Exam M2, solutions

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Exercise 1

1. This is true. The group $\mathrm{Br}\, X$ is a torsion group (Th. 5.6), hence it is finite as soon as it is finitely generated (1 point).

2. This is true. By Theorem 4.19, the groups $H^i(X, \mu_n)$ are finite. Now Kummer exact sequence (cf. Proposition 5.4. for $i = 2, 3$) implies that $H^i(X, \mathbf{G}_m)[n]$ is a quotient of $H^i(X, \mu_n)$, hence it is finite as well (1 point).

3. This is true. Let $U \to X$ be an étale map, then U is smooth over k. In particular it is a reduced scheme, which implies that $\mathcal{O}_U(U)$ is a reduced ring of characteristic p. Thus the equation $x^p = 1$ (i.e. $(x-1)^p = 0$) has 1 as its only solution in $\mathcal{O}_U(U)^*$ (2 points).

4. This is false. Example 2.29 yields an exact sequence for the flat topology

$$0 \to \alpha_p \to \mathbf{G}_a \xrightarrow{\cdot p} \mathbf{G}_a \to 0.$$

Assume $X = \mathrm{Spec}\, A$ affine, then $H^1(X, \mathbf{G}_a) = 0$ by Serre's Theorem and Theorem 3.24 a). Taking the cohomology long exact sequence, we get the equality $H^1_{\mathrm{fppf}}(X, \alpha_p) = A/A^p$. The latter is not necessarily zero, for instance for $X = \mathbf{A}^1_k$ we have $A = k[T]$ and $A/A^p \neq 0$ because T is not a p-power in A (2 points).

Exercise 2

1. Let A be a regular local ring. Then $H^1(A, \mathbf{G}_m) = \mathrm{Pic}\, A = 0$, hence the injectivity property trivially holds. Also $H^2(A, \mathbf{G}_m) = \mathrm{Br}\, A$ injects into $H^2(K, \mathbf{G}_m) = \mathrm{Br}\, K$ by Theorem 5.6., whence the result for the pair $(2, \mathbf{G}_m)$ (1 point).

2. a) The structural morphism $U \to \mathrm{Spec}\, k$ has a section $\mathrm{Spec}\, k \to U$ induced by x. Therefore the canonical map $H^i(k, G) \to H^i(U, G)$ has a retraction, which implies that it is injective (1 point).

b) Take direct limit over all U containing x in a). Using Theorem 3.10 b), we get that the canonical map $H^i(k, G) \to H^i(\mathrm{Spec}\,(\mathcal{O}_{X,x}), G)$ is injective. The injectivity property for (i, G) now tells us that the canonical map $H^i(\mathrm{Spec}\,(\mathcal{O}_{X,x}), G) \to H^i(F, G)$ is injective because $\mathcal{O}_{X,x}$ is a regular local ring (recall that X is smooth over k) with function field F. Hence the composition $H^i(k, G) \to H^i(F, G)$ is injective as well (1.5 point).

3. Let A be a regular local ring with quotient field K. By Proposition 5.4, $H^2(A, \mu_n)$ is a subgroup of $\mathrm{Br}\, A$ because $\mathrm{Pic}\, A = 0$. Since $\mathrm{Br}\, A$ injects into $\mathrm{Br}\, K$ by Theorem 5.6, we get that $H^2(A, \mu_n)$ injects into $H^2(K, \mu_n)$ (1.5 point).

4. To show that the natural morphism $G_A \to j_* G_K$ is an isomorphism, we compare the stalks at a geometric point $\bar{x}$ of $Y := \mathrm{Spec}\, A$ with image x. It is sufficient to show that if $B = \mathcal{O}_{Y,x}^{\mathrm{sh}}$ is the strict henselization of B and $L := \mathrm{Frac}\, B$, the canonical map $G(B) \to G(L)$ is an isomorphism; this follows indeed from the fact that G is finite over k (hence $G \times_k B$ is finite over B by base change) and B is a regular (hence normal) local ring with field of fractions L. Now the injectivity property follows from Leray spectral sequence as in Remark 4.3. (2 points).

Exercise 3

1. a) We apply Theorem 3.27 to the sheaf $j_! F$ on X. Since all terms $H^r(X, j_! F)$ are zero, we get the required result because F and $j_! F$ have same restriction to u by definition of $j_!$ (1 point).

b) Apply Lemma 3.25 and take the corresponding long exact sequence of cohomology. As all groups $H^r(X, j_!(j^* \mathcal{F}))$ are zero, we obtain isomorphisms $H^r(X, \mathcal{F}) \simeq H^r(X, i_* i^* \mathcal{F})$. But we can identify $H^r(X, i_* i^* \mathcal{F})$ with $H^r(x, i^* \mathcal{F})$ because the functor i_* is exact by Corollary 2.37., hence all its higher direct images vanish (2 points).

c) This follows from b) and Example 3.3. b) combined with the fact that $i^* \mathbf{Z} = \mathbf{Z}_x$ by Example 2.33 a) (this can of course also be checked directly, the sheaf $\mathbf{Z}$ being constant), the group scheme $\mathbf{Z}$ being étale over X (1.5 point).

d) Using c), we get $H^0(X, \mathbf{Z}) = \mathbf{Z}$, $H^1(X, \mathbf{Z}) = 0$ and

$$H^2(X, \mathbf{Z}) \simeq H^2(g, \mathbf{Z}) \simeq H^1(g, \mathbf{Q}/\mathbf{Z}) \simeq \mathbf{Q}/\mathbf{Z},$$

the last equality coming from the fact that the absolute Galois group g of the finite field k is isomorphic to $\widehat{\mathbf{Z}}$ (1.5 point).

2. a) This is quite similar to Corollary 3.29. The closed subscheme $U - V$ consists of finitely many closed points $z_1, ..., z_r$ of U. Let (Y_i, y_i) be an étale neighborhood of z_i such that the inverse image of $\{z_i\}$ is y_i, and let Y be the disjoint union of Y_i. By excision, we have

$$H^r_{U-V}(U, \mathcal{F}) \simeq H^r_{\{y_1,\dots,y_r\}}(Y, \mathcal{F}) \simeq \bigoplus_{i=1}^{r} H^r_{y_i}(Y_i, \mathcal{F})$$

for every étale sheaf $\mathcal{F}$ on U. Taking limit over such (Y_i, y_i) yields

$$H^r_{U-V}(U, \mathcal{F}) \simeq \bigoplus_{i=1}^{r} H^r_{z_i}(\mathrm{Spec}\,(\mathcal{O}^h_{U,z_i}), \mathcal{F}) = \bigoplus_{v \in U-V} H^r_v(U_v, \mathcal{F}).$$

It remains to apply the result to $\mathcal{F} = j_! F$ (2 points).

b) Apply Th 3.27 to the open immersion $V \hookrightarrow U$ and the sheaf $j_! F$. We just have to identify $H^{r+1}_{U-V}(U, j_! F)$ to $\bigoplus_{v \in U-V} H^r(K_v, F)$. But this follows from 2.a) and 1.a) (1 point).