Test for the M2 course "Galois Cohomology and Number Theory"

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Every result that has been stated in the course can be used without proof. In each exercise, it is allowed to use the result of a question (even if it has not been solved) to solve a further question. The questions at the end of each exercise are usually the hardest ones.

In the whole test, the word "G-module" (for G a profinite group) is used for *discrete* G-module. The following *approximation theorem* can be used without proof: let k be a number field and let Ω_k be the set of all places of k. Then the image of k^* in $\prod_{v \in \Omega_k} k_v^*$ is dense for the direct product topology.

Exercise 1 : Cohomology of a particular G-module (6 points).

Let G be a profinite group.

1. Let M be a G-module, assumed isomorphic to $\mathbf{Z}^r$ (for some positive integer r) as an abelian group. Let H be an open normal subgroup of G such that H acts trivially on M. Show that $H^1(G, M)$ is isomorphic to $H^1(G/H, M)$.

2. Let M be a G-module isomorphic to $\mathbf{Z}$ as an abelian group. Show that there exists an open normal subgroup H of G, acting trivially on M and such that $[G : H] \le 2$.

3. Fix an open normal subgroup H of G such that $[G : H] = 2$. How many isomorphism classes of G-modules M are there such that M is isomorphic to $\mathbf{Z}$ as an abelian group and H acts trivially on M ?

4. Let H be an open normal subgroup of G with $[G : H] = 2$. Define a G-module M by: M is isomorphic to $\mathbf{Z}$ as an abelian group and the action of G on M is given by $g.x = x$ if $g \in H$, $g.x = -x$ if $g \notin H$.

a) Show that there is an exact sequence of G-modules:

$$0 \to \mathbf{Z} \to \mathbf{Z}[G/H] \to M \to 0$$

b) Compute $H^1(G, M)$.

Exercise 2 : p-adic fields (7 points).

Let p be a prime number and let K be a p-adic field. Set $G_K = \mathrm{Gal}(\overline{K}/K)$. We denote by $K(p)$ the *maximal p-extension* of K : by definition this means that the Galois group $G_K(p) := \mathrm{Gal}(K(p)/K)$ is the biggest pro-p-quotient of

G_K. Let $U_K = \mathcal{O}_K^*$ be the (multiplicative) group of invertible elements in the ring of integers $\mathcal{O}_K$.

The following result (see chapter 5, exercise 4) can be used without proof: for every p-primary $G_K(p)$-module A and every $i \geq 0$, the inflation homomorphism

$$H^i(G_K(p), A) \to H^i(G_K, A)$$

is an isomorphism. We also recall that there exists a finite group $(F, +)$ such that $(U_K, \times)$ is isomorphic to $(F \times \mathbf{Z}_p^N, +)$, where $N := [K : \mathbf{Q}_p]$.

1. In the whole question 1., we assume that K does not contain a primitive p-root of 1.

a) Show that $H^2(G_K(p), \mathbf{Z}/p\mathbf{Z}) = 0$.

b) Using a), compute the p-cohomological dimension of $G_K(p)$.

c) Compute (as a function of the integer N) the dimension of the $\mathbf{Z}/p\mathbf{Z}$-vector space $H^1(G_K(p), \mathbf{Z}/p\mathbf{Z})$ (hint: F is the torsion subgroup of U_K; start with the computation of F/pF).

2. We assume now (until the end of this exercise) that K contains all p-roots of 1.

a) Show that the p-cohomological dimension of $G_K(p)$ is 2.

b) Determine (as a function of N) the dimension of the $\mathbf{Z}/p\mathbf{Z}$-vector space $H^1(G_K(p), \mathbf{Z}/p\mathbf{Z})$.

Exercice 3 : Right or wrong ? (7 points).

Among the following statements, prove the right ones and give a counterexample for the wrong ones (first say whether the statement is right or wrong).

1. Let K be a p-adic field with absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. Let M be a finite G_K-module with cardinality r. Then the cardinality of $H^2(G_K, M)$ is at most r.

2. Let k be a number field. Let v be a place of k and let k_v be the associated completion of k. Then the restriction map $\mathrm{Br}\, k \to \mathrm{Br}\, k_v$ is surjective.

3. Let G be a finite group. Let p be a prime number dividing the cardinality of G. Then for every positive integer i, there exists a finite p-primary G-module M such that $H^i(G, M) \neq 0$ (hint: start with the case $G = \mathbf{Z}/p\mathbf{Z}$).

4. Let k be a number field. Set $G_k := \mathrm{Gal}(\bar{k}/k)$ and consider a finite G-module A. Let Ω_k be the set of all places of k. Then the image of $H^1(k, A)$ in $\prod_{v \in \Omega_k} H^1(k_v, A)$ (the latter equipped with the direct product topology) is a closed subset of $\prod_{v \in \Omega_k} H^1(k_v, A)$.