Midterm exam, algebra, M1 MF (3 hours)

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Every statement in the lecture notes (but not in the TD) can be used without proof. It is allowed to use the result of a question to solve a further question.

The symbol (*) stands for a (a priori) difficult question.

Exercise 1 : Exponent of a group (7 points)

Le G be a finite group (whose group law is denoted multiplicatively). The *exponent* of G (denoted $\exp G$) is by definition the smallest integer $n > 0$ such that $x^n = 1$ for all x in G.

a) Show that for every $x \in G$, the order (denoted $\omega(x)$) of x divides $\exp G$.

b) Let p be a prime number, write $\exp G = p^\alpha m$ with $\alpha \in \mathbf{N}$ and m not divisible by p. Show that there exists $x \in G$ such that $\omega(x)$ is divisible by p^α.

c) Deduce that $\exp G$ is the smallest common multiple of the family $(\omega(x))_{x \in G}$.

d) Compute $\exp G$ when G is the group $\mathcal{A}_5$.

We assume in the remaining part of this exercise that A is an abelian group.

e) Show that if $x_1, ..., x_r$ are elements of A such that the $\omega(x_i)$ are pairwise coprime, then the order of $\prod_{i=1}^r x_i$ is $\prod_{i=1}^r \omega(x_i)$.

f) Deduce that A contains an element of order $\exp A$. Is it still true if A is no longer assumed to be abelian ?

Exercise 2 : 2-Sylow of a finite group (3 points)

Let A be a finite group.

a) Show that the number of 2-Sylow of A is odd.

b) Let G be a 2-group acting on A by automorphisms (that is : for every $g \in G$, the map $x \mapsto g.x$ is an automorphism of the group A). Show that there exists a 2-Sylow S of A such that $g.S = S$ for all $g \in G$.

c) It is a fact (Feit-Thompson Theorem) that a finite group of odd order is solvable. Taking this for granted, show that the normalizer of a 2-Sylow of A is solvable.

Exercise 3 : Prime elements of a domain (11 points)

Let A be an integral domain with fraction field K. A non-zero element p of A is said to be *prime* if $(p) := pA$ is a prime ideal of A.

a) Show that if p is prime, it is irreducible.

b) Denote by T the set of elements of A that are either invertible or can be written

$$x = up_1p_2...p_r$$

with u invertible and all p_i prime. Show that A is a UFD if and only if $T = A - \{0\}$.

c) Let $a, b \in A$ and let p be a prime element of A. Show that if p^m divides ab with $m \in \mathbf{N}^*$, then there exist $n, s \in \mathbf{N}$ such that : $n + s = m$, p^n divides a, and p^s divides b.

(*) d) Deduce that if a and b are in A with $ab \in T$, then a and b are both in T.

e) Let B be the set of elements of K of the form x/y, with $x \in A$ and $y \in T$. Show that B is a subring of K, and that if I is a prime ideal of B, then $I \cap A$ is a prime ideal $\wp$ of A such that $\wp \cap T = \emptyset$.

(*) f) Suppose that there exists $a \neq 0$ in A such that $a \notin T$. Show that there exists a prime ideal $\wp$ of A such that $a \in \wp$ and $\wp \cap T = \emptyset$ (hint : consider the ideal aB of B, then use d) and e)).

g) Deduce the following statement (Kaplansky criterion) : the ring A is a UFD if and only if every non-zero prime ideal of A contains a prime element.