A few remarks

have seen : - languages, structures, formulas
- Theories
- Completeness - Compactness
- Löwenheim Skolem
- elementary embeddings

Compactness very strong property

true because "first-order" languages

cannot express everything.

2 rules • variables represent elements of

the structure (not subsets)

$\forall x\, \varphi(x)$ is true if Forall $m \in M$ $\varphi(m)$

not Forall $X \subset M$. $\varphi(X)$ . . .

cannot say "Forall integer n" unless $M = \mathbb{N}$ or $\mathbb{N}$ is definable. .

- Allow only <u>finite</u> conjonctions (disjonctions)

Compactness $\Longrightarrow$ if Σ has finite models of unbounded cardinality, it has an infinite one., because "finite" cannot be expressed: it would be an infinite disjunction of $\bigvee_{n \in \mathbb{N}} \theta_n$ where θ_n = "there are exactly n elements"...

(*)

This will not prevent us from looking at "infinite conjonctions" or "infinite intersections of definable sets" (in particular <u>Types</u>)

<u>But</u> want first to insist on what we are really interested in:

$\boxed{\text{Definable subsets}}$

(*) **Recall** :

Suppose Σ has models $(M_n)_{n \geq 1}$ where

$|M_n| \geq n$. $\xrightarrow{}$ cardinality of

Consider $\Sigma' = \Sigma \cup \left\{ \exists x_1, \dots, \exists x_n \bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j) \right\}_{n \geq 2}$

Any finite $F \subset \Sigma'$ has a model:

$F \subset \Sigma \cup \{ \exists x_1 \dots \exists x_n \bigwedge x_i \neq x_j \}_{n \leq N_0}$

Take $M_{N_0 + 1}$

By Compactness Σ' has a model

= a model of Σ where for each n there are at least n distinct elements. $\rightarrow$ infinite

Some indispensable definitions.

$\mathcal{L}$ $\quad \mathfrak{M}$ $\quad \mathcal{L}$- structure

Notation: $\quad \mathfrak{M} = \langle M, \underbrace{\ldots\ldots}_{\text{symbols from } \mathcal{L}} \rangle$

$|\mathfrak{M}|$ will denote

the cardinality of M.

Will often denote $\left.\begin{matrix}\mathfrak{M}\\ M\end{matrix}\right\}$ same way.

Definition: $\quad A \subset \mathfrak{M}$, $\quad b \in \mathfrak{M}$ $\quad \bar{a} \subset A$

1) A formula $\varphi(x, \bar{a})$ is algebraic if

$\mathfrak{M} \models$ " $\exists\, n$ elts $b_1, \ldots b_n$ satisfying φ

i.e. s.t. $\mathfrak{M} \models \varphi(b_i, a)$.

Note that if $\quad A \subset \mathfrak{M} \preceq \mathfrak{N}$ $\quad \varphi(x, \bar{a})$ is algebraic in $\mathfrak{M}$

iff $\varphi(x, \bar{a})$ is algebraic in $\mathfrak{N}$.

Say $b \in \mathcal{M}$ is algebraic over A if

$\mathcal{M} \models \varphi(b, \bar{a})$ for some algebraic formula.

Define $acl_{\mathcal{M}}(A) = \{ b \in \mathcal{M} : \text{algebraic over } A \}$.

2) Say b is <u>definable</u> in $\mathcal{M}$ over A

if there is a formula $\varphi(x, \bar{a})$

such that b is the unique element in $\mathcal{M}$

s.t. $\mathcal{M} \models \varphi(b, \bar{a})$

($\{b\}$ is an A-definable set)

Define $dcl_{\mathcal{M}}(A) = \{ b \in \mathcal{M} : \text{definable over } A \}$

$dcl_{\mathcal{M}}(A) \subset acl_{\mathcal{M}}(A)$

- in ACF_0 $\quad acl(A) = \widehat{\langle A \rangle}^{alg}$) field generated by A

$A \subseteq K \models ACF_0 \quad dcl(A) = \langle A \rangle$

- in ACF_p $\quad dcl(A) =$ the perfect closure de $\langle A \rangle = \langle A \rangle^{p^{-\infty}}$

if $x^p = a \quad a \in A \quad x \in dcl(A)$.

Given $\mathcal{L}$, $\mathcal{M}$ an $\mathcal{L}$-structure $A \subset M$

$D \subseteq M^k$ is A-definable
or is definable over A / with parameters in A

if $D = \{ (b_1, \dots, b_k) \in M^k \,;\, \mathcal{M} \models \varphi(b_1, \dots, b_k, \bar{a}) \}$
for $\varphi(x_1, \dots, x_k, \bar{y}) \in \mathcal{L}$
$\bar{a} \subset A$.

In fact one could consider "structures" as being sets M together with a family $\mathrm{Def}(M)$ of subsets ranging over $\bigcup_n M^n$ satisfying a certain number of properties:

- the diagonal is in $\mathrm{Def}(M)$ (some for $=$)
- $\mathrm{Def}(M)$ closed under finite Bool. operations
- ______ projections
- ______ specializations ....

3rd p.

Def(M) — formulas

"Basic subsets" $\longrightarrow$ defined by atomic formulas

$\cap$ $\longrightarrow$ $\wedge$

Compl. $\longrightarrow$ $\neg$

projection $\longrightarrow$ $\exists$

But need formulas in order to work and to know how to extend definable sets in extensions of M.

Note that as in alg. geometry with varieties can regard a formula as a functor

from $\{\mathcal{L}\text{-structures}\} \longrightarrow$ Sets

or

$\{$Models of a theory$\} \longrightarrow$ Sets

$\varphi(x_1 \ldots x_n)$: $M \longrightarrow \varphi(M) := \{a \in M^n ; M \models \varphi(a)\}$

$\precsim$ elementary embeddings are exactly the morphisms needed for this to be a functor to $(\mathrm{Sets}, \subseteq)$.

Let us look at our first emblematic example.

. K an alg. closed field $K \models ACF$ in $\mathcal{L}_{rings}$.

What are the definable sets? over some $k \subset K$

- Zariski closed sets $\rightsquigarrow$ $\wedge$ of atomic formulas with param. in k [or equivalently in $\mathcal{L}(k)$]

$\{x \in K^n : f_1(x) = \dots = f_r(x) = 0\}$

$f_1 \dots f_r \in k[X]$

- Boolean combination of Zar. closed $\rightsquigarrow$ Close under $\wedge, \neg, \vee$

= constructible sets = the formulas without quantifiers

close under Projection?

Chevalley: the family of constructible sets is closed under projections.

close under $\exists$?

Tarski: ACF has Quant. elim.

Q.E. gives a "simple" form for definable subsets

In the case of fields in L_{ring} it is very strong.

<u>Thm</u> (MacIntyre): Let K be an infinite field such that $Th(K)$ in L_{ring} has QE. Then K is algebraically closed.

If the language is richer then the situation is different

ex: $(\mathbb{R}, +, \cdot, -, 0, 1, <)$ has Q.E

$(\mathbb{K}, +, \cdot, -, 0, 1, \delta) \models DCF_0$ diff. closed has Q.E.

Can also consider "infinitely" definable sets

$= \bigcap_{i \in I} D_i$ D_i's are definable.

But must be careful:

$K \models ACF_0$

Consider $\left(D_f = \{x \in K \,;\, f(x) \neq 0\}\right)$ for $\underset{f \neq 0}{f \in \mathbb{Q}[X]}$

then this family of infinite sets
has finite int. property in $\tilde{\mathbb{Q}}$
but $\bigcap_{\substack{f \in \mathbb{Q}[X] \\ f \neq 0}} D_f$ has empty intersection in $\tilde{\mathbb{Q}}$
will have a point in any $K \supsetneq \tilde{\mathbb{Q}}$.

Compactness + diagrams:
if $(D_i)_{i \in I}$ def. in M^k has finite intersection property
(*) then $\bigcap_{i \in I} D_i \neq \emptyset$ in some $N \succ M$.

Plan : • Types
• saturation
• Space of types

Imaginaries and/or Ultraproducts