"Partial" exam M2 "Géométrie algébrique"

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December 3, 2008; 2h; notes allowed

By convention, all rings, fields and algebras are assumed to be commutative. In each exercise, it is allowed to use the result of a question in a further question.

Exercise 1 : Principal and non principal open subsets (7 points).

We recall that if R is a Dedekind ring, then every non-zero ideal I of R has a unique (up to permutation) decomposition $I = \wp_1^{\alpha_1}...\wp_r^{\alpha_r}$, where each $\wp_i$ is a prime ideal of R and $\alpha_i \in \mathbf{N}^*$.

Let $X = \operatorname{Spec} A$ be an affine scheme and let $\wp$ be a closed point of X.

1. Assume that there exists $n > 0$ such that $\wp^n = fA$ for some $f \in A$. Show that $X - \{\wp\}$ is the principal open subset $D(f)$.

2. Assume that A is a Dedekind ring. Show that if $X - \{\wp\}$ is a principal open subset of X, then there exists $n > 0$ and $f \in A$ such that $\wp^n = fA$.

3. In this question we suppose that A is an integral domain of dimension ≥ 2, of finite type (as an algebra) over a field k, and satisfying

$$A = \bigcap_{Q \in \operatorname{Spec} A, \operatorname{ht} Q = 1} A_Q$$

(e.g. A is normal). Show that the open subset $X - \{\wp\}$ cannot be affine.

4. Let A be a valuation ring with quotient field K. Assume that the valuation v is a surjective map from K^* to the field of rational numbers $\mathbf{Q}$. [1] Let $\mathcal{M}$ be the maximal ideal of A (consisting of those $x \in K$ such that $v(x) > 0$). Show that $X - \{\mathcal{M}\}$ is a principal open subset of X, but there is no $n \in \mathbf{N}^*$ such that $\mathcal{M}^n$ is a principal (=generated by one element) ideal of A.

[1] e.g. A is the ring of integers of $\overline{\mathbf{Q}_p}$ with p prime number.

Exercise 2 : Dimension and product (8 points).

We recall that if R is a ring, B and C are R-algebras and S is a multiplicative subset of B, then we have $(B_S \otimes_R C) = (B \otimes_R C)_{S'}$, where S' is the image of S in $(B \otimes_R C)$ ("tensor product commutes with localisation").

Let k be a field. Let X and Y be two k-schemes of respective dimensions r and s (with r, s finite).

1. Assume that X is affine and of finite type over k.

a) Show that there exists a finite and surjective morphism $f : X \to \mathbf{A}^r_k$.

b) Assume further that Y is affine and noetherian. Show that $X \times_k Y$ is of dimension $r + s$.

2. Assume only that X is a scheme of finite type over k and Y is a noetherian k-scheme. Show that we still have $\dim(X \times_k Y) = r + s$.

3. Let $X = \operatorname{Spec}(k(T))$.

a) Let L be a field extension of k. Show that $X \times_k L$ is isomorphic to $\operatorname{Spec} A$, where A is the localisation of $L[T]$ with respect to some multiplicative subset S (and say what S is).

b) Set $Y = X = \operatorname{Spec}(k(T))$. Show that the formula $\dim(X \times_k Y) = \dim X + \dim Y$ does not hold.

Exercise 3 : Examples (5 points).

The answers shoud be thoroughly justified.

1. Give an example of a locally noetherian scheme X such that X is not noetherian.

2. Let $f : X \to Y$ be a morphism of finite type. For $y \in Y$, denote $k(y)$ the residue field of y and $X_y = X \times_Y \operatorname{Spec}(k(y))$ the fibre of f at y. Give an example when the morphism $X_y \to \operatorname{Spec}(k(y))$ (induced by f) is finite for every $y \in Y$, but the morphism f is not finite.

3. Give an example of an integral variety X over the field $\mathbf{R}$, such that the function field of X contains $\mathbf{C}$, but the structural morphism $X \to \operatorname{Spec} \mathbf{R}$ does not factorize through the canonical morphism $\operatorname{Spec} \mathbf{C} \to \operatorname{Spec} \mathbf{R}$.