Exam "Algebraic Geometry" (M2)

Université Paris-Sud (D. Harari)

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By convention, all rings, fields and algebras are assumed to be commutative. In each exercise, it is allowed to use the result of a question in a further question.

The sign (*) advertises a difficult question (from the point of view of the author).

Exercise 1 : Morphisms between curves (7 points).

Let k be a field. Let X and Y be integral k-schemes; assume that X and Y are of dimension 1 and of finite type over k. Let $f : X \to Y$ be a non constant (that is: $f(X)$ does not consist of a single point) morphism of k-schemes Let $x \in X$, set $y = f(x)$.

1. Assume further that Y is normal.

a) Show that the fibre X_y of f at y has dimension 0.

b) Let $k(y)$ be the residue field of y. Show that the morphism $X_y \to \operatorname{Spec}(k(y))$ (induced by f) is finite.

2. Let us not assume anymore that Y is normal. Let $\widetilde{X}$, $\widetilde{Y}$ be the respective normalisations of X, Y (we recall that the corresponding morphisms $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ are finite and surjective).

(*) a) Show that the fibre $\widetilde{X}_y$ of $f \circ p : \widetilde{X} \to Y$ at y has dimension 0.

b) Show that $\widetilde{X}_y = \widetilde{X} \times_X X_y$.

(*) c) Do the results in 1.a) and 1.b) still hold ?

Exercise 2 : An example of regular scheme (5 points).

Let k be a field. Let n be a positive integer. Denote R the polynomial ring (in n^2 variables) $R = k[X_{11}, ..., X_{nn}]$. Define $P \in R$ by the formula $P(X_{11}, ..., X_{nn}) = \det([X_{ij}])$, where $[X_{ij}]$ stands for the (n,n)-matrix (with entries in R) such that the entry that lies in the i-th row and the j-th column is X_{ij}.

The fact that for every $\lambda \in k$, the polynomial $P - \lambda$ is irreducible over k can be used without proof.

Let Y be the closed subsscheme of the affine space $\mathbf{A}_k^{n^2}$ given by the equation $P(X_{11}, ..., X_{nn}) = 1$; in other words $Y = \mathrm{Spec}\,(R/(P-1))$.

1. Show that Y is a regular scheme (hint: expand the determinant along a row or a column).

(*) **2.** Show that Y contains a dense open subset U that is isomorphic (as a k-scheme) to an open subset of the affine space $\mathbf{A}_k^{n^2-1}$ (hint: observe that $P = X_{11}Q + V$, where Q and V do not contain the indeterminate X_{11}).

Exercise 3 : Birational morphisms and $\mathcal{O}_X$-modules (4 points).

Let X and Y be integral and noetherian schemes with respective function fields $K(X)$, $K(Y)$. Assume that Y is normal. Let $f : X \to Y$ be a projective and surjective morphism. Suppose that f is *birational*, that is: the field homomorphism $K(Y) \to K(X)$ (induced by f) is an isomorphism.

1. Assume further (in the whole question 1.) that $Y = \mathrm{Spec}\, A$ is affine. Denote B the A-algebra $\Gamma(Y, f_*\mathcal{O}_X)$.

a) Show that A and B have same fraction field.

b) Deduce from a) that $A = B$.

2. Show that $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Exercise 4 : Cohomology of coherent and locally free sheaves (5 points).

1. Let X be a noetherian and separated scheme. Show that there exists a positive integer N such that $H^i(X, \mathcal{F}) = 0$ for every integer $i > N$ and every quasi-coherent sheaf $\mathcal{F}$ on X.

2. Let X be a projective scheme over a noetherian ring A.

a) Let $\mathcal{F}$ be a coherent sheaf on X. Show that there exists a locally free $\mathcal{O}_X$-module $\mathcal{G}$, with $\mathcal{G}$ of finite rank, such that $\mathcal{F}$ is isomorphic to a quotient of $\mathcal{G}$.

(*) b) Let $d \in \mathbf{N}$. Assume that $H^i(X, \mathcal{G}) = 0$ for every integer $i > d$, and every locally free $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{G}$ is of finite rank. Show that $H^i(X, \mathcal{F}) = 0$ for every integer $i > d$ and every coherent sheaf $\mathcal{F}$ on X.