Exam of the course "Algebraic Geometry"

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January 30, 2009; length : 3h; Notes allowed.

By convention, all rings, fields and algebras are assumed to be commutative. In each exercise, it is allowed to use the result of a question in a further question.

Exercise 1. (4 points)

Questions 1. and 2. are independent.

1. a) Let $X = \operatorname{Spec} A$ be an affine scheme. Show that if x and y are points of X with $x \neq y$, then there exists an open subset U of X such that U contains exactly one of the points x, y (in other words: the cardinality of $U \cap \{x, y\}$ is 1).

b) Does the result of a) still hold for an arbitrary scheme X ?

2. Let $X = \operatorname{Spec} A$ be an affine scheme. Assume that X is connected. Show that the only solutions of the equation $a^2 = a$ in A are $a = 0$ and $a = 1$.

Exercise 2. (5 points)

Say which ones of the following assertions are true and which ones are wrong. Give a proof for the true ones and a counterexample for the wrong ones.

a) Let X and Y be noetherian and regular schemes. Then every surjective morphism $X \to Y$ is smooth.

b) Let $f : X \to Y$ be a surjective morphism of schemes. If X is regular, then Y is normal.

c) Let X and Y be varieties over an algebraically closed field k. If X and Y are regular schemes, then $X \times_k Y$ is regular.

Exercise 3. (9 points)

Let X be an integral and noetherian scheme with function field K. Denote $\mathcal{K}$ the constant sheaf K on X.

1. a) Show that the $\mathcal{O}_X$-module $\mathcal{K}$ is quasi-coherent. Is it coherent ?

b) Show that $\mathcal{O}_X$ is a subsheaf of $\mathcal{K}$.

From now on, the quotient sheaf $\mathcal{K}/\mathcal{O}_X$ is denoted $\mathcal{F}$.

2. Let $U = \operatorname{Spec} A$ be an affine open subset of X. Show that the restriction $\mathcal{F}_{|U}$ is isomorphic (as an $\mathcal{O}_U$-module) to $\widetilde{(K/A)}$, where K/A is viewed as an A-module.

3. Assume further that X is of dimension 1.

a) Show that if Ω is a non empty open subset of X, then $X - \Omega$ is a finite set.

b) Let U be an open subset of X and let $s \in \mathcal{F}(U)$. Show that there exists a finite subset F of U such that for every $x \in (U - F)$, we have $s_x = 0$, where s_x is the restriction of s to the stalk $\mathcal{F}_x$ (hint: start with the case when U is affine).

c) Deduce from b) that there exists an exact sequence of sheaves on X:

$$0 \to \mathcal{O}_X \to \mathcal{K} \to \bigoplus_{x \in X} Q_x \to 0$$

where Q_x is the sheaf defined by $Q_x(U) = K/\mathcal{O}_{X,x}$ if the open subset U contains x, and $Q_x(U) = 0$ if U does not contain x.

4. Let k be a field and let $X = \mathbf{P}^1_k$ be the projective line over k. Let $x_1, ..., x_n$ be pairwise distinct points of X. Let $f_1, ..., f_n \in K$, where K is the function field of X. Show that there exists $f \in K$ such that for every $i \in \{1, ..., n\}$, we have $f - f_i \in \mathcal{O}_{X,x_i}$.

Exercise 4. (4 points)

Let $f : X \to Y$ be a morphism of noetherian and separated schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

1. Show that if f is an affine morphism, then the cohomology groups $H^p(Y, f_*\mathcal{F})$ and $H^p(X, \mathcal{F})$ are isomorphic for every $p \geq 0$ (hint:use Čech cohomology).

2. Assume that Y is not affine and X is affine. Show that there exists a coherent sheaf $\mathcal{G}$ on Y such that $H^1(Y, \mathcal{G})$ and $H^1(X, f^*\mathcal{G})$ are not isomorphic.

3. Show that the result of 1. does not hold in general if f is an open immersion.