# Midterm exam M2 "Géométrie algébrique"

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*By convention, all rings, fields and algebras are assumed to be commutative. In each exercise, it is allowed to use the result of a question in a further question.*

**Exercise 1 : Closed points of a scheme (8 points).**

Let $X$ be a non empty scheme.

**1.** Let $F$ be a minimal irreducible closed subset of $X$ (that is: if $F'$ is an irreducible closed subset of $X$ with $F' \subset F$, then $F' = F$).

a) Let $x$ be a point of $F$. What is the closure $\overline{\{x\}}$ of the point $x$ ?

b) Show that $F$ consists of one single point.

**2.** Let $G$ be a closed subset of $X$. Let $x$ be a closed point of $X$ with $x \in G$. Assume that $G$ contains a point $y$ such that $\overline{\{y\}} = G$.

a) Let $U = \operatorname{Spec} A$ be an open subset of $X$ with $x \in U$. Show that the closed subset $U \cap G$ of $U$ is the closed subset $V(\wp)$ of $\operatorname{Spec} A$, where $\wp$ is the prime ideal of $A$ corresponding to $y$.

b) Deduce that $\mathcal{O}_{X,y}$ is a localization of $\mathcal{O}_{X,x}$.

**3.** Suppose further that $X$ is a noetherian scheme.

a) Show that $X$ contains a closed point $x$.

b) Assume further that for every closed point $x$ of $X$, the ring $\mathcal{O}_{X,x}$ is reduced. Is the scheme $X$ necessarily reduced ?

**Exercise 2 : Fibred products (6 points).**

Let $S$ be a scheme. Let $X$ and $Y$ be $S$-schemes with respective structural morphisms $f : X \to S$ and $g : Y \to S$.

**1.** Assume that the fibred product $X \times_S Y$ is not empty. Show that there exist $x \in X$ and $y \in Y$ such that $f(x) = g(y)$.

**2.** Suppose conversely that there exist $x \in X$ and $y \in Y$ such that $f(x) = g(y)$. Set $s = f(x) = g(y)$.

a) Show that there exists a field $K$ satisfying: there exist morphisms $f_1 : \operatorname{Spec} K \to X$ and $g_1 : \operatorname{Spec} K \to Y$ with $f \circ f_1 = g \circ g_1$.

b) Deduce that $X \times_S Y$ is not empty.

**Exercise 3 : $k$-algebras of finite type (6 points).**

Let $k$ be an algebraically closed field.

**1.** Let $B$ be a $k$-algebra with $B \neq 0$. Let $b$ be a non nilpotent element of $B$ and let $\rho : B \to B_b$ be the localization homomorphism.

a) Show that the localized ring $B_b$ has a maximal ideal $I$.

b) Assume further that $B$ is a finitely generated $k$-algebra. Show that $\rho^{-1}(I)$ is a maximal ideal of $B$. Is it possible that $\rho^{-1}(I)$ contains $b$ ?

**2.** Let $X = \operatorname{Spec} A$ be an affine scheme. Assume that $X$ is reduced and of finite type over $k$. Let $f : X \to \mathbf{A}_k^n$ be a $k$-morphism. Denote $\varphi : k[T_1, ..., T_n] \to A$ the $k$-algebra homomorphism corresponding to $f$. Set $f_i = \varphi(T_i)$ $(i = 1, ..., n)$ and identify the set $\mathbf{A}_k^n(k)$ of $k$-points of the affine space $\mathbf{A}_k^n$ to $k^n$.

a) Show that for every $k$-point $x$ of $X$, the equality $f(x) = (f_1(x), ...f_n(x))$ holds, where $f(x)$ is viewed as an element of $\mathbf{A}_k^n(k) = k^n$ and $f_i(x)$ is the evaluation of $f_i \in A = \mathcal{O}_X(X)$ at $x$.

b) Prove that if an element $\psi$ of $A$ satisfies $\psi(x) = 0$ for every $k$-point $x$ of $X$, then $\psi = 0$.