Partial exam "Algebraic Geometry" (2 hours)

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All rings and algebras are assumed to be commutative.

Exercise 1 : Right or wrong ?

Among the following statements, say which one are right and which one are wrong. Prove the right statements and give a counter-example for the wrong statements.

1. Let k be a field. Let X and Y be two integral k-schemes of finite type. Then the scheme $X \times_k Y$ is integral.

2. Let X be an integral scheme. Then for every non empty open subset U of X, the restriction homomorphism $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is injective.

3. Let $f : X \to Y$ be a morphism of finite type. If for every $y \in Y$, the set $f^{-1}(\{y\})$ is finite, then f is a finite morphism.

4. Le k be a field and let n be a positive integer. Let I be a homogeneous ideal of the graded ring $k[T_0, ..., T_n]$. Then the scheme $\mathrm{Proj}\,(k[T_0, ..., T_n]/I)$ is never affine.

Exercise 2 : Jacobson schemes

When I is an ideal of a ring, its radical is denoted $\sqrt{I}$.

1. Let $X = \mathrm{Spec}\,A$ be an affine scheme. Let $Z = V(I)$ be a closed subset of X, where I is an ideal of A with $\sqrt{I} = I$. The set of closed points of Z is denoted Z_0, and we let $\overline{Z_0}$ denote the closure of Z_0.

a) Let J be the intersection of those maximal ideals $\mathcal{M}$ such that $\mathcal{M} \supset \mathcal{I}$. Show that $\sqrt{J} = J$.

b) Show that $\overline{Z_0} = V(J)$.

c) Deduce from b) that $\overline{Z_0} = Z$ if and only if there exist maximal ideals $(\mathcal{M}_r)$ such that

$$I = \bigcap_r \mathcal{M}_r$$

2. A ring A is said to be a *Jacobson ring* if every prime ideal can be written as the intersection of maximal ideals. We shall say that a scheme X is Jacobson if for every closed subset $Z \subset X$, we have $\overline{Z_0} = Z$, where Z_0 is the set of closed points of Z. Prove that $\operatorname{Spec} A$ is Jacobson if and only if A is a Jacobson ring.

3. Let X be a scheme of finite type over a field k. Show that X is Jacobson.

Exercise 3 : Morphisms and functorial maps.

When X and T are schemes, we let $X(T)$ denote the set of morphisms from T to X. When $f : X \to Y$ is a morphism of schemes and T is a scheme, define a map $f_T : X(T) \to Y(T)$ by the formula $f_T(g) = f \circ g$ (for every morphism $g : T \to X$).

1. Show that if f is an isomorphism, then f_T is a bijection.

2. Prove the converse statement : if f_T is a bijection for every scheme T, then f is an isomorphism (hint : use the morphisms id_X et id_Y).

3. Assume further[1] that the intersection of two affine open subsets of X is affine, and similarly for Y. Suppose that for every *affine* scheme T, the map f_T is a bijection. On suppose que pour tout schéma *affine* T, Show that f is an isomorphism.

4. Let $(A_i)_{i \in I}$ be infinitely many non-zero rings. Set $X_i = \operatorname{Spec} A_i$, and let X be the scheme obtained as the disjoint union of the X_i (in particular each X_i is a open subset of X). Set $A = \prod_{i \in I} A_i$ and $Y = \operatorname{Spec} A$.

a) Show that there exists a unique morphism $u : X \to Y$ whose restriction to each X_i is induced by the projection $A \to A_i$.

b) Prove that u is not an isomorphism.

c) ("Additional question" !) Show that for every affine scheme $T = \operatorname{Spec} B$, the map $g \mapsto g \circ u$ is a bijection from $T_Y = \operatorname{Mor}(Y, T)$ to $T_X = \operatorname{Mor}(X, T)$.

(likely) awards. Exercise 1 : 6 points. Exercise 2 : 7 points. Exercise 3 (without the additional question) : 7 points. Additional question : 2 points.

[1] For example, this hypothesis holds if X and Y are separated.