Test for the M2 course "Class field Theory"

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May 23, 2013; length : 3h; written notes allowed.

Every result that has been stated in the course can be used without proof. In each exercise, it is allowed to use the result of a question (even if it has not been solved) to solve a further question. The questions at the end of each exercise are usually the hardest ones.

Exercise 1 : Right or wrong ? (6 points).

Among the following statements, prove the right ones and give a counterexample for the wrong ones (first say whether the statement is right or wrong).

1. Let K be a p-adic field. Let $n > 0$. Then the groups $H^1(K, \mu_n)$ and $H^1(K, \mathbf{Z}/n)$ are finite and have same cardinality.

2. Let k be a field of characteristic zero. Let K be an algebraic extension of k. Assume $\operatorname{Br} K = 0$. Then there exists a finite extension k' of k, with $k' \subset K$ and such that $\operatorname{Br} k' = 0$.

3. Let G be a finite group. Let A be a G-module. If $H^1(G, A) = 0$, then $H^1(H, A) = 0$ for every subgroup H of G.

4. Let k be a field of characteristic zero with algebraic closure $\bar{k}$. Let i be a positive integer. Assume that the cohomological dimension $\operatorname{cd}(k)$ of k satisfies $\operatorname{cd}(k) \leq i$. Then $H^{i+1}(k, \bar{k}^*) = 0$.

Exercise 2 : Local fields (6 points).

Let K be a p-adic field. Denote by K^{ab} the maximal abelian extension of K. Let U_K^1 be the subgroup of K^* consisting of those x such that $v(1-x) \geq 1$, where v is the valuation on K.

1. Show that the cohomological dimension $\operatorname{cd}(K^{\text{ab}})$ of K^{ab} is ≤ 1.

2. Set $G = \operatorname{Gal}(K^{\text{ab}}/K)$. Show that the p-cohomological dimension $\operatorname{cd}_p(G)$ of G is finite if and only if $\operatorname{cd}_p(U_K^1)$ is finite, where p stands for the characteristic of the residue field of K.

3. Let π be a uniformizing parameter of K. Denote by K_π the corresponding totally ramified extension in Lubin-Tate's theory. Show that the p-primary torsion $\mathrm{Br}\, K_\pi\{p\}$ of $\mathrm{Br}\, K_\pi$ is 0, but $\mathrm{Br}\, K_\pi\{\ell\} \neq 0$ for every prime number $\ell \neq p$.

Exercise 3 : Global fields (8 points).

Let k be a number field. A finite Galois extension K of k is said to be *unramified* if is unramified at every finite place of k and totally split at every archimedean place of k. Let p be a prime number.

Le K be a finite, Galois, unramified extension of k, whose Galois group $G = \mathrm{Gal}(K/k)$ is a p-group. Assume that K has no unramified extensions that are also cyclic of order p. Denote by $\Omega_{K,\infty}$ (resp. $\Omega_{K,f}$) the set of all archimedean (resp. finite) places of K.

1. a) Show that the cardinality $h(K)$ of the class group $\mathrm{Cl}\,(K)$ of K is not divisible by p.

b) Compute $\widehat{H}^q(G, \mathrm{Cl}\,(K))$ for each $q \in \mathbf{Z}$.

2. Let I_K be the idele group of K. Let I_K^1 be the subgroup of I_K defined by

$$I_K^1 := \prod_{w \in \Omega_{K,\infty}} K_w^* \times \prod_{w \in \Omega_{K,f}} U_w$$

where $U_w = \mathcal{O}_w^*$ is the unit group of the ring of integers of K_w^*.

Denote by $C_K = I_K/K^*$ the idele class group of K and let $E_K = K^* \cap I_K^1$ be the group of units of $\mathcal{O}_K$. We recall that $\mathrm{Cl}\,(K)$ is isomorphic to C_K/C_K^1, where $C_K^1 = I_K^1/E_K$ is the image of I_K^1 in C_K.

a) Prove that $\widehat{H}^q(G, I_K^1) = 0$ for every $q \in \mathbf{Z}$ (hint : proceed as in subsection 8.1. in the lecture notes).

b) Show that for all $q \in \mathbf{Z}$, there is an isomorphism $\widehat{H}^q(G, C_K) \simeq \widehat{H}^{q+1}(G, E_K)$.

3. a) Show that the groups $\widehat{H}^0(G, E_K)$ and $\widehat{H}^{-3}(G, \mathbf{Z})$ are isomorphic.

b) Deduce from a) that the rank of $\widehat{H}^{-3}(G, \mathbf{Z})$ (that is : its minimal number of generators) is at most r, where r is the number of archimedean places in k.