Flexion update 1.

Jean Ecalle

This first *Flexion Update* and those soon to follow are meant to flesh out, justify, expand or complement various items in the general Survey titled

The Flexion Structure and Dimorphy: Flexion Units, Singulators, Generators, and the Enumeration of Multizeta Irreducibles.

The Survey[1] in question is systematically referred to as [FLEX].

Contents

[1] In Ann. Scuola Norm. Sup Pisa, 2011, pp 27-211.

1 Essential parity of bialternals.

This section is devoted to establishing the decomposition[2]

$$\mathrm{ARI}^{\mathrm{al/al}} = \mathrm{ARI}^{\dot{\mathrm{a}}\mathrm{l}/\dot{\mathrm{a}}\mathrm{l}} \oplus \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \tag{1.1}$$

of the space $ARI^{al/al}$ of all bialternals into:
(i) a large, regular part $ARI^{\underline{al}/\underline{al}}$, consisting of *even* bimoulds and stable under the *ari*-bracket.
(ii) a small, exceptional part $ARI^{\dot{a}l/\dot{a}l} := BIMU_{1,odd}$, consisting of *odd* bimoulds of length one and endowed with a bilinear mapping *oddari* into $ARI^{\underline{al}/\underline{al}}$.

Everything rests on the following statement.

Proposition 1.1 (Parity of bialternals).
Any bialternal bimould $A^\bullet$ purely of length $r > 1$ is an even function of its double index sequence, i.e. $A^{\boldsymbol{w}} \equiv A^{-\boldsymbol{w}}$.

Proof: Alternality implies invariance under *mantar* $:= -anti.pari$. Bialternality, therefore, implies invariance under *neg.push*, with:

$$\begin{aligned} \mathrm{neg.push} &:= \mathrm{mantar.swap.mantar.swap} \\ &= \mathrm{anti.swap.anti.swap} \end{aligned}$$

[2] See [FLEX] §2.7

The *push* operator, we recall, is idempotent of order $r+1$ when acting on $BIMU_r$, i.e. on bimoulds of length r.

Let us assume that $A^{\boldsymbol{w}}$ is odd in $\boldsymbol{w}$, and show that this implies $A^{\boldsymbol{w}} \equiv 0$.

For an *even* length r, this follows at once from the *neg.push*-invariance:

$$A^{\boldsymbol{w}} = (\text{neg.push})^{r+1}.A^{\boldsymbol{w}} = \text{neg}^{r+1}.\text{push}^{r+1}.A^{\boldsymbol{w}} = \text{neg}.A^{\boldsymbol{w}} = -A^{\boldsymbol{w}} \qquad (1.2)$$

For an *odd* length, the argument is more roundabout. Note first that for $A^{\boldsymbol{w}}$, which we assumed to be odd in $\boldsymbol{w}$, invariance under *neg.push* amounts to invariance under *-push*. Here again, it turns out that the absence of non-trivial solution does not require the full bialternality of $A^\bullet$, but only its alternality and invariance under *-push*. So let us prove this stronger statement:

Lemma 1.1 (Alternality and *push*-invariance).
No non-vanishing bimould $A^\bullet$ purely of length $r > 1$ can be simultaneously alternal and invariant under $-\text{push}$.

Proof: Here again, the statement is obvious for r even. So let us consider an odd length of the form $r = 2\,t + 1 \geq 3$.

Since we shall subject $A^{\boldsymbol{w}}$ to two linear operators, *pus* and *push*, respectively of order r and $r+1$ when restricted to $BIMU_r$, and since *pus* (resp. *push*) reduces to a circular permutation in the *'short'* (resp *'long'*) bimould notation, we shall make use of both. Let us recall the conversion rule:

$$A^{[w_0^*],w_1^*,\dots,w_r^*} \;\; (\textit{long}) \longleftrightarrow A^{w_1,\dots,w_r} \;\; (\textit{short}) \qquad (1.3)$$

with the dual conditions on upper and lower indices:

$$\begin{array}{llcll} u_0^* = -(u_1+\dots u_r) & , & u_i^* & = & u_i \quad \forall i \geq 1 \\ v_0^* \quad \textit{arbitrary} & , & v_i^* - v_0^* & = & v_j \quad \forall i \geq 1 \end{array}$$

To show that $A^\bullet = 0$, we start with the elementary alternality relation:

$$0 = \sum_{\boldsymbol{w} \in \text{sha}(\boldsymbol{w}',\boldsymbol{w}'')} A^{\boldsymbol{w}} \quad \textit{with } \boldsymbol{w}' = (w_1,\dots,w_{2t}) \textit{ and } \boldsymbol{w}' = (w_{2t+1}) \qquad (1.4)$$

which reads:

$$0 = \sum_{1 \leq j \leq 2\,t+1} A^{\overline{w_1,\dots,w_{j-1}},w_{2t+1},\overline{w_j,\dots,w_{2t}}} \qquad (1.5)$$

Due to the invariance of $A^\bullet$ under *-push*, this may be rewritten as:

$$0 = \sum_{1 \leq j \leq 2\,t+1} (-1)^j (\text{push}^{\text{j}}.A)^{\overline{w_1,\dots,w_{j-1}},w_{2t+1},\overline{w_j,\dots,w_{2t}}} \qquad (1.6)$$

In the 'long' notation (of greater relevance here) this becomes:

$$0 \;=\; \sum_{1\le j\le 2t+1} (-1)^j (\mathrm{push}^{\mathrm{j}}.A)^{[w_0],\overline{w_1,\ldots,w_{j-1}},w_{2t+1},\overline{w_j,\ldots,w_{2t}}} \tag{1.7}$$

$$\;=\; \sum_{1\le j\le 2t+1} (-1)^j A^{[w_{2t+1}],\overline{w_j,\ldots,w_{2t}},w_0,\overline{w_1,\ldots,w_{j-1}}} \tag{1.8}$$

Under the exchange $w_0 \leftrightarrow w_{2t+1}$, the last identity becomes:

$$0 = \sum_{1\le j\le 2t+1} (-1)^j A^{[w_0],\overline{w_j,\ldots,w_{2t}},w_{2t+1},\overline{w_1,\ldots,w_{j-1}}} = \sum_{1\le j\le 2t+1} (-1)^j A^{[w_0],\overline{w_j,\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{j-1}}}$$

Or again, reverting to the short notation:

$$0 = \sum_{1\le j\le 2t+1} (-1)^j A^{\overline{w_j,\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{j-1}}} \tag{1.9}$$

On the other hand, alternality implies *pus*-neutrality[3] $\sum pus^j A^{\bullet} \equiv 0$, which reads:

$$0 = \sum_{1\le j\le 2t+1} A^{\overline{w_j,\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{j-1}}} \tag{1.10}$$

From (1.9) and (1.10) we get by addition:

$$0 = \sum_{0\le k\le t} A^{\overline{w_{2k+1},\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{2k}}} \tag{1.11}$$

and by subtraction:

$$0 = \sum_{1\le k\le t} A^{\overline{w_{2k},\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{2k-1}}} \tag{1.12}$$

Under the change $(w_2, w_3, \ldots, w_{2t+1}, w_1) \to (w_1, w_2, \ldots, w_{2t+1})$, (1.12) becomes:

$$0 = \sum_{1\le k\le t} A^{\overline{w_{2k+1},\ldots,w_{2t+1}},\overline{w_1,\ldots,w_{2k}}} \tag{1.13}$$

Subtracting (1.13) from (1.11), we end up with $A^{w_1,..,w_r} \equiv 0$. □.

[3]See [FLEX], §2.4. For a proof, see below, §3.

2 Canonical factorisation of bisymmetrals.

This section is devoted to establishing the factorisation[4]:

$$\mathrm{GARI}^{\mathrm{as/as}} = \mathrm{gari}\big(\mathrm{GARI}^{\dot{\mathrm{a}}\mathrm{s}/\dot{\mathrm{a}}\mathrm{s}}, \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}\big) \tag{2.1}$$

of the set $GARI^{as/as}$ of all bisymmetrals into
(i) a large, regular factor $GARI^{\underline{as}/\underline{as}}$ consisting of *even* bimoulds and stable under the *gari* product
(ii) a small, exceptional factor $GARI^{\dot{a}s/\dot{a}s}$ consisting of special bimoulds derived from so-called *flexion units* and alternately *odd/even*, i.e. invariant under *pari.neg* rather than *neg*.

The proof rests on the construction and properties of the special bisymmetrals $\mathfrak{ess}^\bullet_{\mathfrak{E}}$ (see [FLEX] §4.2) and on the following statement:

Proposition 2.1 (Factorisation of bisymmetrals).
Any bisymmetral bimould $\mathrm{Sa}^\bullet$ *and its swappee simultaneously factor as*

$$\mathrm{Sa}^\bullet \;=\; \mathrm{gari}(\mathrm{Sal}^\bullet, \mathrm{Sar}^\bullet) \;=\; \mathrm{gira}(\mathrm{Sal}^\bullet, \mathrm{Sar}^\bullet) \tag{2.2}$$
$$\mathrm{Si}^\bullet \;=\; \mathrm{gari}(\mathrm{Sil}^\bullet, \mathrm{Sir}^\bullet) \;=\; \mathrm{gira}(\mathrm{Sil}^\bullet, \mathrm{Sir}^\bullet) \tag{2.3}$$

- *with* $\mathrm{Si}^\bullet = \mathrm{swap.Sa}^\bullet$, $\mathrm{Sil}^\bullet = \mathrm{swap.Sal}^\bullet$, $\mathrm{Sir}^\bullet = \mathrm{swap.Sar}^\bullet$
- *with bisymmetral right factors at once* neg- *and* gush-*invariant*
- *with bisymmetral left factors at once* pari.neg- *and* pari.gush-*invariant.*

In other words:

$$\mathrm{Sar}^\bullet, \mathrm{Sir}^\bullet \;\in\; \mathrm{GARI}^{\mathrm{as/as}}_{neg} = \mathrm{GARI}^{\mathrm{as/as}}_{gush} =: \mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}} \tag{2.4}$$
$$\mathrm{Sal}^\bullet, \mathrm{Sil}^\bullet \;\in\; \mathrm{GARI}^{\mathrm{as/as}}_{pari.neg} = \mathrm{GARI}^{\mathrm{as/as}}_{pari.gush} \tag{2.5}$$

The above decompositions are not unique, but two of them stand out, namely the one in which

$$\mathrm{Sal}^\bullet = \mathfrak{ess}^\bullet_{\mathfrak{E}} \quad \textit{with} \; -\frac{1}{2}\,\mathfrak{E}^{w_1} = \mathrm{Sal}^{w_1} = \frac{1}{2}(\mathrm{Sa}^{w_1} - \mathrm{Sa}^{-w_1}) \tag{2.6}$$

and the one in which

$$\mathrm{Sil}^\bullet = \mathfrak{oss}^\bullet_{\mathfrak{O}} \quad \textit{with} \; -\frac{1}{2}\,\mathfrak{O}^{w_1} = \mathrm{Sil}^{w_1} = \frac{1}{2}(\mathrm{Si}^{w_1} - \mathrm{Si}^{-w_1}) \tag{2.7}$$

These 'co-canonical' decompositions involve two conjugate flexion units $\mathfrak{E}$ *and* $\mathfrak{O}$ *and, though distinct, easily translate one into the other under the classical relation between* $\mathfrak{ess}^\bullet_{\mathfrak{E}}$ *and* $\mathfrak{oss}^\bullet_{\mathfrak{O}}$*: see formula (4.63) in §4.2 of [FLEX].*

[4] See [FLEX], §2.8.

Proof: It rests on the Proposition 1.1 of the preceding section, in conjunction with the two following lemmas.

Lemma 2.1 (First components of bisymmetrals).
If the length-one component Sal^{w_1} *of a bisymmetral bimould* $\mathrm{Sal}^\bullet$ *is an even function of* $w_1 = \binom{u_1}{v_1}$*, it may a priori be anything, but if it is an odd function, it is necessarily a flexion unit.*

Proof: Let u_0, u_1, u_2 be constrained by $u_0 + u_1 + u_2 = 0$ and let v_0, v_1, v_2 be defined up to a common additive constant. At length 2, the unique symmetrality relation for $Sal^\bullet$ may be written thus:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}, \begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}, \begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}\right)} \equiv \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} \tag{2.8}$$

Due to Sal^{w_1} being odd, this yields:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} -u_1 \\ -v_{1:0} \end{smallmatrix}, \begin{smallmatrix} -u_2 \\ -v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} -u_2 \\ -v_{2:0} \end{smallmatrix}, \begin{smallmatrix} -u_1 \\ -v_{1:0} \end{smallmatrix}\right)} \equiv \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} \tag{2.9}$$

Likewise, the unique symmetrality relation for $Sal^\bullet$ may be written as:

$$\mathrm{Sil}^{\left(\begin{smallmatrix} -v_{0:2} \\ -u_0 \end{smallmatrix}, \begin{smallmatrix} v_{1:2} \\ u_1 \end{smallmatrix}\right)} + \mathrm{Sil}^{\left(\begin{smallmatrix} v_{1:2} \\ u_1 \end{smallmatrix}, \begin{smallmatrix} -v_{0:2} \\ -u_0 \end{smallmatrix}\right)} \equiv \mathrm{Sil}^{\left(\begin{smallmatrix} v_{1:2} \\ u_1 \end{smallmatrix}\right)} \mathrm{Si}^{\left(\begin{smallmatrix} -v_{0:2} \\ -u_0 \end{smallmatrix}\right)}$$

In the u_i-variables, this translates into:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}, \begin{smallmatrix} -u_{0,1} \\ -v_{0:2} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} -u_0 \\ -v_{0:1} \end{smallmatrix}, \begin{smallmatrix} u_{0,1} \\ v_{1:2} \end{smallmatrix}\right)} \equiv \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} -u_0 \\ -v_{0:2} \end{smallmatrix}\right)}$$

or again, due to imparity and to $\sum u_i = 0$:

$$\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}, \begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} -u_0 \\ -v_{0:1} \end{smallmatrix}, \begin{smallmatrix} -u_2 \\ -v_{2:1} \end{smallmatrix}\right)} \equiv -\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)} \mathrm{Sa}^{\left(\begin{smallmatrix} u_0 \\ v_{0:2} \end{smallmatrix}\right)} \tag{2.10}$$

Let E_1 be the identity obtained by adding the three circular permutations of (2.8) and (2.9), and E_2 the identity obtained by adding the six permutations, circular or anticircular, of (2.10). The left-hand sides of E_1 and E_2 clearly coincide, while their right-hand sides coincide only up to the sign. Equating these right-hand sides, we find:

$$4\left(\mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:0} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:0} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} u_2 \\ v_{2:1} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_0 \\ v_{0:1} \end{smallmatrix}\right)} + \mathrm{Sal}^{\left(\begin{smallmatrix} u_0 \\ v_{0:2} \end{smallmatrix}\right)} \mathrm{Sal}^{\left(\begin{smallmatrix} u_1 \\ v_{1:2} \end{smallmatrix}\right)}\right) \equiv 0 \tag{2.11}$$

which is precisely the symmetrical characterisation of a *flexion unit.* □.

Remark 1: On the face of it, the requirement that the length-1 component be a flexion unit is merely a necessary condition for the existence of a bisymmetral 'continuation'. However, the theory of unit-generated bisymmetrals

$\mathfrak{ess}^\bullet_{\mathfrak{E}}$ shows this condition to be (miraculously) sufficient.[5] This is probably the best *a posteriori* justification for singling out this notion of *flexion unit*, though by no means the only one.

Remark 2: Had we assumed $Sal^\bullet$ to be even, we would have found no constraints at all on the length-1 component – which was only to be expected, since the Lie-exponential of that length-1 component is automatically in $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$.

Remark 3: One should not be too exercised over the presence of the factor 4 in (2.11), but rather observe that it vanishes after the change $Sal^{w_1} = -\frac{1}{2}\mathfrak{E}^{w_1}$ which, as it happens, the construction of $\mathfrak{ess}^\bullet_{\mathfrak{E}}$ quite naturally imposes.

Lemma 2.2 (General and even bisymmetrals).
Though not a group, the set $\mathrm{GARI}^{\mathrm{as/as}}$ *of all bialternals is stable under both* gari- *and* gira-*postcomposition by the group* $\mathrm{GARI}^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$ *of even bisymmetrals, and the identity holds:*

$$\mathrm{gari}(\mathrm{S}_1^\bullet, \mathrm{S}_2^\bullet) \equiv \mathrm{gira}(\mathrm{S}_1^\bullet, \mathrm{S}_2^\bullet) \in \mathrm{as/as} \qquad (\forall \mathrm{S}_1^\bullet \in \mathrm{as/as}\,,\ \forall \mathrm{S}_2^\bullet \in \underline{\mathrm{as}}/\underline{\mathrm{as}}) \tag{2.12}$$

Proof: Here *gira* stands for the pull-back of *gari* under the basic involution *swap*. Both group laws are related as follows[6]:

$$\mathrm{gira}(S_1^\bullet, S_2^\bullet) = \mathrm{ganit}(\mathrm{rash}.S_2^\bullet).\mathrm{gari}(S_1^\bullet, \mathrm{ras}.S_2^\bullet) \tag{2.13}$$

with non-linear operators *ras, rash* defined by:

$$\mathrm{ras}.S_2^\bullet \;=\; \mathrm{invgari.swap.invgari.swap}.S_2^\bullet \tag{2.14}$$
$$\mathrm{rash}.S_2^\bullet \;=\; \mathrm{mu}(\mathrm{push.swap.invmu.swap}.S_2^\bullet, S_2^\bullet) \tag{2.15}$$

But since in Lemma 2.1 the right factor $S_2^\bullet$ is in $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$ and since *gari* and *gira* coincide on $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$ (even as *ari* and *ira* coincide on $ARI^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$), this implies:

$$\mathrm{ras}.S_2^\bullet = \mathrm{invgari.invgira}.S_2^\bullet = S_2^\bullet \tag{2.16}$$

Likewise, any bimould of $\underline{\mathrm{as}}/\underline{\mathrm{as}}$ type is automatically *gush*-invariant (even as any bimould of $\underline{\mathrm{al}}/\underline{\mathrm{al}}$ type is automatically *push*-invariant). See [FLEX], §2.4. This in turn implies:

$$\mathrm{rash}.S_2^\bullet = 1^\bullet \quad \textit{and} \quad \mathrm{ganit}(\mathrm{rash}.S_2^\bullet) = \mathrm{id} \tag{2.17}$$

[5] See [FLEX], §4.2, §11.9, §11.10.

[6] see [FLEX], §2.3. This universal identity holds for *any* factors $S_1^\bullet, S_2^\bullet$.

and establishes (2.12). □.

Remark 4. Thus, $S_2^\bullet$ is the only factor that really matters when comparing $gari(S_1^\bullet, S_2^\bullet)$ and $gira(S_1^\bullet, S_2^\bullet)$. This is less surprising than may appear at first sight, since the *gari* and *gira* products are linear in the *left* factor and violently non-linear in the *right* factor.

We may now return to the **proof of Proposition 2.1**. To define our left factor $Sal^\bullet$ we set:

$$\mathrm{Sal}_r^\bullet := \mathfrak{ess}_{\mathfrak{E}}^\bullet \quad \textit{with} \quad -\frac{1}{2}\mathfrak{E}^{w_1} := \frac{1}{2}(\mathrm{Sa}^{w_1} - \mathrm{Sa}^{-w_1}) \tag{2.18}$$

By the general theory of §4.2 in [FLEX], this left factor is not just bisymmetral, but also invariant under $pari.neg$. Let us now address the construction of the right factor $Sar^\bullet$. For each r, we can construct bimould pairs $(Sa_r^\bullet, sar_r^\bullet)$ by the following induction. For $r = 1$ we set:

$$\mathrm{Sa}_1^\bullet := \mathrm{Sa}^\bullet \tag{2.19}$$

$$\mathrm{sar}_1^\bullet := \frac{1}{2}(\mathrm{Sa}^{w_1} + \mathrm{Sa}^{-w_1}) \tag{2.20}$$

and for $r > 1$ we set:

$$\mathrm{Sa}_r^\bullet := \mathrm{gari}\big(\mathrm{Sa}^\bullet, \mathrm{expari}(-\mathrm{sar}_1^\bullet), \ldots, \mathrm{expari}(-\mathrm{sar}_{r-1}^\bullet)\big) \tag{2.21}$$

$$\mathrm{sar}_r^{w_1,\ldots,w_r} := \mathrm{Sa}_r^{w_1,\ldots,w_r} - \mathrm{Sal}^{w_1,\ldots,w_r} \tag{2.22}$$

$$\mathrm{sar}_r^{w_1,\ldots,w_l} := 0 \quad \textit{if} \quad l \neq r \tag{2.23}$$

Clearly:

$$\mathrm{sar}_r^\bullet \in \mathrm{BIMU}_r \quad \textit{and} \quad \mathrm{Sa}_r^\bullet \equiv \mathrm{Sal}^\bullet \mod \oplus_{r \leq r'} \mathrm{BIMU}_{r'}$$

Let us now check that
(i) each $Sa_l{}^\bullet$ is in $GARI^{\mathrm{as/as}}$;
(ii) each $sar_l{}^\bullet$ is in $ARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$;
(iii) and therefore each $expar(\pm\,\mathrm{sar}_l^\bullet)$ is in $GARI^{\underline{\mathrm{as}}/\underline{\mathrm{as}}}$.

This obviously holds for $l = 1$. If it holds for all $l < r$, then by Lemma 2.1 $Sa_l{}^\bullet$ is also in $GARI^{\mathrm{as/as}}$, as the *gari*-product of an as/as by a string of several $\underline{as}/\underline{as}$. As for $sar_r{}^\bullet$, it is defined as the difference of length-r components of two bisymmetral bimoulds, $\mathrm{Sa}_r^\bullet$ and $\mathrm{Sal}^\bullet$, whose earlier components coincide. It is therefore not just al/al (bialternal) but also, by Lemma 1.1 of the preceding section, $\underline{al}/\underline{al}$ (bialternal *and* even), and its Lie exponential is automatically $\underline{as}/\underline{as}$.

Summing up, we arrive at a factorisation of the announced type (2.2), with a left factor defined by (2.18) and a right factor defined by

$$\mathrm{Sar}^{\bullet} = \lim_{r\to\infty} \mathrm{gari}\big(\mathrm{expari}(\mathrm{sar}_r^{\bullet}), \ldots, \mathrm{expari}(\mathrm{sar}_1^{\bullet})\big) \tag{2.24}$$

The swappee factorisations (2.3) immediately follow, again under (2.13). $\square$