Exam M2 "Géométrie algébrique"

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By convention, all rings, fields and algebras are assumed to be commutative. Every statement given in the lecture notes can be used without proof. In each exercise, it is allowed to use the result of a question in a further question. The most difficult (or with a solution longer to write down) questions according to the author of this exam are signaled by (*). The four exercices are independent.

Exercise 1 : Right or wrong ? (5 points)

Among the following statements, say which ones are right and which ones are wrong. Prove the right statements and give a counterexample for the wrong statements.

1. Let X be a scheme of finite type over an algebraically closed field k. If X is regular, then X is integral.

2. Let X and Y be non empty, notherian, finite-dimensional schemes. Let $f : X \to Y$ be a flat morphism. Then $\dim X \geq \dim Y$.

3. (*) Let X be a scheme of finite type over a field k. Then for every field extension k' of k, the scheme $X \times_k k'$ has same dimension as X.

4. Let X be a projective scheme over a noetherian ring A. Let $i : Y \to X$ be an A-morphism, assume that i is a closed immersion. If $\mathcal{L}$ is an ample invertible sheaf on X, then the sheaf $i^*\mathcal{L}$ is ample on Y.

Exercise 2 : k-points of a k-scheme (6 points)

Let k be a field and let X be a scheme of finite type over k. The scheme X is said to satisfy the property (P) if for every non empty open subset U of X, the set $U(k)$ of k-points of U is non empty (in other terms: U contains a closed point with residue field k).

1. Assume k algebraically closed. Does X always satisfy (P) ? Same question if we assume only k separably closed of positive characteristic.

For the whole exercise, we assume now that X and Y are schemes of finite type over k and we set $Z = X \times_k Y$.

2. Show that the projection $p : Z \to Y$ is a flat morphism.

3. Let m be a k-point of Y. Show that the fibre Z_m of p at m is a k-scheme isomorphic to X.

4. From now on we assume that X and Y satisfy (P).

a) Let U be a non empty open subset of Z. Show that the set $p(U)$ contains a k-point m of Y.

b) Show that Z satisfies (P).

Exercise 3 : An example of sheaf of ideals (6 points)

1. (*) Let $\operatorname{Spec} A$ be an affine scheme, covered by finitely many principal open subsets $D(g_i)$ with $g_i \in A$. Let f and s be two elements of A; denote by f_i and s_i their respective images in the localisation A_{g_i}. Show that if for all i we have $s_i \in f_i A_{g_i}$, then $s \in fA$.

2. (*) Let X be a scheme. Let $f \in \mathcal{O}_X(X)$. Show that one can define a sheaf of ideals $\mathcal{I}_f$ on X as follows : for every affine open subset U, $\mathcal{I}_f(U)$ is the ideal $f_{|U}\mathcal{O}_X(U)$ of $\mathcal{O}_X(U)$ generated by the restriction of f to U.

3. Show that $\mathcal{I}_f$ is quasi-coherent.

4. Assume further that X is noetherian. Is $\mathcal{I}_f$ always coherent ?

5. Give an example of scheme X and of $f \in \mathcal{O}_X(X)$ such that $\mathcal{I}_f$ is not an invertible sheaf.

Exercise 4 : Projective morphisms and cohomology (4 points)

Let r be a positive integer. Let X and Y be noetherian schemes. Let $i : X \to \mathbf{P}^r_Y := \mathbf{P}^r_{\mathbf{Z}} \times_{\mathbf{Z}} Y$ be a closed immersion and $f : X \to Y$ the corresponding projective morphism (f is obtained by composing i with the second projection $\mathbf{P}^r_Y \to Y$). Denote by $g : \mathbf{P}^r_Y \to \mathbf{P}^r_{\mathbf{Z}}$ the first projection and set, for every $n \in \mathbf{Z}$: $\mathcal{O}_Y(n) = g^*\mathcal{O}(n)$ and $\mathcal{O}_X(n) = i^*\mathcal{O}_Y(n)$. Let $\mathcal{F}$ be a coherent sheaf on X, set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

1. Show that exists $n_0 > 0$ such that for every $n \geq n_0$, the natural morphism of sheaves $f^*f_*(\mathcal{F}(n)) \to \mathcal{F}(n)$ is surjective.

2. Show that there exists $n_1 > 0$ such that for every $n \geq n_1$ and every $i > 0$, we have $R^i f_*(\mathcal{F}(n)) = 0$.