Exam of the M2 course "Galois Cohomology and Number Theory"

Université Paris-Sud (D. Harari)

May 19, 2011; length : 3h; written notes allowed.

Every result that has been stated during the course can be used without proof. In each exercise, it is allowed to use the result of a question to solve a further question.

Exercise 1 : Right or wrong ? (5 points).

Among the following statements, prove the right ones and give a counterexample for the wrong ones (first say whether the statement is right or wrong).

a) Let K be a p-adic field with absolute Galois group $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$. Then for every discrete Γ_K-module M and every integer $i \geq 1$, the group $H^i(\Gamma_K, M)$ is finite.

b) Let $r \geq 1$ be an integer. Let k be a field such that $H^{r+1}(k, \mathbf{Z}) = 0$. Then k has strict cohomological dimension $\leq r$.

c) Let G be a profinite group. Let M be a discrete and finitely generated torsion-free G-module. Then $H^1(G, M)$ is finite.

d) Let G be a profinite group. Let p be a prime number. Assume that p divides the order of G (as a supernatural number). Then we have that the strict p-cohomological dimension $\mathrm{scd}_p(G)$ is at least 2.

Exercise 2 : Divisible modules (4 points).

Let G be a profinite group. Let M be a discrete G-module. Assume that M is *divisible*, that is: multiplication by n $M \xrightarrow{\cdot n} M$ is surjective for every integer $n > 0$. For every abelian group A, denote by $A[n]$ the subgroup consisting of those x such that $nx = 0$, and let A/n be the quotient of A by the subgroup of elements of the form ny with $y \in A$.

1. Show that for every postive integers n and i, there is an exact sequence

$$0 \to H^{i-1}(G, M)/n \to H^i(G, M[n]) \to H^i(G, M)[n] \to 0$$

2. Assume that $G = \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of a p-adic field K. Assume further that as an abelian group, M is isomorphic to $(\overline{K}^*)^m$ for some $m \in \mathbf{N}^*$. Show that for every positive integer n, the group $H^i(G, M)[n]$ is finite.

3. Keep the assumptions of 2. and assume further that there exists a finite Galois extension L of K such that as a $\mathrm{Gal}(\overline{K}/L)$-module M is isomorphic to $(\overline{K}^*)^m$. Show that $H^1(G, M)$ is finite.

Exercise 3 : Local and global norms (4 points).

For each finite and separable extension L of a field K, denote by $N_{L|K} : L^* \to K^*$ the norm map from L^* to K^*.

1. Let K be a p-adic field. Let L be a finite and Galois extension of K such that $\mathrm{Gal}\,(L/K)$ is abelian and of cardinality at least 2. Show that $N_{L|K}L^* \neq K^*$.

2. Let k be a number field. Let Ω_k be the set of all places of k. Let F be a finite Galois extension of k with $\mathrm{Gal}\,(F/k)$ cyclic. For each place v of k, denote by k_v the completion of k at v (resp. by F_v the completion of F at some place lying over v). Show that there exists a homomorphism with finite cokernel

$$k^*/N_{F|k}F^* \to \bigoplus_{v \in \Omega_k} k_v^*/N_{F_v|k_v}F_v^*$$

3. Deduce from 2. that for every finite Galois extension F of a number field k such that $\mathrm{Gal}\,(F/k)$ is a non trivial cyclic group, the group $k^*/N_{F|k}F^*$ is infinite.

Exercise 4 : Number fields (7 points).

Let k be a number field with algebraic closure $\bar{k}$ and absolute Galois group $G_k = \mathrm{Gal}\,(\bar{k}/k)$. Let M be a discrete and finitely generated G_k-module. Denote by $\Omega_{\mathbf{R}}$ the set of real places of k; for every place v of k, let k_v be the completion of k at v. For every integer $r \geq 3$, consider the diagonal map

$$\theta^r(M) : H^r(k, M) \to \bigoplus_{v \in \Omega_{\mathbf{R}}} H^r(k_v, M)$$

induced by the restriction maps $H^r(k, M) \to H^r(k_v, M)$.

1. In the whole question 1., we assume that M is torsion-free.

a) Show that for every integer $i \geq 2$, there is an isomorphism

$$H^{i-1}(k, M \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}) \simeq H^i(k, M)$$

b) Show that for every positive integer n, the G_k-module $M \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}$ is finite.

c) Show that for every integer $r \geq 4$, the map $\theta^r(M)$ is an isomorphism.

2. We don't assume anymore that M is torsion-free.

a) Show that there exists an integer s and an exact sequence of G_k-modules

$$0 \to N \to P \to M \to 0$$

with P of the form $\mathbf{Z}[G_k/U]^s \simeq (I_{G_k}^U(\mathbf{Z}))^s$ for some open normal subgroup U of G_k, and N is a finitely generated and torsion-free G_k-module.

b) Suppose that $r \geq 4$. Show that the map $\theta^r(M)$ is an isomorphism.

c) Show that the conclusion of 2.b) still holds for $r = 3$.