Midterm exam M2 "Géométrie algébrique"

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By convention, all rings, fields and algebras are assumed to be commutative. Every statement in the lecture notes can be used without proof. In each exercise, it is allowed to use the result of a question in a further question. The most difficult questions (according to the author of this exam) are usually at the end of each part.

The goal of the problem is to study a few properties of closed points with respect to morphisms between finite type schemes over a field. Parts I and II are independent.

Part I : Examples and counterexamples (9 points).

1. Let k be a field. Let X and Y be finite type schemes over k. Let $f : X \to Y$ be a k-morphism. Show that if x is a closed point of X, then $y := f(x)$ is a closed point of Y (hint: consider the respective residue fields $k(x)$ and $k(y)$ of x, y).

2. Take $k = \mathbf{R}$, $Y = \mathbf{A}^1_k = \operatorname{Spec}(k[t])$, and define the affine scheme X by the equation $t_1^2 + t_2^2 + 1 = 0$, that is: $X = \operatorname{Spec}(k[t_1, t_2]/(t_1^2 + t_2^2 + 1))$. Let $f : X \to Y$ be the k-morphism defined by $(t_1, t_2) \mapsto t_1$, in other words f corresponds to the homomorphism of k-algebras $k[t] \to k[t_1, t_2]/(t_1^2 + t_2^2 + 1)$ that sends t to the class of t_1.

a) Give an example of closed point $x \in X$ whose residue field $k(x)$ is the same as the residue field $k(y)$ of $y := f(x)$.

b) Give an example of closed point $x \in X$ whose residue field $k(x)$ satisfies $[k(x) : k(y)] > 1$.

3. Give an example of morphism of schemes $f : X \to Y$ such that there exists a closed point $x \in X$ such that: $y := f(x)$ is not a closed point of Y.

4. Let X and Y be schemes of finite type over a field k. Denote by p_1, p_2 the projections $X \times_k Y \to X$ and $X \times_k Y \to Y$. Let E be the set of closed points of $X \times_k Y$, E_1 the set of closed points of X, E_2 the set of closed points of Y. Show that the map $E \to E_1 \times E_2$ defined by $m \mapsto (p_1(m), p_2(m))$ is surjective.

Part II : Points with same residue field (13 points).

In what follows, k is a field of characteristic zero. The aim of this part is to show the following statement: Let X and Y be finite type k-schemes. Let $f : X \to Y$ be a k-morphism such that $\dim \overline{f(X)} \geq 1$, where $\overline{f(X)}$ stands for the closure of $f(X)$ in Y. Then f satisfies the following property:

(*) There exists a closed point $x \in X$ such that the residue field $k(x)$ satisfies $k(x) = k(y)$, where $k(y)$ is the residue field of $y := f(x)$.

1. Let $\alpha : X' \to X$, $f : X \to Y$, and $\beta : Y \to Y'$ be morphisms of finite type k-schemes. Let $f' = \beta \circ f \circ \alpha : X' \to Y'$, assume $\dim \overline{f'(X')} \geq 1$.

a) Show that if f' satisfies (*), then so does f.

b) Deduce that to show (*), we can assume further that X is reduced.

c) Show that we can actually reduce to the case when X is integral.

2. From now on we suppose that X is integral. Show that to prove (*), we can assume further that X and Y are affine schemes.

3. Until the end we assume that $X = \operatorname{Spec} B$ with B integral and $Y = \operatorname{Spec} A$; denote by $\varphi : A \to B$ the k-algebra homomorphism associated to f.

a) Prove that if we want to show (*), we can reduce to the case when φ is injective and $\dim Y \geq 1$.

b) Then show that there exists an injective k-homomorphism $k[t] \to A$, and that we can assume $Y = \mathbf{A}^1_k$, that is: $A = k[t]$.

c) Now set $S = A - \{0\} = k[t] - \{0\}$, and let I be a maximal ideal of $S^{-1}B$. Let B' be the image of B in $L := S^{-1}B/I$. Show that the composite map $A \to B \to B'$ is injective. Finally prove (*), assuming that the following algebraic statement is known:

Let C be a finite type k-algebra containing $A := k[t]$ as a subalgebra. Suppose that the field $L := \operatorname{Frac} C$ is a finite extension of $k(t) = \operatorname{Frac} A$. Then there exists a maximal ideal $\wp$ of C such that $A/(\wp \cap A) = C/\wp$.