Exam, M2

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Every statement contained in the lecture notes can be used without proof. In each exercise, it is allowed to assume the result of a question to solve a further question.

Unless explicitely specified, all cohomology groups are étale cohomology groups. The following results can be used without proof: if A is a normal ring with fraction field K and Z is a finite A-scheme, then the canonical map $Z(A) \to Z(K)$ is an isomorphism; if R is a local ring, then $\operatorname{Pic} R = 0$.

Exercise 1 : Right or wrong ? (6 points)

In the following statements, prove the right ones and give a counterexample for the wrong ones (first say whether the statement is right or wrong).

1. Let X be a regular, noetherian, and integral scheme. If $\operatorname{Br} X$ is a finitely generated group, then it is a finite group.

2. Let k be an algebraically closed field of characteristic zero and let X be a smooth k-variety. Then for every positive integers i and n, the n-torsion group $H^i(X, \mathbf{G}_m)[n]$ is finite.

3. Let X be a smooth variety over a field k of characteristic $p > 0$. Then the sheaf on the small étale site of X represented by the group scheme $(\mu_p)_X$ is zero.

4. Let X be a smooth variety over an algebraically closed field k of characteristic $p > 0$. Then the flat cohomology group $H^1_{\text{fppf}}(X, \alpha_p)$ is zero.

Exercise 2 : The injectivity property (7 points)

Let k be a field of characteristic zero. Let G be a smooth and commutative k-group scheme of finite type. Let $i \in \mathbf{N}^*$. We say that *the injectivity property holds for the pair* (i, G) if for every regular local ring A with fraction field K such that $k \subset A$, the natural map $H^i(A, G) \to H^i(K, G)$ is injective (here we write $H^i(A, G)$ for $H^i_{\text{ét}}(\operatorname{Spec} A, G_A)$, where $G_A := G \times_{\operatorname{Spec} k} \operatorname{Spec} A$, and similarly $H^i(K, G) := H^i_{\text{ét}}(\operatorname{Spec} K, G_K)$ with $G_K := G \times_{\operatorname{Spec} k} \operatorname{Spec} K$).

1. Does the injectivity property hold for the pair $(1, \mathbf{G}_m)$? For the pair $(2, \mathbf{G}_m)$?

2. Let X be a smooth k-variety. Assume that X has a k-rational point $x \in X(k)$.

a) Show that for every Zariski open subset U of X with $x \in U$, the canonical map $H^i(k, G) \to H^i(U, G)$ is injective.

b) Assume further that X is integral with function field F and (i, G) has the injectivity property. Show that the canonical map $H^i(k, G) \to H^i(F, G)$ is injective.

3. Show that for every positive integer n, the pair $(2, \mu_n)$ has the injectivity property.

4. Assume that G is a finite k-group scheme. Let A be a regular local ring with fraction field K, denote by $j : \operatorname{Spec} K \to \operatorname{Spec} A$ the inclusion of the generic point. Show that the natural map $G_A \to j_* G_K$ is an isomorphism, then deduce that $(1, G)$ has the injectivity property.

Exercise 3 : Cohomology of local and global fields (9 points)

1. Let R be a henselian discrete valuation ring with fraction field E and perfect residue field k. Let $X = \operatorname{Spec} R$, denote by $x \simeq \operatorname{Spec} k$ the closed point of X and by $u = X - \{x\} \simeq \operatorname{Spec} E$ its generic point. There is a closed immersion $i : x \hookrightarrow X$ and an open immersion $j : u \hookrightarrow X$. Let F be an étale sheaf on u and $r \geq 0$; it is known that $H^r(X, j_! F) = 0$ (this can be used without proof).

a) Show that there is an isomorphism $H^r(u, F) \simeq H^{r+1}_x(X, j_! F)$.

b) Show that for every sheaf $\mathcal{F}$ on X, we have $H^r(X, \mathcal{F}) \simeq H^r(x, i^* \mathcal{F})$.

c) Let $g = \operatorname{Gal}(\bar{k}/k)$ be the absolute Galois group of k. Show that for all $r \geq 0$, there is an isomorphism $H^r(X, \mathbf{Z}) \simeq H^r(g, \mathbf{Z})$.

d) Compute $H^r(X, \mathbf{Z})$ for $r \in \{0, 1, 2\}$ when the field k is finite.

2. Let K be a number field with ring of integers $\mathcal{O}_K$ (e.g. $K = \mathbf{Q}$, $\mathcal{O}_K = \mathbf{Z}$). Let U be a non empty Zariski open subset of $\operatorname{Spec}(\mathcal{O}_K)$. Let $V \subset U$ be a smaller non empty Zariski open subset, denote by $j : V \to U$ the corresponding open immersion. The local ring of a closed point $v \in \operatorname{Spec}(\mathcal{O}_K)$ is denoted by $\mathcal{O}_v$, its henselization by $\mathcal{O}_v^h$, and we set $K_v = \operatorname{Frac}(\mathcal{O}_v^h)$. Let F be an étale sheaf on V.

a) Set $U_v := \operatorname{Spec}(\mathcal{O}_v^h)$. Show that $H^r_{U-V}(U, j_! F) \simeq \bigoplus_{v \in U-V} H^r_v(U_v, j_! F)$.

b) Show that there is a long exact sequence

$$... \to H^r(U, j_! F) \to H^r(V, F) \to \bigoplus_{v \in U-V} H^r(K_v, F) \to ...$$