Convergence de la méthode de Boltzmann sur réseau avec sur-relaxation pour des lois de conservation non linéaires

Denise Aregba-Driollet

Institut de Mathématiques de Bordeaux et Bordeaux-INP

Outlines

- A discrete BGK formalism for systems of conservation laws
- Examples and theoritical results
- A lattice Boltzmann method (LBM)
- Study of the convergence
- Numerical experiments
- Conclusions and perspectives

A discrete BGK formalism

Consider a system of conservation laws

$$\partial_t u + \sum_{d=1}^{D} \partial_{x_d} A_d(u) = 0, \quad u(x,t) \in \mathcal{U} \subset \mathbb{R}^K.$$

Underlying BGK system: A.-D. and Natalini 1998

$$\partial_t f^\varepsilon + \sum_{d=1}^{D} \Lambda_d \partial_{x_d} f^\varepsilon = \frac{1}{\varepsilon} \left(\mathcal{M}(Pf^\varepsilon) - f^\varepsilon \right), \quad f^\varepsilon(x,t) \in \mathbb{R}^N.$$

$\varepsilon > 0$

Λ_d: diagonal matrix with fixed real coefficients

$P \in \mathcal{L}\left(\mathbb{R}^N, \mathbb{R}^K\right)$, and $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_N)$: $\mathcal{U} \to \mathbb{R}^N$.

Compatibility conditions

$$\forall u \in \mathcal{U}, \quad P\mathcal{M}(u) = u, \quad P\Lambda_d \mathcal{M}(u) = A_d(u),\ d = 1, \ldots, D.$$

Notation:

$$\forall f \in \mathbb{R}^N, \quad u = Pf.$$

If f^ε is a solution of the BGK equation then

$$\partial_t u^\varepsilon + \sum_{d=1}^{D} \partial_{x_d} \left(P \Lambda_d f^\varepsilon \right) = 0$$

If $f^\varepsilon \to f$ then $u^\varepsilon \to u$ and $f = \mathcal{M}(u)$:

$$\partial_t u + \sum_{d=1}^{D} \partial_{x_d} A_d(u) = 0$$

Generalisation to parabolic systems and incompressible Navier-Stokes: Bouchut, Guargaglini, Natalini 2000, A.-D., Tang, R. Natalini 2004, Carfora and Natalini 2008, Bouchut, Jobic,Natalini, Occelli, Pavan 2018.

Example: vectorial models

$$\partial_t u + \sum_{d=1}^{D} \partial_{x_d} A_d(u) = 0, \quad u(x,t) \in \mathcal{U} \subset \mathbb{R}^K.$$

$N = KL$, $L \geq 2$:

$$\partial_t f_l^\varepsilon + \sum_{d=1}^{D} v_{ld} \partial_{x_d} f_l^\varepsilon = \frac{1}{\varepsilon} \left(M_l(u^\varepsilon) - f_l^\varepsilon \right), \quad f_l^\varepsilon(x,t) \in \mathbb{R}^K, \quad 1 \leq l \leq L.$$

$$u^\varepsilon = \sum_{l=1}^{L} f_l^\varepsilon$$

Compatibility conditions:

$$\forall u \in \mathcal{U}, \quad \sum_{l=1}^{L} M_l(u) = u, \quad \sum_{l=1}^{L} v_{ld} M_l(u) = A_d(u) \quad (d = 1, \ldots, D).$$

Basic example of vectorial model: 1D Jin and Xin's model (D_1Q_2)

$u = f_1 + f_2 \in \mathbb{R}^K$, $\lambda > 0$,

$$M_1(u) = \frac{u}{2} - \frac{A(u)}{2\lambda}, \quad M_2(u) = \frac{u}{2} + \frac{A(u)}{2\lambda}$$

and

$$\begin{cases} \partial_t f_1^\varepsilon - \lambda \partial_x f_1^\varepsilon = \frac{1}{\varepsilon} \left(M_1(u^\varepsilon) - f_1^\varepsilon \right), \\ \partial_t f_2^\varepsilon + \lambda \partial_x f_2^\varepsilon = \frac{1}{\varepsilon} \left(M_2(u^\varepsilon) - f_2^\varepsilon \right). \end{cases}$$

It is the diagonal form of Jin and Xin's relaxation model (CPAM 1995):

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0 \\ \partial_t v^\varepsilon + \lambda^2 \partial_x u^\varepsilon = \frac{1}{\varepsilon} \left(A(u^\varepsilon) - v^\varepsilon \right). \end{cases}$$

Theoritical results

Convergence of $u^\varepsilon = Pf^\varepsilon$ towards an entropy solution of the system of conservation laws?

Scalar case: u_0: initial condition. We suppose that the Maxwellian functions are monotone nondecreasing functions:

$$\forall u \in [-\|u_0\|_\infty, \|u_0\|_\infty], \quad M_l'(u) \geq 0, \; l = 1, \ldots, L.$$

Example: Jin and Xin's model: $L = 2$,

$$M^-(u) = \frac{-F(u) + au}{2a}, \quad M^+(u) = \frac{F(u) + au}{2a}.$$

M^- and M^+ are increasing functions if and only if the subcharacteristic condition is satisfied:

$$\forall u \in [-\|u_0\|_\infty, \|u_0\|_\infty], \quad -a \leq F'(u) \leq a.$$

Convergence in the scalar case

Theorem 1 R. Natalini, JDE 1998. $u_0 \in L^\infty(\mathbb{R}^D)$ and $f_0 = \mathcal{M}(u_0)$. For all l, M_l is nondecresing on $[-\|u_0 l_\infty, \|u_0 l_\infty]$. For $\varepsilon > 0$ fixed the BGK system has a unique solution $f^\varepsilon \in C([0,\infty[, L^1_{loc} \cap L^\infty)$. Moreover

$$M_l(-\|u_0\|_\infty) \leq f_l^\varepsilon \leq M_l(\|u_0\|_\infty), \quad l = 1, \dots, L.$$

Theorem 2 R. Natalini, JDE 1998. Same assumptions. $u^\varepsilon = Pf^\varepsilon$ converges to the unique entropy solution of the Cauchy problem for the conservation law.

Boundary conditions: V. Milisic, Proc. Amer. Math. Soc. 2003.

Systems

F. Bouchut, J. Stat. Phys. 1999: let E be a set of entropies for the system of conservation laws. For **vectorial BGK models** where $\mathcal{M}(u)$ is a linear combination of $A(u)$ and of u, the obtained solutions are entropic if and only if

$$\forall u \in \mathcal{U}, \quad \sigma(M_l'(u)) \subset [0, +\infty[; \quad l = 1, \dots, L.$$

In this case the Chapman-Enskog expansion is η-dissipative.

This condition is related to the subcharacteristic condition.

See also D. Serre, Ann. Inst. H. Poincaré 2000 for 2x2 1D systems. S. Bianchini, CPAM 2006 for 1D strictly hyperbolic systems (data small in BV).

Numerical methods

Approximation of the BGK model provides numerical schemes for the target system of conservation laws.

- Finite volumes D. A-D. and R. Natalini 2000, D. A.-D. and V. Milisic 2004, D. A.-D. and F. Krantz 2010
- Finite elements T. Katsaounis and C. Makridakis, 2001.
- Discontinuous Galerkin method Coulette et al 2019, D. A.-D. et al 2023
- Lattice Boltzmann method: B. Graille 2014,
  Baty et al 2023,
  R. Hélie, PhD thesis 2023,
  T. Bellotti, PhD thesis 2023.
- Residual distribution schemes: D. Torlo, PhD thesis, 2020.

The lattice Boltzmann method

- Initialization: $f^0 = \mathcal{M}(u^0)$.
- Stream phase : resolution of the free transport equations: obtention of $f^{n+1/2}$.
- Collision phase : resolution of the source-term:

$$(f^\varepsilon)' = \frac{1}{\varepsilon}\left(\mathcal{M}(u^\varepsilon) - f^\varepsilon\right), \quad u^\varepsilon = Pf^\varepsilon$$

We choose here

$$f^{n+1} = (1-\omega) f^{n+\frac{1}{2}} + \omega \mathcal{M}(u^{n+\frac{1}{2}})$$

This is a particular case of the MRT scheme (d'Humières 1992).

Space discretization and stream phase

$$\partial_t f_l + \sum_{d=1}^{D} v_{ld} \partial_{x_d} f_l = 0.$$

Cartesian grid in dimension $D \geq 1$:

$$\Delta x = (\Delta x_1, \dots, \Delta x_D),$$
$$x_\alpha = (\alpha_1 \Delta x_1, \dots, \alpha_D \Delta x_D), \quad \alpha = (\alpha_1, \dots, \alpha_D) \in \mathbb{Z}^D.$$

Velocity scale:

$$\lambda = (\lambda_1, \dots, \lambda_D), \quad \lambda_d \frac{\Delta t}{\Delta x_d} = 1, \quad 1 \leq d \leq D.$$

The set of characteristic velocities is such that $v_{ld} = j_{ld} \lambda_d$, $j_{ld} \in \mathbb{Z}$:

$$f_l(x_\alpha, t + \Delta t) = f_l(x_{\alpha'_l}, t), \quad \alpha'_l = \alpha - j_l.$$

Hence

$$f_{l,\alpha}^{n+\frac{1}{2}} = f_{l,\alpha'_l}^{n}$$

Collision phase

Approximation of $\partial_t f = \frac{1}{\varepsilon}(\mathcal{M}(Pf) - f)$. $\omega > 0$:

$$f_{l,\alpha}^{n+1} = (1-\omega) f_{l,\alpha}^{n+\frac{1}{2}} + \omega \mathcal{M}_l(u_\alpha^{n+\frac{1}{2}}), \quad l = 1, \ldots, L.$$

Interpretation of ω

- $\omega = \frac{\Delta t}{\varepsilon}$: Euler scheme
- $\omega = 1 - \mathrm{e}^{-\frac{\Delta t}{\varepsilon}}$: f_α^{n+1} is the exact solution of the equation. In that case $\omega \in]0, 1[$.
- $\omega = 1$: relaxation limit $\varepsilon \to 0$ of the exact solution
- Optimal ω? ω near 2? equivalent equation method: Dubois 2008, stability for linear equations: Guillon-Hélie-Helluy 2023
- Link between ω and the other parameters? We bring an answer by studying the convergence.

Convergence of the LBM for nonlinear conservation laws

$$\partial_t u + \sum_{d=1}^{D} \partial_{x_d} A_d(u) = 0, \quad u(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}^D$$

$$u(x,0) = u_0(x), \quad u_0 \in \mathrm{L}^1(\mathbb{R}^D) \cap \mathrm{L}^\infty(\mathbb{R}^D) \cap \mathrm{BV}(\mathbb{R}^D)$$

The convergence of LBM is known in the following cases:

- $\omega = 1$: LBM= Finite Volume relaxation with CFL=1. A.-D. and Natalini 2000. Tool: Monotonicity, cf Crandall-Majda 1980.
- $\omega \in]0,1]$ for the D1Q2 model Caetano, Dubois and Graille 2019. Here LBM$\neq$Finite Volume but the same kind of estimates hold.
- $\omega \in]0,2[$ for the D1Q2 model Bellotti 2023. Tool: Monotonicity on a multistep related scheme.

Convergence of the LBM for nonlinear conservation laws

We prove convergence for general multi-D models with equilibrium functions defined by

$$\mathcal{M}_l(u) = a_l u + \sum_{d=1}^{D} b_{ld} A_d(u), \quad l = 1, \dots, L,$$

a_l and b_{ld}: real coefficients.

Compatibility conditions :

$$\sum_{l=1}^{L} a_l = 1, \qquad \sum_{l=1}^{L} b_{ld} = 0, \quad \sum_{l=1}^{L} v_{ld} a_l = 0, \qquad \sum_{l=1}^{L} v_{ld} b_{lj} = \delta_{dj}.$$

Assumption 1: the $\mathcal{M}_l$ are nondecreasing: denoting $\mu_\infty = \|u_0\|_\infty$:

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \mathcal{M}_l'(u) \geq 0, \quad l = 1, \dots, L.$$

Convergence of the exact BGK model has been proved under this condition.

The scheme can be written as

$$\begin{aligned} f_{l,\alpha}^{n+\frac{1}{2}} &= f_{l,\alpha_l'}^{n} && (l = 1, \ldots, L) \\ f_{l,\alpha}^{n+1} &= \mathcal{S}_l(f_{1,\alpha_1'}^{n}, \ldots, f_{L,\alpha_L'}^{n}) && (l = 1, \ldots, L), \\ u_{\alpha}^{n+1} &= \sum_{k=1}^{L} f_{k,\alpha}^{n+1} \end{aligned}$$

with

$$\mathcal{S}_l(f) = (1-\omega)f_l + \omega \mathcal{M}_l \left(\sum_{k=1}^{L} f_k \right)$$

or

$$\mathcal{S}_l(f) = (1-\omega(1-a_l))f_l + \omega a_l \sum_{k \neq l} f_k + \omega \sum_{d=1}^{D} b_{ld} A_d \left(\sum_{k=1}^{L} f_k \right)$$

Note that

$$\mathcal{S}_l(\mathcal{M}(u)) = \mathcal{M}_l(u)$$

Assumption 2: monotonicity of the collision step

$$\forall k \in \{1, \ldots, L\}, \quad \forall l \in \{1, \ldots, L\}, \quad \forall f \in V, \quad \partial_k \mathcal{S}_l(f) \geq 0$$

where

$$V = \prod_{l=1}^{L} [\mathbf{m_l}, \mathbf{M_l}], \quad \mathbf{m_l} = \mathcal{M}_l(-\mu_\infty), \quad \mathbf{M_l} = \mathcal{M}_l(\mu_\infty).$$

Proposition Suppose that assumption 1 is satisfied. For $f \in V$, $u = Pf \in [-\mu_\infty, \mu_\infty]$ and assumption 2 is satisfied if and only if the following condition is satisfied:

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \forall l \in \{1, \ldots, L\}, \quad \omega \leq \frac{1}{1 - a_l - \sum_{d=1}^{D} b_{ld} A'_d(u)}.$$

Moreover if assumptions 1 and 2 are satisfied, then $\omega \in]0, 2[$.

Example in 1D: D1Q2

$$v_2 = -v_1 = \lambda > 0,$$

and

$$\mathcal{M}_1(u) = \frac{1}{2}\left(u - \frac{A(u)}{\lambda}\right), \quad \mathcal{M}_2(u) = \frac{1}{2}\left(u + \frac{A(u)}{\lambda}\right).$$

The $\mathcal{M}_I$ are nondecreasing if and only if

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad |A'(u)| \leq \lambda.$$

The collision step is monotone if moreover

$$\omega \leq \frac{2}{1 + \max\limits_{u \in [-\mu_\infty, \mu_\infty]} \frac{|A'(u)|}{\lambda}}.$$

Note that this condition appears in T. Bellotti's proof.
As a consequence, ω can take all values in $]0, 2[$, provided that λ is large enough.

Example in 2D: A D2Q4 model

$$v^{(1)} = \lambda_1(-1,0), \quad v^{(2)} = \lambda_2(0,-1), \quad v^{(3)} = \lambda_1(1,0), \quad v^{(4)} = \lambda_2(0,1)$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and

$$\mathcal{M}_1(u) = \frac{u}{4} - \frac{A_1(u)}{2\lambda_1}, \quad \mathcal{M}_2(u) = \frac{u}{4} - \frac{A_2(u)}{2\lambda_2},$$
$$\mathcal{M}_3(u) = \frac{u}{4} + \frac{A_1(u)}{2\lambda_1}, \quad \mathcal{M}_4(u) = \frac{u}{4} + \frac{A_2(u)}{2\lambda_2}.$$

The $\mathcal{M}_l$ are nondecreasing if and only if

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad 2|A_d'(u)| \leq \lambda_d, \quad d = 1,2.$$

The collision step is monotone if moreover

$$\omega \leq \frac{4}{3 + 2\frac{|A_d'(u)|}{\lambda_d}}, \quad d = 1,2.$$

Maximal value of ω : $\omega = \frac{4}{3}$.

L^∞ bound

Initialization: $C_\alpha = \prod_{d=1}^{D} \left]x_{d,\alpha_d} - \frac{\Delta x_d}{2}, x_{d,\alpha_d} + \frac{\Delta x_d}{2}\right[$

$\mathcal{V} = \prod_{1 \le d \le D} \Delta x_d$

$$u_\alpha^0 = \frac{1}{\mathcal{V}} \int_{C_\alpha} u_0(x)dx \in [-\mu_\infty, \mu_\infty]$$
$$f_\alpha^0 = \mathcal{M}(u_\alpha^0)$$

Recall that $\mu_\infty = \|u_0\|_\infty$ and

$$V = \prod_{l=1}^{L} [\mathbf{m_l}, \mathbf{M_l}], \quad \mathbf{m_l} = \mathcal{M}_l(-\mu_\infty), \quad \mathbf{M_l} = \mathcal{M}_l(\mu_\infty).$$

If the $\mathcal{M}_l$ are nondecreasing then $f_\alpha^0 \in V$.

L^∞ bound

Proposition If assumptions 1 and 2 are satisfied then for all $n \geq 0$,

$$\forall \alpha \in \mathbb{Z}^D, \quad u_\alpha^n \in [-\mu_\infty, \mu_\infty] \quad \text{and} \quad f_\alpha^n \in V$$

Proof Suppose that the property is true for a given $n \geq 0$. We remark that $\sum_{l=1}^{L} \mathbf{m_l} = -\mu_\infty$. Hence, denoting $\mathbf{m} = (\mathbf{m_1}, \dots, \mathbf{m_L})$:

$$\mathcal{S}_l(\mathbf{m}) = (1-\omega)\mathbf{m_l} + \omega \mathcal{M}_l(-\mu_\infty) = \mathbf{m_l}.$$

Denoting $f_{\alpha'}^n = (f_{1,\alpha_1'}^n, \dots, f_{L,\alpha_L'}^n)$:

$$\begin{aligned} f_{l,\alpha}^{n+1} - \mathbf{m_l} &= \mathcal{S}_l(f_{\alpha'}^n) - \mathcal{S}_l(\mathbf{m}) \\ &= \int_0^1 \sum_{k=1}^{L} \partial_k \mathcal{S}_l(\mathbf{m} + \theta(f_{\alpha'}^n - \mathbf{m}))(f_{k,\alpha_k'}^n - \mathbf{m}_k) d\theta \geq 0 \end{aligned}$$

In the same way, $\mathbf{M_l} - f_{l,\alpha}^{n+1} \geq 0$ so that $f_\alpha^{n+1} \in V$ and hence $u_\alpha^{n+1} \in [-\mu_\infty, \mu_\infty]$.

Contraction property

Lemma Suppose that assumptions 1 and 2 are satisfied.

$$\forall f, g \in V, \quad \sum_{l=1}^{L} |\mathcal{S}_l(g) - \mathcal{S}_l(f)| \leq \sum_{l=1}^{L} |g_l - f_l|.$$

Idea of the proof Denote $u = \sum_{k=1}^{L} f_k$, $v = \sum_{k=1}^{L} g_k$,

$w_\theta = u + \theta(v - u)$.

$$\begin{aligned} |\mathcal{S}_l(g) - \mathcal{S}_l(f)| &\leq \int_0^1 \sum_{k=1}^{L} \partial_k \mathcal{S}_l'(f + \theta(g - f))|g_k - f_k| d\theta \\ &= \omega \int_0^1 \mathcal{M}_l'(w_\theta) \sum_{k=1}^{L} |g_k - f_k| d\theta + (1 - \omega)|g_l - f_l|. \end{aligned}$$

As $\sum_{l=1}^{L} \mathcal{M}_l'(w_\theta) = 1$, summing over l yields the result.

BV and L^1 bounds

Notation

$$f_\Delta^n(x) = \sum_{\alpha\in\mathbb{Z}^D} f_\alpha^n \chi_\alpha(x), \quad u_\Delta^n(x) = \sum_{l=1}^{L} f_{\Delta,l}^n(x)$$

The contraction property is the key tool to obtain:

$$\begin{aligned}
&\mathrm{TV}(f_\Delta^{n+1}) \leq \mathrm{TV}(f_\Delta^n) \leq \mathrm{TV}(u_0)\\
&\mathrm{TV}(u_\Delta^n) \leq \mathrm{TV}(f_\Delta^n) \leq \mathrm{TV}(u_0)\\
&\|g_\Delta^{n+1} - f_\Delta^{n+1}\|_1 \leq \|g_\Delta^n - f_\Delta^n\|_1 \leq \|v_0 - u_0\|_1
\end{aligned}$$

and equicontinuity in time: denote $f_\Delta(x,t) = \sum_{n=0}^{\infty} \sum_{\alpha\in\mathbb{Z}^D} f_\alpha^n \chi_\alpha^n(x,t)$.

$$\forall t' > t > 0, \quad \|f_\Delta(\cdot,t') - f_\Delta(\cdot,t)\|_1 \leq C(t' - t + \Delta t)\mathrm{TV}(u_0).$$

Convergence to equilibrium

Proposition Suppose that assumptions 1 and 2 are satisfied.
There exists a constant $C > 0$ such that

$$\forall n \geq 0, \quad \|\mathcal{M}(u_\Delta^n) - f_\Delta^n\|_1 \leq C\,\mathrm{TV}(u_0)\frac{|1-\omega|}{1-|1-\omega|}\Delta t.$$

Remark If $\omega = 1$ then the collision part of the scheme is just the projection on equilibrium: $\mathcal{M}_l(u_\alpha^n) = f_{l,\alpha}^n$.

Proof

We denote $E^n = \|\mathcal{M}(u^n_\Delta) - f^n_\Delta\|_1$.

$$
\begin{aligned}
E^{n+1} &= \mathcal{V} \sum_{\alpha\in\mathbb{Z}^D} \sum_{l=1}^{L} |\mathcal{M}_l(u^{n+1}_\alpha) - f^{n+1}_{l,\alpha}| \\
&= \mathcal{V}|1-\omega| \sum_{\alpha\in\mathbb{Z}^D} \sum_{l=1}^{L} |\mathcal{M}_l(u^{n+1}_\alpha) - f^{n}_{l,\alpha-j_l}| \\
&\leq \mathcal{V}|1-\omega| \sum_{\alpha\in\mathbb{Z}^D} \sum_{l=1}^{L} \big(|\mathcal{M}_l(u^{n}_{\alpha-j_l}) - f^{n}_{l,\alpha-j_l}| + |\mathcal{M}_l(u^{n+1}_\alpha) - \mathcal{M}_l(u^{n}_{\alpha-
\\
&\leq |1-\omega| \left(E^n + \mathcal{V} \sum_{\alpha\in\mathbb{Z}^D} \sum_{l=1}^{L} |u^{n+1}_\alpha - u^{n}_{\alpha-j_l}| \right) \\
&\leq |1-\omega| \left(E^n + C\,\mathrm{TV}(u_0)\Delta t \right).
\end{aligned}
$$

By recurrence, as $E^0 = 0$, we obtain the result.

Convergence of the LBM

cf Crandall and Majda 1980 for monotone finite volume schemes: let Δt tend to 0, the velocity scale being fixed.

- L^1 and BV estimates : precompactness in L^1_{loc} for each time.
- Equicontinuity in time : convergence of f_Δ in $\mathrm{L}^\infty([0,T],\mathrm{L}^1(\mathbb{R}^D))$ to a function f and convergence of u_Δ to a function u.
- By the last estimate $f = \mathcal{M}(u)$.

The limit u is a weak solution of the conservation law

Idea of the proof: D1Q2 : $\lambda \frac{\Delta t}{\Delta x} = 1$

$$f_{l,\alpha}^{n+\frac{1}{2}} = f_{l,\alpha}^{n} - \frac{\Delta t}{\Delta x}\left(F_{l,\alpha+\frac{1}{2}}^{n} - F_{l,\alpha-\frac{1}{2}}^{n}\right), \quad l = 1, 2$$

$$f_{l,\alpha}^{n+1} = (1-\omega) f_{l,\alpha}^{n+\frac{1}{2}} + \omega \mathcal{M}_l(u_\alpha^{n+1}).$$

with

$$F_{1,\alpha+\frac{1}{2}} = -\lambda f_{1,\alpha+1} \quad F_{2,\alpha+\frac{1}{2}} = \lambda f_{2,\alpha}$$

$$u_\alpha^{n+1} = u_\alpha^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^n_{\alpha+\frac{1}{2}} - \mathcal{A}^n_{\alpha-\frac{1}{2}}\right)$$

with $\mathcal{A}^n_{\alpha+\frac{1}{2}} = F^n_{1,\alpha+\frac{1}{2}} + F^n_{2,\alpha+\frac{1}{2}} = \mathcal{A}(f^n_{1,\alpha+1}, f^n_{2,\alpha})$ and

$$\mathcal{A}\left(\mathcal{M}_1(u), \mathcal{M}_2(u)\right) = A(u).$$

We then follow the same steps as the Lax-Wendroff theorem (Lax and Wendroff 1960).

Convergence to the entropy solution

We recall that the Cauchy problem for the conservation law admits a unique weak entropy solution which is characterized by (Kruzkov 1970)

$$\int\int \left\{ |u-c|\partial_t\varphi + \mathrm{sgn}(u-c)\sum_{d=1}^{D}(A_d(u)-A_d(c))\partial_{x_d}\varphi \right\} dxdt \geq 0$$

for any $c \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R}^D \times (0,T))$, $\varphi \geq 0$, and, for any interval I of $\mathbb{R}^D$:

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_I |u(x,t)-u_0(x)|dx\,dt = 0.$$

We associate to $\eta_c(u) = |u-c|$ the kinetic entropy-entropy flux pair (Natalini 1998):

$$\mathcal{H}_{I,c}(g_I) = |g_I - \mathcal{M}_I(c)|, \quad \mathcal{G}_{I,c}(g_I) = v_{Id}|g_I - \mathcal{M}_I(c)|, \quad d = 1,\dots,D.$$

Discrete entropy inequality: transport part

The transport scheme is monotone so it owns a discrete entropy inequality: for $l = 1, \ldots, L$:

$$\frac{\mathcal{H}_{l,c}(f_{l,\alpha}^{n+\frac{1}{2}}) - \mathcal{H}_{l,c}(f_{l,\alpha}^{n})}{\Delta t} + \sum_{d=1}^{D} \frac{\mathcal{Q}^n_{l,c,\alpha+\frac{e_d}{2}} - \mathcal{Q}^n_{l,c,\alpha-\frac{e_d}{2}}}{\Delta x_d} \leq 0$$

and $\mathcal{Q}_{l,c,\alpha+\frac{e_d}{2}}$ is a function $\mathcal{Q}_{l,c,d}$ of $|j_{ld}|$ variables such that

$$\forall g_l \in \mathbb{R}, \quad \mathcal{Q}_{l,c,d}(g_l, \ldots, g_l) = v_{ld}|g_l - \mathcal{M}_l(c)|.$$

As a consequence, if $g_l = \mathcal{M}_l(u)$, as $\mathcal{M}_l$ is non decreasing:

$$\sum_{l=1}^{L} \mathcal{Q}_{l,c,d}(\mathcal{M}_l(u), \ldots, \mathcal{M}_l(u)) = \operatorname{sgn}(u-c)(A_d(u) - A_d(c)).$$

Discrete entropy inequality: collision part

Lemma Define for $f \in \mathbb{R}^L$: $H_c(f) = \sum_{l=1}^{L} \mathcal{H}_{l,c}(f_l)$.

The following inequality holds:

$$H_c(f_\alpha^{n+1}) \leq H_c(f_\alpha^{n+\frac{1}{2}})$$

Proof Remark that $\mathcal{M}_l(c) = \mathcal{S}_l(\mathcal{M}(c))$. Use contraction property:

$$\begin{aligned}\sum_{l=1}^{L} |f_{l,\alpha}^{n+1} - \mathcal{M}_l(c)| &= \sum_{l=1}^{L} |\mathcal{S}_l(f_\alpha^{n+\frac{1}{2}}) - \mathcal{S}_l(\mathcal{M}(c))| \\ &\leq \sum_{l=1}^{L} |f_{l,\alpha}^{n+\frac{1}{2}} - \mathcal{M}_l(c)| = H_c(f_\alpha^{n+\frac{1}{2}})\end{aligned}$$

Conclusion: convergence to the unique entropy solution.

First numerical experiment : 1D Burgers equation

Test on the shock solution

$$u(x,t) = 1 \quad \text{if} \quad x < \frac{t}{2}, \quad u(x,t) = 0 \quad \text{else.}$$

D1Q2: assumption 1 is satisfied if $\lambda \geq 1$. Our choice: $\lambda = 5$.

Assumption 2:

$$\omega \leq \frac{2}{1 + \frac{1}{\lambda}} = \frac{5}{3} \in]1.66, 1.67[$$

D1Q4: assumption 1 is satisfied if $\lambda \geq \frac{2}{3}$. Our choice: $\lambda = 5$.

Assumption 2:

$$\omega \leq \frac{1}{\frac{3}{4} + \frac{1}{6\lambda}} = \frac{60}{47} \in]1.276, 1.277[$$

Results

ω	D1Q2 $\|u(\cdot, T_{max})\|_\infty$	D1Q4 $\|u(\cdot, T_{max})\|_\infty$
1.28	1.0000000000000000	0.9999997243415302 8
1.30	1.0000000000000000	0.9999998626474997 4
1.67	1.0000000000000004	0.9999999999999988 9
1.70	1.0000814222675634	0.9999999999999769 1
1.80	1.0169571099362162	1.0003640299470273
1.90	1.0362911575372 09	1.0734866415961333

TABLE 1. The values of $\|u\|_\infty$ when ω varies, at time $T_{max} = 0.8$ with 100 points on $[-1, 1]$, $\lambda = 5$.

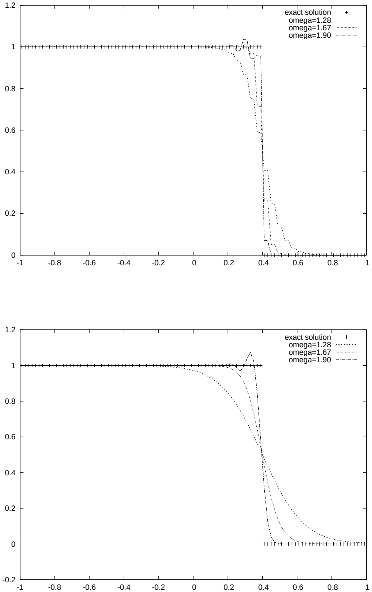

Figure: $\lambda = 5$, 100 points. Top : D1Q2 model. Bottom: D1Q4 model.

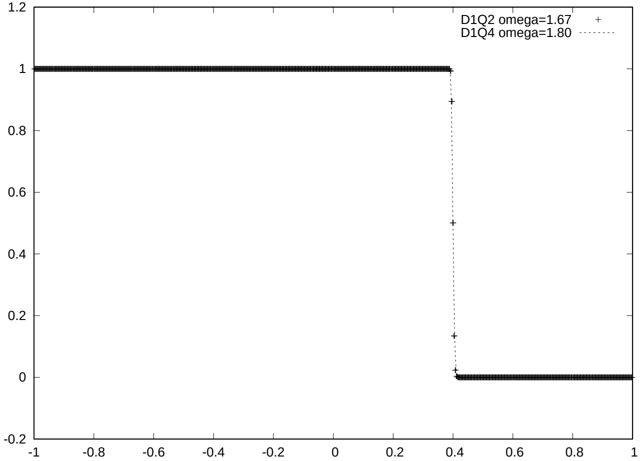

Figure: $\lambda = 5$, 1000 points. D1Q2 $\omega = 1.67$ and D1Q4 $\omega = 1.80$.

2D computations on Burgers equation: D2Q4 and D2Q8

1D solution rotated with an angle $\theta = \frac{\pi}{12}$, 100×100 uniform mesh. As a consequence $\lambda_1 = \lambda_2 = \lambda > 0$.

$\lambda = 10$ satisfies assumption 1 for D2Q4 and D2Q8.

Assumption 2 for D2Q4:

$$\omega \leq \omega_2 = \min \left(\frac{4}{3 + 2\frac{\cos\theta}{\lambda}}, \frac{4}{3 + 2\frac{\sin\theta}{\lambda}} \right)$$

that is $1.252 < \omega_2 < 1.253$.

Assumption 2 for D2Q8:

$$\omega \leq \omega_3 = \frac{8}{7 + 4\frac{\cos\theta + \sin\theta}{3\lambda}}$$

that is $1.116 < \omega_3 < 1.117$.

Results

ω	D2Q4 $\|u(\cdot, T_{max})\|_\infty$	D2Q8 $\|u(\cdot, T_{max})\|_\infty$
1.11	0.999999999999999400	0.999999997648868 27
1.25	0.999999999999999933	0.999999999999974676
1.30	1.0000000000000000	0.999999999999999500
1.40	1.0000000000000004	1.0000000000055236
1.60	1.0000000000000004	1.0000000000013374
1.90	1.0795668257759483	1.0559666526035698

TABLE 2. The values of $\|u\|_\infty$ when ω varies, at time $T_{max} = 0.8$ with 100x100 points on $[-1, 1] \times [-1, 1]$, $\lambda = 10$.

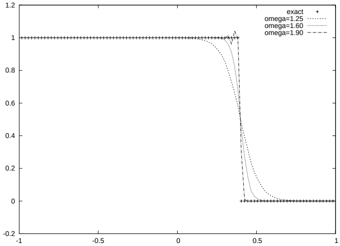

Figure: Shock solution of Burgers equation with rotated data in 2D, at time $T_{max} = 0.8$ with 100×100 points on $[-1, 1] \times [-1, 1]$, $\lambda = 10$. Solution along the axis containing $(0, 0)$ and orthogonal to the direction of propagation of the shock. Left : D2Q4 model. Right: D2Q8 model.

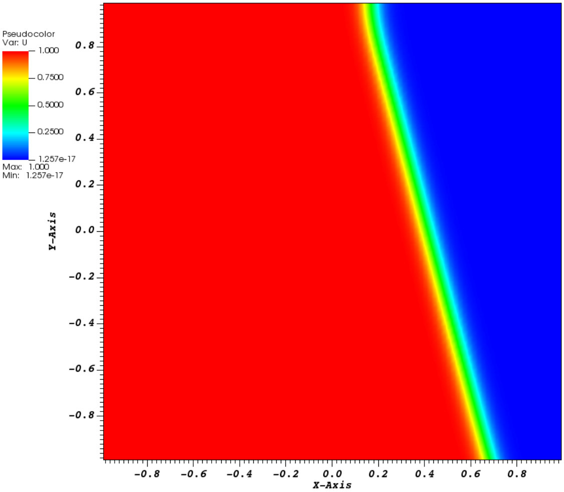

Figure: Shock solution of Burgers equation with rotated data in 2D, at time $T_{max} = 0.8$ with 100×100 points on $[-1, 1] \times [-1, 1]$, $\lambda = 10$, $\omega = 1.60$. Isovalues for the D2Q4 model.

Conclusion and perspectives

- Convergence to the entropy solution of the Cauchy problem for scalar conservation laws has been proved
- Assumptions: link between λ and ω insuring monotonicity properties
- For the D1Q2 model: convergence holds for $\omega \in]0, 2[$, provided λ is large enough
- Numerical tests: monotonicity can be lost when the theoritical conditions are not satisfied
- Perspectives
  - Convergence without monotonicity?
  - Convergence for MRT methods? Work in progress with T. Bellotti
  - Parabolic problems
  - Systems