The Brauer-Manin obstruction for integral points on curves

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Abstract

We discuss the question of whether the Brauer-Manin obstruction is the only obstruction to the Hasse principle for integral points on affine hyperbolic curves. In the case of rational curves we conjecture a positive answer, we prove that this conjecture can be given several equivalent formulations and we relate it to an old conjecture of Skolem. Finally, we show that for elliptic curves minus one point a strong version of the question (describing the set of integral points by local conditions) has a negative answer.

1. *Introduction*

For curves defined over number fields, V. Scharaschkin [**14**] and A. Skorobogatov [**16**] independently raised the question of whether the Brauer-Manin obstruction is the only obstruction to the Hasse principle for rational points, and proved that this is so when the Jacobian has finite Mordell-Weil group and finite Shafarevich-Tate group. The connection with an adelic intersection in the Jacobian was formulated in [**14**]. The question has been discussed also in [**11**] and [**17**] and other papers. The main evidence that this question has a positive answer is the proof of most cases of its function field analogue in [**12**] and its numerical verification in a large number of cases in [**1**].

Parallel to these developments, in [**2**], a Brauer-Manin obstruction for integral points on affine varieties was studied and proved to be the only obstruction to the Hasse principle for integral points on certain homogeneous spaces of linear and simply connected algebraic groups (this has concrete applications to the classical question of representation of an integral quadratic form by another integral quadratic form). A similar result was proved in [**6**] for principal homogeneous spaces of tori.

The purpose of this paper is to put those two threads together and discuss the Brauer-Manin obstruction for integral points on affine curves. For affine subsets of rational curves, we conjecture that the Brauer-Manin obstruction is the only obstruction, in the strong form stating that the set of integral points can be described as the set of adelic points pairing trivially with a suitable Brauer group. We prove that this conjecture can be given several equivalent formulations, including one version in terms of finite abelian descent and another in terms of an adelic intersection in the generalized Jacobian. The latter turns out to be closely related to an old conjecture of Skolem [**15**], which seems to have been largely ignored in the recent literature. We also discuss the case of affine subsets of elliptic curves. We show that for elliptic curves minus one point the question, in its strong form, has a negative answer.

Let k be a field with algebraic closure $\overline{k}$. By *k-curve*, we always mean a smooth, separated and geometrically integral finite type scheme of dimension 1 over Spec k. A k-curve X which is the complement of a reduced effective divisor of degree $d \geq 0$ on a projective curve of genus g is called *hyperbolic* if $2g-2+d>0$.

If X is a geometrically integral k-scheme, we set $\overline{X} = X \times_k \overline{k}$. The pieces of notation $\overline{k}[X]^*$, $\overline{k}(X)^*$, respectively, stand for the group of invertible functions and non-zero rational functions on $\overline{X}$. Define $\operatorname{Br} X := H^2(X, \mathbf{G}_m)$ and $\operatorname{Br}_1 X := \ker[\operatorname{Br} X \to \operatorname{Br} \overline{X}]$: namely $\operatorname{Br} X$ is the Brauer group of X and $\operatorname{Br}_1 X$ is its algebraic part. For every subgroup B of $\operatorname{Br} X$ such that $\operatorname{Br} k$ maps into B, we shall denote by $B/\operatorname{Br} k$ the cokernel of $\operatorname{Br} k \to B$. If X is a curve, then $\operatorname{Br} X = \operatorname{Br}_1 X$ because $\operatorname{Br} \overline{X}$ is a subgroup of the Brauer group of the function field of $\overline{X}$ (cf. [**9**], example III.2.22), and the latter is trivial by Tsen's theorem ([**5**], Theorem 6.2.8.).

Let X be a smooth and geometrically integral scheme over a number field k. Let Ω_k be the set of all places of k. For a place v of k, we denote by k_v the corresponding completion, and if v is non-archimedean, $\mathcal{O}_v$ is its ring of integers. Let $X(\mathbf{A}_k)$ be the set of adelic points of X: if $\mathcal{X}$ is a flat model of X over Spec $\mathcal{O}_k$, then $X(\mathbf{A}_k)$ is defined as the restricted product of the $X(k_v)$ ($v \in \Omega_k$) with respect of the $\mathcal{X}(\mathcal{O}_v)$ (this definition does not depend on the model X). The *Brauer-Manin pairing*

$$X(\mathbf{A}_k) \times \operatorname{Br} X \to \mathbf{Q}/\mathbf{Z}$$

is defined by

$$((P_v), \alpha) \mapsto \sum_{v \in \Omega_k} j_v(\alpha(P_v))$$

where $j_v : \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$ is the local invariant (defined in class field theory). If an adelic point (P_v) is in the closure of the set of rational points $X(k)$ for the "strong" topology (that is: the restricted product topology on $X(\mathbf{A}_k)$), then it is orthogonal to $\operatorname{Br} X$ for the Brauer-Manin pairing by the reciprocity law in global class field theory (see [**16**], II., page 102 for more details). For each subgroup B of $\operatorname{Br} X$, the subset of $X(\mathbf{A}_k)$ consisting of those adelic points orthogonal to B is denoted $X(\mathbf{A}_k)^B$.

2. *The main conjecture*

Let k be a field of characteristic zero. Let X be a nonempty Zariski open affine subset of the projective line $\mathbf{P}^1_k$. Assume that $\mathbf{P}^1_k \setminus X$ does not consist of one single k-point; the curve X can be embedded (via the choice of a k-point on X) into its generalized Jacobian, which is an algebraic k-torus[1] T (see [**13**], section 1 for properties of the Albanese scheme of a variety over an arbitrary field; for a curve, the generalized Jacobian is by definition the Albanese scheme of the curve). Denote by $\widehat{T}$ the Galois module of characters of T.

Lemma 2·1. *Assume that $H^3(k, \mathbf{G}_m) = 0$ (e.g., k is a number field or the completion*

[1] more precisely T is the quotient of $R_{K/k}\mathbf{G}_m$ by the diagonal image of $\mathbf{G}_m$, where K is the separable k-algebra corresponding to $\mathbf{P}^1_k \setminus X$.

of a number field at some place). The group $\operatorname{Br} X/\operatorname{Br} k$ *is isomorphic to* $\operatorname{Br}_1 T/\operatorname{Br} k$ *and to* $H^2(k,\widehat{T})$.

Proof. Since $\overline{T}$ is isomorphic to a product of finitely many copies of $\mathbf{G}_m$, we have $\operatorname{Pic}\overline{T}=0$ and $\operatorname{Pic}\overline{X}=0$ as well. Therefore the complex $[\overline{k}(X)^*/\overline{k}^* \to \operatorname{Div}\overline{X}]$ (concentrated in degrees -1 and 0) is quasi-isomorphic to the complex $[\overline{k}[X]^*/\overline{k}^* \to 0]$, and similarly with T instead of X. By [**7**], Lemma 2.1 this implies that the groups $\operatorname{Br} X/\operatorname{Br} k$ and $\operatorname{Br}_1 T/\operatorname{Br} k$ are respectively isomorphic to $H^2(k,\overline{k}[X]^*/\overline{k}^*)$ and $H^2(k,\overline{k}[T]^*/\overline{k}^*)$. The latter is isomorphic to $H^2(k,\widehat{T})$ because $\overline{k}[T]^*/\overline{k}^*=\widehat{T}$ by Rosenlicht's lemma ([**3**], Corollary 2.2). By definition of the generalized Jacobian variety, the group $\widehat{T}$ is isomorphic to the group of divisors of degree 0 on $\mathbf{P}^1_{\overline{k}}$ supported in $\mathbf{P}^1_{\overline{k}}\setminus\overline{X}$, hence to $\overline{k}[X]^*/\overline{k}^*$ (indeed divisors of degree 0 on $\mathbf{P}^1_{\overline{k}}$ are just principal divisors because $\operatorname{Pic}^0(\mathbf{P}^1_{\overline{k}})=0$). The result follows. □

From now on we assume that k *is a number field.* We fix a finite set S of "bad" places of k; more precisely, we assume the following properties: the set S contains all archimedean places of k, the torus T (resp. the k-variety X) extends to a torus $\mathcal{T}$ (resp. to a smooth scheme $\mathcal{X}$) over the ring of S-integers $\mathcal{O}_S$, and the embedding $X\to T$ extends to an $\mathcal{O}_S$-embedding $\mathcal{X}\to\mathcal{T}$.

If $\mathcal{G}$ is a flat and commutative group scheme over $\mathcal{O}_S$ with generic fibre G, the pieces of notation $H^i(\mathcal{O}_S,\mathcal{G})$ and $H^i(\mathcal{O}_v,\mathcal{G})$ stand for fppf cohomology groups (they coincide with the étale cohomology groups if $\mathcal{G}$ is smooth). Galois cohomology groups $H^i(\operatorname{Gal}(\overline{k}/k),G(\overline{k}))$ will be denoted $H^i(k,G)$ (and similarly with k_v instead of k). We set

$$H^i_S(k,G):=\ker[H^i(k,G)\to\prod_{v\in S}H^i(k_v,G)]$$

and denote by $\mathcal{G}[n]$ the group scheme of n-torsion of $\mathcal{G}$. Using the exact sequence of sheaves for the fppf topology

$$0\to\mathcal{T}[n]\to\mathcal{T}\xrightarrow{\cdot n}\mathcal{T}\to 0$$

we get (for $v\notin S$) canonical embeddings $\mathcal{T}(\mathcal{O}_S)/n\to H^1(\mathcal{O}_S,\mathcal{T}[n])$ and $\mathcal{T}(\mathcal{O}_v)/n\to H^1(\mathcal{O}_v,\mathcal{T}[n])$. The latter is an isomorphism because $H^1(\mathcal{O}_v,\mathcal{T})=0$, by [**8**], Theorem 2 and [**9**], Remark III.3.11.a.

The pullback $\mathcal{Y}_n\to\mathcal{X}$ of the multiplication by n on $\mathcal{T}$ is a $\mathcal{T}[n]$-torsor (for the fppf topology) whose generic fibre $Y_n\to X$ is a $T[n]$-torsor. Let $c\in Z^1(\mathcal{O}_S,\mathcal{T}[n])$ be a Čech cocycle (to make the notation simpler, we shall still denote by c its class in $H^1(\mathcal{O}_S,\mathcal{T}[n])$). Then we can consider the twisted torsor $\mathcal{Y}^c_n\to\mathcal{X}$: its class in $H^1(\mathcal{X},\mathcal{T}[n])$ is $[\mathcal{Y}^c_n]=[\mathcal{Y}_n]-c$. If $a\in H^1(k,\widehat{T}/n)$, the pull-back of a by the structural map $X\to\operatorname{Spec} k$ is an element of $H^1(X,\widehat{T}/n)$; hence the cup-product $(a\cup[Y_n])\in H^2(X,\mathbf{G}_m)=\operatorname{Br} X$ makes sense because $[Y_n]\in H^1(X,T[n])$ and $\widehat{T}/n$ is the Cartier dual of $T[n]$.

LEMMA 2·2. *Let* T_n *be the* T*-torsor under* $T[n]$ *given by multiplication by* n *on* T. *For each* $a\in H^1(k,\widehat{T}/n)$, *the image of* $(a\cup[T_n])\in\operatorname{Br}_1 T$ *in* $H^2(k,\widehat{T})=\operatorname{Br}_1 T/\operatorname{Br} k$ *is the image of* a *by the coboundary map* ∂ *associated to the multiplication by* n *exact sequence:*

$$0\to\widehat{T}\xrightarrow{\cdot n}\widehat{T}\to\widehat{T}/n\to 0 \qquad (2\cdot 1)$$

Proof. Let $[E]$ be the class of the extension (2·1) in $\operatorname{Ext}^1_k(\widehat{T}/n,\widehat{T})$. Apply [**16**], Theorem

2.3.6 to T. Since $\operatorname{Pic} \overline{T} = 0$, we have $H^1(T, T[n]) = \operatorname{Ext}^1_k(\widehat{T}/n, \overline{k}[T]^*)$ and the image of $[T_n]$ in $\operatorname{Ext}^1_k(\widehat{T}/n, \widehat{T})$ (via the quotient map $\overline{k}[T]^* \to \overline{k}[T]^*/\overline{k}^* = \widehat{T}$) is $[E]$. Therefore the image in $H^2(k, \widehat{T})$ of the cup-product $(a \cup [T_n]) \in H^2(k, \overline{k}[T]^*)$ is obtained as the image of $(a, [E])$ by the canonical pairing

$$H^1(k, \widehat{T}/n) \times \operatorname{Ext}^1_k(\widehat{T}/n, \widehat{T}) \to H^2(k, \widehat{T}).$$

By definition this is $\partial(a)$. □

LEMMA 2·3. *The maps*

$$\varinjlim_n H^1(k, \widehat{T}/n) \to H^2(k, \widehat{T})$$

$$\varinjlim_n H^1_S(k, \widehat{T}/n) \to H^2_S(k, \widehat{T})$$

induced by the coboundary ∂ (associated to the exact sequence (2·1)) are isomorphisms (here the transition maps $H^1(k, \widehat{T}/n) \to H^1(k, \widehat{T}/m)$ and $H^1_S(k, \widehat{T}/n) \to H^1_S(k, \widehat{T}/m)$ for $n|m$ are induced by the multiplication by m/n map on $\widehat{T}$).

Proof. Consider the exact commutative diagram

$$\begin{array}{ccccccc}
H^1(k, \widehat{T})/n & \longrightarrow & H^1(k, \widehat{T}/n) & \longrightarrow & H^2(k, \widehat{T})[n] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{v \in S} H^1(k_v, \widehat{T})/n & \longrightarrow & \bigoplus_{v \in S} H^1(k_v, \widehat{T}/n) & \longrightarrow & \bigoplus_{v \in S} H^2(k_v, \widehat{T})[n] & \longrightarrow & 0
\end{array}$$

and observe that for each $v \in S$, we have

$$\varinjlim_n H^1(k_v, \widehat{T})/n = H^1(k_v, \widehat{T}) \otimes \mathbf{Q}/\mathbf{Z} = 0$$

because $H^1(k_v, \widehat{T})$ is finite. Similarly $\varinjlim_n H^1(k, \widehat{T})/n = 0$. The result now follows from the fact that $\varinjlim$ is an exact functor. □

Let $\overline{\mathcal{T}(\mathcal{O}_S)}$ be the closure of $\mathcal{T}(\mathcal{O}_S)$ in $\prod_{v \notin S} \mathcal{T}(\mathcal{O}_v)$ (the latter is equipped with the product of the v-adic topologies).

Let $(P_v) \in \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \subset \prod_{v \notin S} \mathcal{T}(\mathcal{O}_v)$. Define the subgroup $B_S(X) \subset \operatorname{Br} X$ as the kernel of the diagonal map

$$\operatorname{Br} X \to \prod_{v \in S} \operatorname{Br}(X \times_k k_v)/\operatorname{Br} k_v.$$

There is a Brauer-Manin-like pairing

$$\prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \times B_S(X) \to \mathbf{Q}/\mathbf{Z} \tag{2·2}$$

defined by

$$((P_v), \alpha) \mapsto \sum_{v \in \Omega_k} j_v(\alpha(P_v))$$

where $P_v \in X(k_v)$ is chosen arbitrarily for $v \in S$ (the definition of the pairing does not depend on this choice because for $v \in S$, the restriction of α to $\operatorname{Br}(X \times_k k_v)$ is constant). Observe that $(P_v)_{v \notin S}$ is orthogonal to $B_S(X)$ for the pairing (2·2) if and only if the equality $\sum_{v\notin S} j_v(\alpha(P_v)) = 0$ is satisfied for every $\alpha \in \ker[\mathrm{Br}\, X \to \prod_{v\in S} \mathrm{Br}(X \times_k k_v)]$; this equality holds in particular if $(P_v)_{v\notin S}$ is in the diagonal image of $\mathcal{X}(\mathcal{O}_S)$, which is finite by Siegel's theorem.

We define the subgroup $B'_S(X)$ of $\mathrm{Br}\, X$ to be the subgroup generated by $\mathrm{Br}\, k$ and the cup-products $a \cup [Y_n]$ for all $a \in H^1_S(k, \widehat{T}/n)$ and all $n > 0$.

THEOREM 1.

(1) The group $B_S(X)$ coincides with the subgroup $B'_S(X)$ of $\mathrm{Br}\, X$*.*

(2) The following are equivalent:

a) The point $(P_v)_{v\notin S}$ belongs to $\overline{\mathcal{T}(\mathcal{O}_S)}$.

b) For every $n > 0$, the image of (P_v) in $\prod_{v\notin S} \mathcal{T}(\mathcal{O}_v)/n$ belongs to the image of $\mathcal{T}(\mathcal{O}_S)/n$ by the diagonal map.

b') For every $n > 0$, there exists c in the image of the map $\mathcal{T}(\mathcal{O}_S) \to H^1(\mathcal{O}_S, \mathcal{T}[n])$ such that the twisted torsor $\mathcal{Y}^c_n$ contains a point $(Q_v) \in \prod_{v\notin S} \mathcal{Y}^c_n(\mathcal{O}_v)$ that maps to (P_v).

c) For every $n > 0$, the image of (P_v) in $\prod_{v\notin S} H^1(\mathcal{O}_v, \mathcal{T}[n])$ belongs to the diagonal image of $H^1(\mathcal{O}_S, \mathcal{T}[n])$.

d) The point $(P_v)_{v\notin S}$ is orthogonal (for the pairing (2·2)) to $B_S(X)$.

For example, take the curve X defined by the affine equation $x + y = 1, xy \neq 0$ and $T = \mathbf{G}_m \times \mathbf{G}_m$. The torsor Y_n is given by the equation $x^n + y^n = 1, xy \neq 0$. Let $b = (u, v) \in \mathcal{O}^*_S \times \mathcal{O}^*_S$ and let c be the image of b in $H^1(\mathcal{O}_S, \mu_n \times \mu_n)$. Then the twist Y^c_n is given by the equation $ux^n + vy^n = 1, xy \neq 0$.

Remark 2·4. In general b) and c) are not equivalent for a given $n > 0$ because the group $H^1(\mathcal{O}_S, \mathcal{T})[n]$ can be non-zero. The point is that by finiteness of $H^1(\mathcal{O}_S, \mathcal{T})$ ([**10**], Theorem II.4.6.a) the inverse limit (over n) of the $H^1(\mathcal{O}_S, \mathcal{T})[n]$ is always zero.

Conjecture 2. Let $X \subset \mathbf{P}^1$ be an affine hyperbolic curve. Let S be a finite set of places of k (containing all archimedean places) and let $\mathcal{X}$ be a flat model of X over $\mathcal{O}_S$. Then the subset of $\prod_{v\notin S} \mathcal{X}(O_v)$ consisting of those points that are orthogonal (for the pairing (2·2)) to $B_S(X)$ coincides with the image of $\mathcal{X}(\mathcal{O}_S)$ in $\prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$.

More generally, let X be an arbitrary hyperbolic curve with flat model $\mathcal{X}$ over $\mathcal{O}_S$. Let $B_S(X)$ be the kernel of $\mathrm{Br}\, X \to \prod_{v\in S} \mathrm{Br}\,(X \times_k k_v)/\mathrm{Br}\, k_v$. We will say that *Brauer-Manin suffices* for $\mathcal{X}(\mathcal{O}_S)$ if the subset of $\prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$ orthogonal (for the pairing (2·2)) to $B_S(X)$ coincides with the image of the finite set $\mathcal{X}(\mathcal{O}_S)$. In particular, the conjecture states that Brauer-Manin suffices for hyperbolic open subsets of $\mathbf{P}^1$. We will discuss this question in Section 3.

Remarks 2·5. Skolem conjectured in [**15**] that an "exponential diophantine equation" has a solution if and only if the corresponding congruences have a solution for all moduli. In other words, let Γ be a finitely generated subgroup of k^* and $a_1, \ldots, a_m \in k$; let S be a finite set of places of k such that a_i and the elements of Γ are in $\mathcal{O}^*_S$. Then the equation $\sum a_i x_i = 0$ has a solution with $x_i \in \Gamma$ if and only if for all ideals I of $\mathcal{O}_S$, the equation $\sum a_i x_i \in I$ has a solution with $x_i \in \Gamma$. If $m = 3$ and $\Gamma = \mathcal{O}^*_S$, this conjecture is equivalent to requiring that when $X = \mathbf{P}^1$ minus three points (equipped with its embedding into $T = \mathbf{G}_m \times \mathbf{G}_m$) in conjecture 2, and the intersection of $\prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$ with $\overline{\mathcal{T}(\mathcal{O}_S)}$ is nonempty, then $\mathcal{X}(\mathcal{O}_S)$ is nonempty. By Theorem 1, this is a consequence of conjecture 2.

As evidence in favour of conjecture 2, C.-L. Sun [**18**] has verified its function field analogue in some cases and A. Moore (unpublished) has numerically verified Skolem's conjecture for $n = 3$ for many cases with a_i small integers and Γ subgroups of $\mathbf{Q}^*$ of rank one or two generated by small primes.

Proof of Theorem 1.

(1) The group $B'_S(X)$ is the image in $\operatorname{Br} X$ of the subgroup of $\operatorname{Br}_1 T$ generated by $\operatorname{Br} k$ and the cup-products $(a \cup [T_n])$, $a \in H^1_S(k, \widehat{T}/n)$, $n > 0$. For each $a \in H^1(k, \widehat{T}/n)$, the image of $(a \cup [T_n]) \in \operatorname{Br}_1 T$ in $H^2(k, \widehat{T}) = \operatorname{Br}_1 T/\operatorname{Br} k$ is the image of a by the coboundary map ∂ (Lemma 2·2). By Lemma 2·3, the group $H^2_S(k, \widehat{T})$ is the direct limit of the images by ∂ of $H^1_S(k, \widehat{T}/n)$, hence we obtain that $B'_S(X)/\operatorname{Br} k$ is the image of $H^2_S(k, \widehat{T}) \subset \operatorname{Br}_1 T/\operatorname{Br} k$ in $\operatorname{Br}_1 X/\operatorname{Br} k$. The conclusion now follows from Lemma 2·1 (applied to the number field k and the completions k_v for $v \in S$) and from the definition of $H^2_S(k, \widehat{T})$.

(2) Let us show the equivalence of the four assertions:

a) $\Rightarrow$ b): If the assumption a) is satisfied, then the image (P_v^n) of (P_v) in the compact space $\prod_{v \notin S} \mathcal{T}(\mathcal{O}_v)/n$ belongs to the closure of the image of $\mathcal{T}(\mathcal{O}_S)/n$. Since $\mathcal{T}(\mathcal{O}_S)/n$ is finite, this implies b).

b) $\Leftrightarrow$ b'): By definition of the torsor $\mathcal{Y}_n$, the image (P_v^n) of (P_v) in $\prod_{v \notin S} H^1(\mathcal{O}_v, \mathcal{T}[n])$ is $([\mathcal{Y}_n](P_v))$. Therefore b) is equivalent to the fact that $([\mathcal{Y}_n](P_v))$ is in the image of $\mathcal{T}(\mathcal{O}_S)$ in $\prod_{v \notin S} H^1(\mathcal{O}_v, \mathcal{T}[n])$. The equivalence now follows from the fact that for $c \in H^1(\mathcal{O}_S, \mathcal{T}[n])$ and $v \notin S$, the property $[\mathcal{Y}_n^c](P_v) = 0$ means that $[\mathcal{Y}_n](P_v)$ is the image of c in $H^1(\mathcal{O}_v, \mathcal{T}[n])$.

b) $\Rightarrow$ c): this follows from the commutative diagram:

$$\begin{array}{ccc} \mathcal{T}(\mathcal{O}_S)/n & \longrightarrow & \prod_{v \notin S} \mathcal{T}(\mathcal{O}_v)/n \\ \downarrow & & \downarrow \\ H^1(\mathcal{O}_S, \mathcal{T}[n]) & \longrightarrow & \prod_{v \notin S} H^1(\mathcal{O}_v, \mathcal{T}[n]) \end{array}$$

c) $\Rightarrow$ d): As above, we know that the image (P_v^n) of (P_v) in $\prod_{v \notin S} H^1(\mathcal{O}_v, \mathcal{T}[n])$ is $([\mathcal{Y}_n](P_v))$. Thus assumption c) implies that there exists $\beta \in H^1(k, T[n])$ whose image in $\prod_{v \notin S} H^1(k_v, T[n])$ coincides with $([Y_n](P_v))$. Choose $P_v \in X(k_v)$ arbitrarily for $v \in S$. Then for every $a \in H^1_S(k, \widehat{T}/n)$:

$$\sum_{v \in \Omega_k} j_v((a \cup [Y_n])(P_v)) = \sum_{v \notin S} j_v((a \cup [Y_n])(P_v)) = \sum_{v \notin S} j_v(a_v \cup \beta_v) = \sum_{v \in \Omega_k} j_v((a \cup \beta)_v) = 0$$

by the reciprocity law (here a_v denotes the image of a in $H^1(k_v, \widehat{T}/n)$, and similarly for $\beta, (a \cup \beta)$). Using (1), this means exactly that $(P_v)_{v \notin S}$ is orthogonal to $B_S(X)$.

d) $\Rightarrow$ a): For each abelian group B, denote by B^D its dual $\operatorname{Hom}(B, \mathbf{Q}/\mathbf{Z})$. Recall (cf. [**6**], p. 7) that there is a canonical map:

$$\theta : \prod_{v \notin S} \mathcal{T}(\mathcal{O}_v) \to H^2_S(k, \widehat{T})^D$$

defined by the formula

$$\theta((P_v)).t = \sum_{v \notin S} j_v(t_v \cup P_v)$$

for each $t \in H^2_S(k, \widehat{T})$. Here the cup-products $(t_v \cup P_v)$ correspond to the local pairings

$$T(k_v) \times H^2(k_v, \widehat{T}) \to H^2(k_v, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}$$

Assumption d) now means that for every $n > 0$, the image of $\theta((P_v))$ in $H^1_S(k, \widehat{T}/n)^D$ is zero. By Lemma 2·3, the map

$$\varinjlim_n H^1_S(k, \widehat{T}/n) \to H^2_S(k, \widehat{T})$$

induced by the coboundary ∂ is an isomorphism, hence $\theta((P_v)) = 0$. Therefore (P_v) belongs to $\overline{\mathcal{T}(\mathcal{O}_S)}$ by [**6**], Proposition 3 (take the projective limit over the finite S' as in this proposition).

□

3. *Further results*

Theorem 3. *Let X be a hyperbolic k-curve. Let U be a k-curve equipped with a quasi-finite morphism $f : U \to X$ such that all k-fibres of f are of degree at most four. Suppose that f extends to a map between flat models $\mathcal{U}$, $\mathcal{X}$ of U, X over $\mathcal{O}_S$ (where S is a finite set of places of k containing all archimedean places) and assume that Brauer-Manin suffices for $\mathcal{X}(\mathcal{O}_S)$. If the subset of $\prod_{v \notin S} \mathcal{U}(O_v)$ consisting of those points that are orthogonal (for the pairing (2·2)) to $B_S(U)$ is nonempty, then $\mathcal{U}(\mathcal{O}_S)$ is also nonempty. Moreover, if U is a subset of X and f is the inclusion, then Brauer-Manin suffices for $\mathcal{U}(\mathcal{O}_S)$ also.*

Proof of Theorem 3.

Let $(P_v)_{v \notin S}$ (with $P_v \in \mathcal{U}(\mathcal{O}_v)$) be orthogonal to $B_S(U)$. The points $f(P_v)$ are points of X. Moreover we have $f^* : B_S(X) \to B_S(U)$ which has the usual functorial properties. It follows that $(f(P_v))_{v \notin S}$ is orthogonal to $B_S(X)$. It follows from Siegel's theorem, which states that a hyperbolic affine curve has only finitely many S-integral points, that $\mathcal{X}(\mathcal{O}_S)$ is finite. Therefore our hypotheses imply that there exists $Q \in \mathcal{X}(\mathcal{O}_S)$ such that $Q = f(P_v)$ for $v \notin S$. It is a simple fact of algebraic number theory (relying on Cebotarev's density theorem) that a zero-dimensional k-scheme of degree at most four with points locally almost everywhere has a global point. That is, there exists a rational point $P \in U$ with $f(P) = Q$. It follows that $P \in \mathcal{U}(\mathcal{O}_S)$. If, in addition U is a subset of X, then $P = P_v$ for all $v \notin S$, completing the proof. □

Remarks 3·1. J. Tate suggested the following example. Consider the zero-dimensional k-scheme formed by the union of the roots of an irreducible cubic with Galois group S_3 and the roots of $x^2 - D$, where D is the discriminant of the cubic. It has points locally almost everywhere but no global point. Thus, the proof of theorem 3 cannot be extended to degree five (or more).

The theorem applies in particular when U is an open subset of a projective curve X of genus at least two. It follows that, in this case, if (for a given S) Brauer-Manin suffices for a projective curve, the same holds for all its affine open subsets.

The following variant is due to Colliot-Thélène (personal communication) in the case when S consists of archimedean places. His proof inspired our proof of Theorem 3.

Theorem 4. *Let E/k be an an elliptic curve with $E(k)$ and Ш(E/k) both finite, then Brauer-Manin suffices for any affine open subset U of E and any S (containing all archimedean places).*

Proof. Using the same argument as in Theorem 3 (in the special case when U is an open subset of E), we just have to check that Brauer-Manin suffices for E (with S arbitrary). Using the finiteness of $E(k)$, this is a consequence of the Cassels-Tate-like exact sequence (obtained by taking the projective limit over the finite S' in [**6**], Proposition 3)

$$0 \to \overline{E(k)} \to \prod_{v \notin S} E(k_v) \to B_S(E)^D$$

where $\overline{E(k)}$ is the closure of $E(k)$ in $\prod_{v \notin S} E(k_v)$ and the map $\prod_{v \notin S} E(k_v) \to B_S(E)^D$ is given by the pairing (2·2) (see [**7**] p.22). □

Let now $U = E \setminus E[2]$ where $E[2]$ is the two-torsion on the elliptic curve E. The function x on a short Weierstrass equation for E is a map from U to $\mathbf{P}^1$ of degree two such that the image is also the complement of a reduced effective divisor of degree 4. So conjecture 2 implies that theorem 3 applies to U.

We end by giving an example of an affine hyperbolic curve for which Brauer-Manin does not suffice.

Let $E/\mathbf{Q}$ be the (complete) elliptic curve given by $y^2 = x^3 + 3$ and X the affine curve given by the same equation (i.e. $E \setminus \{0\}$). This equation defines a model $\mathcal{X}$ of X over Spec $\mathbf{Z}$. Let $P = (1, 2)$. Then $E(\mathbf{Q}) = \mathbf{Z}P$ and $\mathcal{X}(\mathbf{Z}) = \{P, -P\}$. This is standard effective diophantine geometry, and was checked e.g., as part of the calculations described in [**4**]. Consider the sequence $\{pP\}$ where p runs over primes $\equiv 3 \mod 8$. Extract a convergent subsequence in $E(\mathbf{A_Q})$ and let (P_v) be the limit. We claim that $P_v \in \mathcal{X}(\mathbf{Z}_v)$ for non-archimedean v. Indeed, if this is not the case, then $pP = 0 \mod v$ for arbitrarily large primes p and, since P is integral, it follows that P has order $p \mod v$, and this forces $p \le v + 1 + 2\sqrt{v}$. (Note that this works also for the places $v = 2, 3$ of bad reduction for E, as the point P reduces to a smooth point modulo these places). This proves the claim. We now claim that $(P_v) \notin \mathcal{X}(\mathbf{Z})$. Indeed we will show that a subsequence of the sequence $\{pP\}, p \equiv 3 \mod 8$ cannot converge to $\pm P$ 2-adically. If pP is close to $\pm P$ 2-adically then $(p \pm 1)P$ needs to be close to 0 2-adically and (using the 2-adic elliptic logarithm) this forces $(p \pm 1)$ to be close to 0 2-adically, which is impossible as $p \equiv 3 \mod 8$. Finally we claim that (P_v) is orthogonal under the Brauer-Manin pairing to $B_S(X), S = \{\infty\}$. First notice that $B_S(X) \subset \operatorname{Br} X = \operatorname{Br} E$ (the latter by [**9**], III, ex. 2.22 a) and [**5**], Theorem 6.4.4). As (P_v) is the limit of a sequence in $E(\mathbf{Q})$, we get that (P_v) is orthogonal to $\operatorname{Br} E$ and this produces the desired counterexample.

In this example, Brauer-Manin (or abelian descent obstructions) do not suffice but, it can be shown that, assuming Conjecture 2, non-abelian descent obstructions (as defined in [**16**], II, section 5.3)) suffice. This is similar to the situation of Skorobogatov's famous example (see [**16**], II, section 8.1), and (if we still assume Conjecture 2) the non-abelian descent obstructions also suffice for $E \setminus D$ when $\deg D = 2$. In the latter case, it is not clear whether or not Brauer-Manin alone suffices.

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