Exam, M2

D. Harari

May 15, 2023; 3h; written notes allowed

Every statement contained in the lecture notes can be used without proof. In each exercise, it is allowed to assume the result of a question to solve a further question.

Unless explicitely specified, all cohomology groups are étale cohomology groups.

Exercise 1 (6 points)

In the following statements, prove the right ones and give a counterexample for the wrong ones (first say whether the statement is right or wrong).

a) Let k be a perfect field of characteristic $p > 0$. Then the sheaf μ_p is trivial on $\operatorname{Spec} k$ for the flat topology.

b) Let X be a smooth variety over the field $\mathbf{R}$ of real numbers. Let n be a positive integer. Then the cohomology groups $H^i(X, \mu_n)$ are finite for all $i \geq 0$.

c) Let X be the projective $\mathbf{C}$-variety defined by the equation

$$a_0 x_0^2 + ... + a_n x_n^2 = 0$$

in the projective space $\mathbf{P}^n_{\mathbf{C}}$, where $a_0, ..., a_n$ are non-zero complex numbers and $n \geq 3$. Then $\operatorname{Br} X = 0$.

d) Let X be an integral, noetherian and regular scheme with function field K. Let G be a commutative group scheme over X and $G_K := G \times_X \operatorname{Spec} K$. Then the canonical map $H^1(X, G) \to H^1(K, G_K)$ is injective.

Exercise 2 (5 points)

Let X be a scheme. Let Z be a closed subscheme of X, set $U = X - Z$. Denote by $i : Z \to X$ the corresponding closed immersion and by $j : U \to X$ the open immersion.

1. a) Let $\mathcal{F}$ be a sheaf on $U_{\text{ét}}$. Show that there is an isomorphism $j^* j_* \mathcal{F} \simeq \mathcal{F}$.

b) Deduce that there is an exact sequence of sheaves on $X_{\text{ét}}$:

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \xrightarrow{u} i_*i^*j_*\mathcal{F} \to 0. \qquad (1)$$

2. Assume further that $i^*j_* : S(U) \to S(Z)$ has an exact left-adjoint and that the map $H^0(X, j_*\mathcal{F}) \to H^0(X, i_*i^*j_*\mathcal{F})$ induced by u is an isomorphism.

a) Show that $H^0(X, j_!\mathcal{F}) = 0$.

b) Suppose that $\mathcal{F}$ is injective in the category of sheaves on $U_{\text{ét}}$. Show that $H^r(X, j_!\mathcal{F}) = 0$ for all $r \geq 0$.

Exercise 3 (6 points)

Let k be a field with separable closure $\bar{k}$. Let X be a geometrically integral variety over k. Set $\overline{X} = X \times_k \bar{k}$ and $\bar{k}[X]^* = H^0(\overline{X}, \mathbf{G}_m)$. Define $\mathrm{Br}_1 X := \ker[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}]$ and $U(X) = \bar{k}[X]^*/\bar{k}^*$.

1. Show that there is an exact sequence

$$0 \to H^1(k, \bar{k}[X]^*) \to \mathrm{Pic}\, X \to H^0(k, \mathrm{Pic}\, \overline{X}) \to H^2(k, \bar{k}[X]^*) \to \mathrm{Br}_1 X \to H^1(k, \mathrm{Pic}\, \overline{X}).$$

From now on, we assume that the set $X(k)$ of k-points of X is not empty.

2. a) Show that the inclusion $\bar{k}^* \hookrightarrow \bar{k}[X]^*$ induces an injective map $\mathrm{Br}\, k \to H^2(k, \bar{k}[X]^*)$.

b) Deduce that there is an isomorphism $H^1(k, \bar{k}[X]^*) \simeq H^1(k, U(X))$.

3. Assume further that $\mathrm{Pic}\, \overline{X} = 0$. Show that $\mathrm{Br}_1 X / \mathrm{Br}\, k$ is isomorphic to $H^2(k, U(X))$.

Exercise 4 (4 points)

Let $Y = \mathrm{Spec}\, A$, where A is a discrete valuation ring with fraction field K. Let $j : \eta = \mathrm{Spec}\, K \to Y$ be the inclusion of the generic point. Let G be a proper and commutative group scheme over Y, set $G_K = G \times_A K = G \times_Y \mathrm{Spec}\, K$.

a) Show that there is an isomorphism $G \simeq j_*G_K$ of étale sheaves on Y.

b) Deduce that the canonical map $i : H^1(Y, G) \to H^1(K, G_K)$ is injective.

c) Let B be a smooth and proper (not necessarily commutative) group scheme over Y. Show that the map $H^1(Y, B) \to H^1(K, B_K)$ still has trivial kernel.