<u>IMAGINARIES</u>

Deal with two problems which will turn out to have the same solution.

1) quotients?

We are interested, given an $\mathcal{L}$-structure $\mathcal{M}$ in the definable subsets, but in particular in looking at "algebraic structures" definable in $\mathcal{M}$:

<u>Def</u>: . A map $f: M^n \longrightarrow M^k$ is <u>definable</u> if its graph is definable.

. A group $(G, \cdot)$ is definable in M if $\underset{\text{definable}}{G \subset M^k}$ and $\cdot : G \times G \to G$ is def. map.

Example in $K \models ACF$

if G is linear group over K then $G(K)$ the K-rational points of G is clearly a definable group.

In fact any algebraic group can be considered as a definable group ...

if $H \trianglelefteq G$ both definable, what about G/H?

it is not in general a definable set.

More generally, if $E \subset M^{2n}$ is any $\emptyset$-definable equivalence relation on M^n, what about M^n/E?

One solution (always works) extend the language and the model M, by adding in an "innocuous" way coset spaces and cosets = imaginary elements

($\mathcal{L} \to \mathcal{L}^{eq}$ = for each E eq. relat def. on M^n,

add a new sort (or predicate) M^n/E.

$\tilde{M} = M \dot{\cup} M^n/E$, add also $\pi_E : M^n \to M^n/E$, $a \to a/E$ ...)

Before looking at examples, see how this solves

<u>2nd question</u>: Given a definable set $D \subset M^n$

D can be defined by different formulas

$\varphi_1(x, a_1), \ldots, \varphi_k(x, a_k) \ldots$

is there a "canonical" way to define D?

Suppose M elim. imaginaries

Let $D = \{x \in M^n ; \varphi(x, \bar{a})\}$ $\quad \bar{a} \subset M$.

Consider the relation $\bar{a} \sim \bar{a}'$ if

$M \models \forall x \; \varphi(x, \bar{a}) \leftrightarrow \varphi(x, \bar{a}')$

$\sim$ is a definable eq. relation

and $\bar{a}/\sim$ the class of $\bar{a}$ has the property that any autom. of M fixes $\bar{a}/\sim$ pointwise iff it fixes D setwise.

We call $\tilde{a}/\sim$ a <u>canonical parameter</u> for D (or for $\varphi(x, \bar{a})$)

If $\mathcal{M}$ eliminates imaginaries then every definable subset has a canonical parameter in $\mathcal{M}$.

<u>Examples</u> The infinite set in $\mathcal{L}_{\{=\}}$ does not completely eliminate imaginaries

Consider $a \neq b \in M$

$\{a, b\}$ is definable by the formula $\{(x = a \wedge y = b) \vee (x = b \wedge y = a)\}$

but has no can. parameter.

- ACF eliminates imaginaries
- DCF_0 ——————

in ACF : for a Zariski closed set $V \subseteq K^n$

the canon. parameter will be the smallest field of definition i.e k such that $I(V)$ is generated (in $K[x]$) by polynomials in $k[x]$ and smallest such. Then k is fixed pointwise by automorph. g iff g fixes V setwise.

What about types or more generally infinitely definable sets?

Let $p \in S_n(M)$, M suff. homogeneous, saturated, "big"

<u>Idea of Definition</u>:

$A \subset M$ is a <u>canonical basis</u> for p, if A is small and if for all $f \in Aut(M)$ f fixes A pointwise iff $f(p) = p$

$\left[\begin{array}{l} \text{if } p = \{\varphi_i(x, \bar{m}) ; i \in I\} \\ f(p) = \{\varphi_i(x, f(\bar{m})) ; -\} \end{array} \right.$

But there are cases when it is not necessary because all the E-classes already exist in M itself

<u>Definition</u> We say $\mathcal{M}$ has (=strong) <u>elimination of imaginaries</u> if for all definable equivalence relation on M^n, there is some definable map $f_E : M^n \to M^k$ such that $\mathcal{M} \models x E y \leftrightarrow f_E(x) = f_E(y)$.

Then $f_E(x)$ "represents" the class of E and in fact an automorphism will fix the class of x globally iff it fixes $f_E(x) \in M^k$ pointwise.

When $\mathcal{M}$ eliminates imaginaries (or $Th(\mathcal{M})$ does) then any M^n/E can be considered as a definable set

$$M^n/E \approx \{ f_E(x) \, ; \, x \in M^n \}$$

Does not exist in general.

Does exist for stable theories.

Notion will be adapted to make sense in ACFA.

As we will see in further lectures.