

GOOD ELLIPTIC OPERATORS ON CANTOR SETS

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ABSTRACT. It is well known that a purely unrectifiable set cannot support a harmonic measure which is absolutely continuous with respect to the Hausdorff measure of this set. We show that nonetheless there exist elliptic operators on (purely unrectifiable) Cantor sets in \mathbb{R}^2 whose elliptic measure is absolutely continuous, and in fact, essentially proportional to the Hausdorff measure.

Key words: Harmonic measure, Green function, Elliptic operator, Counterexample, Cantor set.

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1. INTRODUCTION

Starting with the seminal 1916 work of F. and M. Riesz [RR], considerable efforts over the century culminating in the past 10-20 years, identified necessary and sufficient geometric conditions on the domains for which harmonic measure is absolutely continuous with respect to the Hausdorff measure of the boundary. In particular, it was established that purely unrectifiable sets are those for which harmonic measure is necessarily singular with respect to the Hausdorff measure of the boundary of the domain. Moreover, it was shown that a similar statement holds for operators reasonably close to the Laplacian.

It turns out, however, that even for a purely unrectifiable set there may exist a “good” elliptic operator. The main result of this paper is the construction of elliptic operators L associated to the four corners Cantor set K of dimension 1 in the plane, whose elliptic measure ω_L is essentially proportional to the one-dimensional Hausdorff measure on K . To the best of our knowledge, this is the first result of this nature.

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We shall concentrate mainly on the emblematic Garnett-Ivanov Cantor set of dimension 1 in the plane, also known as the four-corners Cantor set (see Figure 2), because it is probably the most celebrated example of a one-dimensional, Ahlfors regular set, such that the harmonic measure and the Hausdorff measure \mathcal{H}^1 are mutually singular on K , but we will explain in Section 5 that our construction is fairly flexible.

The constructed operators $L = \operatorname{div} A \nabla$ are of divergence form. Moreover, they are scalar, that is, we can write them as

$$(1.1) \quad L = \operatorname{div} a \nabla,$$

where a is a continuous scalar function on \mathbb{R}^2 (as opposed to a general matrix-valued function A) such that $C^{-1} \leq a(X) \leq C$ for $X \in \Omega = \mathbb{R}^2 \setminus K$.

Even though for the Cantor set the question whether there exists an elliptic operator for which elliptic measure is absolutely continuous with respect to the Hausdorff measure on the boundary was also open for general elliptic operators $L = \operatorname{div} A \nabla$, we aimed to have a solution in the smaller class of isotropic operators $L = \operatorname{div} a \nabla$ as above which seem to be more clearly geometrically relevant. Unfortunately, this is also harder: if one agrees to work with a general case of matrix-valued A the theory of quasiconformal mappings becomes an ally.

Let us briefly discuss the history of known results pertaining to the following general question. Given a domain $\Omega \subset \mathbb{R}^n$, bounded by $E = \partial\Omega$, and a divergence form operator $L = \operatorname{div} A \nabla$, where $A = A(X)$ is an elliptic matrix defined for $X \in \Omega$, when can we say that the elliptic measure ω_L associated to E (on Ω) is absolutely continuous with respect to the surface measure on E or more generally, when E is not assumed to be smooth, to the relevant Hausdorff measure or an Ahlfors regular measure σ living on E ?

Consider first the case of the Laplacian. In the positive direction, many results give the absolute continuity of ω_Δ when E is rectifiable and some sort of topological connectedness condition is satisfied; see for instance [RR, L, KL] in the plane, and [Dah, DJ, Se, HM] in higher dimensions, and for instance [Bad] in the absence of Ahlfors regularity. In the converse direction, it was long known that the harmonic measure on the Garnett-Ivanov Cantor K is singular with respect to the natural measure, but the precise more general results are much more recent. In particular, rectifiability was identified as a necessary condition for the absolute continuity of harmonic measure only in 2016 [AHM3TV], but even then the exact necessary topological (connectedness) assumptions remained elusive. Finally, in 2018, the sharp necessary and sufficient conditions were established in [AHMMT]. Many of these results generalize to operators L other than the Laplacian but morally close to it, for instance, those satisfying a suitable square Carleson measure condition. See [KP, HMMTZ].

Concerning Cantor sets, it has been known since [Car] that in the case of K , the (usual) harmonic measure ω_Δ is singular with respect to the Hausdorff measure, and even that it lives on a subset of dimension strictly smaller than 1. The same thing is true even in larger dimensions, and Carleson's result was later generalized to larger classes of fractal sets; see for instance [Bat, BZ]. These results are quite delicate (and the condition that ω_Δ is supported by a set of dimension < 1 is also significantly stronger than the mere singularity). See also [Az] for a more recent result with much less structure.

The present papers deals with operators L that are *not* close to the Laplacian. For its authors, the issue arose when they tried to define reasonable elliptic operators on $\mathbb{R}^n \setminus E$, where E is an Ahlfors regular set of dimension $d < n - 1$. In such a context, it was shown in [DFM1, DFM2] that L should rather be of the form $L = \operatorname{div} A \nabla$, where the product of A by $\operatorname{dist}(X, E)^{n-d-1}$ satisfies the usual ellipticity conditions. These operators cannot be close to the Laplacian (except in spirit). What was perhaps more surprising, is that in order to prove the absolute continuity of the elliptic measure, we had to work with a very particular choice of coefficients, not the one driven by the powers of the Euclidean distance. And indeed, if

$$D_\alpha(X) = \left(\int_E |X - y|^{-d-\alpha} d\mathcal{H}_{|E}^d(y) \right)^{-\frac{1}{\alpha}}, \quad X \in \mathbb{R}^n \setminus E, \quad \alpha > 0,$$

and $L_\alpha = \operatorname{div} D_\alpha(X)^{-n+d+1} \nabla$ then we can prove that the elliptic measure is absolutely continuous (and given by an A_∞ weight) when E is uniformly rectifiable; see [DFM3, DM, Fen].

For these results the converse is not known. In fact, the authors of [DEM] have discovered a “magical” counterexample which really brings us close to the subject of the present paper. It turns out that when $d < n - 2$ and $\alpha = n - d - 2$, the elliptic measure for the operator L_α defined above is absolutely continuous with respect to the Hausdorff measure for *any Ahlfors regular set of dimension d* . The dimension does not have to be an integer, and the set in question does not have to be rectifiable or carry any other geometric characteristics on top of Ahlfors regularity. Of course, in this case the coefficients depend on the set via a particular distance function.

In view of all these results the following question is quite natural. Even in the classical case of sets with $n - 1$ dimensional boundary, given a bad (totally unrectifiable set) like K above, is it possible to find elliptic operators $L = \operatorname{div} A \nabla$ on $\mathbb{R}^n \setminus K$ such that ω_L is nonetheless absolutely continuous (possibly even proportional) to the natural measure on K ?

The present paper shows that the answer is yes for the Cantor set $K \subset \mathbb{R}^2$ described near (3.1), and quite a few other ones. In fact, we establish the stronger result that the elliptic measure and Hausdorff measure are roughly proportional, in the sense that if we take a pole $X \in \mathbb{R}^2 \setminus B(0, 1)$ (that is, far enough from K), then

$$(1.2) \quad C^{-1} \mathcal{H}_{|K}^1 \leq \omega^X \leq C \mathcal{H}_{|K}^1.$$

See Theorem 4.2 for the precise statement, and Section 5 for other sets K .

The basic idea is very simple: we shall be able to construct the Green function with a pole at ∞ . Usually, one does not dream of computing the Green function explicitly, except in the very simple cases of the Laplacian on a disc or a half-space, or when some unexpected miracle happens (see, e.g., the aforementioned example [DEM]). But here the situation is different: we construct the Green function G first, subject to suitable constraints, and then compute the coefficients of L in terms of G . This will require some care because we want G to be a solution of $\operatorname{div} a \nabla G = 0$ with $C^{-1} \leq a \leq C$, and a is very far from unique given G , but we have a fair chance.

We shall use the fact that we work in dimension 2 because G , and locally a conjugate function, will be computed from their level lines and the way they cross. In particular, we will start from a family of level lines and construct the orthogonal curves, and this is certainly

easier in the plane. Also, we are not sure that there is a good enough notion of conjugate function in higher dimensions. Of course we can create examples in \mathbb{R}^n , for instance on the product of K with \mathbb{R}^{n-2} , but in terms of construction this is obviously cheating.

In the course of the proof we discovered a number of the properties of the equation $Lu = 0$, with L as in (1.1), in dimension 2, concentrating on a somewhat less conventional direction from solution to the coefficients of the PDE. Some of these are probably very well known, particularly in connection with the so-called Calderón inverse problem, but they were of considerable help to us for understanding how to find coefficients a such that $\operatorname{div} a \nabla G = 0$ for a given function G . We explain this in Section 2.

The definition of K and main construction, with the level curves and the pictures, will be done in Section 3. As we just said, the level sets of G will be constructed so that G is a solution of some equation $LG = 0$, but we will need to make sure that the function a computed from G has uniform bounds. For this, the fractal nature of K will be useful, because it will allow us to prove the desired estimates only on some fundamental domains, and then glue the different pieces.

We shall then explain, in a short Section 4 the relation between the constructed Green function and the elliptic measure associated to L ; this is also the section where our main theorem will be stated (hopefully with no surprise).

In the last Section 5, we explain that our construction is not so rigid. For instance it works essentially with no modification for a non-fractal variant of K where we are also allowed to rotate the squares independently. This is interesting, because for these Cantor sets, as far as we know the fact that the harmonic measure ω_Δ lives on a set of smaller dimension has been established only recently, with a very complicated proof, while the case of K was treated quite some time ago [Car], using the fractal nature of K .

For the operator L that we construct, the Green function $G(X)$ is equivalent (for X close to K and the other pole far away) to the distance $\operatorname{dist}(X, K)$; it is amusing that for slightly different functions a (but this time degenerate elliptic), or, alternatively, for similar Cantor sets of different dimensions, we get Green functions G that are equivalent to different powers of $\operatorname{dist}(X, K)$ (and ω_L is still proportional to the Hausdorff measure). This is related to the invariance properties of the equation $Lu = 0$, with L as in (1.1), but also to the reason why we prefer to have a scalar function a .

As we mentioned above, another reason for emphasizing scalar coefficients (and for the difficulties this entails) comes from the theory of quasiconformal mappings. Consider the similar question where K is instead a snowflake of dimension $\alpha > 1$ in \mathbb{R}^2 . If you want to find an elliptic operator $L = \operatorname{div} A \nabla$ such that the associated elliptic measure ω_L is absolutely continuous with respect \mathcal{H}_K^α , this is in fact (too) easy: we use a quasiconformal mapping $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps the line ℓ (or a circle, depending on whether K is unbounded or bounded) to K , and then use ψ to move the Laplacian on a component of $\mathbb{R}^2 \setminus \ell$ to the desired component of $\mathbb{R}^2 \setminus K$. It is known that the conjugated operator is an elliptic operator $L = \operatorname{div} A \nabla$, and the absolute continuity of ω_L with respect to \mathcal{H}^α follows directly from the corresponding absolute continuity result for ℓ and Δ .

Of course in the case of the Cantor set K , we cannot do that, even if we allow general elliptic matrices A , but this says that the class of elliptic operators is really too large. As far as the authors know, the same question for a snowflake, but with a scalar operator L as

in (1.1), is open. Possibly the ideas in this paper could help, but one would need to write down level sets in a very careful way.

2. ABOUT THE EQUATION $\operatorname{div} a \nabla u = 0$ IN THE PLANE

In this section we try to see how a given function u on the plane can be seen as a solution of an equation $Lu = 0$, with L as in (1.1) and a function a that we could compute in terms of u .

We want to do this in a geometrical way, in terms of the level sets of u , by introducing a second function v , which is related to u (as is conventional, we will say conjugated in analogy with harmonic functions) and satisfies a similar equation but with the function a^{-1} . We proceed locally in an open set U , where we assume that u is of class C^4 and with $\nabla u \neq 0$. There could be a brutal analytic way to find v and a , but the geometry of level sets seems much easier to understand.

Associated to u are the level sets $\gamma_s = u^{-1}(\{s\})$, and its nonvanishing gradient. We can use the gradient as a vector field, and then solve the equation $z'(t) = \nabla u(z(t))$ to get a family of orthogonal curves Γ_θ . To be more precise, we first parameterize one of the level curves, say, γ_0 , call $\theta \rightarrow \gamma_0(\theta)$ this parameterization, and then solve the equation $z'(t) = \nabla u(z(t))$ with the initial condition $z(0) = \gamma_0(\theta)$ to get the curve Γ_θ . We can extend Γ_θ in both direction, as long as it stays in the domain U , and we know from the uniqueness of solutions that the curves Γ_θ never cross. They may be periodic, though.

Locally they cover the space, in the sense that if z lies in some Γ_{θ_0} , it is easy to see (by running along the vector field backwards) that every point of a small ball around z lies in the union of the Γ_θ , θ close to θ_0 . Indeed, we can run along Γ_{θ_0} backwards starting from z (and up to the point $\gamma_0(\theta_0)$). For z' close to z , we can run the same vector field, and get a curve that stays close to Γ_{θ_0} . In fact, the solution is a C^2 function of z , because ∇u is of class C^3 (we are not aiming for optimal regularity here). Also, since Γ_{θ_0} meets γ_0 transversally at $\gamma_0(\theta_0)$, we can apply the implicit function theorem to prove that there is a unique θ , θ close to θ_0 , that contains z' . Thus we get a C^2 mapping v , defined near z , which to z associates the unique θ near θ_0 such that $z' \in \Gamma_\theta$. In fact, the uniqueness is global (as long as the parameterization of γ_0 is injective; we would need to be more specific when γ_0 is a loop), because the different Γ_θ do not cross. Finally, $\nabla v(z) \neq 0$, because it is obtained from the implicit function theorem.

At this point, we have, in an open set that contains γ_0 , two functions u and v that can be used as coordinates. The pair (u, v) satisfies the desirable orthogonality relation

$$(2.1) \quad \nabla u(z) \perp \nabla v(z).$$

This does not mean that the change of variables defined by (u, v) is conformal, because although the gradients (or the level lines) are orthogonal, the two lengths $|\nabla u(z)|$ and $|\nabla v(z)|$ are different. In fact, if we wanted to be sure that (u, v) defines a conformal change of variables, we should require u to be harmonic (and then we'll see that v is harmonic too). So for example, replacing (u, v) with $(u + v, u - v)$ gives another pair that does not satisfy (2.1) in general.

Our next step is to use this relation to show that, in any domain where u, v are functions of class C^2 in an open set, where $\nabla u \neq 0$ and $\nabla v \neq 0$, and (2.1) holds,

$$(2.2) \quad \operatorname{div} a \nabla u = 0, \quad \text{with } a(z) = \frac{|\nabla v(z)|}{|\nabla u(z)|}.$$

Of course the same computation will show that v satisfies the twin equation $\operatorname{div} \frac{1}{a} \nabla v = 0$.

Let us write this in coordinates and check this near a point z , but avoid to mention the argument z when we don't need to. The vector $w = (\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x})$ is perpendicular to ∇v and has the same length as ∇v , so since ∇u is proportional to w (by (2.1)), in fact $a \nabla u = \pm a w$ (because the two vectors have the same length). The sign is locally constant, so we may now compute

$$(2.3) \quad \operatorname{div} a \nabla u = \operatorname{div} w = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0,$$

as promised.

Here we shall be interested in making sure that $C^{-1} \leq a \leq C$, or in other words that $C^{-1} |\nabla u| \leq |\nabla v| \leq C |\nabla u|$. In terms of level curves for u and v , assuming that a nice parameterization of γ_0 was chosen, the gradient of G is (locally) proportional to the inverse of the distance between the level sets (divided by an increment δ_s), and similarly the gradient of v corresponds to the inverse of the distance between the Γ_θ (divided by a δ_θ); then we should compare these quantities and say whether they stay between constants. On the picture, it means that if we use roughly equal increments δ_x and δ_θ , we should see small rectangles that are not too thin. In particular, it is all right if the two types of level sets become too sparsely or too densely spaced, provided that they do this in essentially the same way.

Example 2.4. Let us even describe an example where u and v are conjugated harmonic functions, so that we can even take $a = 1$. Identify R^2 with \mathbb{C} , write $z = x + iy$, also use polar coordinates, and take

$$(2.5) \quad u(x, y) = \operatorname{Re}(z^4) = r^4 \cos 4\theta \quad \text{and} \quad v(x, y) = \operatorname{Im}(z^4) = r^4 \sin 4\theta.$$

It is known (and easy to compute) that u and v are harmonic and $|\nabla u(x, y)| = |\nabla v(x, y)| = 2|z|^2$. Of course we are not exactly in the case described above, because u and v have a critical point at the origin, but we do not have to divide by $|\nabla u|$ to know that u and v are harmonic. See Figure 1 for a clumsy attempt to describe the level sets of u (in green) and its gradient lines (in red). Notice that at the origin, u increases at maximal speed when $\cos 4\theta = 1$, i.e., when $\theta = k\pi/2$, along the axes, and u decreases at maximal speed when $\cos 4\theta = -1$, along the diagonals. All these lines are special red curves (oriented differently) and are separated by green lines where $4\theta = \frac{\pi}{2} + k\pi$.

The advantage of this example is that the uniform bounds on $|\nabla u|^{-1} |\nabla v|$ are obvious. We may always replace u and v with the new functions G and R , where

$$(2.6) \quad G(z) = G(0) + \beta u(z) \quad \text{and} \quad R(z) = R(0) + \beta v(z),$$

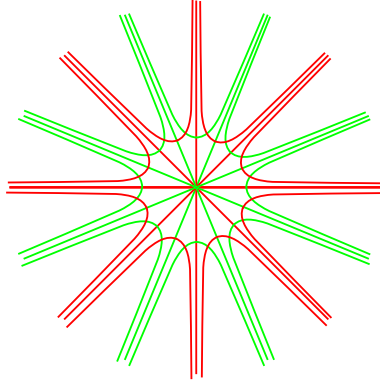


FIGURE 1. Level lines for $u(r, \theta) = r^4 \cos 4\theta$ (in green), and the conjugate function v (in red)

where we can choose $G(0)$, $R(0)$, and $\beta > 0$ as we like (we prefer $\beta > 0$ because this way we preserve the direction of the arrows). This does not change the level lines, just the way they are labelled.

Remark 2.7. If we are given the level lines of u , we can deduce the direction of the gradient, so we can also draw the level lines for v . We do not change the picture when we relabel the level lines of u , i.e., compose with a function f and replace u with $w = f \circ u$, or when we change the parameterization of γ_0 , i.e., in effect, replace v by some $g \circ v$.

Another way to see this is to observe that when u satisfies $\operatorname{div} a \nabla u = 0$, then $w = f \circ u$ satisfies the related equation $\operatorname{div} b \nabla w = 0$, with $b(z) = a(z)/f'(u(z))$, just because $b(z) \nabla (f \circ u)(z) = b(z) f'(u(z)) \nabla u(z) = a(z) \nabla u(z)$, and then the divergence is the same.

This is an interesting flexibility that we have with the equation $\operatorname{div} a \nabla u = 0$, at least if we allow ourselves to play with a .

We could also replace the function v with another function $g \circ v$; this does not change the level sets of v , nor the orthogonality condition (2.1), but as before it changes $a = \frac{|\nabla v|}{|\nabla u|}$.

This last remark is important because it helps us understand that given u , we have a large choice of functions a such that $\operatorname{div} a \nabla u = 0$, but some of them are equivalent for geometric reasons. Even if u is harmonic, we can find lots of operators $L = \operatorname{div} a \nabla$ such that $Lu = 0$, in particular $L = \Delta$ and $L = 2\Delta$, but not only. The geometry will help us choose good functions a .

3. LEVEL SETS OF G

Let us first define our Cantor set. We start with the square $K_0 = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ of sidelength 1. We replace K_0 with a set K_1 , composed of four squares $K_{1,j}$ contained in K_0 , situated at the four corners, and of sidelength 4^{-1} . Then we define the sets K_k , $k \geq 1$ by induction, to be the set $K_k \subset K_{k-1}$, composed of 4^k squares of sidelength 4^{-k} , and obtained by replacing each of the squares of sidelength 4^{1-k} that compose K_{k-1} by four squares situated at the four corners. See Figure 2. Our final set is

$$(3.1) \quad K = \bigcup_{k \geq 0} K_k \subset K_0.$$

It is a compact set of dimension 1, such that $0 < \mathcal{H}^1(K) < +\infty$, which is also Ahlfors regular and has been known in particular to be a simple example of a compact set such that $\mathcal{H}^1(K) > 0$ and with a vanishing analytic capacity; see [Ga, Iv]. It is also known that $\mathcal{H}^1|_K$ and the harmonic measure on K are mutually singular.

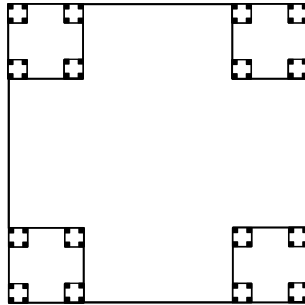


FIGURE 2. The set K_3 (three generations of the construction of K)

Our goal for this section is to construct a function G on $B_0 = B(0, 1)$, which we will decide is the restriction to B_0 of the desired Green function. We can already decide that

$$(3.2) \quad G(z) = 0 \text{ on } K, \text{ and } G(z) = 1 \text{ on } \partial B(0, 1).$$

We will also define G in a self-similar way, so we consider the four centers z_j of the four squares Q_j that compose K_1 . Take for instance $z_1 = (\frac{7}{16}, \frac{7}{16})$ and turn in the trigonometric direction, i.e., take $z_2 = (-\frac{7}{16}, \frac{7}{16})$, $z_3 = (-\frac{7}{16}, -\frac{7}{16})$, and $z_4 = (\frac{7}{16}, -\frac{7}{16})$. Then, for $i = 1, \dots, 4$, set $B_j = B(z_j, 1/4)$. Thus B_j is the analogue for Q_j of B_0 for K_0 . The radius $r_0 = 1$ was chosen sufficiently large for K_0 to be contained in B_0 (and Q_j in B_j), but small enough for B_j to stay away from the axes, i.e., to be contained in the same quadrant as Q_j . We will in fact construct G on the annular region

$$(3.3) \quad A_0 = B_0 \setminus \left[\bigcup_{j=1}^4 B_j \right],$$

and in order to be able to glue easily, we will make sure that

$$(3.4) \quad G(z) = \frac{1}{4} \text{ for } z \in \bigcup_{j=1}^4 \partial B_j$$

In addition, let us use the symmetries of our set. Call \mathfrak{S} the collection of symmetries with respect to the two axes and the two diagonals; we decide that

$$(3.5) \quad G(\sigma(z)) = G(z) \text{ for } z \in A_0 \text{ and } \sigma \in \mathfrak{S};$$

and because of this it will be enough to define G on the smaller region

$$(3.6) \quad A_{00} = \{(x, y) \in A_0; x \geq 0, y \geq 0, \text{ and } x \geq y\}.$$

See Figure 3 for a first sketch of the level lines of G , in green, and its gradient lines (or the level lines of the conjugated function), in red. Observe before we start that because of

the symmetries, we need to have a critical point at the origin; the part of the axes and the diagonals that lie in A_0 are really gradient lines of G , and they part at the critical point 0.

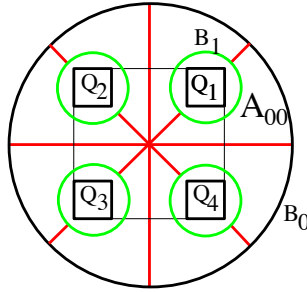


FIGURE 3. The cubes Q_j , the balls B_0 and B_1 , and the annuli A_0 (in the big ball, outside of the green ones) and A_{00} (one eighth of the picture)

We will fill the picture little by little; we will need to pay a special attention to what happens near the critical point, because this is the place where it is not obvious that our curves have a uniform behavior.

We start with the definition of G near the boundary circles, where we slightly prefer G to be harmonic. So pick a radius ρ just a little bit smaller than $r_0 = 1$, and set

$$(3.7) \quad G(z) = 1 + \ln(|z|) \text{ for } z \in B_0 \setminus B(0, \rho).$$

This respects the symmetries and (3.2); in principle, we should only have taken (3.7) in $A_{00} \cap B_0 \setminus B(0, \rho) = A_{00} \setminus B(0, \rho)$, but this is the same. We do the same thing near the B_j , i.e., choose a radius ρ_1 just a tiny bit larger than $1/4$ and set

$$(3.8) \quad G(z) = \frac{1}{4} + \frac{1}{4} \ln(4|z - z_1|) \text{ for } z \in A_{00} \setminus B(z_1, \rho_1)$$

the other balls B_j would be taken of by symmetry.

Next we take care of the situation near the origin. We decide that in a small ball $B(0, \rho_2)$, we take the functions G and its conjugate R according to the formula (2.6), where we can still choose $G(0) \in (\frac{1}{4}, 1)$ and $R(0)$ so that the picture looks nicer (we shall return to this issue when we glue). In the meantime we can at least draw the green curves and their orthogonal red curves in $B(0, \rho_2)$ (without labelling them yet). This gives something like Figure 4 when we restrict to A_{00} (as before our choice of G preserves the symmetry). In particular, (3.4) is satisfied.

So we have already drawn the level sets of G and its conjugated function R (i.e., the gradient lines) in three small regions of A_{00} ; now we complete the red lines, subject to the following constraints, that we claim are easy to implement:

$$(3.9) \quad \text{the unit vector field } V_R \text{ tangent to the red lines is smooth on } A_{00} \setminus B(0, \rho_2/2)$$

(there is indeed a singularity at the origin, but it is well controlled), and in addition

$$(3.10) \quad \text{the intersection of the diagonal and the } x\text{-axis with } \overline{A_{00}} \text{ are red curves,}$$

and even slightly more, the red vector field V_R is parallel to the straight parts of the boundary of A_{00} in the neighborhood of the corresponding parts of the boundary, and consequently

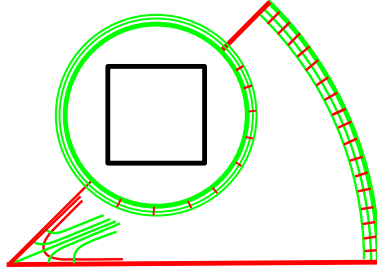


FIGURE 4. The function G (level lines in green, gradient lines in red) in the three special regions. The lower left part is a copy of part of Figure 1

when we derive the Green lines, they will meet the straight part of the boundary transversally, and of course (since we mentioned vector fields), the red lines do not cross. Finally, we need to patch the different red lines, for instance in the zone below and to the right of B_1 , in such a way that

$$(3.11) \quad \begin{aligned} & \text{the red curve } \Gamma_\theta \text{ that leaves from } \partial B_1 \text{ at the point } z_1 + \frac{1}{4}e^{-i\frac{\pi}{4}+i\theta}, 0 \leq \theta \leq \pi, \\ & \text{leaves } B_0 \text{ at the point } e^{i\theta/4}. \end{aligned}$$

That is, we have a mapping from a part (a half) of ∂B_1 to a part (one eighth), obtained by associating the two endpoints of each red curve, and we want it to be not only bijective, but run at constant speed. Here the two arcs have the same length, so this also means that the mapping is locally isometric (for the arclength). This is a little unpleasant to do practically, so let us reduce this to a simpler problem. We use a mapping ψ to send A_{00} to a rectangle R , so that the two circular pieces of the boundary, namely $\partial B_1 \cap \partial A_{00}$ and $\partial B_0 \cap \partial A_{00}$, are sent to two opposite edges of R , which we call I_1 and I_3 , and even at constant speed 1. The two arcs have the same length, so this is possible. We can also make sure that the mapping is a smooth diffeomorphism (with a bounded derivative for the inverse), except at the origin where we can make it equal to $z \mapsto z^4$.

We already constructed (pieces of) our red curves Γ_θ in a small neighborhood in \bar{A}_{00} of ∂A_{00} , and we consider their images $\tilde{\Gamma}_\theta = \psi(\Gamma_\theta)$ in R . First of all, notice that the singularity at 0 of our collection of red curves disappears, so we really have pieces of curves that are smooth (where they are defined) and leave I_1 and arrive on I_3 perpendicularly. Those that are globally defined, along the remaining edges I_2 and I_3 , also do not cross, and follow nicely the edges. In addition, we may modify them a little near the middle of I_2 and I_4 , if needed, so that the the endpoint $y(\theta)$ of $\tilde{\Gamma}_\theta$, which is defined for θ near 0 and near π , goes along I_3 at constant speed. Along I_1 and I_3 , our red curves touch the boundary perpendicularly. By restricting to a smaller neighborhood of ∂R (or equivalently ∂A_{00}), we can make sure that they are graphs of Lipschitz functions with a small constant (assuming that I_1 and I_3 are vertical). At this point we claim that it is easy to extend the curves so that they fiber R and $y(\theta)$ runs at constant speed 1 between the two vertices of I_3 . Then the inverse images by ψ solve our initial problem. We kindly leave the verification to the reader.

When all this is done, we get a picture like Figure 5. The collection of red curves, labeled by θ , defines a function R on the region A_{00} . The green curves that are drawn in Figure 6, are just the orthogonal curves, obtained as in Section 2 by following a vector field V_G perpendicular to the previous one. In $B(0, 1/4)$ we already had what we needed, so we don't need to solve a singular vector field problem, just notice that the two definitions patch. The smoothness of V_R gives the existence and regularity of the green lines. A continuity argument shows that they start from the upper part D_1 of the first diagonal, perpendicularly to D_1 , and then end up along the rest of the straight boundary, on the union of the lower part D_2 of the first diagonal, followed by the piece of the x -axis. The perpendicular landing comes from (3.10), the rest comes from the existence, uniqueness, and smoothness for the integral curves.

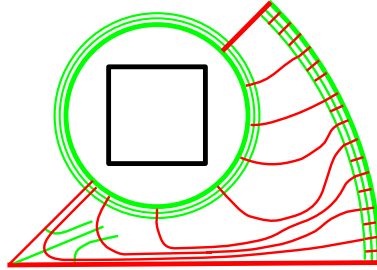


FIGURE 5. The completed red lines, subject to the constraints above. We make them turn in the region in the right so that the two arcs of circle connect at equal speeds

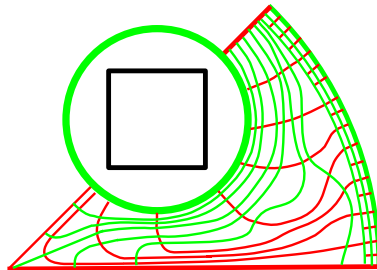


FIGURE 6. The level lines of G (in green). Notice that a principle once stated by B. Dahlberg verifies: in spite of the existence and regularity theorem, it is very hard to draw orthogonal green curves so that they look nice.

Remark 3.12. Finally the reader may wonder why we asked for a matching condition on the extremities of the red curves. Near ∂B_0 and ∂B_1 , we decided a formula for G , and our construction ensures that R is a conjugate function to G , as in the orthogonality condition (2.1). But there is a dangerous pitfall here, which we try to explain so that the reader does not make the same mistake as some of the authors.

It is true that G is harmonic near ∂B_0 and ∂B_1 , so we know for sure that it satisfies $LG = 0$, with $L = \Delta$ and $a = 1$. But many other choices of a are possible, for instance $a = 1$

near ∂B_0 and $a = 12$ near ∂B_1 , not to mention more exotic choices. Here it is important that a should be defined globally, which for the moment means on the whole A_{00} , and for this the function R is very useful, because the computations of Section 2 show that we can take $a = \frac{|\nabla R|}{|\nabla G|}$. So we have less choice about a than one could believe: we already defined G near both circles, with $|\nabla G| = 1$ on both circles, and we want to take $a = 1$ near ∂B_0 and ∂B_1 , so we need to make sure that $|\nabla R| = |\nabla G|$ in both regions, so in particular $|\nabla R| = 1$ on both circles. This is what we ensured with (3.11), and now the fact that $|\nabla R| = |\nabla G|$ near the circles follows from the fact that G is harmonic there, the red curves are known and parameterized in the right way, so R has to be the usual conjugation of G that we don't even need to compute.

At this point we have two functions R and G , where R is entirely determined by the red curves and the parameterization from (3.11), and we have a little bit of latitude for G , because we only decided about its values in three precise regions. This is not too shocking, because we could decide to replace G with a composed function like $f \circ G$ (with a nice f); we can lift the ambiguity by parameterizing one of the red curves Γ_θ by $s \in (\frac{1}{4}, 1)$ and deciding that $G(z) = s$ along the green curve that contains $\Gamma(s)$.

As was just explained, we want to take

$$(3.13) \quad a(z) = |\nabla G(z)|/|\nabla R(z)| \text{ on } A_{00},$$

because this is our way of making sure that

$$(3.14) \quad \operatorname{div} a \nabla G = 0 \text{ on } A_{00},$$

even in the strong sense (because G is smooth). Let us check that

$$(3.15) \quad C^{-1} \leq a(z) \leq C \text{ on } A_{00}.$$

Away from 0, this is clear because all our functions are smooth, and we can ensure lower bounds on their gradients. In the small ball $B = B(0, \rho_2)$ near 0, we have a precise formula for G , which determines the red curves but not how they are labelled. This is what gives the picture of Figure 4, copied from a piece of Figure 1. There is a good choice R_0 for R , coming from the formula (2.6) and that yield $a = 1$, but multiplying R_0 by a constant will only lead to a different constant value of a , which is fine too. In fact, if we do not pay attention, the values of R when we enter B will (by smoothness) be equivalent to the values of R_0 , in the sense that $C^{-1}|\nabla R_0| \leq |\nabla R| \leq C|\nabla R_0|$. This will be enough for ellipticity, but we also promised that a would be continuous, so we modify slightly the way our red curves $\tilde{\Gamma}_\theta$ were organized near I_2 (or I_4) to make sure that for those curves that enter $B(0, \rho_2)$, the parameterization speed is proportional to what we would get for R_0 . This way we obtain that

$$(3.16) \quad a \text{ is constant on } A_{00} \cap B(0, \rho_2),$$

which is a brutal way to ensure that a will be continuous at 0. Let us also record that

$$(3.17) \quad a(z) = 1 \text{ in } A_{00} \cap \left((B_0 \setminus B(0, \rho)) \cup (B(z_1, \rho_1) \setminus B_1) \right).$$

This ends our construction of R and G on A_{00} . We extend G and a to A_0 by symmetry. Notice that a is continuous on A_0 ; it is possible that we could make it smooth, by choosing

the red curves near ∂A_{00} even more carefully so that the normal derivative of a vanishes there, but we were not courageous enough to try. Also, G is C^1 across the the straight part of the boundary of ∂A_{00} , because its normal derivative vanishes (this part of the boundary is a red curve). Then it is also of class C^2 , and by symmetry (or reflection principle), (3.14) still holds in A_0 . What happens near 0 is even simpler, because G is harmonic and (3.16) holds there. At this point we have a picture that looks like Figure 7.

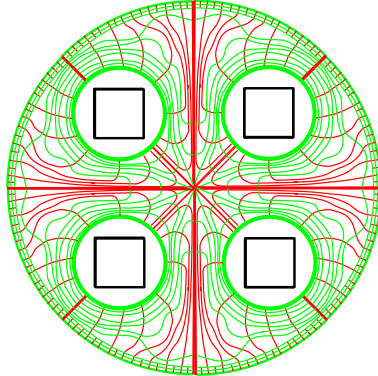


FIGURE 7. The level and gradient lines of G in A_0 , completed by symmetry

It is now time to glue G to itself in a fractal way. For each generation $k \geq 0$, denote by $\mathcal{Q}(k)$ the set of 4^k squares that compose K_k . For each square $Q \in \mathcal{Q}(k)$ we denote by $z(Q)$ the center of Q and F_Q the obvious affine mapping that sends Q to $Q_0 = K_0$. Thus

$$(3.18) \quad F_Q(z) = 4^k(z - z(Q)) \quad \text{for } z \in \mathbb{R}^2.$$

We define an annular region $A(Q) = F_Q^{-1}(A_0)$, notice that the $A(Q)$, $Q \in \cup_k \mathcal{Q}(k)$, form a partition of $B_0 \setminus K$, and define functions on $B_0 \setminus K$ by

$$(3.19) \quad G(z) = 4^{-k}G(F_Q(z)), \quad \text{and } a(z) = a(F_Q(z)) \quad \text{for } z \in A(Q), Q \in \mathcal{Q}(k).$$

Notice that

$$(3.20) \quad 4^{-k-1} \leq G \leq 4^{-k} \quad \text{on } A(Q) \quad \text{when } Q \in \mathcal{Q}(k),$$

with

$$(3.21) \quad G(z) = 4^{-k} \quad \text{on the exterior boundary } F_Q^{-1}(\partial B_0),$$

which is also (when $k \geq 1$) the interior boundary of the cube of the previous generation that contains Q . That is, G is continuous across the boundaries, and we even claim that

$$(3.22) \quad G \text{ is harmonic across } F_Q^{-1}(\partial B_0).$$

This requires a small verification of normal derivatives, or we can directly use the formulas. Let us just do the verification when Q is the cube Q_1 of generation 1; in the general case, we would need to compose both functions with an additional affine transform. Near ∂B_1 ,

but at the exterior, we decided in (3.8) that $G(z) = \frac{1}{4} + \frac{1}{4} \ln(4|z - z_1|)$. But on the interior part, we use the formula

$$(3.23) \quad G(z) = \frac{1}{4}G(F_{Q_1}(z)) = \frac{1}{4}G(4(z - z_1)) = \frac{1}{4}(1 + \ln(4|z - z_1|)).$$

by (3.7). This is the same formula (we made it on purpose!), so $G(z)$ is indeed harmonic across the circle.

We finally obtained our scale invariant elliptic coefficient a , and a function G which is L -harmonic (i.e., satisfies the equation $\operatorname{div} a \nabla u = 0$) on $B_0 \setminus K$, and is equivalent to $\operatorname{dist}(z, K)$ (by (3.20)). This is essentially all we needed. For the fun of it, we check on Figure 8 that when we put four copies of Figure 8, reduced by a factor of 4, in their correct place in the B_j , we get a description of G in a larger region that looks coherent. With our additional patching law, the continuations of the red curves arrive to 16 small circles with the same distances as when they left ∂B_0 , even though the picture seems to say something different.

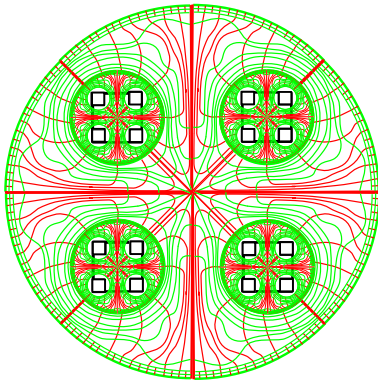


FIGURE 8. The level and gradient lines of G on a larger region A_0 , completed by fractality

In the next section we finally state the main theorem and explain why we essentially proved it already.

4. A STATEMENT OF THE MAIN THEOREM

Let us recall some of the notation for the main theorem. Let $K \subset \mathbb{R}^2$ be the Garnett-Ivanov Cantor set that was described near (3.1), and let μ denote the canonical probability measure on K , defined by the fact that $\mu(Q) = 4^{-k}$ for each of the squares of sidelength 4^{-k} that compose K_k (so that we don't even need to define \mathcal{H}^1).

For each operator $L = \operatorname{div} a \nabla$, with a measurable and such that

$$(4.1) \quad C^{-1} \leq a(z) \leq C \text{ for } z \in \mathbb{R}^2 \setminus K$$

(so that L is elliptic), and each point $z \in \mathbb{R}^2 \setminus K$, there is a probability measure ω_L^z on K , called the elliptic measure for L on K , and which for instance allows one to solve the Dirichlet problem. We can also let z tend to ∞ , and by an argument that uses the comparison principle, ω_L^z tends to a probability measure ω_L^∞ , which we call the harmonic measure with pole at ∞ . We introduce it here because it gives a cleaner statement.

Theorem 4.2. *Retain the notation above. There exists an absolute constant $C \geq 1$ and a continuous function $a : \mathbb{R}^2 \setminus K \rightarrow (0, +\infty)$, such that (4.1) holds, $a(z) = 1$ for $z \in \mathbb{R}^2 \setminus B(0, 1)$, and if ω_L^∞ denotes the harmonic measure, with pole at ∞ , associated to the operator $L = \operatorname{div} a \nabla$ on the domain $\mathbb{R}^2 \setminus K$, then $\omega_L^\infty = \mu$. Also,*

$$(4.3) \quad C^{-1} \mu(A) \leq \omega_L^z(A) \leq C \mu(A) \text{ for } z \in \mathbb{R}^2 \setminus B(0, 1) \text{ and } A \subset K \text{ measurable.}$$

We shall discuss a few variants and improvements in the next section. In the meantime we prove the theorem. Let a be the function that was constructed in the last section, which we extend by taking $a = 1$ on $\mathbb{R}^2 \setminus B_0$. Then let G be the L -harmonic function above, which we extend by taking $G(z) = 1 + \ln(|z|)$ for $|z| > 1$. That is, we just extend the formula (3.7), and of course G is harmonic on $\mathbb{R}^2 \setminus B_0$. We claim that G is (a constant times) the Green function for L , with the pole at ∞ , essentially because it is L -harmonic, positive, and vanishes at the boundary. Here it is even equivalent to $\operatorname{dist}(z, K)$ near K , but the usual Hölder continuity would be enough.

Now we don't want to use the usual relation between the normal derivative of the Green function and the Poisson kernel directly on K , because K is irregular, so the simplest seems to approximate K by the circular variant of K_k and compute there.

Denote by $D_k = \cup_{Q \in \mathcal{Q}(k)} F_Q^{-1}(\overline{B}_0)$ this approximation. Here we use the notation near (3.18), D_k is just a union of balls of radius 4^{-k} with the same centers as the pieces of K_k , and $E_k = \partial D_k$ is the corresponding union of circles. Notice that $K \subset D_k$, $\Omega_k = \mathbb{R}^2 \setminus D_k$ is a nice smooth domain, G is still L -harmonic on Ω_k , and $G_k = G - 4^{-k}$ is clearly (a multiple of) the Green function on Ω_k , with pole at ∞ . We can even compute the normal derivative $g_k = \frac{\partial G_k}{\partial n}$ on ∂D_k , because we have the explicit formulas (3.19) and (3.8). We find that g_k is a constant, that does not even depend on k , so the elliptic measure at ∞ associated to L on Ω_k (call it ω_k) is equal to the invariant measure μ_k on ∂D_k . We write the reproducing formula $f(\infty) = \int_{E_k} f d\omega_k = \int_{E_k} \mu_k$ for continuous bounded L -harmonic functions, let k tend to $+\infty$, observe that μ_k tends to μ (look at the effect on any finite union of sets $F_Q^{-1}(\overline{B}_0)$), prove that μ gives an good reproducing formula on K , and conclude.

The second part of our statement is an easy consequence of the first one and the change of poles formula (or the comparison principle); if the reader does not like to take a pole at ∞ , they can also look at the argument above, notice that our function G is equivalent to the Green function with the pole X , and then derive (4.3) with the same argument as sketched above for the Green function at infinity. \square

Here we decided to go for the simplest statement, but variants and extensions are possible, some of which we describe in the next section.

5. VARIANTS AND EXTENSIONS

We start with rotated Cantor sets. Suppose that, in the description of the construction of K_k from K_{k-1} , when we replace each cube Q with four squares at its corners, we also allow to rotate Q (or now the four cubes) by any angle that depends on Q . See Figure 9 for a hint; we leave the reader draw themselves the analogue of Figure 8 in this case. We get a new, apparently more complicated set K^* ; this manipulation makes it much more difficult to control the usual harmonic measure on K^* (and prove that it is very singular), or control the

measure of the projections, but here the effect on our construction is just null: the domains A_0 and A_{00} are different, but the important information is that we have a (rotation invariant) formula for G on the exterior and interior boundaries, so that we can glue the pieces. Of course the function a depends on the rotations.

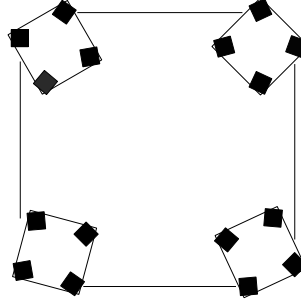


FIGURE 9. The third iteration of a rotating version of the Cantor set

Next consider differently scaling Green functions. The simplest way to do this is to take a function of the function $G(z)$ constructed above, such as $G(z)^\alpha$. This gives a solution of $\tilde{L} = \operatorname{div} \tilde{a} \nabla$, where $\tilde{a}(z) = \alpha a(z) G(z)^{\alpha-1} \sim \operatorname{dist}(z, K)^{\alpha-1}$. The Green function for \tilde{L} is $G(z)^\alpha \sim \operatorname{dist}(z, K)^\alpha$. This is not too surprising: we trade a different scaling for G against a different scaling for the degenerate elliptic operator \tilde{L} . Notice that in this case too, the corresponding harmonic measure at ∞ is still the invariant measure μ (what else?).

There is no special difficulty with replacing K by different Cantor sets K_β constructed in the same way, but a dimension $\beta \neq 1$. That is, instead of taking squares of sidelength 4^{-k} at the k th generation, we take squares of sidelength λ^k for some $\lambda \neq \frac{1}{4}$. The dimension β is limited by the fact that we want the balls B_j , $1 \leq j \leq 4$ to contain the corresponding cubes Q_j and be disjoint.

Something interesting happens in the construction, which is related to Remark 3.12. When we construct the function G , we decide that it is equal to 1 on ∂B_0 , and to some constant $\gamma < 1$ on the analogue of ∂B_1 . Then we choose the function R so that the transfer map from ∂B_0 to ∂B_1 goes at a constant speed τ . The length to cover is $\pi/4$ on ∂B_0 and $\lambda\pi$ on ∂B_1 , so the gradient of R is 4λ times smaller on ∂B_1 .

At the same time, if we want to use a fractal formula like (3.19), we should write it down as

$$(5.1) \quad G(z) = \gamma^k G(F_Q(z)),$$

where the derivative of $F_Q(z)$ is now λ^{-k} . Hence $|\nabla G|$ is $\gamma\lambda^{-1}$ times larger on ∂B_1 than on ∂B_0 and $a = |\nabla R|/|\nabla G|$ is 4γ times larger on ∂B_1 .

If we want to construct an elliptic operator, we should be able to get $a = 1$ on both circles, so $\gamma = 1/4$. This means that for points of generation k , the distance to K is like λ^k , but the size of the Green function is now like $\gamma^k = 4^{-k}$, which is also roughly the measure in K of a ball of radius λ^k . In other words, $G(z) \sim \operatorname{dist}(z, K)^d$, where d is the dimension of the Cantor set.

In general, we can pick $\gamma \neq 1/4$, get a Green function with essentially any other homogeneity, but then our operator $L = \operatorname{div} A \nabla$ will be degenerate elliptic, with $|a(z)| \sim \operatorname{dist}(z, K)^\alpha$ for some $\alpha \neq 0$.

We can pursue all this a little further, and replace the balls in the construction of G with objects with a different shape (in fact our first constructions were like that) but then we are no longer allowed to rotate the squares.

In fact (as in [Bat], for instance), we can let the dilation ratio depend on the scale (as long as we keep some uniformity), or take Cantor sets based on dividing each box into more than 4 pieces, and probably combine all of the above. See Figure 10 for a hint.

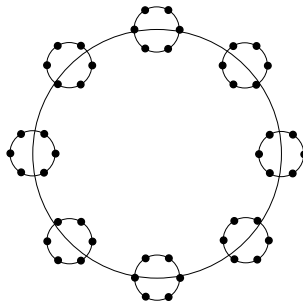


FIGURE 10. The third iteration of a variable scale/multiplicity analogue of K ; there is no point in trying to draw polygons in this case

It would be nice to have an operator like the one above, with the special form (1.1) (otherwise this is too easy), associated to the standard snowflake, or some general Reifenberg flat domains. This is tempting, but one would have to be careful with the construction and the verification that $C^{-1} \leq a \leq C$. We can also ask the same questions in higher dimensions, i.e., for instance for a product of three Cantor sets of dimension $1/3$ in \mathbb{R}^3 .

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