Algebra exam, M1

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Every statement contained in the lecture notes can be used without proof (but the same does not hold for statements that have only been seen in the "TD" sessions). In each exercise, it is allowed to assume the result of a question to solve a further question.

Exercise 1 : Groups (4 points)

Recall that if G is a group, a subgroup H of G is said to be *characteristic* if H is stable under the action of all automorphisms of G.

a) Let $n \geq 5$ be an integer. Show that every normal subgroup of $\mathcal{S}_n$ is characteristic.

b) Let p be a prime number, set $G = (\mathbf{Z}/p\mathbf{Z})^n$ with $n \geq 2$. Show that there exists a subgroup H of G such that H is not characteristic.

c) Let p be a prime number. Let H be a p-Sylow subgroup of a finite group G. Is it possible that H is normal in G, but H is not characteristic in G ?

Exercise 2 : Rings (4 points)

Let A be a commutative ring. The ring A is said to be *of dimension* 0 if every prime ideal of A is maximal.

a) Prove that if A is an integral domain, then A is of dimension 0 if and only if A is a field.

b) Give (with justification) an example of a non-zero commutative ring A such that A is of dimension 0 but A is not a field.

c) Let A be a commutative ring of dimension 0. Show that for every ideal I of A, the quotient ring A/I is of dimension 0

Exercise 3 : Modules (7 points)

Let A be a non-zero commutative ring. Let M be an A-module and N a submodule of M. One says that N is a *direct summand* of M if there exists a submodule P of M such that $M = N \oplus P$.

a) Let $x \in M$. Show that if there exists a morphism of A-modules $u : M \to A$ such that $u(x) = 1$, then the submodule $A.x$ is a direct summand of M.

b) Show that if N is a direct summand of M and N is a free A-module with $N \neq \{0\}$, then there exists a morphism of A-modules $u : M \to A$ such that $1 \in u(N)$.

From now on in this exercise, we assume that A is a principal ideal domain and we take $M = A^n$ with $n \in \mathbf{N}^*$.

c) Let $a = (a_1, ..., a_n) \in M$, with $a \neq (0, 0, ..., 0)$. Set $N = A.a$. Prove that the submodule N is a direct summand of M if and only if $\gcd(a_1, ..., a_n) = 1$.

d) Show that $a = (a_1, ..., a_n)$ can be made the first element of a basis $(a, e_2, ..., e_n)$ of $M = A^n$ if and only if $\gcd(a_1, ..., a_n) = 1$.

e) Let $(f_1, ..., f_r)$ be a family of linearly independent elements of the A-module $M = A^n$. Denote by $B \in M_{n,r}(A)$ the matrix whose columns consist of the coordinates of $f_1, ..., f_r$. Give a necessary and sufficient condition on the minors of B to have the following property : the submodule of M generated by $(f_1, ..., f_r)$ is a direct summand of M.

Exercise 4 : Fields and Galois Theory (6 points+2 for the bonus question)

Let K be a field. Let L be a finite Galois extension of K, set $G = \mathrm{Gal}\,(L/K)$. Let F_1 and F_2 be two field extensions of K with $K \subset F_i \subset L$ for $i \in \{1, 2\}$. Define $G_i := \mathrm{Gal}\,(L/F_i)$ for $i \in \{1, 2\}$. Thus G_1 and G_2 are subgroups of G.

a) Assume that $F_1 \cap F_2 = K$. Show that the subset $G_1 \cup G_2$ generates the group G.

b) Show conversely that if $G_1 \cup G_2$ generates G, then $F_1 \cap F_2 = K$.

From now on, we denote by F the subfield of L generated by $F_1 \cup F_2$.

c) Prove that $F = L$ if and only if $G_1 \cap G_2$ is the trivial group.

d) Assume that F_1 and F_2 are Galois extensions of K with $F_1 \cap F_2 = K$ and $F = L$. Show that the group G is isomorphic to the direct product $G_1 \times G_2$ (hint: first show that G is a semi-direct product of G_2 by G_1).

e) Show that the result in d) does not necessarily hold if F_1 et F_2 are not assumed to be Galois over K.

f) (Bonus question) Assume again that F_1 and F_2 are arbitrary intermediate extensions between K and L. Prove that there exists a surjective morphism of K-algebras from $F_1 \otimes_K F_2$ to F, but these K-algebras are not necessarily isomorphic.