Partial exam "Algebraic Geometry" (2 hours)

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December 13, 2006; typed or manuscript notes allowed.

All rings and algebras are assumed to be commutative.

Exercise 1 : Right or wrong ? (6 points)

Among the following statements, say which one are right and which one are wrong. Prove the right statements and give a counter-example for the wrong statements.

1. Let X be a proper scheme with X of finite type over a field k. Then the ring $\mathcal{O}_X(X)$ is isomorphic to k.

2. Let X be a non empty algebraic variety over an algebraically closed field k. Then the set $X(k) = \mathrm{Mor}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k, X)$ of k-points of X is non empty.

3. Let X and Y be affine and integral schemes. Let Z be a dense subset of X. Let f and g be morphisms from X to Y; assume that for every point $z \in Z$, we have $f(z) = g(z)$. Then the morphisms f and g coincide.

4. Let $f : X \to Y$ be a morphism of schemes and let U be an open subset of Y. If $f(X) \subset U$, then there exists a morphism of schemes $g : X \to U$ such that $f = i \circ g$, where $i : U \to Y$ is the open immersion corresponding to U.

Exercise 2 : "Retract-rational" varieties (14 points)

It is allowed to admit the result of a question to solve a following question.

1. Let A be a local ring with maximal ideal $\mathcal{M}$. Let x be the closed point of $\mathrm{Spec}\, A$ (corresponding to $\mathcal{M}$). Show that for every point $y \in \mathrm{Spec}\, A$, the point x belongs to the closure $\overline{\{y\}}$ of y.

2. Let k be a field and let A be a local k-algebra [1] with maximal ideal $\mathcal{M}$ and residue field L. For each k-scheme X, consider the "reduction modulo $\mathcal{M}$ map" [2]

$$\varphi_{X,A} : X(A) \to X(L)$$

a) Show that if X is the affine space $\mathbf{A}^n_k$, then $X(A) = A^n$.

b) Let X be a k-scheme and let U be a non empty open subset of X. Show that if $\varphi_{X,A}$ is surjective, then $\varphi_{U,A}$ is surjective (use **1.**).

c) Let U be an integral k-variety. Assume that the following condition holds :

(*) There exist an integer n, a non empty open subset Y of the affine space $\mathbf{A}^n_k$, and k-morphisms $f : U \to Y$ and $g : Y \to U$, such that $g \circ f$ is the identity morphism id_U of U.

Show that $\varphi_{U,A}$ is surjective.

An integral k-variety X is said to be *retract-rational* (this notion is due to D. Saltman) if there exists a non empty open subset U of X satisfying (*).

3. Let X be an integral k-variety. Suppose that there exists a non empty open subset V of X such that for every local k-algebra A, the map $\varphi_{V,A} : V(A) \to V(L)$ (where L is the residue field of A) is surjective. The goal of the following questions is to prove that X is retract-rational.

a) Show that it is sufficient to prove this result for $V = X$ and $X = \operatorname{Spec} B$, where $B = k[T_1, ..., T_n]/\wp$, with $\wp$ prime ideal of the polynomial ring $k[T_1, ..., T_n]$.

b) Set $R = k[T_1, ..., T_n]$ and $A = R_\wp$; denote by $L = \operatorname{Frac} B$ the function field of X, which also is the residue field of A. Show that there exist a non empty and affine open subset W of $\mathbf{A}^n_k$ and k-morphisms $g : \operatorname{Spec} L \to W$ and $h : W \to X$ such that $h \circ g : \operatorname{Spec} L \to X$ is the canonical morphism (corresponding to the inclusion map $B \to L$).

c) Deduce from b) that there exist a non empty open subset U of X and a k-morphism $f : U \to W$ such that $h \circ f$ is the open immersion $U \to X$.

d) Prove that X is retract-rational.

[1] this just means that as a ring, A is local.

[2] Namely if $u \in X(A) = \mathrm{Mor}_{\operatorname{Spec} k}(\operatorname{Spec} A, X)$ is an A-point of X, then its image by $\varphi_{X,A}$ is the L-point of X defined by the k-morphism $u \circ \pi : \operatorname{Spec} L \to X$, where $\pi : \operatorname{Spec} L \to \operatorname{Spec} A$ is the k-morphism associed to the canonical surjection $A \to L$.