Comparison Methods in Riemannian Geometry

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1 Introduction

In this essay, we will give an account on some well known results of Riemannian Geometry focusing on problems related to the so called comparison methods. The word "comparison" usually stands for comparison of curvature related quantities between two manifolds. Often, the problem is to infer some topological properties of the manifold (like the structure of its fundamental group or whether it is homeomorphic to some well known space, etc) from geometric information such as bounds on sectional or Ricci curvature. We have tried to distinguish clearly between the results about Ricci curvature and those about sectional curvature.

Given some constraints on the curvature tensor and in order to get some insight on the structure of the manifold, one can compare locally or globally the manifold to some other one, more standard, for which much more is known. This leads two the first comparison results such as Bonnet and Myers' theorem and Rauch's theorems.

Furthermore, by refining the techniques of comparing Jacobi fields, used in the latter theorems, one can obtain a volume comparison theorem, which give the comparison of the Riemannian volume of geodesic balls while assuming bounds on Ricci curvature. This result happens to be the main tool for studying the fundamental group of the manifold and obtain results concerning its growth or finiteness results like Anderson's theorem.

On the other hand, the comparison of sectional curvature, following Rauch's theorems, leads to the triangle comparison theorem, also called Toponogov's theorem. This allows to give a complete description of the comparison between side lengths and angles of triangles in two different manifolds, assuming inequalities between the sectional curvature of the two manifolds considered. The applications of such a result are numerous. In particular, this gives an easy proof of Grove and Shiohama's generalized sphere theorem, which, in turn, gives a proof of the classical sphere theorem of Riemannian Geometry. Bounds on sectional curvature, such that the δ-pinched condition, can force the manifold to be homeomorphic to the standard sphere. Finally, the essay ends with some further easy applications of the techniques developped so far.

2 Preliminaries

We will assume the reader familiar with the basic notion of Riemannian Geometry, such as geodesics and the curvature tensor. However, in the next few paragraphs, we set the notations and recall some basic facts, which will prove useful in the sequel.

2.1 Notations

M is an n-dimensional connected real smooth manifold on which we consider a Riemannian metric $\langle , \rangle$. The associated norm is denoted by $\|\cdot\|$.

The tangent space at a point p in M in denoted by M_p.

Any piecewise smooth curve from some closed interval $[a, b]$ into M is called a *path* in M.

TM is the tangent bundle, and SM the unit sphere bundle, while SM_p denotes the unit sphere in M_p.

The Levi-Civita connection associated to the metric is denoted by $\nabla_X Y$ for two vector fields X and Y.

The curvature tensor is denoted by $R(x, y)z$, for any three tangent vectors x, y, z. For any three vector fields X, Y and Z.

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Sectional curvature is defined by

$$K(x, y) = \frac{\langle R(x, y)y, x\rangle}{\|x \wedge y\|}$$

Ricci curvature is defined by

$$Ric(x, y) = \sum_{i=1}^{n} \langle R(e_i, x)y, e_i\rangle$$

where $(e_i)_i$ is an orthonormal basis of M_p.

Scalar curvature is defined by

$$Scal(p) = \sum_{i,j} K(e_i, e_j) = \sum_{i=1}^{n} Ric(e_i)$$

$\mathbb{M}_\kappa$ will be the model space of constant curvature κ (i.e. the simply connected Riemannian manifold of constant curvature κ).

We will use the notation $K_M \geq \kappa$ (resp. $\leq, >, <, =$) to say that for all orthonormal tangent vectors x, y we have $K(x, y) \geq \kappa$. Similarly, we will write

$Ric \geq (n-1)\kappa$ (resp. $\leq, >, <, =$) to say that, for all unit tangent vectors x, $Ric(x,x) \geq (n-1)\kappa$.

We will denote the Riemannian measure on M by $d\nu$. For any measurable subset F of M we set $vol(F)$ to be the Riemannian measure (or volume) of F.

If $c(t)$ is a smooth path in M, we usually denote by T its tangent vector field and if V is any vector field along c, we denote $\nabla_T V$ by V'. The coordinates of V' in a parallel frame along c are precisely the derivatives of that of V.

We denote by $B_p(r)$ the ball with center p and radius r in M as a metric space endowed with the Riemannian distance.

Also, the cut locus in M at p will be denoted by C_p, and U_p will be the star-shaped domain on M_p on which $\exp_p$ is a diffeomorphism onto $M \backslash C_p$.

2.2 Geodesics and Riemannian distance

2.2.1 Geodesics

We recall, that the length of a path (i.e. piecewise smooth curve) $c : [a,b] \to M$ is defined by

$$L[c] = \int_a^b \|c'(t)\|\, dt$$

This is of course independent of the parametrization of c.

Consider now a smooth path c. A **smooth variation** of the smooth path c is a family of paths $\{c_s\}_{|s|<\varepsilon}$ given by $c_s(t) = \alpha(s,t)$, where $\alpha : (-\varepsilon, \varepsilon) \times [a,b] \to M$ is smooth and $c_0 = c$. Then, it is easy to compute the first derivative of $L[c_s]$. If we denote by $T = c'(t)$ and $V = \frac{\partial}{\partial s}\alpha(s,t)_{|s=0}$ the corresponding vector fields along c, we have the **first variation formula**

$$\frac{dL[c_s]}{ds}_{|s=0} = \frac{1}{L[c_0]} \left(\langle V, T \rangle \,|_a^b - \int_a^b \langle V, \nabla_T T \rangle\, dt \right) \tag{1}$$

The geodesics are critical smooth paths of L. More precisely, they are smooth paths c in M such that for all t

$$\nabla_{c'(t)} c'(t) = 0 \tag{2}$$

They are of constant speed (i.e. $\|c'(t)\| = cste$), and if this speed is 1, they are called **normal geodesics**. As this is a second order ordinary differential equation, geodesics are determined by their initial value and initial speed.

We also recall that the exponential map at a point p is defined by

$$\begin{array}{rcl} \exp_p & : & M_p \to M \\ \xi & \mapsto & \gamma_\xi(1) \end{array}$$

where γ_ξ is the geodesic with initial value p and initial derivative ξ. This function may not be defined everywhere on M_p, since the solution of the differential equation 2 may not be defined everywhere on $\mathbb{R}$. However, it is defined on some neighbourhood of 0. By the theorem on the regularity of the flow of a differential equation, $\exp_p$ is a smooth function and $d_0 \exp_p = id$, hence $\exp_p$ is a local diffeomorphism at the origin.

The exponential map is not a local isometry in general, but it has the following important property called the Gauss lemma,

Lemma 2.1 *(**Gauss lemma**) Let v and w be in M_p with $\langle v, w\rangle = 0$ and t such that $\exp_p$ is defined along sv, $s \in [0,t]$, then $\left\langle d_{tv}\exp_p(w), d_{tv}\exp_p(v)\right\rangle = 0$ where v and w are considered as vectors in $(M_p)_{tv}$.*

2.2.2 Riemannian distance

The distance between two points p and q in M is defined to be

$$d(p,q) = \inf\{L[c] | c \in \Omega(p,q)\}$$

where $\Omega(p,q)$ is the set of paths from p to q.

A geodesic between p and q is said to be **minimal** if its length is equal to $d(p,q)$. Then we have the following important results,

Theorem 2.1 *We have the following fundamental properties,*

1. *d is a distance, which induced topology coincide with that of M.*

2. *(**Hopf-Rinow**) If $\exp_p$ is defined on all of M_p for some p, then it is defined on all M_p for all p. Also, this condition is equivalent to saying that M is a complete metric space for the distance d. Finally, in case M is complete, any two points in M can be joined by a minimal geodesic.*

Let's note that the compact subsets of M are the closed and bounded subsets. Indeed, consider a closed bounded set F in M and a sequence $(q_i)_{i\geq 0}$ in F and fix $p \in M$. Then consider minimal geodesics γ_i from p to q_i. As SM_p is compact, we can take a subsequence such that $\gamma_i'(0)$ converges to some ξ. But, since $diam(F) < \infty$, we must have $L[\gamma_i] \leq diam(F) < \infty$. Therefore we can take again a subsequence such that $L[\gamma_i]$ converges to some positive real l. But then $\gamma_i(L[\gamma_i]) = q_i \to \gamma_\xi(l) \in F$. Since M is a metric space, this shows the compacity of F.

2.3 The curvature tensor and Jacobi fields

2.3.1 Jacobi fields

Consider a geodesic c and a smooth variation $\{c_s\}_{|s|<\varepsilon}$ of this geodesic **through geodesics** given by $c_s(t) = \alpha(s,t)$, where $\alpha : (-\varepsilon, \varepsilon) \times [a,b] \to M$ is smooth, $c_0 = c$, and c_s is a geodesic for all s. A **Jacobi field** is the variation field of α, i.e. $V = \frac{\partial}{\partial s}\alpha(s,t)_{|s=0}$. It is a vector field along c, and in fact, the Jacobi fields are precisely those vector fields V along c which satisfy the Jacobi equation

$$\nabla_T \nabla_T V = R(T,V)T$$

Equivalently, if $V(t) = (v_i(t))_i$ are the coordinates of a vector field V along c, taken in an orthonormal parallel frame $(E_i)_i$, then V is a Jacobi field iff

$$V'(t) = R(t)V(t)$$

where $R(t)$ is the matrix $(\langle R(T,E_i)T, E_j\rangle)_{i,j}$. Therefore, a Jacobi field is determined by its initial value and its first derivative initial value.

When M has constant curvature κ, then $R(t) = -\kappa I$. Therefore, if we denote by $S_\kappa(t)$ (resp. $C_\kappa(t)$) the solution of the linear differential equation $y'' + \kappa y = 0$, with initial values $S_\kappa(0) = 0$ (resp. $C_\kappa(0) = 1$), $S_\kappa'(0) = 1$ (resp. $C_\kappa'(0) = 0$), then the Jacobi fields are of the form

$$\sum_{i=1}^{n} \left(a_i S_\kappa(t) + b_i C_\kappa(t)\right) E_i(t)$$

Also, the Jacobi field J such that $J(0) = 0$ and $J'(0) = v \in M_p$ is given by

$$J(t) = d_{tv} \exp_p \cdot vt \tag{3}$$

where the second v is considered in $(M_p)_{tv}$. It is easy to compute the first derivatives of V at 0. In particular, we have

$$J(0) = 0, J'(0) = v, J''(0) = 0, J'''(0) = R(\xi, v)\xi \tag{4}$$

Consider a geodesic γ between two points p and q in M. Recall that the point q is said to be **conjugate** to p along γ, if it is a singular value of $\exp_p$. This is equivalent to the existence of a non-zero Jacobi field along γ which vanishes at end points. Thus q is conjugate to p iff p is conjugate to q. If p and q are not conjugate along γ, then a Jacobi field along γ is determined by its values at p and q.

Similarly, consider $\xi \in SM_p$ and γ_ξ is a geodesic from p to q. Then, we say that q is **a focal point** of p along γ_ξ if it is a singular value of $\exp_{p|\mathbb{R}\xi}$, i.e. if there is $\lambda \in \mathbb{R}$ with $\exp_p(\lambda\xi) = q$ and such that $d_{\lambda\xi} \exp_p \cdot\xi = 0$. This is equivalent (by 3) to the existence of a non-zero Jacobi field along γ_ξ which vanishes at end points p and q and whose initial first derivative is in $\mathbb{R}\xi$.

2.3.2 The Index form

We now consider a smooth variation $\{c_{v,w}\}_{|v|+|w|<\varepsilon}$ of a geodesic c with respect to two parameters $c_{v,w}(t) = \alpha(v,w,t)$, through geodesics (i.e. $t \mapsto \alpha(v,w,t)$ is a geodesic for all v and w). Then, let $V = \frac{\partial}{\partial v}\alpha(v,w,t)_{|v,w=0}$ and $W = \frac{\partial}{\partial w}\alpha(v,w,t)_{|v,w=0}$ and suppose V and W vanish at the end points of c. We have the **second variation formula,**

$$\frac{\partial^2 L[c_{v,w}]}{\partial w \partial v}_{|v,w=0} = \int_a^b \langle \nabla_T V, \nabla_T W\rangle + \langle R(W,T)V, T\rangle$$

The right hand-side is taken as a definition of the **index form** $I(V,W)$ of two piecewise smooth vector fields V and W along c (not necessarily Jacobi fields). This is a symmetric bilinear form on the space of such vector fields, which can also be written, after integration by parts, as

$$I(V,W) = \langle \nabla_T V, W\rangle\,|_a^b - \int_a^b \langle \nabla_T \nabla_T V, W\rangle - \langle R(T,V)T, W\rangle \tag{5}$$

for any smooth vector fields V and W along c.

We now state the Index lemmas, which are of fundamental importance for the sequel.

Lemma 2.2 ***(First Index Lemma)*** *Let γ be a geodesic from p to q such that there is no conjugate points of p along γ. Let W be a piecewise smooth vector field on γ and V the unique Jacobi field such that $V(p) = W(p) = 0$ and $V(q) = W(q)$. Then $I(V,V) \leq I(W,W)$ and equality holds iff $V = W$.*

Lemma 2.3 ***(Second Index Lemma)*** *Let γ be a geodesic from p to q such that there is no focal point of p along γ. Let W be a piecewise smooth vector field on γ. Then there is a unique Jacobi field V such that $V'(p) = 0$ and $V(q) = W(q)$. And we have $I(V, V) \leq I(W, W)$ with equality iff $V = W$.*

An important consequence of the first Index Lemma is that the geodesics no longer minimize locally after the first conjugate point.

2.3.3 The cut locus

Recall that the **cut locus** C_p of a point p in M is the set of all cut points q, that is to say points $q = \gamma_\xi(t_0)$ where $\gamma_\xi : [0, \infty) \to \mathbb{R}$ is a geodesic on M (complete) such that $\gamma'_\xi(0) = \xi$ and $t_0 = \min\{t \geq 0 | d(p, \gamma(t)) = t\}$. Then the distance function $f(p, \xi) = d(p, q)$ is a continuous fonction on SM. As a consequence, C_p is of null Riemannian measure. Also, the set of all $v \in M_p$ such that $d(p, \exp_p(v)) < f(p, \xi)$ is a star-shaped open set U_p such that $\exp_p$ is a diffeomorphism from U_p to $M \backslash C_p$.

Finally, recall that the injectivity radius $inj(M)$ is defined to be the lower bound of the distances from any p in M to C_p.

2.4 Riemannian volume

Consider a chart (U, x) at some point p in M. By definition, the local density of the Riemannian measure $d\nu$ is given by

$$\sqrt{g_x} dx_1 ... dx_n$$

where $g_x = \det(\langle \partial_i, \partial_j \rangle_{i,j})$. This is easily seen to be independent of the given chart. Next, we will need to investigate what this expression becomes when one considers the polar coordinate system $x = \exp_p(r\xi)$, $r \geq 0$, $\xi \in SM_p$. The inversed exponential map $\exp_p^{-1}$ is a chart from $M \backslash C_p$ to the star-shaped domain U_p in the tangent space M_p as described above. Therefore, the local density of the Riemannian measure is given by an expression of the form $\lambda(r, \xi) dr d\xi$ where $\lambda(r, \xi)$ is some function defined on U_p and $d\xi$ is the usual surface measure on the sphere S^{n-1}. More precisely, if we set $\lambda(r, \xi) = 0$ outside U_p, then for any integrable function f on M, since the cut locus is of measure zero, we have

$$\int_M f d\nu = \int_0^{+\infty} \int_{S^{n-1}} f(\exp_p(r\xi)) \lambda(r, \xi) d\xi dr \tag{6}$$

It remains to compute $\lambda(r, \xi)$.

Proposition 2.1 *Let $J_2, ..., J_n$ be Jacobi fields along the geodesic $\gamma_\xi(t) = \exp_p(\xi t)$, with $J_i(0) = 0$, and $(J_2'(0), ..., J_n'(0))$ linearly independent. Then for $r\xi \in U_p$,*

$$\lambda(r, \xi) = \frac{\det(J_2(r), ..., J_n(r))}{\det(J_2'(0), ..., J_n'(0))} \tag{7}$$

where the determinant is taken with respect to an orthonormal frame $(E_i(t))_t$ along the geodesic γ_ξ.

Proof : Fix $\xi \in SM_p$, and consider a coordinate system $\xi(u)$ around $\xi = \xi(0)$ in SM_p ($u = (u_2, ..., u_n) \in \mathbb{R}^{n-1}$). Then $(t, u) \mapsto \exp_p(t\xi(u))$ is a smooth variation of γ_ξ through geodesics. Therefore the variation fields $J_i(t) = \partial_i \exp_p(t\xi(u)) = d_{t\xi} \exp_p \cdot (t\partial/\partial u_i)$ for $i = 2, ..., n$ are Jacobi fields with initial value zero and derivative at zero $J_i'(0) = \partial_i \xi_{|u=0}$. Conversely, given the Jacobi fields as above, we can find a coordinate system $\xi(u)$ with $\partial_i \xi_{|u=0} = J_i'(0)$. Now, $(t, u) \mapsto \exp_p(t\xi(u))$ is also a coordinate system around $\exp_p(r\xi) \in M$, if $r\xi \in U_p$. Thus, the Riemannian density on M about this point must be

$$\sqrt{g} du_1 du_2 ... du_n$$

where $g = \det(\langle \partial/\partial u_i, \partial/\partial u_j \rangle_{i,j})$ at $\exp_p(r\xi)$, $u_1 = t$, and $\partial/\partial u_i = J_i(r)$ for $i \geq 2$. But, since γ_ξ is a normal geodesic, $\langle \partial/\partial u_1, \partial/\partial u_1 \rangle = 0$. Also, by the Gauss' lemma, we have $\langle \partial/\partial u_1, \partial/\partial u_i \rangle = 0$ for $i \geq 2$. Then

$$g = \det(\langle \partial/\partial u_i, \partial/\partial u_j \rangle_{i,j \geq 2}) = \det(\langle J_i(r), J_j(r) \rangle_{i,j}) = (\det(J_2(r), ..., J_n(r)))^2$$

On the other hand, the Riemannian density on the sphere SM_p is $d\xi = \sqrt{g'} du_2 ... du_n$ where $g' = \det(\langle \partial_i \xi, \partial_j \xi \rangle_{i,j \geq 2})$ at $\xi = \xi(0)$ and $\partial_i \xi = J_i'(0)$. So $g' = (\det(J_2'(0), ..., J_n'(0)))^2$. But, by definition $\lambda(r, \xi) dr d\xi = \sqrt{g} du_1 du_2 ... du_n = \sqrt{\frac{g}{g'}} dr d\xi$. Thus we get the result we wished 7 with an absolute value. But, by continuity and since we stay in U_p, we can get rid of this absolute value.■

In order to illustrate the last proposition we give next an asymptotic formula for the volume of a geodesic ball.

Proposition 2.2 *Let $B_p(r)$ be a ball of radius r in M at p and let V_n be the volume of the unit ball in the n-dimensional euclidian space. We have*

$$vol(B_p(r)) = r^n V_n \left(1 - \frac{scal(p)}{6(n+2)} r^2 + o(r^2)\right)$$

Proof : Consider $\xi \in SM_p$, an orthonormal parallel frame $(E_i)_i$ along γ_ξ with $E_1 = T$, and the corresponding Jacobi fields $J_i(t)$ with $J_i'(0) = E_i$. Then, by the Taylor-Young formula and 4, we get

$$J_i(t) = tE_i - \frac{t^3}{6} R(E_i, T)T + o(t^3)$$

We also have, from the last proposition, $\lambda(t, \xi) = t^{n-1} \det A(t, \xi)$, where $A(t, \xi)$ is the matrix with column vectors $J_2(t)/t, ..., J_n(t)/t$. Then, $\det A = 1 + t(\det A)'(0) + \frac{t^2}{2}(\det A)''(0) + o(t^2)$. But for any $A(t) \in M_{n-1}(\mathbb{R})$, $(\det A)' = (\det A) tr(A^{-1}A')$. Hence, we compute $(\det A)'(0) = 0$ and $(\det A)''(0) = tr(A'') = -\frac{1}{3}\sum \langle R(E_i, \xi)\xi, E_i\rangle (0)$. From 6 we now get,

$$\begin{aligned} vol(B_p(r)) &= \int_0^r \int_{S^{n-1}} t^{n-1} \left(1 - \frac{t^2}{6} \sum \langle R(E_i, \xi)\xi, E_i\rangle (0) + o(t^2)\right) d\xi dt \\ &= \int_0^r t^{n-1} vol(S^{n-1}) \left(1 - \frac{t^2}{6}\frac{1}{n} scal(p) + o(t^2)\right) dt \\ &= \frac{r^n}{n} vol(S^{n-1}) \left(1 - \frac{r^2}{6}\frac{1}{n+2} scal(p) + o(r^2)\right) \end{aligned}$$

And we're done, since $V_n = \frac{1}{n} vol(S^{n-1})$.■

3 First comparison results

In this section, we will show the first results among the comparison theorems in Riemannian Geometry. These are Bonnet-Myers' theorem on the diameter of manifolds with a positive lower bound on Ricci curvature, and Rauch's theorems, which compare the size of Jacobi fields in M and in the model spaces $\mathbb{M}_\kappa$ if we assume an upper bound on sectional curvature.

3.1 Bonnet and Myers' theorem

Theorem 3.2 *(**Bonnet and Myers**) Let M be a complete Riemannian manifold with $Ric \geq (n-1)\kappa$. Then every geodesic of length greater or equal to $\frac{\pi}{\sqrt{\kappa}}$ has conjugate points and $diam(M) \leq \frac{\pi}{\sqrt{\kappa}}$.*

Proof : Fix a normal geodesic γ of length l in M without conjugate points and consider an orthonormal parallel frame $(E_i)_i$ along γ with $E_1 = T$. Then let

$H_i(t) = s_i(t)E_i(t)$ where $s_i(t) = \sin(\frac{\pi}{l}t)$. Then, by 5 we get, since $H_i(0) = H_i(l) = 0$,

$$\begin{aligned} I(H_i, H_i) &= -\int_0^l \langle \nabla_T \nabla_T H_i + R(H_i, T)T, H_i \rangle \\ &= \int_0^l (\sin(\frac{\pi t}{l}))^2 \left(\frac{\pi^2}{l^2} - \langle R(E_i, T)T, E_i \rangle \right) \end{aligned}$$

But $Ric(T,T) = \sum_{i=2}^{n} \langle R(E_i, T)T, E_i \rangle$ so,

$$\sum_{i=2}^{n} I(H_i, H_i) = \int_0^l (\sin(\frac{\pi t}{l}))^2 \left((n-1)\frac{\pi^2}{l^2} - Ric(T,T) \right) \leq 0$$

Therefore for some i at least, we have $I(H_i, H_i) \leq 0$. Since there are no conjugate points on γ, the only Jacobi field vanishing at $\gamma(0)$ and $\gamma(l)$ is the zero Jacobi field. Thus by the first Index Lemma, we must have $I(H_i, H_i) \geq 0$, so $I(H_i, H_i) = 0$. But the equality case in the first Index Lemma implies $H_i \equiv 0$, which is a contradiction.

Finally, if $diam(M) > \frac{\pi}{\sqrt{\kappa}}$, then we can find two points p and q with $d(p,q) > \frac{\pi}{\sqrt{\kappa}}$ and, since M is complete, we can also find a minimal normal geodesic from p to q. But since there must be conjugate points on γ, γ cannot minimize form all the way from p to q (cf. 1.3.2.). Therefore $diam(M) \leq \frac{\pi}{\sqrt{\kappa}}$.■

Corollary 3.1 *With the hypothesis of the last theorem, we get that M is compact with compact universal cover and consequently finite fundamental group.*

Proof : M has finite diameter hence is compact (cf. 1.2.2.). Also we can endow the universal cover $\widetilde{M}$ of M with the pull-back metric, such that the canonical projection $\pi : \widetilde{M} \to M$ is a Riemannian cover. But we must also have $Ric \geq (n-1)\kappa$ in $\widetilde{M}$. Hence $\widetilde{M}$ is compact. As a consequence, the fibers of π are discrete, so $\pi_1(M)$ is finite.■

We next move on to Rauch's theorem. We must emphasize the fact that these theorems make an assumption on sectional curvature, whereas Myers' theorem assumed a lower bound on Ricci curvature only.

3.2 Rauch's comparison theorems

Theorem 3.3 *(**Rauch I**) Let M and M_0 be Riemannian manifolds with $\dim M_0 \geq \dim M$ and γ (resp. γ_0) be normal geodesics of length l in M (resp. M_0). Assume that for all $t \in [0, l]$ and $x \in M_{\gamma(t)}$ (resp. $x_0 \in M_{\gamma_0(t)}$) we have $K(x, T) \geq K(x_0, T_0)$ and that $\gamma_0(t)$ is not conjugate to $\gamma_0(0)$ along γ_0 for all $t < l$. Let J (resp. J_0) be a Jacobi field along γ (resp. γ_0) such that $J(0) \in \mathbb{R}T(0)$ (resp. $J_0(0) \in \mathbb{R}T_0(0)$) and*

$$\|J(0)\| = \|J_0(0)\|, \ \langle T(0), J'(0)\rangle = \langle T_0(0), J_0'(0)\rangle \text{ and } \|J'(0)\| = \|J_0'(0)\|$$

Then for all $t \in [0, l]$,

$$\|J(t)\| \geq \|J_0(t)\|$$

In this theorem, we usually take M_0 to be a space of constant curvature κ, though to get the last result, we only need lighter assumptions. The first direct consequence of Rauch's theorem is that if $J_0(t)$ does not vanish, then $J(t)$ does not vanish either. Therefore, if there are no conjugate points to $\gamma_0(0)$ on γ_0, there will be no conjugate points of $\gamma(0)$ on γ. We will see a direct application of this fact in Klingenberg's lower bound for the injectivity radius.

Proof : First, we reduce to the case where J and J_0 are such that

$$J(0) = \langle T(0), J'(0)\rangle = 0 \tag{8}$$
$$J_0(0) = \langle T_0, J_0'(0)\rangle = 0 \tag{9}$$

This is easy by putting $\widetilde{J}(t) = J(t) - \langle T, J(t)\rangle T$ (resp. $\widetilde{J}_0(t) = J_0(t) - \langle T_0, J_0(t)\rangle T_0$), because these are again Jacobi fields which satisfy the above conditions. So assuming we've proved the theorem for these Jacobi fields, we get $\left\|\widetilde{J}(t)\right\| \geq \left\|\widetilde{J}_0(t)\right\|$ and we're done since $\langle T, J(t)\rangle = \langle T, J(0)\rangle + \langle T, J'(0)\rangle t = \langle T_0, J_0(t)\rangle$.

So assume 8. Then consider an orthonormal parallel frame $(E_i)_i$ along γ with $E_1 = T$, and a similar one $(E_i^0)_i$ along γ_0. Then $J(t) = \sum_{i=2}^{n} J_i(t)E_i(t)$ for some functions $J_i(t)$ and similarly for J_0. We will actually show that the ratio $\frac{\|J(t)\|}{\|J_0(t)\|}$ is non-decreasing and tends to 1 as t tends to 0. The last assertion is clear by l'Hospital's rule, since

$$\lim_{t\to 0} \frac{\|J(t)\|^2}{\|J_0(t)\|^2} = \frac{\|J'(0)\|^2}{\|J_0'(0)\|^2} = 1$$

by assumption. Now it suffices to show that for all t

$$\frac{d}{dt}\left(\frac{\|J(t)\|^2}{\|J_0(t)\|^2}\right) \geq 0$$

or equivalently for all $t_1 \leq l$

$$\frac{\langle J'(t_1), J(t_1)\rangle}{\langle J(t_1), J(t_1)\rangle} \geq \frac{\langle J_0'(t_1), J_0(t_1)\rangle}{\langle J_0(t_1), J_0(t_1)\rangle}$$

We fix $t_1 \leq l$ and consider $H(t) = J(t)/\|J(t_1)\|$ (resp. $H_0(t) = J_0(t)/\|J_0(t_1)\|$) for all $t \leq t_1$. These are again Jacobi fields orthogonal to T. Thus we can find an isometry $I : M_{\gamma(t_1)} \to M_{\gamma_0(t_1)}$ such that $I(H(t_1)) = H_0(t_1)$ and $I(T(t_1)) = T_0(t_1)$. Then set $E_i^0(t_1) = I(E_i(t_1))$ for all i and extend backwards these orthonormal vectors to an orthonormal parallel frame along γ_0. Next, we write $H(t) = \sum_{i=2}^{n} H_i(t)E_i(t)$. Also, we consider the vector field $\widetilde{H_0}$ along γ_0 defined by $\widetilde{H_0}(t) = \sum_{i=2}^{n} H_i(t)E_i^0(t)$. From our construction of E_i^0 we get $H_0(t_1) = \widetilde{H_0}(t_1)$. Then we compute

$$\begin{aligned}
\frac{\langle J', J\rangle}{\langle J, J\rangle}_{|t=t_1} &= \langle H', H\rangle(t_1) \\
\frac{\langle J_0', J_0\rangle}{\langle J_0, J_0\rangle}_{|t=t_1} &= \langle H_0', H_0\rangle(t_1)
\end{aligned}$$

But for any Jacobi field V along γ with $V(0) = 0$

$$\begin{aligned}
\langle V', V\rangle(t_1) &= \int_0^{t_1} \langle V', V\rangle' \\
&= \int_0^{t_1} \langle V', V'\rangle + \langle V'', V\rangle \\
&= \int_0^{t_1} \langle V', V'\rangle - K(T, V)\|V\|^2 = I(V, V)
\end{aligned} \tag{10}$$

We now have, applying 10 to H and since $\langle H', H'\rangle = \left\langle \widetilde{H_0}', \widetilde{H_0}'\right\rangle$ and $\|H\|^2 = \left\|\widetilde{H_0}\right\|^2$ we get,

$$\langle H', H\rangle(t_1) = \int_0^{t_1} \langle H', H'\rangle - K(T, H)\|H\|^2$$

$$\geq \int_0^{t_1} \left\langle \widetilde{H_0}', \widetilde{H_0}' \right\rangle - K(T, \widetilde{H_0}) \left\| \widetilde{H_0} \right\|^2 = I(\widetilde{H_0}, \widetilde{H_0})$$

The last inequality follows from the assumption on sectional cuvature. Then the first Index Lemma applies, since $H_0(0) = \widetilde{H_0}(0) = 0$ and $H_0(t_1) = \widetilde{H_0}(t_1)$

$$I(\widetilde{H_0}, \widetilde{H_0}) \geq I(H_0, H_0)$$

Applying again 10 to H_0 we finally obtain what we wanted,

$$\frac{\langle J', J \rangle}{\langle J, J \rangle}_{|t=t_1} = \langle H', H \rangle (t_1) \geq I(\widetilde{H_0}, \widetilde{H_0}) \geq I(H_0, H_0) = \langle H_0', H_0 \rangle (t_1) = \frac{\langle J_0', J_0 \rangle}{\langle J_0, J_0 \rangle}_{|t=t_1}$$

■

A completely analogous statement is valid for focal points under the same assumptions. This is known as the second Rauch's theorem, but was first stated by Berger.

Theorem 3.4 *(**Rauch II**) Let M and M_0 be Riemannian manifolds with $\dim M_0 \geq \dim M$ and γ (resp. γ_0) be normal geodesics of length l in M (resp. M_0). Assume that for all $t \in [0, l]$ and $x \in M_{\gamma(t)}$ (resp. $x_0 \in M_{\gamma_0(t)}$) we have $K(x, T) \geq K(x_0, T_0)$ and that $\gamma_0(t)$ is not a focal point of $\gamma_0(0)$ along γ_0 for all $t < l$. Let J (resp. J_0) be a Jacobi field along γ (resp. γ_0) such that $J'(0) \in \mathbb{R}T(0)$ (resp. $J_0'(0) \in \mathbb{R}T_0(0)$) and*

$$\|J(0)\| = \|J_0(0)\|, \ \langle T(0), J(0) \rangle = \langle T_0(0), J_0(0) \rangle, \ \|J'(0)\| = \|J_0'(0)\|$$

Then for all $t \in [0, l]$,

$$\|J(t)\| \geq \|J_0(t)\|$$

The proof of this is the same as the last one, except that we use the second Index Lemma instead of the first.

3.3 Consequences of Rauch's theorems

We now present an important corollary of Rauch II, which will be used in the proof of Toponogov's theorem.

Corollary 3.2 *Consider two Riemannian manifolds M and M_0 with $\dim M_0 \geq \dim M$ and $K_M \geq K_{M_0}$. Consider further two normal geodesics γ in M and γ_0 in M_0 of length l, and let E (resp. E_0) be a parallel unit vector field along γ (resp. γ_0) such that $\langle E, T\rangle(s) = \langle E_0, T_0\rangle(s)$ $\forall s \in [0, l]$. Then let $\phi : [0, l] \to \mathbb{R}$ be a smooth function and set $c(s) = \exp_{\gamma(s)}(\phi(s)E(s))$ (resp. $c_0(s) = \exp_{\gamma_0(s)}(\phi_0(s)E_0(s))$). Assume that $\gamma(s)$ has no focal points along the geodesic $t \mapsto \exp_{\gamma(s)}(tE(s))$, $t \in [0, \phi(s)]$. Then*

$$L[c] \geq L[c_0]$$

Proof : Let $\alpha(s,t) = \exp_{\gamma(s)}(t\phi(s)E(s))$ (resp. $\alpha^0(s,t) = \exp_{\gamma_0(s)}(t\phi(s)E_0(s))$). The curves $c_s = \alpha(s,\cdot)$ form a smooth variation through geodesics, hence $J_s(t) = \frac{\partial}{\partial s}\exp_{\gamma(s)}(t\phi(s)E(s))$ (resp. $J_s^0(t) = \frac{\partial}{\partial s}\exp_{\gamma_0(s)}(t\phi(s)E_0(s))$) are Jacobi fields along the geodesics $t \mapsto \alpha(s,t)$. As

$$L[c] = \int_0^1 \|J_s(1)\|\, ds$$

we only need to show that $\|J_s(1)\| \geq \|J_s^0(1)\|$ for all s. This would be a direct consequence of Rauch II if the geodesics $t \mapsto \alpha(s,t)$ were normal geodesics. But this is not the case, so we need to introduce the normal geodesics $\widetilde{c_s} : t \mapsto \exp_{\gamma(s)}(tE(s))$, $t \in [0, \phi(s)]$ and their corresponding Jacobi field $H_s(t) = \frac{\partial}{\partial s}\exp_{\gamma(s)}(tE(s))$. We have $H_s(0) = \gamma'(s) = T(s)$ and $H_s'(0) = 0$ and, since $\langle E, T\rangle(s) = \langle E_0, T_0\rangle(s)$, we can apply Rauch II to $H_s(t)$ to get

$$\|H_s(t)\| \geq \left\|H_s^0(t)\right\|$$

Also, we set $K_s(t) = \frac{\partial}{\partial t}\exp_{\gamma(s)}(tE(s))$ the tangent vector to geodesics $\widetilde{c_s}$. Then we compute

$$J_s(t) = H_s(t\phi(s)) + \phi'(s)K_s(\phi(s))$$

and a similar expression for $J_s^0(t)$. As the geodesics $\widetilde{c_s}$ are normal, $\|K_s(t)\| = 1$ for all t and s (resp. $\|K_s^0(t)\| = 1$). So we get

$$\|J_s(1)\|^2 = \|H_s(\phi(s))\|^2 + (\phi'(s))^2 + 2\phi'(s)\langle H_s(\phi(s)), K_s(\phi(s))\rangle \qquad (11)$$

and the corresponding expression for $J_s^0(1)$. Now, we have $L[\widetilde{c_s}] = \phi(s)$ and, by the first variation formula 1 we get

$$\frac{dL[\widetilde{c_s}]}{ds} = \phi'(s) = \frac{1}{\phi(s)}\left(\langle H_s(\phi(s)), K_s(\phi(s))\rangle - \langle T(s), E(s)\rangle\right)$$

And a similar expression in M_0. But as $\langle T(s), E(s)\rangle = \langle T_0(s), E_0(s)\rangle$ by assumption, we get from 11 that

$$\|J_s(1)\|^2 \geq \left\|J_s^0(1)\right\|^2 \Leftrightarrow \|H_s(\phi(s))\|^2 \geq \left\|H_s^0(\phi(s))\right\|^2$$

so we are done.■

Of course we also have the analogous statement for conjugate points.

As an direct application of Rauch I, we get the following well-known result,

Theorem 3.5 *(**Cartan-Hadamard**) Let M be a complete Riemannian manifold with $K_M \leq 0$. Then for any $p \in M$, $\exp_p : M_p \to M$ is a covering map and the universal cover of M is diffeomorphic to $\mathbb{R}^n$.*

Proof : By Rauch I, we see that there are no conjugate points to p in M along any geodesic. Hence $\exp_p : M_p \to M$ has nonsingular differential. Therefore we can endow M_p with the pull-back metric from $\exp_p$. Then, by the Hopf-Rinow theorem, M_p is complete with this metric and $\exp_p$ is a local isometry, hence a covering map.■

Finally, we'll give one last application of Rauch I, to the celebrated Klingenberg's lower bound for the injectivity radius.

Theorem 3.6 *(**Klingenberg**) Suppose M is a Riemannian manifold with $K \geq K_M \geq H > 0$ and let l be half the length of the shortest closed geodesic in M. Then*

$$inj(M) = \min\{\frac{\pi}{\sqrt{K}}, l\}$$

Proof : By Myers' theorem, M is compact, so we can find $p \in M$ and $q \in C_p$ such that $d(p,q) = inj(M)$. If q is conjugate to p along a minimal normal geodesic γ from p to q, then by Rauch I, applied to M and $M_0 = \mathbb{M}_K$, we must have $L[\gamma] \geq \frac{\pi}{\sqrt{K}}$. If q is not conjugate to p along any minimal geodesic, then we claim that they are precisely two minimal geodesics from p to q, say γ and σ, and that they are such that $\gamma'(p) = -\sigma'(p)$. Granting the claim, we can apply it by reversing p and q because p is not a conjugate point of q either and p is a point of C_q closest to q. Then we obtain, by unicity of the two geodesics, $\gamma'(q) = -\sigma'(q)$ and therefore γ and σ form a closed geodesic in M, so $d(p,q) \geq l$.

We now prove the claim. As $q \in C_q$ there are at least two distinct minimal geodesics from p to q. Choose two of them γ and σ. If $\gamma'(p) \neq -\sigma'(p)$ one can find $v \in M_p$ such that

$$\langle v, -\gamma'(p)\rangle > 0,\, \langle v, -\sigma'(p)\rangle > 0$$

Consider a geodesic τ with initial speed vector v. As q is not conjugate to p along γ or σ, we can find smooth variations $\{\gamma_s\}$ and $\{\sigma_s\}$ of γ and σ going from p to $\tau(s)$. By the first variation formula 1, we get

$$\frac{dL[\gamma_s]}{ds}_{|s=0} < 0, \frac{dL[\sigma_s]}{ds}_{|s=0} < 0$$

and hence for s small enough, we have

$$L[\gamma_s] \leq L[\sigma_s] < L[\gamma] = L[\sigma]$$

But this would imply the existence of a cut point of σ_s between p and $\tau(s)$ and gives a contradiction. So we must have $\gamma'(p) = -\sigma'(p)$ and therefore γ and σ are the only two minimal geodesics from p to q, which ends the proof of the claim.$\blacksquare$

4 Volume comparison

4.1 Comparison of Riemannian densities in polar coordinates

In this section, we will refine the proof of Myers' theorem, using ideas similar to that of the proof of Rauch's theorem, in order to obtain estimates of the volume of a geodesic ball, when we assume a lower bound on Ricci curvature or an upper bound on sectional curvature. Recall that we denote by M a complete Riemannian manifold of dimension n, and by $\mathbb{M}_\kappa$ the model space of constant curvature κ. If $p \in M$, U_p is the star-shaped domain in M_p to which $\exp_p^{-1}$ is a chart from $M \backslash C_p$ (cf. 1.). The main comparison result is the following :

Proposition 4.3 *Fix $\xi \in SM_p$ and let $\lambda(r)$ $(= \lambda(r, \xi)$, cf. 1.), resp. $\lambda_\kappa(r)$, be the Riemannian density for M, resp. $\mathbb{M}_\kappa$, in spherical coordinates. Then if $r\xi \in U_p \backslash \{0\}$,*

1. *If $Ric \geq (n-1)\kappa$, then*

$$\frac{\lambda'(r)}{\lambda(r)} \leq \frac{\lambda'_\kappa(r)}{\lambda_\kappa(r)}$$

In particular $\lambda(r) \leq \lambda_\kappa(r)$.

2. *If $K_M \leq \kappa$, then*

$$\frac{\lambda'(r)}{\lambda(r)} \geq \frac{\lambda'_\kappa(r)}{\lambda_\kappa(r)}$$

In particular $\lambda(r) \geq \lambda_\kappa(r)$.

Proof : Let γ_ξ be the geodesic in M with initial vector ξ, and consider a parallel orthonormal frame $(E_i(t))_i$ along γ_ξ with $E_1(t) = T(:= \gamma'_\xi(t))$. Similarly consider a normal geodesic γ^κ in $\mathbb{M}_\kappa$ together with a orthonormal parallel frame $(E_i^\kappa(t))_i$ along it. Next, we let $J_i(t)$ be the Jacobi field along γ_ξ such that $J_i(0) = 0$ and $J_i(r) = E_i(r)$ (such a J_i exists because $r\xi \in U_p$ and $\exp_p$ is not singular at any point of U_p). Then, according to 1.,

$$\lambda(t) = C \det(J_2(t), ..., J_n(t)) = C\sqrt{d(t)}$$

where C is some constant and $d(t) = \det(A(t))$ with $A(t) = \langle J_i(t), J_j(t)\rangle_{i,j\geq 2}$. Hence $\lambda'/\lambda = \frac{1}{2}d'/d$. It is straightforward to compute $d'(t) = d(t)\mathrm{tr}(A^{-1}(t)A'(t))$. But since $A(r) = id$, we get

$$d'(r) = 2\sum_{i=2}^{n} \langle J_i(r), J'_i(r)\rangle$$

However, from 1. we have $I(J,J) = \langle J, J'\rangle(r) - \langle J, J'\rangle(0)$ for all Jacobi fields J, where I is the Index form along the geodesic. Therefore,

$$\frac{\lambda'(r)}{\lambda(r)} = \sum_{i=2}^{n} \langle J_i(r), J'_i(r)\rangle = \sum_{i=2}^{n} I(J_i, J_i) \tag{12}$$

Now we consider case 1. and put $H_i(t) = \frac{S_\kappa(t)}{S_\kappa(r)}E_i(t)$ and $H_i^\kappa(t) = \frac{S_\kappa(t)}{S_\kappa(r)}E_i^\kappa(t)$ where $S_\kappa(t)$ is the fonction defined in 1.3.1. By definition of S_κ, H_i^κ is a Jacobi field in $\mathbb{M}_\kappa$. Also, H_i and J_i have the same values at end points when $t = 0$ and $t = r$. Therefore, by the Index lemma 1., we get for all i

$$I(J_i, J_i) \leq I(H_i, H_i) \tag{13}$$

But (cf.)

$$I(H_i, H_i) = \int_0^r (\frac{S_\kappa(t)}{S_\kappa(r)})^2(\kappa - \langle R(T, E_i)T, E_i\rangle)dt + \langle H_i(r), H_i'(r)\rangle \qquad (14)$$

and summing over the $i = 2, ..., n$ we get

$$\sum_{i=2}^n I(H_i, H_i) = \int_0^r (\frac{S_\kappa(t)}{S_\kappa(r)})^2((n-1)\kappa - Ric(T, T))dt + \sum_{i=2}^n \langle H_i(r), H_i'(r)\rangle \qquad (15)$$

From 12 and 15 applied to the Jacobi field H_i^κ in $\mathbb{M}_\kappa$ we get

$$\sum_{i=2}^n \langle H_i(r), H_i'(r)\rangle = \sum_{i=2}^n \langle H_i^\kappa(r), H_i^{\kappa\prime}(r)\rangle = \sum_{i=2}^n I_\kappa(H_i^\kappa, H_i^\kappa) = \frac{\lambda_\kappa'(r)}{\lambda_\kappa(r)}$$

Then 13 and condition 1. in 15 imply

$$\frac{\lambda'(r)}{\lambda(r)} = \sum_{i=2}^n I(J_i, J_i) \leq \sum_{i=2}^n I(H_i, H_i) \leq \sum_{i=2}^n \langle H_i(r), H_i'(r)\rangle = \frac{\lambda_\kappa'(r)}{\lambda_\kappa(r)}$$

which gives what we wanted.

Similarly for case 2., we denote by J_i^κ the vector field along γ^κ in $\mathbb{M}_\kappa$ whose coordinates in the orthonormal frame $(E_i^\kappa)_i$ are the same as the coordinates of J_i in $(E_i)_i$. Then, from 1.

$$I(J_i, J_i) = \langle J_i(r), J_i'(r)\rangle = \int_0^r (\langle J_i', J_i'\rangle - \langle R(J_i, T)T, J_i\rangle)\, dt \qquad (16)$$

Hence

$$\begin{aligned} I(J_i, J_i) &\geq \int_0^r (\langle J_i', J_i'\rangle - \kappa \langle J_i, J_i\rangle)\, dt \\ &\geq \int_0^r (\langle J_i^{\kappa\prime}, J_i^{\kappa\prime}\rangle - \kappa \langle J_i^\kappa, J_i^\kappa\rangle)\, dt \\ &\geq I_\kappa(J_i^\kappa, J_i^\kappa) \end{aligned}$$

Then the Index Lemma applied to γ^κ in $\mathbb{M}_\kappa$ gives

$$I(J_i, J_i) \geq I_\kappa(J_i^\kappa, J_i^\kappa) \geq I_\kappa(H_i^\kappa, H_i^\kappa)$$

And by summing over $i = 2, ..., n$ we get from 12 the desired result,

$$\frac{\lambda'(r)}{\lambda(r)} = \sum_{i=2}^n I(J_i, J_i) \geq \sum_{i=2}^n I_\kappa(H_i^\kappa, H_i^\kappa) = \frac{\lambda_\kappa'(r)}{\lambda_\kappa(r)}$$

■

4.2 Comparison of the volume of geodesic balls

Consider now a geodesic ball $B_p(r) = \left\{x = \exp_p(t\theta) | \theta \in S^{n-1}, 0 \leq t \leq r\right\}$ for $p \in M_p$ and arbitrary r and let's $vol(B_p(r))$ be its Riemannian volume. As an easy consequence of the latter property, we get the celebrated

Theorem 4.7 *(**Bishop-Gunther–Cheeger-Gromov**) Let $B_p(r)$ (resp. $B_p^\kappa(r)$) be a geodesic ball in M (resp. $\mathbb{M}_\kappa$).*

1. *If $Ric \geq (n-1)\kappa$, then*
$$r \mapsto \frac{vol(B_p(r))}{vol(B_p^\kappa(r))}$$
is a non-increasing function, which tends to 1 as r goes to 0. In particular $vol(B_p(r)) \leq vol(B_p^\kappa(r))$.

2. *If $K_M \leq \kappa$ and if $B_p(r)$ does not meet the cut locus C_p, then*
$$r \mapsto \frac{vol(B_p(r))}{vol(B_p^\kappa(r))}$$
is a non-decreasing function, which tends to 1 as r goes to 0. In particular $vol(B_p(r)) \geq vol(B_p^\kappa(r))$.

Proof : From 6 we have $vol(B_p(r)) = \int_0^r \int_{S^{n-1}} \lambda(r,\xi) d\xi dr$. Consider $f(r) = \log(\frac{vol(B_p(r))}{vol(B_p^\kappa(r))})$, then $f'(r) = a(r)/\int_0^r a(t)dt - b(r)/\int_0^r b(t)dt$ where $a(t) = \int_{S^{n-1}} \lambda(t,\xi)d\xi$ and $b(t) = \int_{S^{n-1}} \lambda_\kappa(t,\xi)d\xi$. To get what we want we only have to check the sign of $f'(r)$. But this is the same as that of

$$\int_0^r (a(r)b(t) - b(r)a(t))\, dt$$

So we need to check whether $\frac{a(t)}{b(t)}$ is monotone. But

$$\frac{a(t)}{b(t)} = \int_{S^{n-1}} \frac{\lambda(t,\xi)}{\lambda_\kappa(t,\xi)} d\xi$$

as $\lambda_\kappa(t,\xi)$ does not depend on ξ. Hence we need to consider the variations of $\frac{\lambda(t,\xi)}{\lambda_\kappa(t,\xi)}$. But

$$\frac{d}{dt}\log(\frac{\lambda(t,\xi)}{\lambda_\kappa(t,\xi)}) = \frac{\lambda'(t)}{\lambda(t)} - \frac{\lambda'_\kappa(t)}{\lambda_\kappa(t)}$$

And the sign of this expression is given by the last proposition in both cases when $t\xi \in U_p$. However, in the first case, if $t\xi \notin U_p$ then $\lambda(t,\xi) = 0$ so $t \mapsto \frac{\lambda(t,\xi)}{\lambda_\kappa(t,\xi)}$ is still decreasing beyond U_p. But it is necessary to have the restriction on r in the second case.

Finally, the statement about $\lim f(r)$ when $r \to 0$ is a direct consequence of the asymptotic formula computed in 1.■

Now we will discuss the case of equality in the previous results and we'll deduce Cheng's theorem about maximal diameter rigidity, which clarifies the case of equality in Myers' theorem.

4.3 Maximal diameter rigidity

First, we show that the equality case in Bishop et allii's theorem implies that the spherical cap considered is isometric to that of a model space of constant curvature κ.

Proposition 4.4 *If, for some $r > 0$ (such that $B_p(r)$ does not meet the cut locus C_p, only if we are in case 2) we have equality in prop. 3.1., or if $vol(B_p(r)) = vol(B^\kappa(r))$ in theorem 3.1., or if $\frac{vol(B_p(r))}{vol(B_{\overline{p}}^\kappa(r))} = \frac{vol(B_p(r_1))}{vol(B_{\overline{p}}^\kappa(r_1))}$ for some $r_1 < r$, $r \in [0, \max_{v \in U_p} \|v\|]$, then $B_p(r)$ is isometric to $B^\kappa(r)$.*

Proof : By checking the equality case of all inequalities and in both cases 1 and 2 of prop.3.1., or theorem 3.1., we see that we must have $\lambda(t,\xi) = \lambda_\kappa(t,\xi)$ for all $\xi \in SM_p$ and for all $t \leq r$. In case 2, we see directly from 16 that $\langle R(J_i,T)T, J_i\rangle = \kappa \langle J_i, J_i\rangle$ for all i. So $K(T,x) = \kappa$ for all $x \in M_{\gamma_\xi(t)}$ (recall $T = \gamma_\xi'(t)$). In the first case, we get equality in 13 and therefore, by the equality case in the Index Lemma 1., we have $J_i = H_i$. Then $\langle H_i(r), H_i'(r)\rangle = \langle J_i(r), J_i'(r)\rangle = I(J_i,J_i) = I(H_i,H_i)$. Now, by 14 we also get $\langle R(E_i,T)T, E_i\rangle = \kappa$, i.e. $K(T,x) = \kappa$ for all $x \in M_{\gamma_\xi(t)}$.

Now, we'll show that the latter property implies that the two balls are isometric. We fix $\overline{p} \in \mathbb{M}_\kappa$ and we consider $\phi : B_p(r) \to B_{\overline{p}}^\kappa(r)$ given by $\phi = \exp_{\overline{p}} \circ \exp_p^{-1}$. This is a diffeomorphism (because of the assumption on $B_p(r)$) and we'll show that it is in fact an isometry. It will be enough to see that $\|d_q\phi(x)\| = \|x\|$ for all $q \in B_p(r)$ and $x \in M_q$. So, take $q \in B_p(r)$ and consider the minimal geodesic γ_ξ from p to $q = \gamma_\xi(t_0)$. Then $\phi(\gamma_\xi)$ is also a minimal geodesic from $\overline{p} = \phi(p)$ to $\phi(q)$. Then let J be the Jacobi field along γ_ξ such that $J(0) = 0$ and $J(t_0) = x$, for an arbitrary x in M_q. Consider two orthonormal frames $(E_i)_i$ and $(E_i^\kappa)_i$ along γ_ξ and $\phi(\gamma_\xi)$ such that $E_1 = T = \gamma'_\xi$ and $E_1^\kappa = (\phi(\gamma_\xi))'$. Then let J^κ be the vector field along $\phi(\gamma_\xi)$ with same coordinates in $(E_i^\kappa)_i$ as J in $(E_i)_i$. Then, by the hypothesis $K(T,x) = \kappa$ for all $x \in M_{\gamma_\xi(t)}$, and from the very definition of Jacobi fields (cf. 1.) we see that J^κ is also a Jacobi field. Moreover $\|J(t)\| = \|J^\kappa(t)\|$ for all $t \leq t_0$, so it is enough to show that $J^\kappa(t) = d\phi(J(t))$.

But this is easy, since (from 1. for example)

$$J(t) = d_{t\xi} \exp_p(tJ'(0))$$
$$J^\kappa(t) = d_{td\phi(\xi)} \exp_{\overline{p}}(tJ^{\kappa\prime}(0))$$

and $J^{\kappa\prime}(0) = J'(0)$ by definition of J^κ. So

$$J^\kappa(t) = d_{td\phi(\xi)} \exp_{\overline{p}}(tJ'(0)) = (d_{td\phi(\xi)} \exp_{\overline{p}}) \circ (d_{t\xi} \exp_p)^{-1}(J(t)) = d\phi(J(t))$$

which ends the proof.■

We can now give an easy proof of Cheng's theorem (with the sectional curvature condition, instead of the Ricci curvature condition, this was first proved by Toponogov, by use of his theorem on triangles, cf. below).

Theorem 4.8 ***(Cheng)*** *Let M be a complete Riemannian manifold with $Ric \geq (n-1)\kappa$ and $\kappa > 0$. By Myers' theorem, we know that $diam(M) \leq \frac{\pi}{\sqrt{\kappa}}$. But suppose $diam(M) = \frac{\pi}{\sqrt{\kappa}}$, then M is isometric to the standard sphere of constant curvature κ.*

Proof : Since $diam(M) = \frac{\pi}{\sqrt{\kappa}}$ we know that M is compact and, therefore, we can pick two points x and y in M such that $d(x,y) = \frac{\pi}{\sqrt{\kappa}}$. Then, by theorem 3.1., we have

$$\frac{vol(B_x(\frac{\pi}{2\sqrt{\kappa}}))}{vol(B^\kappa(\frac{\pi}{2\sqrt{\kappa}}))} \geq \frac{vol(B_x(\frac{\pi}{\sqrt{\kappa}}))}{vol(B^\kappa(\frac{\pi}{\sqrt{\kappa}}))} = \frac{vol(M)}{vol(\mathbb{M}_\kappa)} \tag{17}$$

So

$$vol(B_x(\frac{\pi}{2\sqrt{\kappa}})) \geq \frac{1}{2} vol(M)$$

and also

$$vol(B_y(\frac{\pi}{2\sqrt{\kappa}})) \geq \frac{1}{2} vol(M)$$

But, $B_x(\frac{\pi}{2\sqrt{\kappa}}) \cap B_y(\frac{\pi}{2\sqrt{\kappa}}) = \emptyset$ since $d(x,y) = \frac{\pi}{\sqrt{\kappa}}$ therefore

$$\begin{aligned} vol(B_x(\frac{\pi}{2\sqrt{\kappa}})) &= \frac{1}{2}vol(M) \\ vol(B_y(\frac{\pi}{2\sqrt{\kappa}})) &= \frac{1}{2}vol(M) \end{aligned}$$

So we have equality in 17 and therefore $M = B_x(\frac{\pi}{\sqrt{\kappa}})$ and $B^\kappa(\frac{\pi}{\sqrt{\kappa}}) = \mathbb{M}_\kappa$ are isometric by the last proposition.■

5 Growth and finiteness of fundamental groups

5.1 Exponential and polynomial growth

As an interesting application of the volume comparison results we discussed in the last section, we will here show how one can get some insight of the fundamental group of the manifold by using those methods.

Definition 5.1 *Let Γ be a finitely generated group and $G = \{\gamma_1, ..., \gamma_k\}$ a system of generators. Let also $N_G(t)$ be the number of different elements of Γ which can be expressed as a word in G of length at most $t \in \mathbb{N}$. We say that Γ has **exponential growth** if*

$$\lim_{t\to\infty} \frac{1}{t}\log(N_G(t)) > 0$$

*Also, we say that Γ has **polynomial growth** of at most degree $n \geq 1$ if*

$$\lim_{t\to\infty} \sup \frac{1}{t^n} N_G(t) < \infty$$

Two remarks have to be done after this definition. First, as it is easy to check, both definitions are independent of the choice of G. Secondly, the limit does exist as shows an easy subadditivity argument ($N_G(t+s) \leq N_G(t)N_G(s)$).

We then recall that the fondamental group Γ of a Riemannian manifold M acts isometrically on its universal cover $\widetilde{M}$ endowed with the pull-back metric. The canonical projection $\pi : \widetilde{M} \to M$ is a Riemannian covering (i.e. covering map and local isometry). Then we have the following important result :

Proposition 5.5 *The fundamental group Γ acts properly discontinuously on $\widetilde{M}$.*

Proof : We have to show that for every compact K in $\widetilde{M}$, $\{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\}$ is finite. Suppose it is not the case. Then there exist sequences (y_n) in some compact K in $\widetilde{M}$ and (γ_n), consisting of distinct elements of Γ, such that $\gamma_n(y_n) \in K$. Then, by taking a subsequence if necessary, we can assume that $\gamma_n(y_n)$ converges in K and that (y_n) also converges to some $y \in K$. Then, because γ_n are isometries, we get that $\gamma_n(y)$ converges in K. But this is impossible, since there exists $r > 0$ such that all balls $\gamma(B_y(r))$, $\gamma \in \Gamma$ are disjoint, for $\widetilde{M}$ is a Riemannian covering of M.■

If M is compact, then one can find a compact set K in $\widetilde{M}$ such that $\pi(K) = M$ (just use compactness of M and local compactness of $\widetilde{M}$). But one can even take K to have same volume as M. In fact, if we fix $x \in \widetilde{M}$, and set

$$K = \left\{ z \in \widetilde{M} | d(x,z) \leq d(\gamma(x), z) \forall \gamma \in \Gamma \right\} \tag{18}$$

then $\pi(K) = M$ and π maps ∂K on a set of measure zero in M, so $vol(K) = vol(M)$. Given this, it is easy to prove the next proposition :

Proposition 5.6 *If M is a compact Riemannian manifold, then Γ is finitely generated. More precisely, if K is a compact set in $\widetilde{M}$ such that $\pi(K) = M$ and $G = \{\gamma \in \Gamma | d(K, \gamma(K)) \leq diam(K) + \varepsilon\}$, with $\varepsilon > 0$ and if*

$$d(K, \gamma(K)) \leq \varepsilon t \tag{19}$$

for some $t \in \mathbb{N}$, then γ can be written as a product of t elements of G. In particular G generates Γ.

Proof : First, we see by the last proposition that G is finite. By assumption, we can pick $x \in K$ and $y \in \gamma(K)$ such that $d(x,y)) \leq \varepsilon t$. Then choose $y_1, ..., y_{t+1}$ in $\widetilde{M}$ such that $\forall i = 1, ..., k$

$$d(y_i, y_{i+1}) \leq \varepsilon$$

where we've set $y_{t+1} = y$ and $x = y_1$. Then we can find $x_i \in K$ and $\gamma_i \in \Gamma$ such that $\gamma_i(x_i) = y_i$ for $i = 1, ..., t+1$. As we can take $\gamma_1 = id$, we have

$$\gamma = (\gamma_1^{-1}\gamma_2)(\gamma_2^{-1}\gamma_3)...(\gamma_t^{-1}\gamma_{t+1})$$

and will suffice to show that the $\gamma_i^{-1}\gamma_{i+1}$ are all in G. This is clear, since we have

$$\begin{aligned} d(\gamma_i^{-1}\gamma_{i+1}(x_{i+1}), x_{i+1}) &= d(y_{i+1}, \gamma_i(x_{i+1})) \\ &\leq d(y_{i+1}, y_i) + d(x_{i+1}, x_i) \\ &\leq \varepsilon + diam(K) \end{aligned}$$

■

We can now prove the main result of this paragraph, where Bishop's theorems are crucial.

Theorem 5.9 ***(Milnor)*** *Let M be a compact Riemannian manifold.*

1. *If $Ric > 0$, then its fundamental group Γ has polynomial growth of degree at most $n = \dim M$.*
2. *If $K_M < 0$, then its fundamental group Γ has exponential growth.*

Proof : As M is compact, we now know that Γ is finitely generated. In the first case, let G be any finite set of generators. Fix $x \in \widetilde{M}$, and put $l = \max_{\gamma \in G} d(x, \gamma(x))$. Now, if γ is a product of t elements of G, then $d(x, \gamma(x)) \leq lt$. But, since π is a Riemannian covering, we can find $r > 0$ such that all balls $B_{\gamma(x)}(r)$, $\gamma \in \Gamma$ are disjoint. Hence

$$N_G(t) vol(B_x(r)) \leq vol(B_x(lt + r))$$

But, as M is compact and $Ric > 0$, there exists $\kappa > 0$ such that $Ric > (n - 1)\kappa$. Then we can apply Bishop's theorem 3.2.(1) to get that $vol(B_x(lt+r)) \leq vol(B^\kappa(lt + r))$. Therefore, for some constant $c_\kappa > 0$, we have

$$N_G(t) \leq \frac{c_\kappa}{vol(B_x(r))}(lt + r)^n$$

So Γ have polynomial growth of degree at most n.

In the second case, consider the set of generators G defined in the preceding proposition. Fix $x \in K$, and choose any $y \in B_x(\varepsilon t)$. There exists $\gamma \in \Gamma$ such that $y \in \gamma(K)$. Thus this γ satisfies condition 19 and therefore γ can be written as a product of t elements of G. In other words, $B_x(\varepsilon t)$ is covered by $N_G(t)$ sets of the form $\gamma(K)$. As γ is an isometry, $vol(\gamma(K)) = vol(K)$. Hence,

$$vol(B_x(\varepsilon t)) \leq N_G(t) vol(K)$$

On the other hand, as $K_M \leq -\kappa$ for some $\kappa > 0$ because $K_M < 0$ and M is compact, we deduce from theorem 3.2.(2) applied to $\widetilde{M}$ that

$$vol(B_x(\varepsilon t)) \geq vol(B^{-\kappa}(\varepsilon t)) \geq c \exp((n-1)\varepsilon t)$$

for some constant $c > 0$ (cf. 3.2.). So, Γ has exponential growth (note that this is true for any $\varepsilon > 0$).■

In fact, to go further, one has the beautiful result of Avez (cf. [Bal]), which states that if $K_M \leq 0$, then either Γ has exponential, or $K_M = 0$.

5.2 A finiteness result

We will end this section proving a finiteness result for the fundamental group, which is another application of theorem 3.2. Let $\mathcal{N}(D, v, \kappa, n)$ be the set of all compact Riemannian manifolds M which satisfy the following conditions

$$\begin{aligned} Ric &\geq (n-1)\kappa \\ vol(M) &\geq v \\ diam(M) &\leq D \end{aligned}$$

Then we shall prove

Theorem 5.10 ***(Anderson)*** *There are only finitely many fundamental groups among the manifolds in $\mathcal{N}(D, v, \kappa, n)$ for fixed D, v, κ and n.*

Before going into the proof of this theorem, we will need an important lemma.

Lemma 5.4 *Let M be a compact Riemannian manifold with fundamental group Γ and $x \in \widetilde{M}$. Then one can find a finite set of generators $G = \{\gamma_1, ..., \gamma_m\}$ of Γ such that all relations for Γ in these generators are of the form $\gamma_i \gamma_j \gamma_k^{-1} = 1$, and such that $d(x, \gamma_i(x)) \leq 2diam(M)$ for all $i = 1, ..., m$.*

Proof : As M is compact, one can find a triangulation T of M, such that the distance between two adjacent vertices is less than $\varepsilon > 0$ and such that any loop in M is homotopic to a loop in the 1-skeleton of T. Then, let $\{v_1, ..., v_q\}$ be the vertices of that T and $\{e_{ij}\}$ the corresponding edges. Then fix $a = \pi(x)$ and consider Γ as $\pi_1(a, M)$. Furthermore, we set σ_i $(i = 1...q)$ to be a minimal geodesic from a to v_i, and $\sigma_{ij} = \sigma_i e_{ij} \sigma_j^{-1}$ a loop at a. We have $d(\sigma_{ij}(x), x) \leq d(a, v_i) + d(a, v_j) + \varepsilon \leq 2diam(M) + \varepsilon$. Then all loops in the 1-skeleton of T are products of the σ_{ij}, and hence so is any loop in $\pi_1(a, \Gamma)$. Now if v_i, v_j, v_k are adjacent vertices, then they span a triangle in M. Thus the loop $\sigma_{ij}\sigma_{jk}\sigma_{ik}^{-1}$ is homotopically trivial. Conversely, for any homotopically trivial loop σ based at a, σ can actually be retracted through the 2-skeleton of T, and therefore the corresponding homotopy is given by a collection of triangles in T. Thus the relation $\sigma = 1$ in Γ, is in fact the product of the corresponding relations for those triangles. Hence $G = \{\sigma_{ij}\}$ fits our requirements, except we need to get rid of the ε. But this easy, for Γ is discrete.$\blacksquare$

Proof of Anderson's theorem : Given a set $G = \{\gamma_1, ..., \gamma_m\}$ of generators as in the previous lemma, we see that there are only finitely many possible groups Γ generated by G (at most 2^{m^3}). Thus we only need to show that m is bounded under our hypothesis. Choose K as in 18, with $\pi(K) = M$ and $vol(K) = vol(M)$. Then, with this definition of K we see that the $\gamma(K)$, $\gamma \in \Gamma$ are disjoint up to sets of measure zero, and all have same volume. By the previous lemma, we have that the $\gamma_i(K)$ all lie in the ball $B_x(4D)$. Thus, by Bishop's theorem 3.2. we get

$$m \times v \leq m \times vol(F) \leq vol(B_x(4D)) \leq vol(B^\kappa(4D))$$

Hence m is bounded by a constant derived from D, v, κ and n.$\blacksquare$

There are many other results about the fundamental group of Riemannian manifolds and how to deduce topoological information from geometrical hypothesis on curvature, diameter or volume. For instance, by use of Toponogov's theorem (see next section) we will see that the number of elements needed to generate $\pi_1(M)$ for manifolds of non-negative curvature, is bounded by a universal constant, which depends only on the dimension. Also, one of the most famous result in this direction is Gromov's bound for Betti numbers which shows the existence of an upper bound for the $b_i(M)$, which depends only on the dimension (cf. [Kli] or [Che]).

6 Toponogov's theorem

We will here show Toponogov's theorem, which is one of the major comparison theorems in the Riemannian Geometry. This result is about comparing triangles in an arbitrary complete Riemannian manifold with triangles in one of the model spaces of constant curvature, by assuming a lower bound on sectional curvature. We will present later some interesting applications.

We first recall the definition of a geodesic triangle and a hinge in a Riemannian manifold.

Definition 6.2 *A **geodesic triangle** is a triangle whose sides are normal geodesics with lengths satisfying the three triangle inequalities. A **hinge** is a configuration with two normal geodesics γ, γ' with lengths l, l' such that $\gamma(0) = \gamma'(0)$.*

If γ, γ' is a hinge, we denote by α the angle between the tangent vectors ξ and ξ' (taken in $[0, \pi]$) as indicated in the following diagram (we define $\alpha_1, \alpha_2, \alpha_3$ similarly for any geodesic triangles).

We also denote by $\mathbb{M}_\kappa$ the model surface of constant curvature κ (if $\kappa > 0$ the sphere of radius $\frac{1}{\sqrt{\kappa}}$, if $\kappa = 0$ the euclidean plane, and if $\kappa < 0$ the corresponding hyperbolic disk).

Theorem 6.11 ***(Toponogov)****Let M be a complete Riemannian manifold with sectional curvature $K_M \geq \kappa$.*

***A**. Let $(\gamma_1, \gamma_2, \gamma_3)$ be the three sides of a geodesic triangle in M with lengths l_1, l_2, l_3. Suppose γ_1, γ_3 are minimal, and if $\kappa > 0$ suppose $l_2 \leq \pi/\sqrt{\kappa}$. Then there exists a geodesic triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ in $\mathbb{M}_\kappa$ such that $l_i = \overline{l_i}$ for all i, $\overline{\alpha_1} \leq \alpha_1$ and $\overline{\alpha_3} \leq \alpha_3$ with obvious notations. Except in the case where $l_i = \frac{\pi}{\sqrt{\kappa}}$ for some i and $\kappa > 0$, the triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ is uniquely determined.*

B. *Let γ, γ' be a hinge in M with angle α such that γ is minimal and $l' \leq \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$. Then, for any hinge $\overline{\gamma}, \overline{\gamma}'$ in M_κ with same angle and same sides length,*

$$d(A, B) \leq d(\overline{A}, \overline{B})$$

Note In the sequel, whenever a condition involving $\frac{\pi}{\sqrt{\kappa}}$ appears, it should be ignored if $\kappa \leq 0$.

Remark 6.1 *Basically, this theorem shows that the more curved the manifold is, the more convergent its geodesics will be.*

Note that in $\mathbb{M}_\kappa$ any geodesic with length less or equal to $\frac{\pi}{\sqrt{\kappa}}$ is minimal. Similarly, by Myers' theorem, every minimal geodesic in M ($K_M \geq \kappa$) is of length less or equal to $\frac{\pi}{\sqrt{\kappa}}$.

Before explaining the proof of the last theorem, we must recall some basic facts about triangles in the model spaces of dimension 2.

We first recall the **Law of Cosines** in the model spaces (see [Ber]). Consider a geodesic triangle in $\mathbb{M}_\kappa$ with lengths a, b, c and let α be the angle opposite the side of length a. Then (with obvious modifications if $\kappa \notin \{0, -1, 1\}$),

1. spherical geometry $(\kappa = 1)$: $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$
2. hyperbolic geometry $(\kappa = -1)$: $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$
3. euclidean geometry $(\kappa = 0)$: $a^2 = b^2 + c^2 - 2bc \cos \alpha$

As a result, any geodesic triangle in $\mathbb{M}_\kappa$ (with sides of length $< \frac{\pi}{\sqrt{\kappa}}$) is uniquely determined (up to isometry) by the length of its sides (unicity part of statement **A.**).

Moreover, a is an increasing function of α. We easily deduce the following important lemma :

Lemma 6.5 *Let γ, γ' be a hinge in $\mathbb{M}_\kappa$ with fixed sides length ($l, l' \leq \frac{\pi}{\sqrt{\kappa}}$) and varying angle α. We set $f(\alpha) = d(A, B)$ (see figure). Then, f is continuous and strictly increasing on $[0, \pi]$ unless l or l' equals $\frac{\pi}{\sqrt{\kappa}}$, in which case f is constant.*

As a direct consequence of this lemma, we have that the two conditions **A.** and **B.** in the above theorem are equivalent. More precisely, in the situation of **A.,** if **B.** is true for the hinges γ_1, γ_2 and γ_3, γ_2 (γ_1 and γ_3 being the minimal ones of statement **B.**) then **A.** is true for $(\gamma_1, \gamma_2, \gamma_3)$. To see this, consider a hinge of lengths l_1, l_2 and angle α_3 in $\mathbb{M}_\kappa$, then by **B.** $f(\alpha_3) \geq l_3$ but by the triangular inequality, $f(0) = |l_1 - l_2| \leq l_3$, hence ($f$ is continuous) we can find $\overline{\alpha_3} \leq \alpha_3$ with $f(\overline{\alpha_3}) = l_3$, which proves the existence of $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ in $\mathbb{M}_\kappa$. Then, apply **B.** to the other hinge of lengths l_3, l_2 to get $\overline{\alpha_1} \leq \alpha_1$ (in fact we also have $\overline{\alpha_2} \leq \alpha_2$). Conversely, given a hinge γ, γ' like in **B.**, we can draw (M is complete) a minimal geodesic γ_3 from A to B. Since γ and γ_3 are minimal, by **A.**, we get an corresponding triangle in $\mathbb{M}_\kappa$ with same length of sides and angle $\overline{\alpha} \leq \alpha$ between γ and γ_3. The last lemma shows that $d(A, B) = L[\gamma_3] = f(\overline{\alpha}) \leq f(\alpha)$, which proves **B.**

We next proceed to the proof of Toponogov's theorem.

Proof of Toponogov's theorem : We first show statement **B.** for a particular type of hinges. Consider a hinge γ, γ' in M and let $E(t)$ be the unit parallel field along γ, with $E(0) = \xi'$ at O (see figure above). Also consider the corresponding hinge $\overline{\gamma}, \overline{\gamma}'$ (same lengths, same angle) in $\mathbb{M}_\kappa$ and corresponding parallel field $\overline{E}$. We say that the hinge γ, γ' is **thin** if

1. There exists a fonction $\phi : [0, l] \to \mathbb{R}$ such that

$$\overline{c}(s) := \exp_{\overline{\gamma}(s)}(\phi(s)\overline{E}(s))$$

is a minimal path from $\overline{A}$ to $\overline{B}$.

2. $\forall s \in [0,l]$ $\gamma(s)$, in M, has no focal points along the geodesic $t \mapsto \exp_{\gamma(s)}(t\phi(s)E(s))$, $t \in [0,1]$.

The path $c(s) = \exp_{\gamma(s)}(\phi(s)E(s))$ in M satisfies the hypothesis of the corollary to Rauch's theorem for focal points (cf.), therefore we get statement **B.** in this situation, because,

$$d(A,B) \leq L[c] \leq L[\overline{c}] = d(\overline{A},\overline{B})$$

Now consider an arbitrary hinge γ, γ' in M, suppose that the conditions of **B.** are satisfied, and in case $\kappa > 0$, suppose

$$\max_{t\in[0,l']} d(A,\gamma'(t)) < \frac{\pi}{\sqrt{\kappa}} \tag{20}$$

Claim : There exists a subdivision $0 = t_0 < t_1 < ... < t_n = l'$ and minimal geodesics $(\gamma_i'')_{0\leq i\leq n}$ from $\gamma'(t_i)$ to A such that $\gamma_i'', \gamma'_{|[t_i,t_{i+1}]}$ and $\gamma_i'', \left(\gamma'_{|[t_{i-1},t_i]}\right)^{-1}$ are thin hinges in M.

By compacity of $[0,l']$, it suffices to check that for all $t \in [0,l']$, $\exists \varepsilon > 0$ such that $\forall s \in (t-\varepsilon, t+\varepsilon)$ and for all minimal geodesics γ'' from $\gamma'(s)$ to A, $\gamma'', \gamma'_{|[s,t+\varepsilon]}$ and $\gamma'', \left(\gamma'_{|[t-\varepsilon,s]}\right)^{-1}$ are thin hinges in M. To see this, notice that for all $t \in [0,l']$, for all geodesics v_t from $\gamma'(t)$ to A, and for all points p on v_t, there exists $r > 0$, such that $d\exp_p$ has no critical points in the ball of radius r. Moreover, by compacity (in particular, by compacity of the set of elements in the unit sphere bundle which give rise to a minimal geodesic from $\gamma'(t)$ to A), one can choose a single $r > 0$ suitable for all s in a neighboorhood $(t-\varepsilon, t+\varepsilon)$ of t, all minimal geodesics from $\gamma'(s)$ to A and all points on those geodesics. For each v_s, we then consider the corresponding hinges in $\mathbb{M}_\kappa$ $\overline{v_s}, \overline{\gamma}'_{|[s,t+\varepsilon]}$ (and $\overline{v_s}$, $\left(\overline{\gamma}'_{|[t-\varepsilon,s]}\right)^{-1}$). By taking a smaller ε if necessary we can write the minimal geodesic from $\overline{\gamma}'(t+\varepsilon)$ to A (there is only one because of 20) as $\exp_{\overline{v_s}(x)}(\phi(x)\overline{E}(x))$ for some function ϕ such that $|\phi(x)| \leq r$. Hence conditions 1 and 2 of thin hinges are satisfied by both hinges $\gamma'', \gamma'_{|[s,t+\varepsilon]}$ and γ'', $\left(\gamma'_{|[t-\varepsilon,s]}\right)^{-1}$, which proves the claim.■

Next, note that $l' \leq l + d(\overline{A}, \overline{B})$, thus if $l' \geq l + d(A, B)$ then **B.** is true. So we can assume $l' \leq l + d(A, B)$. Therefore, we easily get by induction that the three triangle inequalities are verified for all intermediate triangles $(\gamma_i'', \gamma'_{|[t_i,t_j]}, (\gamma_j'')^{-1})$ for all $i < j$.

By the claim, **B.** is valid for hinges $\gamma, \gamma'_{|[0,t_1]}$ and $\gamma_1'', (\gamma'_{|[0,t_1]})^{-1}$, hence **A.** is true for the geodesic triangle $(\gamma, \gamma'_{|[0,t_1]}, \gamma_1'')$ (cf. consequence of the last lemma). Therefore, we can build the corresponding triangle in $\mathbb{M}_\kappa$ $(\overline{\gamma}, \overline{\gamma}_1', \overline{\gamma}_1'')$ given by **A.** and we glue on it the hinge $\overline{\gamma}_1'', \overline{\gamma}_2'$ corresponding to the hinge $\gamma_1'', \gamma'_{|[t_1,t_2]}$ in M (see the following diagram).

Since the hinge $\gamma_1'', \gamma'_{|[t_1,t_2]}$ is thin by the claim, we get $d(\overline{A}, \overline{B}) \geq L[\gamma_2'']$. But $\overline{\beta_1} \leq \beta_1$, so $\alpha_1 = \overline{\alpha_1} \leq \pi - \overline{\beta_1}$, hence the lemma applied to the hinge $\overline{\gamma}_1'', \overline{\gamma}_2'$ in $\mathbb{M}_\kappa$ gives $f(\pi - \overline{\beta_1}) \geq f(\alpha_1) = d(\overline{A}, \overline{B}) \geq L[\gamma_2'']$. This is saying that **B.** holds for the hinge $\gamma, \gamma'_{|[0,t_2]}$ in M.

Similarly, we show that **B.** holds for $\gamma_2'', (\gamma'_{|[0,t_2]})^{-1}$, then A. holds for the bigger triangle $(\gamma, \gamma'_{|[0,t_2]}, \gamma_2'')$. Step by step, we get that **B.** holds for the initial hinge γ, γ'.

To get rid of assumption 20, one has only to consider the case $\kappa > 0$. Then take any κ' with $0 < \kappa' < \kappa$. As the diameter of M is less or equal to $\frac{\pi}{\sqrt{\kappa}}$ by Myers' theorem, we automatically get 20 for κ'. Thus **B.** is true when the comparison triangle is in $\mathbb{M}_{\kappa'}$. Letting κ' tend to κ we obviously get the result for κ.■

7 A generalized sphere theorem

In this section, we will expose Grove and Shiohama's generalized sphere theorem, which is a clever application of Toponogov's theorem. Let's state the theorem first.

Theorem 7.12 *(**Grove-Shiohama's sphere theorem**) Let M be a complete riemaniann manifold. Suppose $K_M \geq \kappa > 0$ and $diam(M) > \frac{\pi}{2\sqrt{\kappa}}$. Then M is homeomorphic to a sphere.*

The main idea introduced by Grove and Shiohama is the notion of critical point of distance function. They observed that with a suitable notion of critical point of the distance from a given point, the isotopy lemma of Morse theory still holds. With these observations, the pattern of proof of the above theorem is clear : as in the Reeb's theorem of Morse theory (cf. [Mil]) we just have to show that the hypothesis on M prevent it from having more than 2 critical points. We denote $\rho_p(x) = \overline{xp}$ the distance between the points p and x.

Definition 7.3 *A point q is a **critical point** of ρ_p (we also say critical point of p) if $q = p$, or if for any vector $v \in M_q$ there exists a minimal geodesic γ from q to p such that $\angle(v, \gamma'(0)) \leq \frac{\pi}{2}$ (unoriented angle between the two vectors).*

Note that every critical point lies in the cut locus $C(p)$ of p. We next turn to the fondamental fact about critical points, that is the Isotopy lemma. We set $B_a(p) = \{x | \overline{xp} < a\}$ for $a > 0$ in a complete Riemannian manifold M.

Lemma 7.6 *(**Isotopy lemma**) Let $a < b < \infty$, and suppose $\overline{B_b(p)} \backslash B_a(p)$ contains no critical points of p. Then $\overline{B_b(p)} \backslash B_a(p)$ is homeomorphic to $\partial B_a \times [b - a]$. Furthermore, $\overline{B_a(p)}$ is a deformation retract of $\overline{B_b(p)}$, so the inclusion is a homotopy equivalence.*

Proof : Consider a non-critical point x, then one can find w in the tangent space M_x such that $\measuredangle(w, \gamma'(0)) < \frac{\pi}{2}$ for all minimal geodesics γ from x to p. By continuity, there exists an extension of w to a vector field W_x on a neighbourhood U_x of x, such that for any $y \in U_x$ and any minimal geodesic σ from y to p, $\measuredangle(W_x(y), \sigma'(0)) < \frac{\pi}{2}$. Now, take a locally finite open cover of $\overline{B_b(p)} \backslash B_a(p)$ by sets U_{x_i} and a smooth partition of unity, $\sum \phi_i = 1$, subordinate to it. Set $W = \sum \phi_i W_{x_i}$, which does not vanish on $\overline{B_b(p)} \backslash B_a(p)$.

Let K be the compact $\overline{B_b(p)} \backslash B_a(p)$, then $SK \subset SM$ (the unit sphere bundle) is compact. Therefore, there exists $\alpha > 0$ such that for all $y \in K$ and all minimal geodesics σ from y to p, $\langle W(y), \sigma'(0)\rangle \geq 2\alpha$. Take any b' with $a < b' < b$ and set $K = \overline{B_{b'}(p)}\backslash B_a(p)$.Take an arbitrary $x \in K$. As K is compact, W vanishes outside a compact set, so its flow is defined over $\mathbb{R}$ and so form a one-parameter group of diffeomorphisms on M.

Consider the integral curve associated to W at x, $\psi_t(x)$ (with $\psi_0(x) = x$) take a minimal geodesic σ from x to p, and take a smooth variation of it through curves $(c_s)_{|s|\leq\varepsilon}$ starting at $\psi_s(x)$ and ending at p. Then, the first variation formula (cf.) gives

$$\left(\frac{dL[c_s]}{ds}\right)_{s=0} = -\langle\sigma'(0), W(x)\rangle \leq -2\alpha$$

Hence, on a small interval $|s| < \eta$,

$$\frac{dL[c_s]}{ds} \leq -\alpha$$

Thus, for all $t \in [0, \eta]$,

$$\overline{p\psi_t(x)} \leq L[c_t] \leq L[c_0] - t\alpha = \overline{px} - t\alpha$$

Then, for small $t > 0$, $\psi_t(x)$ is in K. Consequently, by compacity, we see that $\overline{p\psi_t(x)}$ is α-linearly decreasing as long as $\psi_t(x)$ remains in K. As $t \mapsto \overline{p\psi_t(x)}$ is continuous, there exists a unique $t(x) > 0$ such that $\overline{p\psi_{t(x)}(x)} = a$. Moreover, if $x_n \to x$, $|t(x_n) - t(x)| \leq \alpha|\overline{p\psi_{t(x_n)}(x_n)} - \overline{p\psi_{t(x)}(x_n)}| \to 0$ by continuity of the flow, so $t(x)$ is a continuous fonction of x.

Now write for $s \in [0, 1]$, $r_s : \overline{B_b(p)} \to \overline{B_b(p)}$ which sends $x \mapsto x$ if $x \in B_a(p)$ and $x \mapsto \psi_{st(x)}(x)$ if $x \in \overline{B_b(p)}\backslash B_a(p)$. Then $(r_s)_{0\leq s\leq 1}$ is clearly a deformation retract from $\overline{B_b(p)}$ to $\overline{B_a(p)}$.

Furthermore, the map $K \to \partial B_a \times [b-a]$ which sends x to $(\psi_{t(x)}(x), \overline{p\psi_{t(x)}(x)})$ is continuous and injective, from what we have seen above, hence a homeomorphism (K is compact).■

Now, we will need one more result, that was observed by Berger before the introduction of the "critical points" by Grove and Shiohama, and which is a key step in the original proof of the Sphere Theorem of Riemannian Geometry (see below).

Lemma 7.7 ***(Berger)*** *Let q be a local maximum of $r \mapsto \overline{pr}$. Then q is a critical point of p.*

Proof : Suppose q is not a critical point of p, then there exist a unit vector $w \in M_q$ such that $\measuredangle(w, \gamma'(0)) < \frac{\pi}{2}$ for all minimal geodesics γ from q to p. Consider the geodesic $\sigma(t) = \exp_q(-wt)$. By continuity, there exists a small interval $[0, \varepsilon]$ ($\varepsilon > 0$) such that for all $t \in [0, \varepsilon]$, any minimal geodesic γ_t from $\sigma(t)$ to p verifies $\langle \sigma'(t), \gamma_t'(0) \rangle < 0$. But, as in the proof of the last lemma, we see that for any smooth variation of γ_t through curves $(c_s)_{|s| \leq \varepsilon}$ starting at $\sigma(t-s)$ and ending at p, the first variation formula (cf.) gives

$$\left(\frac{dL[c_s]}{ds} \right)_{s=0} = \langle \sigma'(t), \gamma_t'(0) \rangle < 0$$

Hence for small $s > 0$,

$$\overline{p\sigma(t-s)} \leq L[c_s] < L[c_0] = \overline{p\sigma(t)}$$

Therefore $t \mapsto \overline{p\sigma(t)}$ is striclty decreasing for $t \in [0, t]$, which contradicts the fact that $\overline{pr}$ has a local maximum at $r = q$.■

Knowing these properties of critical points, we can now give the proof of Grove and Shiohama's generalized sphere theorem stated at the beginning of this section. This will now be an easy application of Toponogov's theorem.

Proof of Grove and Shiohama's theorem : As $K_M \geq \kappa > 0$, we know by Myers' theorem that M is compact, hence we can find p and q in M such that $\overline{pq} = diam(M)$. By Berger's lemma, q is a critical point of p. We claim that p and q are the only critical points of p. Granting the claim, we can apply the Isotopy lemma with $a > 0$ small enough so that $\exp_p$ is an embedding on $B_a(p)$ and $b < \overline{pq}$ large enough so that $\exp_q$ is an embedding on $B_{(\overline{pq}-b)}(q)$.

Therefore, M is homeomorphic to two spherical caps glued together with $\partial B_a(p) \times [a,b]$, which gives a sphere.

So it remains to show the claim. Suppose $x \neq p, q$ were critical. Let γ_3 be a minimal normal geodesic from q to x. As x is critical, there exists a minimal normal geodesic γ_1 from x to p such that

$$\alpha_2 = \measuredangle(-\gamma_3'(l_3), \gamma_1'(0)) \leq \frac{\pi}{2}$$

But as p and q are mutually critical, we can find normal minimal geodesics γ_1 and $\widetilde{\gamma_1}$ from p to q, such that,

$$\begin{aligned} \alpha_3 &= \measuredangle(-\gamma_1'(l_1), \gamma_2'(0)) \leq \frac{\pi}{2} \\ \widetilde{\alpha_1} &= \measuredangle(-\widetilde{\gamma}_2'(l_2), \gamma_3'(0)) \leq \frac{\pi}{2} \end{aligned}$$

Since the geodesics are minimal and M is complete with $K_M \geq \kappa$, we can

apply Toponogov's theorem **A**. to both triangles $(\gamma_1, \gamma_2, \gamma_3)$ and $(\gamma_1, \widetilde{\gamma_2}, \gamma_3)$. Since a geodesic triangle in the model space $\mathbb{M}_\kappa$ is uniquely determined by its side lengths, we obtain the same triangle $(\overline{\gamma_1}, \overline{\gamma_2}, \overline{\gamma_3})$ in $\mathbb{M}_\kappa$. Moreover $\overline{\alpha_2} \leq \alpha_2 \leq \frac{\pi}{2}$, $\overline{\alpha_3} \leq \alpha_3 \leq \frac{\pi}{2}$ and $\overline{\alpha_1} \leq \widetilde{\alpha_1} \leq \frac{\pi}{2}$, so all angles are less than $\frac{\pi}{2}$. By the spherical law of cosines (cf.), we easily get that the sides lengths $\overline{l_i}$ must be less or equal to $\frac{\pi}{2\sqrt{\kappa}}$. But, since $\frac{\pi}{2\sqrt{\kappa}} < \overline{pq} = l_2 \leq \overline{l_2}$, we get a contradiction.■

Remark 7.2 *This theorem is the best we can have of its kind, for the lower bound on the diameter is achieved for real projective spaces.*

8 Other applications of Rauch's and Toponogov's theorems

In this section, we will give some further applications of Toponogov's theorem mainly to non-negatively curved manifolds and, in particular, we will present the celebrated Sphere Theorem of Riemannian geometry as a consequence of Grove and Shiohama's generalized sphere theorem.

8.1 Number of generators of $\pi_1(M)$

As an easy and straigthforward application of Toponogov's theorem, we have the following result,

Theorem 8.13 *There exists a constant $C(n) \geq 0$ such that for every complete Riemannian manifold M of dimension n with $K_M \geq 0$, $\pi_1(M)$ can be generated by less than $C(n)$ elements.*

Proof : Fix $x \in \widetilde{M}$ and let act $\Gamma = \pi_1(M)$ as isometries on $\widetilde{M}$. Inductively, order Γ by choosing $\gamma_1, \gamma_2, ... \in \Gamma$ such that

$$\begin{aligned} d(x, \gamma_1(x)) &= \min_{\gamma \in \Gamma} d(x, \gamma(x)) \\ d(x, \gamma_2(x)) &= \min_{\gamma \in \Gamma \setminus \{\gamma_1\}} d(x, \gamma(x)) \\ &\ \ \cdots \end{aligned}$$

Then join x to each $\gamma_k(x)$ by minimal geodesics σ_k. Fix $l > 0$ and consider any integer $k < l$. Next, apply Toponogov's theorem **B.** to the hinges σ_k, σ_l. We get, from the law of cosines in the euclidean plane,

$$\begin{aligned} d(\gamma_k(x), \gamma_l(x))^2 &\leq d(x, \gamma_k(x))^2 + d(x, \gamma_l(x))^2 - 2d(x, \gamma_k(x))d(x, \gamma_l(x)) \cos \alpha_{k,l} \\ &\leq d(x, \gamma_l(x))^2 + d(x, \gamma_k(x))^2 (1 - 2 \cos \alpha_{k,l}) \end{aligned}$$

where $\alpha_{k,l} = \measuredangle(\sigma_k'(0), \sigma_l'(0))$. But, by the construction of the γ_i, we get

$$d(\gamma_k(x), \gamma_l(x)) = d(x, \gamma_k^{-1}\gamma_l(x)) \geq d(x, \gamma_l(x))$$

or $\gamma_k^{-1}\gamma_l = \gamma_i$ for some $i < l$ and therefore γ_l can be written as a product of some γ_i's $i < l$. Therefore, if we assume that γ_l is not such a product, we must have from the last two inequalities

$$\cos \alpha_{k,l} \leq \frac{1}{2}$$

or equivalently,

$$\alpha_{k,l} \geq \frac{\pi}{3}$$

But this is possible for only a limited number of γ_k's. For each such γ_l we have a disjoint ball of radius $\frac{\pi}{6}$ on the standard sphere of radius π and dimension $n-1$. Therefore, we have at most

$$C(n) = \frac{vol(B(n-1,\pi))}{vol(B(n-1,\frac{\pi}{6}))}$$

such γ_k's and the others can be written as products of the first (where $B(n,r)$ is the ball of radius r in the sphere of dimension n).■

8.2 Geodesics which go to infinity in non-negatively curved spaces

The following result about non-compact positively curved manifolds is a straightforward application of Toponogov's theorem.

Theorem 8.14 *Let M be a complete Riemannian manifold with non-negative sectional curvature ($K_M \geq 0$) and let $\gamma, \sigma : [0,\infty) \to M$ be geodesics such that $\gamma(0) = \sigma(0)$. Furthermore, we assume that $d(\gamma(0),\gamma(t)) = t$ for all $t \geq 0$ and $\measuredangle(\gamma'(0),\sigma'(0)) < \frac{\pi}{2}$, then $\lim d(\sigma(0),\sigma(t)) = \infty$.*

Proof : For all $t \leq s$ positive real numbers, we consider a minimal geodesic $c_{s,t}$ from $\gamma(s)$ to $\sigma(t)$. Then, since γ and $c_{s,t}$ are minimal and $t \leq s$, the geodesic triangle $(\gamma_{|[0,s]}, \sigma_{|[0,t]}, c_{s,t})$ satisfies the hypothesis of Toponogov's theorem **A.** Since $K_M \geq 0$, we can compare M with the euclidean plane. We therefore get a triangle in the plane with side lengths s, t, $L[c_{s,t}]$ and an angle $\overline{\alpha} = \measuredangle(\overline{\gamma}'(0),\overline{\sigma}'(0)) < \frac{\pi}{2}$. By the standard law of cosines

$$L[c_{s,t}]^2 = t^2 + s^2 - 2st\cos\overline{\alpha}$$

Then

$$s - L[c_{s,t}] = \frac{2ts\cos\overline{\alpha} - t^2}{s + L[c_{s,t}]} \tag{21}$$

But we have $d(\sigma(t),\sigma(0)) \geq s - L[c_{s,t}]$ because γ is minimal and $L[c_{s,t}] \leq s+t$, because $c_{s,t}$ is minimal. Now fix t and allow s to tend to infinity. The last two inequalities show that $L[c_{s,t}]/s \to 1$. Therefore, by 21 we get for s large enough

$$d(\sigma(t), \sigma(0)) \geq s - L[c_{s,t}] \geq \frac{t}{2} \cos \overline{\alpha}$$

Since this is true for arbitrary t, we are done.■

8.3 The Sphere Theorem

We will now discuss the celebrated Sphere Theorem, which was a great achievement of the theory in the early sixties.

Theorem 8.15 *(**The Sphere Theorem**) Let M be a compact simply connected Riemannian manifold such that there exists a $\kappa > 0$ with $0 < \frac{\kappa}{4} < K_M \leq \kappa$. Then M is homeomorphic to a sphere.*

A Riemannian manifold such that $\kappa_1 \leq K_M \leq \kappa_2$ is called $\frac{\kappa_2}{\kappa_1}$-pinched, so this theorem states that **δ-pinched** manifolds with $\delta > \frac{1}{4}$ are homeomorphic to a sphere.

This theorem is a direct consequence of the generalized sphere theorem discussed in the last section. As M is compact, we see that there exists a constant $\kappa_0 > 0$ such that $K_M \geq \kappa_0 > \frac{\kappa}{4}$. From Grove and Shiohama's generalized sphere theorem, it suffices to prove the hypothesis $diam(M) > \frac{\pi}{2\sqrt{\kappa_0}}$. This is a consequence of the theorem on the injectivity radius of $\frac{1}{4}$-pinched manifolds :

Theorem 8.16 *Let M be a simply connected Riemannian manifold with a constant $\kappa_1 > 0$ such that $\frac{\kappa_1}{4} \leq K_M \leq \kappa_1$. Then $inj(M) \geq \frac{\pi}{\sqrt{\kappa_1}}$.*

Indeed, we'll have $diam(M) \geq inj(M) \geq \frac{\pi}{\sqrt{\kappa}} > \frac{\pi}{2\sqrt{\kappa_0}}$. For a nice and self-contained proof of the last theorem, we refer the reader to [Kli].

The Sphere Theorem was proved in the fifties before Grove and Shiohama's theorem (1977). In fact, it is fairly easy to see, using Toponogov's theorem and Berger's lemma, that the Sphere Theorem reduces to the latter theorem on the injectivity radius. But this theorem was not known, or only in special cases like in even dimension, until it was proved by Klingenberg in the sixties. As we have explained the proof of Grove and Shiohama's theorem in detail, we shall not go into the details of the original proof of the Sphere Theorem.

A natural question to ask is whether M is diffeomorphic to a sphere with its standard differential structure. In general the answer to that question is no.

As it is well known, for some dimensions there exist on the sphere other differential structures than the standard one (exotic spheres). However, there is an equivalent "Differentiable Sphere Theorem", which states that a simply connected 0.76-pinched compact Riemannian manifold is diffeomorphic to the sphere with its standard differentiable structure. It is not known whether one can improve this to the $\frac{1}{4}$-pinched manifolds. In fact, there is no known example of a Riemannian metric of strictly positive curvature on an exotic sphere. Nevertheless, Gromoll and Meyer gave the following famous example of an exotic sphere with sectional curvature ≥ 0.

Let $Sp(n)$ be the n-dimensional symplectic group realized by $n \times n$ matrices A with quaternionic entries such that $A^*A = I$. $Sp(n)$ is a compact Lie group, and therefore carries a bi-invariant metric which is of non-negative sectional curvature. Then consider the action of $Sp(1)$ on $Sp(2)$ given by

$$(q, A) \mapsto \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} A \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

Then, this action is free and the quotient turn out to be an exotic 7-sphere.

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