Easy quantum actions ?

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Action!

Introducing the main character

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- $C(\mathbb{G})$ unital C*-algebra ;
- $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$;
- $\Delta \circ (\mathrm{id} \otimes \Delta) = (\mathrm{id} \otimes \Delta) \circ \Delta$;
- $\overline{\mathrm{Span}}\,\{\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))\} = C(\mathbb{G}) \otimes C(\mathbb{G})$.

Definition

An *action* of $\mathbb{G}$ on A is a $*$-homomorphism $\alpha : A \to A \otimes C(\mathbb{G})$ such that

1. $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$;
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- The action is *ergodic* if $A^{\alpha} = \{x \in A \mid \alpha(x) = x \otimes 1\} = \mathbb{C}.1_A$
- Classically, ergodic $\Leftrightarrow$ minimal.

Plot twist : it's about categories !

Theorem (TANNAKA-KREIN-WORONOWICZ)

A CQG is a unitary tensor functor $\mathscr{F} : \mathrm{Rep} \to \mathrm{Hilb}_f$, *where* Rep *is a rigid C*-tensor category.*

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We need a *weird* tensor functor ...

Take it easy

Idea : Start with a simpler yet interesting setting :

- $(\mathbb{G}, u)$ OCMQG, i.e. $u = \overline{u}$ and $\overline{\langle u_{ij} \mid 1 \leqslant i, j \leqslant N \rangle} = C(\mathbb{G})$;
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Let $(\mathbb{G}, u)$ be an OCMQG and $(H_n)_{n\in\mathbb{N}}$ Hilbert spaces with maps

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2. $P_{k+\ell,m} \circ P_{k,\ell+m} = P_{k,\ell,m}$.

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2. $P_{k+\ell,m} \circ P_{k,\ell+m} = P_{k,\ell,m}$.

Then, this is the restriction of $\mathscr{F}_\alpha$ for some ergodic action α of $\mathbb{G}$.

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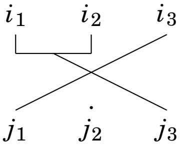
$p =$ 

- We then define an operator $T_p : \left(\mathbb{C}^N\right)^{\otimes 3} \to \left(\mathbb{C}^N\right)^{\otimes 3}$ by the formula

$$T_p(e_{i_1} \otimes e_{i_2} \otimes e_{i_3}) = \delta_{i_1,i_2} \sum_{j_2=1}^{N} e_{i_3} \otimes e_{j_2} \otimes e_{i_1}.$$

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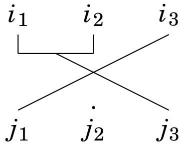

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Theorem (Jones, Martin)

Set $V = \mathbb{C}^N$ with the permutation representation of S_N. Then, for any k, ℓ,

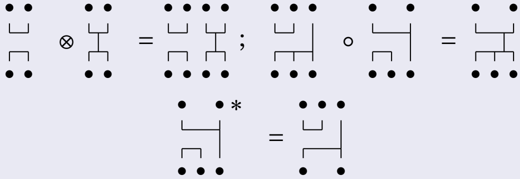
$$\mathrm{Mor}_{S_N}\left(V^{\otimes k}, V^{\otimes \ell}\right) = \mathrm{Span}\left\{T_p \mid p \in P(k,\ell)\right\}$$

where $P(k,\ell)$ is the set of partitions of $\{1,\cdots,k+\ell\}$.

Take it easy

Definition

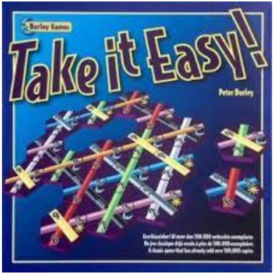
A *category of partitions* is a set $\mathscr{C}$ of partitions containing $|$ which is stable under



TANNAKA-KREIN-WORONOWICZ : Given $\mathscr{C}$, there exists a unique OCMQG $\mathbb{G}_N(\mathscr{C})$ such that

$$\mathrm{Mor}_{\mathbb{G}_N(\mathscr{C})}\left(\left(\mathbb{C}^N\right)^{\otimes k}, \left(\mathbb{C}^N\right)^{\otimes \ell}\right) = \mathrm{Span}\left\{T_p \mid p \in \mathscr{C}(k, \ell)\right\}.$$

Easy actions



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A *module of projective partitions* over $\mathscr{C}$ is a set $\mathscr{P}$ of partition, writing $\mathscr{P}_k = \mathscr{P} \cap P(k,k)$,

1. For all $p \in \mathscr{P}$, $p \circ p = p = p^*$;
2. $\mathscr{P}_k \odot \mathscr{P}_l \subset \mathscr{P}_{k+l}$ for all $k, l \in \mathbb{N}$;
3. For any $r \in \mathscr{C}(k,l)$ and $p \in \mathscr{P}_k$, $rpr^* \in \mathscr{P}_l$;
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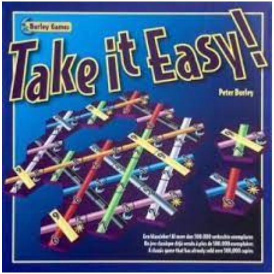

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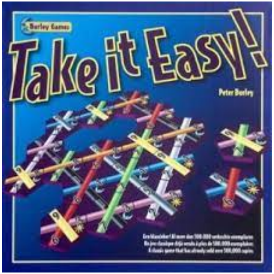


Theorem (F.-Taipe-Wang)

Let $\mathscr{C}$ be a category of partitions, let $\mathscr{P}$ be a module of projective partitions over $\mathscr{C}$ and let N be an integer. Then, the spaces

$$K_n^{\mathscr{P}} = \mathrm{Span}\{T_p(e_1^{\otimes n}) \mid p \in \mathscr{P}_n\} \subset (\mathbb{C}^N)^{\otimes n}$$

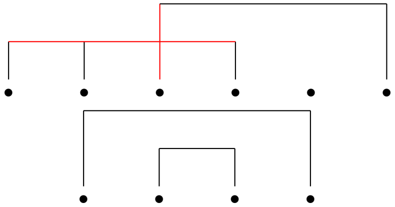
and the maps $\varphi_{k,\ell}(r) : T_p(e_1^{\otimes k}) \mapsto T_{rpr^*}(e_1^{\otimes \ell})$ satisfy the assumptions of the Theorem.

The non-crossing case

Definition

A partition is *non-crossing* if it can be drawn without letting the strings cross.

$p =$



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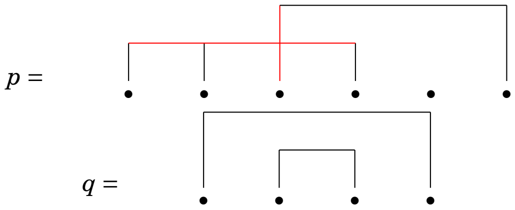

Theorem (F.-Taipe-Wang)

If $\mathscr{C}$ *is non-crossing and* $\mathscr{P} = \mathrm{Proj}_{\mathscr{C}} = \{p \in \mathscr{C} \mid p \circ p = p = p^*\}$*, then the previous action is the standard action on the first column space.*

Examples : $O_N^+ \curvearrowright S_+^{N-1,\mathbb{R}}$; $S_N^+ \curvearrowright \mathbb{C}^N$; $H_N^+ \curvearrowright \mathbb{C}^{2N}$.

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- $S_N^+ = \mathbb{G}_N(NC)$;
- $X_N = N$ points $\rightsquigarrow C(X_N) = \mathbb{C}^N$;
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Theorem (Sh. Wang)

The action α_N is universal.

Action from 2012

Question : Can S_N^+ act non-trivially on a connected space ?

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Let Y be a connected compact space and $Z \subset Y$ a non-empty proper closed subset. Then, S_N^+ acts non-trivially on $Y^{\amalg N}/Z$.

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Lemma

If $\mathscr{F}_\alpha(P_n^{k,k}) \neq 0$, *then*

$$H_n \subset \mathrm{Sym}(H_k \otimes H_k) \cup \mathrm{ASym}(H_k \otimes H_k)$$

Proof of the Theorem :

1. Set $k_0 = \min\{k \mid A_k \neq \{0\}\}$;
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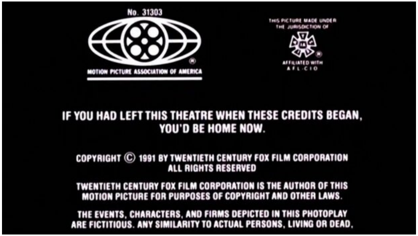
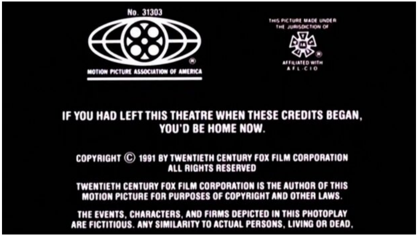
6. ... hence has a non-trivial projection !

Post-credit scene





Some open questions :

- Can S_N^+ act ergodically on a space which is not totally disconnected ?
- Can S_N^+ act ergodically on a space which is infinite ?
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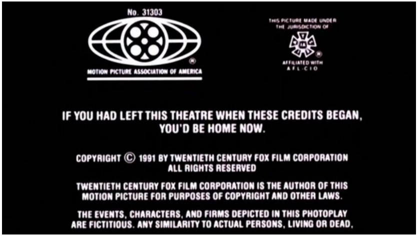
Post-credit scene



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What about other quantum groups ?

- H_N^+ : same status as S_N^+ ;
- O_N^+ : does not act non-trivially on any classical space !


The End