Exercise 1 (6 points)

a) This is wrong. Let $A = k[T]/(T^p)$ and $Y = \operatorname{Spec} A$. Then Y is an objet of the flat site of $\operatorname{Spec} k$, but $\mu_p(Y)$ is not trivial, e.g. $T + 1$ is a non-trivial element of $\mu_p(Y)$ (1 point).

b) This is true. Set $X_{\mathbf{C}} = X \times_{\mathbf{R}} \mathbf{C}$ and $\Gamma = \operatorname{Gal}(\mathbf{C}/\mathbf{R})$. By Theorem 5.17, the groups $H^s(X_{\mathbf{C}}, \mu_n)$ are finite for all $s \geq 0$. Now Hochschild-Serre spectral sequence yields the result (see also Remark 5.10), given that $H^r(\Gamma, M)$ is finite for every finite Γ-module M and all $r \geq 0$ (2 points).

c) This is true. Write $a_i = b_i^2$ for some $b_i \in \mathbf{C}$ and set $y_i = a_i x_i$. Then X (which is obviously smooth) is isomorphic to the projective variety

$$Y : y_1^2 + y_2^2 + ... + y_n^2 = 0.$$

Set $z_1 = y_1 + iy_2$ and $z_2 = y_1 - iy_2$, we obtain that Y is isomorphic to the variety

$$Z : z_1 z_2 = -(y_3^2 + ... + y_n^2)$$

and the latter is obviously birational to the projective space because its equation can be rewritten

$$z_1 = -\frac{y_3^2 + ... + y_n^2}{z_2}.$$

Now apply Corollary 6.18 (2 points).

d) This is wrong. Take $G = \mathbf{G}_m$, then $H^1(K, G_K) = 0$ by Hilbert 90, but $H^1(X, G) = \operatorname{Pic} X$ can be non zero, for example for an elliptic curve over an algebraically closed field or the spectrum of a Dedekind ring which is not a UFD (1 point).

Exercise 2 (5 points)

1. a) Let f : $j^* j_* \mathcal{F} \simeq \mathcal{F}$ be the natural adjunction map. It is sufficient to show that it is an isomorphism on the geometric stalks. Let $\bar{x}$ be a geometric point of $U \subset X$ with image x. By Proposition 2.38, we have

$$(j^* j_* \mathcal{F})_{\bar{x}} = (j_* \mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}},$$

the latter equality coming the fact that $x \in U$ (1.5 point).

b) Apply Lemma 2.44 to the sheaf $j_* \mathcal{F}$, and replace $j^* j_* \mathcal{F}$ by $\mathcal{F}$ thanks to a) (1 point).

2. a) The long exact sequence of cohomology associated to (1) gives an exact sequence

$$0 \to H^0(X, j_! \mathcal{F}) \to H^0(X, j_* \mathcal{F}) \to H^0(X, i_* i^* j_* \mathcal{F}),$$

whence the result with the second assumption (1 point).

b) The same long exact sequence

$$H^0(X, i_* i^* j_* \mathcal{F}) \to H^1(X, j_! \mathcal{F}) \to H^1(X, j_* \mathcal{F}) \to H^1(X, i_* i^* j_* \mathcal{F}) \to H^2(X, j_! \mathcal{F})...$$

shows that it is sufficient to show that $H^r(X, j_* \mathcal{F})$ and $H^r(X, i_* i^* j_* \mathcal{F})$ are both zero for $r \geq 1$. We know that j_* takes injectives to injectives (since it has an exact left adjoint j^* and the same holds for $i^* j_*$ (by the first assumption) and for i_* (for the same reason). Whence the result (1.5 point).

Exercise 3 (6 points)

1. This is just the exact sequence of the first terms of Hochschild-Serre spectral sequence

$$H^r(k, H^s(\overline{X}, \mathbf{G}_m)) \Rightarrow H^{r+s}(X, \mathbf{G}_m)$$

(1 point).

2. a) A rational point $x \in X(k)$ induces a Galois-equivariant retraction $\bar{k}[X]^* \to \bar{k}^*, f \mapsto f(x)$ of the inclusion $\bar{k}^* \to \bar{k}[X]^*$. This implies that the natural map $\operatorname{Br} k = H^2(k, \bar{k}^*) \to H^2(k, \bar{k}[X]^*)$ has a retraction, hence it is injective (1.5 point).

b) The exact sequence

$$0 \to \bar{k}^* \to \bar{k}[X]^* \to U(X) \to 0$$

induces an exact sequence

$$0 \to H^1(k, \bar{k}^*) \to H^1(k, \bar{k}[X]^*) \to H^1(k, U(X)) \to \operatorname{Br} k \to H^2(k, \bar{k}[X]^*).$$

Now apply a) and Hilbert 90 (1.5 points).

3. Using 1. and the assumption, we get an isomorphism $H^2(k, \bar{k}[X]^*) \to \operatorname{Br}_1 X$. By 2., there is an exact sequence

$$0 \to \operatorname{Br} k \to H^2(k, \bar{k}[X]^*) \to H^2(k, U(X)) \to H^3(k, \bar{k}^*) \to H^3(k, \bar{k}[X]^*).$$

But the map $H^3(k, \bar{k}^*) \to H^3(k, \bar{k}[X]^*)$ is injective (same argument as in question 2.a), whence

$$\operatorname{Br}_1 X / \operatorname{Br} k \simeq H^2(k, \bar{k}[X]^*)/\operatorname{Br} k \simeq H^2(k, U(X))$$

(2 points).

Exercise 4 (4 points)

a) The adjunction map $G \to j_* j^* G$ and the canonical map $j^* G \to G_K$ (cf. Example 2.37) induce a map $G \to j_* G_K$. Let us show that it induces an isomorphism on every geometric stalk $\bar{x} \to Y$ with image x. This is obvious if x is the generic point of Y. If x is the closed point, the respective stalks are $G(A^{\mathrm{sh}})$ and $G(\mathrm{Frac}\, A^{\mathrm{sh}})$, which are the same because G is proper over A (1.5 point).

b) This follows from a) and the first two terms of the exact sequence associated to Leray spectral sequence (1 point).

c) A class in the kernel corresponds to a Y-torsor Z under G such that the generic fibre of Z has a K-point. But Z is proper over Y (as G is proper over Y and Z becomes isomorphic to G after base change by a fppf covering), hence Z has an A-point by the valuative criterion of properness, which means that its class in $H^1(Y, G)$ is trivial (1.5 point).

Remark: The assumptions of Exercise **2**, question **2.**, are actually satisfied if $X = \mathrm{Spec}\, A$ is the spectrum of a henselian discrete valuation ring with fraction field K and perfect residue field k and we take for $i : \{x\} \to X$ the inclusion of the closed point of X (resp. for $j : \{u\} \to X$ the inclusion of the generic point of X). See Milne, *Arithmetic duality theorems*, Prop 2.1.1.