Exam M2 "Algebraic Geometry" (D. Harari).
January 31, 2007; 3 hours; typed or manuscript notes allowed

All rings are assumed to be commutative. In each exercise, it is allowed to admit the result of a question to solve a further question.

Exercise 1 (10 points).

We shall say that a morphism of schemes $f : X \to Y$ is *constant* if $f(x) = f(y)$ for all points $x, y \in X$. The following result (consequence of Hilbert's Nulstellensatz) can be used without proof : if X and Y are varieties over an algebraically closed field k with X reduced, and f, g are k-morphisms $X \to Y$, then $f = g$ as soon as $f(x) = g(x)$ for all k-points x of X.

1. Let $f : X \to Y$ be a morphism of schemes with X integral. Assume that for every affine open subset U of X, the morphism $f_{|U} : U \to Y$ (obtained by restriction of f) is constant. Show that f is constant.

2. Let $Y = \operatorname{Spec} A$ be an affine scheme and let $f : X \to Y$ be a morphism of schemes with X integral. Show that for every affine open subset $U = \operatorname{Spec} B$ of X, the homomorphism $A \to B$ (associated to $f_{|U} : U \to Y$) takes values in the subring $\Gamma(X, \mathcal{O}_X)$ of B.

3. Let X be a projective and geometrically integral scheme over a field k. Let Y be an affine scheme. Prove that every morphism $f : X \to Y$ is constant.

4. Let X, Y, Z be varieties over an algebraically closed field k. Let $p_1 : X \times_k Y \to X$ and $p_2 : X \times_k Y \to Y$ be the projections. Assume that X and Y are integral, with X projective over k. For each k-point y of Y, the k-morphism $x \mapsto (x, y)$ from X to $X \times_k Y$ is denoted φ_y; namely $p_1 \circ \varphi_y : X \to X$ is the identity map, and $p_2 \circ \varphi_y : X \to Y$ is the constant map with image y.

Let $f : X \times_k Y \to Z$ be a k-morphism satisfying the following condition : there exists $y_0 \in Y(k)$ such that the morphism $f \circ \varphi_{y_0} : X \to Z$ is constant with image a k-point z_0 of Z.

a) Let V be an affine open subset of Z with $z_0 \in V$. Set $F = Z - V$. Show that $p_2(f^{-1}(F))$ is a closed subset of Y with $p_2(f^{-1}(F)) \neq Y$.

b) Deduce from a) that there exists a non empty open subset U of Y such that for every $y \in U(k)$, the morphism $f \circ \varphi_y : X \to Z$ takes values in V; then show that $f \circ \varphi_y$ is constant.

c) Let $x_0 \in X(k)$. Denote by ψ_0 the morphism $y \mapsto (x_0, y)$ from Y to $X \times_k Y$. Prove that the morphisms $f \circ \psi_0 \circ p_2$ and f coincide on $X \times_k U$.

d) Deduce from c) that there exists a k-morphism $g : Y \to Z$ such that $f = g \circ p_2$ ("rigidity lemma").

Exercise 2 (5 points).

Let X be a projective and geometrically integral scheme over a field k. Let K be the function field of X.

1. Let D be a Cartier divisor on X. Assume that there exist two effective Cartier divisors D_1, D_2 on X such that $D \simeq D_1$ and $-D \simeq D_2$.

a) Show that D can be written $D = (U_i, f_i)$, where (U_i) is an open cover of X and $f_i \in K^*$, with the condition : there exist $u, v \in K^*$ such that each f_i satisfies $f_i.u \in \mathcal{O}_X(U_i)$ and $f_i^{-1}v \in \mathcal{O}_X(U_i)$.

b) Show that uv is a constant element $\lambda \in k^*$, and deduce from this that $D \simeq 0$.

2. Let $\mathcal{L}$ be an invertible sheaf on X. Prove that if both $H^0(X, \mathcal{L})$ and $H^0(X, \mathcal{L}^{-1})$ are $\neq \{0\}$, then $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$.

3. Let $\bar{k}$ be an algebraic closure of k; set $\overline{X} = X \times_k \bar{k}$. Denote by π the projection $\overline{X} \to X$. The inverse image $\mathcal{L} \to \pi^*\mathcal{L}$ defines a homomorphism $\theta : \operatorname{Pic} X \to \operatorname{Pic} \overline{X}$. Show that θ is injective.

Exercise 3 (5 points).

Let A be a ring. Recall that if $0 \to M \to M' \to M'' \to 0$ is an exact sequence of A-modules with M'' flat, then M is flat if and only if M' is flat. In this exercise, we consider a noetherian ring A and a coherent $\mathcal{O}_X$-module $\mathcal{F}$ on $X = \mathbf{P}^n_A = \operatorname{Proj}(A[T_0, ..., T_n])$. We assume that for every affine open subset U of X, the A-module $\Gamma(U, \mathcal{F})$ is flat.

1. Let $\mathcal{U}$ be the standard open cover of X given by $D_+(T_0)$,...,$D_+(T_n)$. Set $U_{i_0,...,i_p} = D_+(i_0) \cap ... \cap D_+(i_p)$. Show that for every $p \geq 0$ and every $m \in \mathbf{Z}$, the A-module $C^p(\mathcal{U}, \mathcal{F}(m)) = \prod_{i_0<...<i_p} \mathcal{F}(m)(U_{i_0,...,i_p})$ is flat.

2. Prove that there exists $m_0 \in \mathbf{N}$ such that for all $m \geq m_0$, the sequence

$$0 \to H^0(X, \mathcal{F}(m)) \to C^0(\mathcal{U}, \mathcal{F}(m)) \to C^1(\mathcal{U}, \mathcal{F}(m)) \to ... \to C^n(\mathcal{U}, \mathcal{F}(m)) \to 0$$

is exact.

3. Deduce from this that if we assume further that A is a local ring, then $H^0(X, \mathcal{F}(m))$ is a free and of finite type A-module for all $m \geq m_0$.

4. Keep the assumption A local. Let $B = A[T_0, ..., T_n]$. Show that there exists a graded B-module M with M free as an A-module, and such that $\mathcal{F}$ is isomorphic to the sheaf $\widetilde{M}$ on X.