, b be two distinct elements of $\mathbf{C}^*$. Let V be the $\mathbf{C}$-variety defined in the affine space $\mathbf{A}^4$ by the equation*

$$y^2 - tz^2 = (x^2 + a)(1 + t^2 - t(x^2 + b)).$$

Then $(t, x^2 + a)$ is a non-zero element of $\mathrm{Br}_{\mathrm{nr}}\,(\mathbf{C}(V))$.

This yields an example of a $\mathbf{C}$-variety which is unirational (it is dominated by the rational variety obtained via the change of variables $t = u^2$) but not stably rational.

Proof (sketch of): (See [6]). To show $(t, x^2 + a) \neq 0$, observe that V is fibered over the affine plane (via t, x), the generic fibre being a conic X over $F := \mathbf{C}(x, t)$ with function field $F(X) = \mathbf{C}(V)$. Then apply Remark 7.11 to obtain that $(t, x^2 + a)$ is not in the kernel of $\mathrm{Br}\, F \to \mathrm{Br}\,(F(X))$ because it is not zero and distinct from $(t, (x^2 + a)(1 + t^2 - t(x^2 + b))$ in $\mathrm{Br}\, F$ (equivalently, $(t, x^2 + a)$ and $(t, (1 + t^2 - t(x^2 + b))$ are both non zero in $\mathrm{Br}\, F$, which is not difficult to check).

To show that $(t, x^2 + a)$ is in $\mathrm{Br}_{\mathrm{nr}}\,(\mathbf{C}(V))$, the method is similar to Proposition 7.12: it consists of proving that all residues are zero thanks to Proposition 7.10.

□

7.3. The Brauer-Manin obstruction

Let k be a number field. Denote by Ω the set of all places of k and by k_v the completion of k at v. Local class field theory gives a one-to-one homomorphism $\mathrm{inv}_v : \mathrm{Br}\, k_v \to \mathbf{Q}/\mathbf{Z}$, which is an isomorphism if v is not archimedean. Global class field theory yields an exact sequence

$$0 \to \mathrm{Br}\, k \to \bigoplus_{v \in \Omega} \mathrm{Br}\, k_v \xrightarrow{\sum \mathrm{inv}_v} \bigoplus_{v \in \Omega} \mathbf{Q}/\mathbf{Z} \to 0.$$

Now let X be a k-variety. The above exact sequence implies that for every rational point $M \in X(k)$ with image $M_v \in X(k_v)$, we have for any $\alpha \in \mathrm{Br}\, X$:

$$\sum_{v \in \Omega} \mathrm{inv}_v(\alpha(M_v)) = 0.$$

Therefore, if for all families of local points $(P_v) \in \prod_{v \in \Omega} X(k_v)$, there exists $\alpha \in \operatorname{Br} X$ such that $\sum_{v \in \Omega} \operatorname{inv}_v(\alpha(P_v)) \neq 0$, we have $X(k) = \emptyset$: this is the so-called *Brauer-Manin obstruction* to the Hasse principle, introduced by Manin in 1970. Here is an example of this obstruction:

Theorem 7.15 (Iskovskih, 1970) *Let V be the smooth $\mathbf{Q}$-variety defined in the affine space by the equation*

$$y^2 + z^2 = (x^2 - 2)(3 - x^2) \neq 0.$$

Then V has points in every completion of $\mathbf{Q}$ but $V(\mathbf{Q}) = \emptyset$. The same holds for every smooth and projective model X of V.

Proof (sketch of): The property that V has points everywhere locally is easy to check via Hensel's lemma. Then the element $\alpha := (-1, x^2 - 2) \in \operatorname{Br}(\mathbf{Q}(V))$ actually belongs to $\operatorname{Br}_{\mathrm{nr}}(\mathbf{Q}(V)) \simeq \operatorname{Br} X$ by Proposition 7.12 and Theorem 6.17. Local computations then show that for every local point $P_v \in V(Q_v)$, we have $\alpha(P_v) = 0$, except if v is the finite place 2 where $\alpha(P_v) \neq 0$. Hence $V(\mathbf{Q}) = \emptyset$ thanks to the Brauer-Manin obstruction associated to α. The same argument works for X because V (which is smooth) is isomorphic to a Zariski open subset of X, so $V(\mathbf{Q}_v)$ is dense in $X(\mathbf{Q}_v)$ by the implicit function Theorem.

□

Remark 7.16 For a smooth and proper variety X, there is also a Brauer-Manin obstruction to *weak approximation*, namely: a family of local points $(P_v) \in \prod_{v \in \Omega} X(k_v)$ such that

$$\sum_{v \in \Omega} \operatorname{inv}_v(\alpha(P_v)) \neq 0$$

cannot be in the closure of $X(k)$ (for the product of v-adic topologies). [6] gives examples (similar to the one of Theorem 7.14) of Brauer-Manin obstruction (to the Hasse principle as well as to weak approximation) given by a *transcendental* element of $\operatorname{Br} X$ and not detected by algebraic elements (i.e. elements of $\operatorname{Br}_1 X$).

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