Classifiable theories without finitary invariants

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Abstract: It follows directly from Shelah's structure theory that if T is a classifiable theory, then the isomorphism type of any model of T is determined by the theory of that model in the language $L_{\infty,\omega_1}(d.q.)$. Leo Harrington asked if one could improve this to the logic $L_{\infty,\aleph_\epsilon}(d.q.)$ In [Sh 04] S. Shelah gives a partial positive answer, showing that for T a countable superstable NDOP theory, two $\aleph_\epsilon$-saturated models of T are isomorphic if and only if they have the same $L_{\infty,\aleph_\epsilon}(d.q)$-theory. We give here a negative answer to the general question by constructing two classifiable theories, each with $2^{\aleph_1}$ pairwise non-isomorphic models of cardinality $\aleph_1$ which are all $L_{\infty,\aleph_\epsilon}(d.q.)$-equivalent: a shallow depth 3 ω-stable theory and a shallow NOTOP depth 1 superstable theory. In the other direction, we show that in the case of an ω-stable depth 2 theory, the $L_{\infty,\aleph_\epsilon}(d.q)$-theory is enough to describe the isomorphism type of all models.

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1 Introduction

Shelah's classification theory divides countable first order theories into two classes, the "classifiable" theories and the "unclassifiable" ones, in terms of certain precise properties of the theory (stability, superstability, NDOP, NOTOP, ordinal depth).
The unclassifiable theories are shown to have models coding second-order information, and as a consequence to have many non-isomorphic models in all uncountable cardinalities. On the other hand the number of models of classifiable theories is controlled by showing that any model is the prime model over a free amalgam - along a tree - of countable models. This reduces the structure of arbitrary models to that of countable models.

Many parts of the classification are stronger, in that they relate only to finitely generated substructures and their algebraic closures, rather than arbitrary countable structures. This led Leo Harrington to ask whether it was possible to improve Shelah's theory and to describe the models in terms of such "finitary" substructures. We show here that this is not the case; classifiability leaves the possibility of some residual but genuinely infinite set theory in the isomorphism type of models.

In more precise language, we consider three different logics. We will begin at the base with a complete, classifiable theory in a countable first-order language. In particular the theory is superstable and we can define *dimension quantifiers*; see [Sh 90, XIII,1.2, page 624]). Our three logics will all include dimension quantifiers and permit sentences of arbitrary recursive depth. The difference between them will be in the size of the sets one is allowed to quantify over and, hence, in the size of the sets over which the types whose dimension we consider are based : the logic $L_{\infty,\omega}(d.q.)$ allows quantification over (enumerated) finite sets, the logic $L_{\infty,\aleph_\epsilon}(d.q.)$ quantifies over (enumerated) algebraic closures of finite sets and finally $L_{\infty,\omega_1}(d.q.)$ allows quantification over arbitrary (enumerated) countable sets.

Harrington's question can now be phrased in the following way. It follows directly from Shelah's structure theory ([Sh 82a], [Sh 82b], [Sh 85], [BuSh 89]) that if T is a classifiable theory, then the isomorphism type of any model of T is determined by the theory of that model in $L_{\infty,\omega_1}(d.q.)$. Can one improve this to the logic $L_{\infty,\aleph_\epsilon}(d.q.)$?

Alternatively, the question can be phrased in terms of back-and-forth systems, i.e. of winning strategies in Ehrenfeucht-Fraisse games. If T is a classifiable theory, two models of T are isomorphic if and only if there is a back-and-forth system of partial elementary isomorphisms with domains the countable subsets and which respects dimensions of (regular) types over countable subsets. Is it enough to have such a back-and-forth system of partial elementary isomorphisms with domains the algebraic closures of finite subsets and which respects dimension of types over such subsets (see section 1.1 for a precise definition)?

We will show in the positive direction that for ω-stable theories of depth at most 2, $L_{\infty,\aleph_\epsilon}(d.q.)$ does determine the isomorphism type. However in general we show that the answer is negative even in the omega-stable world. For superstable but not omega-stable theories, the answer can be negative even in depth 1.

A different partial positive answer was given by S. Shelah in [Sh 04]. If one considers only models whose $L_{\omega_1,\aleph_\epsilon}$-theory is trivial (i.e. equals that of the universal domain), the answer is positive. Indeed he proves that, if T is a complete countable superstable theory with elimination of imaginaries and NDOP, then any two $\aleph_\epsilon$-saturated models of T are isomorphic if and only if they are equivalent for the Logic $L_{\infty,\aleph_\epsilon}(d.q.)$.

We will construct two theories: in Section 3.1 we describe an ω-stable NDOP theory with $2^{\aleph_1}$ pairwise non isomorphic models of cardinality $\aleph_1$, which are all $L_{\infty,\aleph_\epsilon}(d.q.)$-equivalent. This theory is a minimal counterexample in the following sense : it is a shallow theory of depth 3 and in Section 4 we show that the result holds in the case of ω-stable theories of depth 2. In Section 3.2, we construct, along similar lines, a superstable theory, NDOP, NOTOP, of depth 1 this time, again with $2^{\aleph_1}$ pairwise non-isomorphic models of cardinality $\aleph_1$, which are all $L_{\infty,\aleph_\epsilon}(d.q.)$-equivalent.

Let us point out that, of course, in order for the logic $L_{\infty,\aleph_\epsilon}(d.q.)$to suffice to describe the isomorphism type of models, it is essential to work in T^{eq}, so that types over algebraically closed sets are stationary.

Before launching ourselves into the somewhat lengthy descriptions of the counterexamples, we will remark (section 1.2) that even in the ω-stable case, the Logic $L_{\infty,\omega}(d.q.)$ cannot suffice, and one must quantify over algebraic closures of finite sets and not just finite sets themselves.

First, let us state the precise definition of the back-and-forth which we will take as our defintion of $L_{\infty,\epsilon}(d.q)$-equivalence.

1.1 ϵ-finite sets and $L_{\infty,\aleph_\epsilon}(d.q.)$-equivalence

Let $T\,(=T^{eq})$ be a **complete superstable theory in a countable language with elimination of imaginaries**.
As usual, we suppose that we are working inside a monster model $\mathbb{C}$ of T which is saturated and that all other models we consider are elementary substructures of $\mathbb{C}$ of cardinality strictly smaller than $|\mathbb{C}|$.

We are not going to give a precise syntactic definition of the logic $L_{\infty,\aleph_\epsilon}(d.q.)$, but just say a few words. Consider $L_{\infty,\aleph_\epsilon}$, which is strictly included in $L_{\infty,\aleph_1}$: one is allowed arbitrary conjunctions and disjunctions but may quantify only over countable sequences of variables which are contained in the algebraic closure of a finite subset. More precisely one only allows formulas of the form $\exists\bar{x}\phi(\bar{x})$, for $\bar{x}=(x_i)_{i<\alpha}$, for $\alpha<\omega_1$, if they contain a sub-formula of the form:

$$\exists\bar{y}\ \bigwedge_{i<\alpha}\left[(\theta_i(x_i,\bar{y}))\ \wedge\ (\exists^{<\aleph_0}z\ \theta_i(z,\bar{y}))\right]$$

for some finite $\bar{y}$ and the θ_i's in $L_{\omega,\omega}$. Then one should close under dimension quantifiers (see [Sh 90], XIII, 1.2, p:624). But there are various difficulties involved in such a definition, in particular in choosing how to define dimension quantifiers, difficulties which we will avoid here, by taking as our definition for $L_{\infty,\aleph_\epsilon}(d.q.)$, the existence of the back-and-forth described below.

Definition: We say that a subset a of M is *ϵ-finite* if there is a_0 finite, $a_0\subseteq a$, such that $a\subseteq acl(a_0)$. Note that by superstability, if $b\subseteq a$ for a an ϵ-finite set, then b is also ϵ-finite .
We say that a is *ϵ-closed* if a is ϵ-finite and algebraically closed.
We say that $p\in S(A)$ is an *ϵ-type* if $p=t(a/A)$ for some *enumerated* ϵ-closed set a, that is $p=((a_i)_{i<\omega})$, where $a=\{a_i;i<\omega\}$ is ϵ-finite .
In our notation, we will make no difference between enumerated ϵ-finite sets and finite tuples.

Definition: Let M,N be two models of T, and let $c\subseteq M$, $c'\subseteq N$ range over enumerated ϵ-closed subsets.
We define by induction on ordinals α

$$(M,c)\equiv^\epsilon_\alpha(N,c').$$

- $(M,c) \equiv_0^\epsilon (N,c')$ if there is a partial elementary isomorphism f, from M into N such that $f(c)=c'$ (i.e. if c and c' realize the same type)
- for δ limit ordinal, $(M,c) \equiv_\delta^\epsilon (N,c')$ iff for every $\alpha<\delta$, $(M,c) \equiv_\alpha^\epsilon (N,c')$
- $(M,c) \equiv_{\alpha+1}^\epsilon (N,c')$ if $(M,c) \equiv_\alpha^\epsilon (N,c')$ and
  - for any ϵ-type q over c, if $E=\{d_i; i\in I\}$ is a maximal Morley sequence in M for q, there is some E', maximal Morley sequence in N for the type q', conjugate of q over c', and a one-to-one correspondence h between E and E', such that for every $i\in I$, $(M, acl(d_i c)) \equiv_\alpha^\epsilon (N, acl(h(d_i)c'))$
  - conversely, for any ϵ-type q' over c', if $E'=\{d'_i; i\in I\}$ is a maximal Morley sequence in N for q', there is some E, maximal Morley sequence in M for the type q, conjugate of q' over c, and a one-to-one correspondence h' between E' and E, such that for every $i\in I$, $(M, acl(d'_i c)) \equiv_\alpha^\epsilon (N, acl(h'(d'_i)c'))$.

We say that:

$$(M,c) \equiv_\infty^\epsilon (N,c') \text{ if } (M,c) \equiv_\alpha^\epsilon (N,c') \text{ holds for all ordinals } \alpha,$$

and that

$$M \text{ and } N \text{ are } \equiv_\infty^\epsilon\text{-equivalent } (M \equiv_\infty^\epsilon N), \text{ if } (M, acl(\emptyset)) \equiv_\infty^\epsilon (N, acl(\emptyset)).$$

Lemma 1.1 First properties of the relation $\equiv_\infty^\epsilon$
Let M and N be two models of T, $c\subseteq M$, $c'\subseteq N$, ϵ-closed , such that $(M,c) \equiv_\infty^\epsilon (N,c')$.

1) If q is any ϵ-type over c and if $E=\{d_i; i\in I\}$ is a maximal Morley sequence in M for the type q, then there is some E', maximal Morley sequence in N for the type q', conjugate of q over c', and a one-to-one correspondence h between E and E', such that for every $i\in I$, $(M, acl(d_i c)) \equiv_\infty^\epsilon (N, acl(h(d_i)c'))$.

2) For any ϵ-closed d, $c\subset d\subset M$, there is $d'\subset N$, $c'\subset d'$, such that

$$(M,cd) \equiv_\infty^\epsilon (N,c'd').$$

Proof: 1) By definition for every ordinal α, there is some E'_α maximal Morley sequence in N for q' and a one-to-one correspondence h_α between E and E'_α, such that for every $i\in I$, $(M, acl(d_i c)) \equiv_\alpha^\epsilon (N, acl(h_\alpha(d_i)c'))$. This is taking

place in a fixed model N, hence for cardinality reasons, the same E'_α and h_α must appear cofinally.
2) follows from the definition of $\equiv^\epsilon_\infty$ and 1). □

Remark: We have chosen the above family of partial isomorphisms as representing, in the context of superstable theories the logic $Ł_{\infty,\epsilon}(d.q.)$. We do not restrict ourselves, in the definition, to considering only types for which there is a well defined notion of dimension (regular types). There are many possible ways to define families of partial isomorphism which could describe $L_{\infty,\aleph_\epsilon}(d.q.)$. Shelah, in [Sh 04] fixes a set of apparently weaker conditions which he then shows to be enough to actually give the full isomorphism type for the class of $\aleph_\epsilon$-saturated models of classifiable theories in particular. Similarly, when we prove the positive result, in the second half of this paper, for the case of depth 2 ω-stable theories, we use a back-and-forth which is apparently weaker than the one described in this section.

1.2 Finite sets do not suffice

It is easy to see (and not very surprising) that the introduction of infinite countable sets which are contained in the algebraic closure of finite sets (or in other words the introduction of strong types) is unavoidable when dealing with models of superstable not ω-stable theories. In the case of ω-stable theories the finiteness of the number of strong types over finite sets might lead one to think that there could be a really finitary description, at least for $\aleph_0$-saturated models, that is, that, in fact, the Logic $L_{\infty,\omega}(d.q.)$ might suffice. It is not the case either.

To see that for superstable not ω-stable theories the Logic $L_{\infty,\omega}(d.q.)$ does not suffice, consider the theory T of infinitely many equivalence relations $(E_n)_{n<\omega}$ where each E_n has exactly 2^n classes and E_{n+1} refines each E_n-class into 2 infinite classes. Working in T^{eq} means that we have names for the equivalence classes in the algebraic closure of the empty set. Say that $xE_\infty y$ if xE_ny for every $n<\omega$. Given a model M, choose an enumeration of $acl^{eq}(\emptyset) = \{a_\eta : \eta \in \bigcup_{n<\omega} 2^n\}$ with the obvious conditions that $a_{\eta\hat{}0}$ and $a_{\eta\hat{}1}$ are the two E_{n+1}-classes refining the E_n-class a_η. Then assign to M the map f_M from 2^ω to cardinals smaller or equal to the cardinality of M, $f_M(\nu) := |\{x \in M : x \in a_{\nu(n)},\text{ for all } n<\omega\}|$. Then M and N are isomorphic if and only if there is a permutation σ of 2^ω such that $f_N = \sigma \circ f_M$.

But suppose κ is any uncountable cardinal and M and N are two models such that

– for every n, every E_n-class has cardinality κ,

– every E_∞ class which is realised in the model has cardinality κ and the set of E_∞-classes realised in the model is dense in 2^ω.

It is easy to check that M and N are $L_{\infty,\omega}(d.q.)$ equivalent, noting that in this theory, for every type, the dimension is just the cardinality of the set of realisations. But M and N need not be isomorphic.

Here is now an ω-stable example. Consider a structure M with infinitely many sorts: Q, R_n for $n \in \omega$. There is for each n a map g_n from R_n onto Q, such that for each n the inverse images of the elements of Q in R_n are pairwise disjoint and infinite; for $a \in Q$ and $n < \omega$ we denote by $R_n(a)$ the inverse image of a by g_n. For each n, there is an equivalence relation E_n on R_n, with two infinite classes such that, for each $a \in Q$, E_n divides $R_n(a)$ into two infinite sets. The theory of M is ω-stable, ω-categorical, NDOP, shallow of depth 2. In particular all models are $\aleph_0$-saturated. Consider the set of models of cardinality $\aleph_1$ where Q has cardinality $\aleph_1$, for each n and for each $a \in Q$ $R_n(a)$ is divided by E_n into one countable set and one of cardinality $\aleph_1$. In such a model, after fixing for each n a bijection between R_n/E_n and $\{0, 1\} := \mathbf{2}$, one can associate to each element a in Q an element $s(a)$ in $\mathbf{2}^\omega$: if the classes of E_n are denoted $C(n, 0)$ and $C(n, 1)$, then $s(a)(n) = 0$ iff $C(n, 0) \cap R_n(a)$ is countable. Suppose furthermore that in our models, for each $s \in \mathbf{2}^\omega$, if there is some $a \in Q$ such that $s = s(a)$ then there are $\aleph_1$ many such a's. Then one can check that if M and N are models such that $\{s(a); a \in Q(M)\}$ and $\{s(a); a \in Q(N)\}$ are dense in 2^ω, M and N are equivalent for the logic $L_{\infty,\omega}(d.q.)$. In fact, then, for each n, the truncated models $M_n = Q(M) \cup R_0 \cup \ldots \cup R_n$ and N_n are isomorphic. But in order for M and N to be isomorphic, $S(M) = \{s \in 2^\omega;\ \exists a \in Q(M)\ s = s(a)\}$ and $S(N)$ must be equal up to coordinate-wise permutations of $\mathbf{2}^\omega$.

2 Preliminaries

2.1 Dimensional Order Property and Depth

We give only basic definitions and properties. The reader is referred to [Sh 90], [Ba],[Ha 87], [HaMa 85],[La 87], where complete or partial expositions can be found.

Let T be a complete countable superstable theory. When we talk of $p \in S(A)$ being regular or strongly regular, we always mean that p is stationary.

Definition: 1) We say that T *does not have the Dimensional order property*, or that T has *NDOP* if for all A, B_1, B_2, C, with $A \subset B_1 \cap B_2$, B_1 and B_2 independent over A, $B_1 \cup B_2 \subseteq C$ and for all regular type $p \in S(C)$, if p is not orthogonal to $B_1 \cup B_2$, then p is not orthogonal to B_1 or p is not orthogonal to B_2.

2) Let T have NDOP, let $p \in S(A)$ be regular. We define *the depth* of p, denoted $d(p)$, by induction:

- $d(p) \geq 0$,
- if β is a limit ordinal, $d(p) \geq \beta$ if for all $\alpha < \beta$, $d(p) \geq \alpha$,
- $d(p) \geq \alpha + 1$ if there is a realizing p, $C \supseteq A \cup \{a\}$ and $q \in S(C)$, q orthogonal to A but not orthogonal to $A \cup \{a\}$, with $d(q) \geq \alpha$.

We let $d(p) = \infty$ if $d(p) \geq \alpha$ for all ordinals α. Otherwise we let $d(p) = \alpha$ where α is the first ordinal such that $d(p) \geq \alpha$ and $d(p) \not\geq \alpha + 1$.
The *depth* of T, $d(T)$ is defined to be the $sup\{d(p) + 1; A \subseteq M, M \models T, p \in S(A)\}$. If $d(T) = \infty$ we say that T is *deep*, otherwise we say that T is *shallow*.

3) We say that T *has the Omitting types order property (OTOP)* if there is a type $p(\bar{x}, \bar{y}, \bar{z})$ such that for every ordinal λ and every binary relation R on λ, there is a model M of T and $(\bar{a}_\alpha)_{\alpha<\lambda}$ in M such that: for any $\alpha < \beta < \lambda$, the type $p(\bar{a}_\alpha, \bar{a}_\beta, \bar{z})$ is realized in M iff $\alpha R \beta$.

If T has NDOP, then the class of $\aleph_\epsilon$-saturated of T is *classifiable*, that is admits a good class of isomorphism invariants. If T is ω-stable and has NDOP, then the class of all uncountable models of T is classifiable. If T (superstable not ω-stable) has NDOP and NOTOP (does not have the OTOP), then the class of all uncountable models of T is classifiable.

3 The counterexamples

3.1 An ω-stable depth 3 theory

We are going to construct in stages a complete ω-stable theory T, with NDOP, shallow of depth 3 and with $2^{\aleph_1}$ non-isomorphic models of cardinality $\aleph_1$ which are all $\equiv^{\epsilon}_{\infty}$- equivalent.
Notation : if X is a sort or a relation of arity k in in our language, and if M is a structure for this language, we denote by $X(M)$ the set $\{\bar{m} \in M^k; M \models X(\bar{m})\}$.

3.1.1 First language and axioms

We describe a first language L^0 with finitely many sorts, and a first theory T_0 in L^0, complete, ω-stable, NDOP and shallow of depth 2.
In L^0 we have three sorts E, C, A, a map r_0 from A to $E \times C$, a binary relation $R \subset C \times C$, a ternary relation $g \subset C \times C \times E$, and finally a 5-ary relation $f \subset E \times C \times C \times A \times A$.
If M is an L^0-structure, M will be a model of T_0 if M satisfies the following first-order conditions, (1) to (5):

1. R is irreflexive, symmetric, and has no closed cycles.

2. $E(M)$ is infinite. For each $c \in C(M)$, $g(c, y, z)$ induces a bijection, which we denote g_c, between the set of vertices adjacent to c in $C(M)$ in M (i.e. the set $\{y \in C(M); M \models R(c, y)\}$) and $E(M)$, such that if $R(c, c')$ holds, then $g_c(c') = g_{c'}(c)$. If $g_c(c') = e$ we will say that *the edge between* c *and* c' *has label* e.
   This gives rise to an induced action on $C(M)$ of the free group on $E(M)$ with relations $\{e^2 = 1, e \in E(M)\}$, which is sharply transitive on orbits, if we let for $e \in E(M)$, $c = ec'$ iff $R(c, c')$ and $g(c, c', e)$.

3. The map r_0 is surjective from $A(M)$ onto $E(M) \times C(M)$ and such that, if $A(e, c)$ denotes ${r_0}^{-1}(e, c)$, then all the $A(e, c)'s$ are infinite.
   This says that $A(M)$ is an infinite cover of $E(M) \times C(M)$.

4. For each $e \in E(M)$, for every distinct $c, c' \in C(M)$, $f(e, c, c', y, z)$, induces a bijective map, denoted $f_{ecc'}$, from $A(e, c)$ to $A(e, c')$, such that $(f_{ecc'})^{-1} = f_{ec'c}$.

5. We must now say how the maps $f_{ecc'}$ behave with respect to composition, for fixed $e \in E(M)$. We want that if two products of the $f_{ecc'}$'s and their inverses (with same domain and range) are not formally equal modulo the relations
$$(f_{ecc'})^{-1} = f_{ec'c}$$
then they should differ on every element of the domain. This is an infinite scheme of axioms.

We will be using the following notation: if $c_1, c_2, \ldots, c_n$ is an n-tuple from $C(M)$, then we denote by $j(e, c_1, \ldots, c_n)$ the map from $A(e, c_n)$ to $A(e, c_1)$, $(f_{ec_2c_1} \circ \cdots \circ f_{ec_nc_{n-1}})$. Note that for any $n+1$-tuple $c_0, c_1, \ldots, c_n$ from C, then $j(e, c_0, c_1, \ldots, c_n, c_0)$ is a permutation of $A(e, c_0)$.

We leave it to the reader to check that this first theory T_0 is a consistent complete ω-stable theory, NDOP, shallow of depth 2. More precisely, we have two orthogonal types over the empty set, $E(x)$ and $C(x)$. If M is a model of T_0, if $e, c \in E(M) \times C(M)$, we have the type "$x \in A(e, c)$", over ec, which is orthogonal to c, but, by the existence of the maps $f_{ecc'}$'s, is not orthogonal to e.
Any two elements of $C(M)$ are independent if and only if they are not in the same connected component for the graph structure, the type $C(x)$ has U-rank equal to ω. We call *the C-dimension of M* the number of components in the graph structure on $C(M)$. The type $E(x)$ is trivial of U-rank one and its dimension is equal to its cardinality. For $c_0 \in C(M)$, the dimension of the type "$x \in A(e, c_0)$" (which is also trivial of rank one) is equal to the number of orbits in M modulo the action of the group generated by the permutations $j(e, c_0, c_1, \ldots, c_{n-1}, c_0)$.

We are going to extend the language and the theory in stages, and at the last stage we will explicitly construct a model of the final theory (see Claim 3.5).

We will now consider an extension of the language,

$$L^1 = L^0 \cup \{B\} \cup \{r_1(x)\} \cup \{p_i(x, y), q_i(x, y); i < \omega\} \cup \{\gamma_i(x, y, z, z'); i < \omega\}$$

where B is a new sort, r_1 is a map from B to A, $p_i, q_i \subset A \times B$ and $\gamma_i \subset A \times A \times B \times B$. Let M be an L^1-structure, model of T_0. We add the following conditions :

1. $B(M)$ is an infinite cover of $A(M)$, that is, r_1 is a surjection such that for each $a \in A(M)$, $r_1^{-1}(a)$, denoted $B(a)$, is infinite.

2. for each $a \in A(M)$, for every $i < \omega$, $p_i(a, M), q_i(a, M) \subset B(a)$ and the $p_i(a, M), q_i(a, M)$'s are all infinite and pairwise disjoint.

3. if $e \in E(M)$, $c, c' \in C(M)$ distinct, $a \in A(e, c)$ and $a' = f_{ecc'}(a)$, $\gamma_i(a, a', z, z')$ induces a bijective map, denoted $\gamma_{i,aa'}$, from $[(p_i(a, M) \cup q_i(a, M)]$ onto $[p_i(a', M) \cup q_i(a', M)]$.
For each $i < \omega$, either $\gamma_{i,aa'}$ maps $p_i(a, M)$ onto $p_i(a', M)$ and $q_i(a, M)$ onto $q_i(a', M)$, or $\gamma_{i,aa'}$ makes a switch, that is, it maps $p_i(a, M)$ onto $q_i(a', M)$ and $q_i(a, M)$ onto $p_i(a', M)$.
We require that $\gamma_{i,a'a} = \gamma_{i,a'a}$, and that ***!!!!! completeness

Now, for $a \in A(e, c)$, and $a' = f_{ecc'}(a)$, we will arrange that the choice of whether $\gamma_{i,aa'}$ makes the switch or not, depends exclusively on part of the type of (ecc') in M.
For this we need some more notation :
Let us fix some e in $E(M)$. For any distinct $c, c' \in C(M)$, let $d(c, c')$ be the distance between c and c' in the graph structure on $C(M)$ if c and c' are in the same component, and infinity otherwise. If $d(c, c') < \infty$, let $d_e(c, c')$ denote the e-distance between c and c', that is, the number of edges with label e on the path between between c and c' (recall that if c and c' are adjacent in $C(M)$, i.e. if $R(c, c')$, then the edge between c and c' has label e if $g_c(c') = g_{c'}(c) = e$). Now let $\delta_e(c, c')$ be equal to $2d(c, c') - d_e(c, c')$ if c and c' are in the same component. The following can be checked easily:

- If c and c' are distinct elements of $C(M)$ in the same component, then: $2\ d_e(c, c') \leq (d(c, c') + 1)$ (because $e^2 = 1$ or, equivalently, two adjacent edges must have different labels); $\delta_e(c, c') = \delta_e(c', c)$ and $\delta_e(c, c') \geq d(c, c') \geq 1$.
- For each $n \geq 1$, and for each $c \in C(M)$ there is some $c' \in C(M)$ such that $\delta_e(c, c') = n$.

We add the following condition (an infinite scheme of first-order sentences):

4. for every $e \in E(M)$, for all $c, c' \in C(M)$, for all $a \in A(e,c)$, if $a' = f_{ecc'}(a)$, then for $i < \omega$,

   (a) if $i+1 < d(c,c')$, then $\gamma_{i,aa'}$ maps $p_i(a,M)$ to $p_i(a',M)$ (i.e. there is no switch)

   (b) if $d(c,c')$ is finite, then

      - for $i < (\delta_e(c,c') - 1)$, $\gamma_{i,aa'}$ maps $p_i(a,M)$ to $p_i(a',M)$ (i.e. there is no switch)
      - for $i \geq (\delta_e(c,c') - 1)$, $\gamma_{i,aa'}$ maps $p_i(a,M)$ to $q_i(a',M)$ (i.e. there is a switch)

Let M be an L^1-structure, we say that M is a model of T_1 if it is a model of T_0 and satisfies the above conditions (1) to (4).
Note that, by 4.(a) if M is a model of T_1, and c, c' in $C(M)$ are not in the same component, for each $e \in E(M)$, each $a \in A(e,c)$, if $a' = f_{ecc'}(a)$, then, for each $i < \omega$, $\gamma_{i,aa'}$ does not make the switch.

We leave the following claim to the reader:

Claim 3.1 *The theory T_1 is complete, ω-stable, NDOP, shallow of depth 3. More precisely, if M is a model of T_1, if $(e,c) \in E(M) \times C(M)$, if $a \in A(e,c)$, then the types $p_i(a,x)$ and $q_i(a,x)$, defined over aec, are trivial types of U-rank one which are orthogonal to ec. Hence the type "$y \in A(e,c)$" has depth 1, and the type $E(x)$ has depth 2.*

From now on we will consider only models M of T_1 of C-dimension 1, that is such that $C(M)$ consists of only one R-component.

Definition and notation: For convenience, we denote by $\gamma_{a,a'}$ the union of the definable maps $\gamma_{i,aa'}$, for $i < \omega$. And rather than $\gamma_{aa'}$, we will consider the permutation it induces on the set of pairs $\{(p_i, q_i) : i < \omega\}$. So we attach to each map $\gamma_{aa'}$ an element of $((\mathbb{Z}/2\mathbb{Z})^\omega, +)$, $s[\gamma_{aa'}]$: if $\gamma_{aa'}$ maps $p_i(a,M)$ to $p_i(a',M)$, we let $s[\gamma_{aa'}](i) = 0$, and if $\gamma_{aa'}$ maps $p_i(a,M)$ to $q_i(a',M)$, we let $s[\gamma_{a,a'}](i) = 1$.

Condition (4) above now becomes:

- for $i < \delta_e(c,c') - 1$, $s[\gamma_{aa'}](i) = 0$
- for $i \geq \delta_e(c,c') - 1$, $s[\gamma_{aa'}](i) = 1$.

In fact, if $a_1, a_2 \in A(e,c)$, and $a'_1 = f_{ecc'}(a_1)$, $a'_2 = f_{ecc'}(a_2)$, then $s[\gamma_{a_1a'_1}] = s[\gamma_{a_2a'_2}]$, as the switching depends only on ecc' in M. Hence we can forget about the choice of a_1, a_2 and denote

$$s[\gamma_{a_1a'_1}] = s[\gamma_{a_2a'_2}] = s[e,c,c'](= s[e,c',c]).$$

Any composition $j(e, c_0, c_1, \ldots, c_n)$ will induce a map $\gamma_{aa'}$ between $B(a)$ and $B(a')$, for $a' = j(e, c_0, c_1, \ldots, c_n)(a)$. The corresponding permutation of the pairs (p_k, q_k), which we denote by $s[e, c_0, c_1, \ldots, c_n]$ is equal to $s[e, c_n, c_{n-1}] + \cdots + s[e, c_2, c_1] + s[e, c_1, c_0]$.

Now, for $e \in E(M)$, let $\Gamma(e)$ denote the subgroup of $(\mathbb{Z}/2\mathbb{Z})^\omega$ generated by $\{s[e,c,c'];$ for distinct $c, c' \in C(M)\}$. Fix any $c_0 \in C$, let $\Gamma_0(e, c_0)$ denote the subgroup of $\Gamma(e)$ generated by

$$\{s[e, c_0, c_1, \ldots, c_k, c_0];\ \text{for all } c_1, \ldots, c_k \in C(M), \quad \text{for all } k < \omega\}.$$

We also leave the checking of the following claim to the reader:

Claim 3.2 *Let M be a model of T_1 of C-dimension one.*

- *For each $e \in E(M)$,*

$$\Gamma(e) = \{s \in (\mathbb{Z}/2\mathbb{Z})^\omega; \exists n < \omega, \forall m, m \geq n, s(m) = s(n)\}.$$

- *For each $e \in E(M)$ and each $c_0 \in C(M)$, for any $c_1, c_2, \ldots, c_n \in C(M)$, $(s[e, c_1, c_2] + s[e, c_3, c_4] + \cdots + s[e, c_n, c_{n+1}]) \in \Gamma_0(e, c_0)$ iff $\sum_{k=1}^n \delta_e(c_k, c_{k+1})$ is even.*

- *For each $e \in E(M)$ and each $c_0 \in C(M)$, $\Gamma_0(e, c_0)$ is a subgroup of index 2 of $\Gamma(e)$.*

It follows in particular that $\Gamma(e)$ does not depend on e, that $\Gamma_0(e, c_0)$ does not depend on e, c_0, and that they are also independent of the choice of the model M, as long as M has C-dimension one. Hence from now on, we will use the notation Γ and Γ_0.

We will from now on restrict our attention to a certain class of models of T_1.
Definition: Let M be a model of T_1. We say that M is a **one-dimensional model of T_1 of cardinality** $\aleph_1$, if it satisfies the following conditions (a) to (e):

- (a) $E(M)$ has cardinality $\aleph_1$
- (b) M has C-dimension one, that is, all elements of $C(M)$ are in one R-component for the graph structure on $C(M)$
- (c) for every (e, c) in $E(M) \times C(M)$, $A(e, c)$ has dimension one, that is, consists of just one orbit modulo the action of the group generated by the permutations $\{j(e, c, c_1, \ldots, c_k, c), c_1, \ldots, c_k \in C(M)\}$
- (d) for every (e, c) in $E(M) \times C(M)$, for every $a \in A(e, c)$ and for each $i < \omega$, either
  - $p_i(a, M)$ has cardinality $\aleph_0$ and $q_i(a, M)$ has cardinality $\aleph_1$
  - or $p_i(a, M)$ has cardinality $\aleph_1$ and $q_i(a, M)$ has cardinality $\aleph_0$.
- (e) for every $a \in A(M)$, $B(a) = \bigcup_{i<\omega}(p_i(a, M) \cup q_i(a, M))$.

Condition (d) enables us to describe the induced action of the $s[e, c, c']$'s on the pairs (p_k, q_k) in a particularly convenient way : fix some $e \in E(M)$. By (d), for any $c \in C(M)$, we can associate to each element $a \in A(e, c)$ an element $v[a]$ in $(\mathbb{Z}/2\mathbb{Z})^\omega$ in the following way

$$v[a](n) = 0 \text{ iff the cardinality of } p_n(a, M) \text{ is } \aleph_0.$$

It is easy to check that if $a, a' \in \bigcup_{c \in C(M)} A(e, c)$, and $a' = j(e, c_1, \ldots, c_m)(a)$, then $v[a'] = v[a] + s[e, c_1, \ldots, c_m]$.
We can now add another condition on the class of models we want to consider:

- (f) for every $(e, c) \in E(M) \times C(M)$, for every $a \in A(e, c), v[a] \in \Gamma$ (i.e. $v(a)$ is eventually constant, by Claim 3.2).

Definition: We say that a model M of T_1 is a **Γ-model** if it is one-dimensional of cardinality $\aleph_1$, and satisfies condition (f).

Claim 3.3 *Let M be any one-dimensional model of T_1 of cardinality $\aleph_1$.*

- *Condition (f) is equivalent to : for each $e \in E(M)$, there is some $c_0 \in C(M)$ and some $a_0 \in A(e, c_0)$, such that $v[a_0] \in \Gamma$*

- *Let $v_1 \in \Gamma \setminus \Gamma_0$ and let $V(e,c)$ denote the following subset of $(\mathbb{Z}/2\mathbb{Z})^\omega$,*

$$V(e,c) = \{v[a] \ ; \ a \in A(e,c)\}.$$

Then $V(e,c) = \Gamma_0$ or $V(e,c) = v_1 + \Gamma_0$.
If $c, c' \in C(M)$, and $\delta_e(c,c')$ is odd, then

$$V(e,c) = \Gamma_0 \text{ iff } V(e,c') = v_1 + \Gamma_0.$$

If $\delta_e(c,c')$ is even (in particular if $d_e(c,c') = 0$), then $V(e,c) = V(e,c')$.

Proof of the Claim: For any other a' in $A(e,c_0)$, then $a' = j(e,c_0,c_1,\ldots,c_0)(a_0)$ by condition (c). So, as remarked above, $v[a'] = v[a_0] + s[e,c_0,c_1,\ldots,c_0]$, that is, $v[a']$ must be in $v[a_0] + \Gamma_0$. For any $a' \in A(e,c')$, with $c' \neq c_0$, then there is some $a'' \in A(e,c_0)$ such that $a' = f_{ec_0c'}(a'')$. This means that $v[a'] = v[a''] + s[e,c_0,c']$, hence $v[a'] \in \Gamma$. Then condition (f) above says that for all e,c, either $V(e,c) = \Gamma_0$ or $V(e,c) = v_1 + \Gamma_0$ for $v_1 \in \Gamma \setminus \Gamma_0$. The last statement is clear by Claim 3.2. □

3.1.2 The example with infinitely many sorts

Now we can finally get to the theory we want to consider. We consider a language L with infinitely many sorts :

$$E, (C_j)_{j<\omega}, (A_j)_{j<\omega}, (B_j)_{j<\omega}.$$

For each $j < \omega$, the restriction of our models to the sorts E, C_j, A_j, B_j is a model of T_1, and there is no other link between any of the sorts. We call this theory T_2.

Let us denote by Res_n the restriction of the language L to the sorts E, C_n, A_n, B_n and, if M is a model of T_2, $M(Res_n) = E(M) \cup C_n(M) \cup A_n(M) \cup B_n(M)$, in particular for each n, $M(Res_n)$ is an L^1-structure that satisfies T_1. It is straightforward to check the following:

Lemma 3.4 *Let M, N be models of T_2, let $e_1, \ldots, e_k, \in E(M)$, and for each n, let d_n be a finite tuple, $d_n \subset M(Res_n) \setminus E(M)$, $e'_1, \ldots, e'_k, \in E(N)$ and $d'_n \subset N(Res_n) \setminus E(N)$. Then*

$$(M, e_1, \ldots, e_k, d_0, \ldots, d_n, \ldots) \equiv (N, e'_1, \ldots, e'_k, d'_0, \ldots, d'_n, \ldots) \ (\textit{in } L)$$

if and only if, for every n,

$$(M(Res_n), e_1, \ldots, e_k, d_n) \equiv (N(Res_n), e'_1, \ldots, e'_k, d'_n) \ \ (\textit{in } L^1).$$

The theory T_2 is complete, ω-stable, NDOP and shallow of depth 3.

Definition: We say that a model M of T_2 is a **Γ-model** if, for each $j < \omega$, the restriction of M to the sorts E, C_j, A_j, B_j is a Γ-model of T_1.

For convenience we may use the notation $A(M)$ for the union of the $A_j(M)$'s or $B(M)$ for the union of the $B_j(M)$'s. We will also consider the set $B(a) = B_j(a)$, for $a \in A_j(M)$, or the set $A(e,c) = A_j(e,c)$ where $e \in E(M)$ and $c \in C_j(M)$, if there is no risk of ambiguity.

We can associate to each Γ-model M of T_2 a quasi-invariant in the following way :
for each $i < \omega$, for each couple $(e, c_i) \in E(M) \times C_i(M)$ let $\Delta(e, c_i) \in \{0, 1\}$ be defined in the following way ($V(e, c_i)$ is defined as in Claim 3.3):

- $\Delta(e, c_i) = 1$ if $V(e, c_i) = \Gamma_0$
- $\Delta(e, c_i) = 0$ if $V(e, c_i) \neq \Gamma_0$.

Now if, for each sort C_i, we fix an element $c_i \in C_i(M)$, then, to each element $e \in E(M)$, we can associate an element $\Delta(e) \in \mathbf{2}^\omega$ (depending on the chosen sequence $(c_i)_{i \in \omega}$) by setting : $\Delta(e)(i) = \Delta(e, c_i)$. Given e and $\Delta(e)$, we know what $V(e, c'_i)$ must be for any other $c'_i \in C_i(M)$, by Claim 3.3, hence we know $\Delta(e, c'_i)$.
Given a sequence $(c_i)_{i \in \omega}$ the above induces an associated map Δ from $E(M)$ into $\mathbf{2}^\omega$.

Claim 3.5 *Given any subset of $\mathbf{2}^\omega$ of cardinality $\aleph_1$, F, there is a Γ-model M of T_2, and a choice of sequence $(c_i)_{i \in \omega}$ in M, such that if Δ denotes the associated map, then $\Delta(E(M)) = F$.*

Proof of the Claim : Take E a set of cardinality $\aleph_1$, and any surjective map Δ from E onto F. Fix some $i \in \omega$.
Take C_i a set of cardinality $\aleph_1$ also, on which the free group on E, with relations $e^2 = 1$ for all e, acts regularly. Let A_i be again of cardinality $\aleph_1$. Choose r_i, a map from A_i onto $E \times C_i$ such that for all $(e, c) \in E \times C_i$, the $r_i^{-1}(e, c)$'s are of cardinality $\aleph_1$. As before denote $r_i^{-1}(e, c)$ by $A(e, c)$.

We must now construct the maps $f_{ecc'}$'s. Pick one element $c_0 \in C_i$. Fix some $e \in E$. For each $c_\alpha \neq c_0$ in C_i, pick a bijection between $A(e, c_0)$ and $A(e, c_\alpha)$, this will be $f_{ec_0c_\alpha}$. Let $f_{ec_\alpha c_0} = (f_{ec_0c_\alpha})^{-1}$. Now let G be the free group on $\aleph_1$ generators, G generated by $\{g_{\alpha\beta} : 0 < \alpha < \beta < \aleph_1\}$. Let G act regularly on the set $A(e, c_0)$, which so far has no structure. We decide that the permutation $j(e, c_0, c_\alpha, c_\beta, c_0)$ will act on $A(e, c_0)$ like $g_{\alpha\beta}$. This determines $f_{ec_\alpha c_\beta}$ for all $\alpha < \beta$,

$$f_{ec_\alpha c_\beta} = (f_{ec_0c_\beta} \circ g_{\alpha\beta} \circ f_{ec_\alpha c_0}).$$

It is fairly straightforward to check that these maps behave as they should. For each $a \in A$, let $B(a)$ again be a set of cardinality $\aleph_1$, $B(a)$ will be the union of pairs $(p_k(a, x), q_k(a, x))$, for $k < \omega$, such that for each k, exactly one of $p_k(a, x)$ or $q_k(a, x)$ will have cardinality $\aleph_1$ and the other, cardinality $\aleph_0$. Pick some v_0 in Γ_0, and some v_1 in $\Gamma \setminus \Gamma_0$. Fix some a in $A(e, c_0)$. Recall that we denoted by $v[a]$ the element of Γ associated to a in the following way:

$$v[a](n) = 0 \text{ iff the dimension of } p_n(a, x) \text{ is } \aleph_0.$$

If $\Delta(e)(i) = 0$, then let $v[a] = v_0$, if $\Delta(e)(i) = 1$, let $v[a] = v_1$. By using the above correspondence between $v[a]$ and the cardinalities of the $p_k(a, x)$'s, this determines completely the way $B(a)$ is constructed. For any other $a' \in A(e, c)$, for any $c \in C_i$, by Claim 3.3, $v[a']$ is determined, and hence also $B(a')$.
We follow the same construction for every $e \in E$, and then for every $i \in \omega$. It is clear that we get a model M of T_2, with all the right properties, Δ being the map associated to the sequence $(c_{0_i})_{i<\omega}$, $c_{0_i} \in C_i$. □

Claim 3.6

1. *If M and N are Γ-models of T_2, then M and N are isomorphic if and only if there are sequences $(c_i)_{i<\omega}$ in M, and $(d_i)_{i<\omega}$ in N, such that the corresponding maps Δ from $E(M)$ to $\mathbf{2}^\omega$, and Δ' from $E(N)$ to $\mathbf{2}^\omega$ commute in the following sense : there exists a bijection f from $E(M)$ to $E(N)$ such that $\Delta = (\Delta' \circ f)$.*

2. *If M is a Γ-model of T_2, $(c_i)_{i<\omega}$ and $(d_i)_{i<\omega}$ are sequences such that, for each $i < \omega$, $c_i, d_i \in C_i(M)$, then there exists at most countably many elements $e \in E(M)$ such that for some i, $\Delta(e, c_i) \neq \Delta(e, d_i)$.*

3. *If M and N are two isomorphic Γ-models of T_2, with associated maps Δ_M and Δ_N, then the symmetric difference between $\Delta_M(E(M))$ and $\Delta_N(E(N))$ is countable.*

Proof of the Claim : 1) One direction is obvious. For the converse, in order, given the bijection f, to construct the isomorphism, it suffices to note that as we have a regular action of the free group generated by $E(M)$ modulo $\{e^2 = 1; e \in E(M)\}$ on each $C_i(M)$, then, if one lets $f(c_i) = d_i$, there is a unique way to extend f to each $C_i(M)$ which respects the action. Similarly as $\Delta(e, f(c)) = \Delta(f(e), d)$, choose some $a \in A(e, c_i)$, and some $a' \in A(f(e), d_i)$ such that $v[a] = v[a']$, let $f(a) = a'$, there again is only one way to extend f to the whole of $A(e, c)$, and then to the other $A(e, c')$'s, by Claim 3.3, and this way works.
2) Fix an $i < \omega$. Recall that $C_i(M)$ consists of just one component. By Claim 3.3 again, if $\Delta(e, c_i) \neq \Delta(e, d_i)$, this means that there is an edge with label e on the path between c_i and d_i. This happens only for finitely (at most $d(c_i, d_i)$) many distinct $e \in E(M)$.
3) This is a direct consequence of 1) and 2). □

Now let us add one more (and last) restriction to our class of models. We want to consider only Γ-models, together with a choice of $(c_i)_{i\in\omega}$, such that the associated map Δ satisfies:

- (g) $\Delta(E)$ is an $\aleph_1$-dense subset of $\mathbf{2}^\omega$, that is, for every open subset O in $\mathbf{2}^\omega$, $O \cap \Delta(E)$ has cardinality $\aleph_1$.

Note that condition (g) holds of Γ-models independently of the choice of the sequence $(c_i)_{i\in\omega}$, by Claim 3.6.
Definition: We say that a Γ-model of T_2 is **good** if it satisfies condition (g).

We are going to prove that :

Proposition 3.7 *Any two good Γ-models of T_2 are $\equiv^\epsilon_\infty$-equivalent . There are $2^{\aleph_1}$ non-isomorphic good Γ-models of T_2.*

The second statement follows directly from Claims 3.5 and 3.6 and the fact that there are $2^{\aleph_1}$ $\aleph_1$-dense subsets of $\mathbf{2}^\omega$, of cardinality $\aleph_1$, which pairwise have uncountable symmetric difference.

It remains only to prove the $\equiv^\epsilon_\infty$-equivalence.

Claim 3.8 *1. For any* Γ*-model* M*, for any* $e_1, \ldots, e_n$ *in* $E(M)$*, for any* $i < \omega$*, for any choice of* $\epsilon_1, \ldots, \epsilon_n$ *in* $\{0, 1\}$*, there is* $c \in C_i(M)$ *such that, for all* k*,* $1 \leq k \leq n$*,* $\Delta(e_k, c_i) = \epsilon_k$*.*

2. Let $\Delta : E \to \mathbf{2}^n$*,* $\Delta' : E' \to \mathbf{2}^n$ *have the following properties:*

- E *and* E' *have cardinality* $\aleph_1$
- *for each* $s \in \mathbf{2}^n, \Delta(s)^{-1}$ *and* $\Delta'(s)^{-1}$ *have cardinality* $\aleph_1$*.*

Let $e_1, \ldots, e_k \in E$ *and* $e'_1, \ldots, e'_k \in E'$ *be such that for all* i*,* $1 \leq i \leq k$*,* $\Delta(e_i) = \Delta'(e'_i)$*.*
Then there is a bijection f *from* E *to* E' *such that*

$$\text{for all } i, 1 \leq i \leq k, f(e_i) = e'_i \text{ and } \Delta = (\Delta' \circ f).$$

Proof of the Claim : straightforward. □

Notation : For $n < \omega$, let L_n denote the restriction of our language with infinitely many sorts to the first n sorts, that is to

$$E, (C_i)_{i \leq n}, (A_i)_{i \leq n}, (B_i)_{i \leq n}.$$

If M is a model of T_2, let M_n denote the restriction of M to the language L_n.

Remark: If M and N are good Γ-models of T_2, then for all n, M_n and N_n are isomorphic: for each $i \leq n$, pick some $c_i \in C_i(M)$, and some $d_i \in C_i(N)$. Let Δ from $E(M)$ in $\mathbf{2}^n$ be defined by $\Delta(e)(i) = \Delta(e, c_i)$, and Δ' from $E(N)$ in $\mathbf{2}^n$ be defined by $\Delta'(e) = \Delta(e, d_i)$. By Claim 3.8, there is a bijection f from $E(M)$ onto $E(N)$ such that $\Delta = (\Delta' \circ f)$. Now exactly the same proof as in Claim 3.6, but restricted to M_n and N_n gives the isomorphism.
But having such isomorphisms is not enough a priori to conclude that M and N are $\equiv^{\epsilon}_{\infty}$. For this we need these isomorphisms to be "finitely compatible". This is exactly what the next proposition says.

Proposition 3.9 *Let* M*,* N *be good* Γ*-models of* T_2*. Let* $n < \omega$ *and let* $D \subset M_n$ *be a definably closed countable subset such that* $D \cap E(M)$ *is finite. Let* h *be a partial* L_n*-elementary isomorphism from* M_n *to* N_n *with domain* D *such that:*

- *if $e \in D \cap E(M)$, $c \in D \cap C_i(M)$, for some $i \leq n$, then $\Delta(e, c) = \Delta(h(e), h(c))$*

- *if $e \in D \cap E(M)$, $c \in D \cap C_i(M)$, for some $i \leq n$, and $a \in D \cap A(e, c)$, then $v[a] = v[h(a)]$.*

Then, for all $k \geq n$, h extends to a full isomorphism from M_k onto N_k.

Note that if h is the restriction to D of a full isomorphism between M_n and N_n, then certainly the two conditions in the above proposition are satisfied. Note also that it follows from the above proposition that, if h and D satisfy the assumptions, then D and $h(D)$ must have same type in M and N respectively, that is $(M, D) \equiv_0^\epsilon (N, h(D))$.

Proof of Proposition 3.9 : Note first that by the condition that D is definably closed, if $a \in A_i \cap D$, $a \in A_i(e, c)$, then $e \in D$ and $c \in D$. Similarly, if $b \in B_i \cap D$, $b \in B_i(a)$, then $a \in D$.
For each $i \leq k$ pick some $c_{i,0} \in C_i(M)$ and some $d_{i,0} \in C_i(N)$ in the following way:

1. if $D \cap C_i(M) \neq \emptyset$, pick any $c \in D \cap C_i(M)$, this will be $c_{i,0}$, and let $d_{i,0} = h(c_{i,0})$

2. if $D \cap C_i(M) = \emptyset$, pick any element $c_{i,0}$ in $C_i(M)$. As $D \cap E(M)$ is finite, by Claim 3.8, find $d_{i,0} \in C_i(N)$ such that for each $e \in D \cap E(M)$, $\Delta(e, c_{i,0}) = \Delta(h(e), d_{i,0})$.

Now, again by Claim 3.8, extend $h|_{E(M) \cap D}$ to a bijection f from $E(M)$ onto $E(N)$ such that

$$\text{for each } i \leq k,\ \Delta(e, c_{i,0}) = \Delta(f(e), d_{i,0}).$$

Now let $f|_D = h$, $f(c_{i,0}) = d_{i,0}$, for all $i \leq k$.
Note that as the action of the group generated by $E(M)$ is regular, there is a unique way to extend f to each $C_i(M)$ which respects this action. As D was supposed definably closed, all this is coherent.
Now for each $e \in E(M)$, each $i \leq k$, choose some a_i in $A(e, c_{i,0}) \cap D$ if there is one, any $a_i \in A(e, c_{i,0})$ if there is not. Let $f(a_i) = h(a_i)$ if $a_i \in D$. Otherwise by the condition that $\Delta(e, c_{i,0}) = \Delta(f(e), f(c_{i,0}))$, find some $a'_i \in A(f(e), f(c_{i,0}))$ such that $v[a_i] = v[a'_i]$, and let $f(a_i) = a'_i$.

For every $i \leq k$, by the existence of the $f_{ecc'}$'s there is a unique way to extend f in an elementary way (for the theory T_1) to $\bigcup_{e \in E(M), c \in C_i(M)} A(e, c)$.
By Claim 3.3, it will follow that for all $a \in A(e, c)$, $v[a] = v[f(a)]$. It follows that, for all a, for all $j < \omega$, $p_j(a, M)$ (respectively $q_j(a, M)$) has same cardinality as $p_j(f(a), N)$ (respectively $q_j(f(a), N)$). Hence it is easy to extend f to $B(a)$, in such a way that f coincides with h on $B(a) \cap D$ and is L_k-elementary. $\square$

Recall that a set D is said to be ϵ-closed if $D = acl^{eq}(D_0)$, for some finite D_0.
We now need to see how ϵ-closed sets behave in our models.

Claim 3.10 *Let M be a good Γ-model of T_2.*

1. *If $D \subset M^{eq}$ is ϵ-closed , then there is $n < \omega$ and $D_0 \subset M_n$, D_0 finite, such that $D \subset M_n^{eq}$, and in M_n^{eq}, $D = acl^{eq}(D_0)$.*

2. *$E(M)$ is an indiscernible set over $\emptyset$. If $D \subset M^{eq}$, is ϵ-closed , then $E_0 = D \cap E(M)$ is finite and $E(M) \setminus E_0$ is indiscernible over D.*

3. *If $D \subset M_n^{eq}$ is ϵ-closed, there is some $D_1 \subset M$, countable such that $D \subseteq dcl^{eq}(D_1)$, and $dcl(D_1) \cap E(M)$ is finite.*

Proof of the Claim : 1) This is clear by the way the theory T_2 is defined.
2) By Lemma 3.4, the type of a subset of $E(M)$ is determined by its respective types in the restrictions $M(Res_n)$. In each $M(Res_n)$, $E(M)$ is a strongly minimal trivial set where any two elements are independent. This remains true in the full model M, and the rest follows.
3) As D is ϵ-closed , $D = acl^{eq}(F_0)$ for some finite $F_0 \subset M_n$. Suppose to simplify notation that $n = 0$. Choose some $c_0 \in C_0(M) \cap acl(F_0)$ if there is one, any $c_0 \in C_0(M)$ otherwise. Let $D_0 = acl(F_0 c_0)$. We claim that $D \subseteq dcl^{eq}(D_0)$. By 2) $E_0 = E(M) \cap D_0$ is finite. Let z be an element of D. Then there is some finite tuple $y \in M_0$ such that $z \in dcl^{eq}(y)$. Consider the possible cases for y:
- $y \subset E(M)$, and $y \not\subset E_0$. Then $z \in dcl^{eq}(E_0 \cup \{e_1, \ldots, e_k\} \setminus dcl^{eq}(D_0))$, with $e_j \notin E_0$, for $1 \leq j \leq k$. This is impossible: by 2) $E(M) \setminus E_0$ is indiscernible over D_0, in particular any two k-tuples in $E(M)$ have the same type over $acl^{eq}(D_0)$ and $z \in acl^{eq}(D_0)$.

- $y \subset E(M) \cup C(M)$. For each $c \in C(M)$, there is a unique sequence $(e_1, \ldots, e_k)$ in $E(M)$, definable over $\{c_0, c\}$, such that $c = (e_1 \ldots e^k)c_0$. Hence $z \in dcl^{eq}(E_0 \cup E_1 \cup \{c_0\}) \setminus dcl^{eq}(D_0)$, for some $E_1 \subset E(M)$. By the previous case, $z \notin dcl^{eq}(E_0 \cup E_1)$. Consider, for each $e \in E_0 \cup E_1$, some element $a(e) \in A(e, c_0)$. Then $e, c_0 \in dcl(a(e))$, hence $z \in dcl^{eq}(D_0 \cup \{a(e); e \in E_0 \cup E_1\} \setminus dcl^{eq}(D_0)$. Again this is impossible: any two a, a' in $A(e, c_0)$ and not in $acl(D_0)$ have the same strong type over D_0, hence the same type over $acl^{eq}(D_0)$.

- the other cases are worked out similarly. $\square$

Claim 3.11 *Let M and N be two good Γ-models of T_2. For all ordinals α, if $A \subset M_n^{eq}$, $A' \subset N_n^{eq}$, A, A' ϵ-closed, are such that there is an L_n-isomorphism f from M_n^{eq} to N_n^{eq}, such that $f(A) = A'$, then $(M^{eq}, A) \equiv_\alpha^\epsilon (N^{eq}, A')$.*

Proof of the Claim : By Claim 3.10, there is some $A_1 \subset M_n$ countable such that $A \subseteq dcl^{eq}(A_1)$, A_1 is definably closed in M_n, and $A_1 \cap E(M)$ is finite. Let $A'_1 = f(A_1)$; then by Proposition 3.9, for every k, $f_{\restriction A_1}$ extends to an L_k-isomorphism from M_k onto N_k. It follows that for every k, $(M_k^{eq}, dcl^{eq}(A_1)) \equiv_0^\epsilon (N_k^{eq}, dcl^{eq}(A'_1))$ and hence that

$$(M^{eq}, A) \equiv_0^\epsilon (N^{eq}, A').$$

We now proceed by induction on α. The limit case is clear. Consider $\alpha + 1$. Let q be any ϵ-type over A, and d any realization of q. By Claim 3.10, there is some $m \geq n$ such that $acl^{eq}(Ad) \subset M_m^{eq}$. Consider f_m the isomorphism of L_m extending $f_{\restriction A}$. Consider $E = \{d_i; i \in I\}$ a maximal Morley sequence in M_m for q, and $E' = f_m(E)$. Then by induction, for every $i \in I$,

$$(M^{eq}, acl^{eq}(Ad_i) \equiv_\alpha^\epsilon (N^{eq}, (A' f_m(d_i)).$$

It is also clear by Prop. 3.9 that E and E' remain maximal Morley sequences in M^{eq} for q and $f_m(q)$ respectively. By definition of $\equiv_\alpha^\epsilon$, it follows that $(M^{eq}, A) \equiv_{\alpha+1}^\epsilon (N^{eq}, A')$. $\square$

Now in order to prove Proposition 3.7, just take any isomorphism from M_0 to N_0. By the previous claim, we have

$$(M^{eq}) \equiv_\infty^\epsilon (N^{eq}).$$

Remark: It is easy to see that the assumption that the models M and N are of C-dimension one is essential in what we did. It might in fact be true for all ω-stable theories T that if one restricts oneself to "infinite dimensional" models of T, these are isomorphic if and only if they are equivalent for $\equiv^{\epsilon}_{\infty}$, where a model M is said to be infinite dimensional if for all $A \subset M^{eq}$, A ϵ-closed, for all $q \in S(A)$, strongly regular, if q is realized in M, then it has infinite dimension in M.

3.2 A superstable $NOTOP$ nonmultidimensional theory

This second example is based on the same principle as the previous one, hence our account of the construction will be less detailed than the first one. The main difference with the previous example is that here some of the group actions present in the previous one become definable. As before, we begin by a description of the language and the basic axioms and then we construct the models we work with.

3.2.1 First language and axioms

We start with the same three sorted language L^0 as in the previous example:

$$L^0 = \{E, C, A, r_0, R, g, f\}$$

and consider L^0-structures M satisfying the first four conditions 1) to 4) listed at the beginning of the first section.

So we have an action on C of the free group on E with relations $\{e^2 = 1; e \in E(M)\}$, acting sharply transitively on orbits, in the language where, for $e \in E(M)$, $c, c' \in C(M)$,

$$c = ec' \text{ iff } R(c, c') \text{ and } g(e, c, c').$$

$A(M)$ is a cover of $E(M) \times C(M)$ via the map r_0, with infinite fibers, denoted $A(e, c)$.

For each $e \in E(M)$, for every distinct $c, c' \in C(M)$, we have a bijective map $f_{ecc'}$ from $A(e, c)$ to $A(e, c')$, with $(f_{ecc'})^{-1} = f_{ec'c}$. Before we say how the maps $f_{ecc'}$ behave with respect to composition, we need to introduce more structure.

We increase our language to

$$L^2 = L^0 \cup \{F, +_F, G, +_G, h, (i_n)_{n \geq 3}, t, (G_n)_{n<\omega}\}$$

where F and G are new sorts, $+_F \subset F^3$, $+_G \subset G^3$, h is a map from $E \times C^{[2]}$ into F ($C^{[2]}$ denotes the 2-elements subsets of C) , $i_n \subset E \times F \times G$, $t \subset E \times C \times G \times A \times A$ and for each n, $G_n \subset G$.

We add the following axioms :

5. $(F(M), +_F)$ is an Abelian group of exponent 2.

   We want $F(M)$ to contain the free Abelian group of exponent 2 generated by $E(M) \times C^{[2]}(M)$.
   We express this in the language via the map h, with an infinite scheme of axioms :

6. h is injective and $Im(h)$ is an independent subset in $F(M)$.

   Now, for $n \geq 3$, for $e \in E(M)$, let $F(n, e)$ denote the following definable subset of $F(M)$

   $$\{h(e, c_0, c_1) + h(e, c_1, c_2) + \cdots + h(e, c_{n-1}, c_n);$$

   $$c_0, \ldots, c_n \in C(M), c_0 = c_n, c_i \neq c_j \text{ for } 0 < i < j \leq n\}.$$

   For convenience we will denote by $F_\omega(M)$ the (non-definable) subgroup of $F(M)$ generated by $(\bigcup_{n\geq 3,\ e\in E} F(n, e))$.

7. $(G(M), +_G)$ is an Abelian group of exponent two.

8. For each $n \geq 3$, i_n induces a map denoted $i_{n,e}$ from $F(n, e)$ into $G(M)$ which is one-to-one and such that $\bigcup_{n\geq 3,\ e\in E} i_{n,e}$ induces a one-to-one group homomorphism from the subgroup $F_\omega(M)$ into $G(M)$.

9. for each $e \in E(M)$, for each $c \in C(M)$, t induces a definable regular action, denoted t_{ec}, of the group $G(M)$ on the set $A(e, c)$.

   We will denote the groups $F(M)$ and $G(M)$ additively and use the usual additive notation for the affine G-sets $A(e, c)$'s.
   Now we can say that:

10. for each $(e, c, c') \in E(M) \times C(M) \times C(M)$, the map $f_{ecc'}$ is an isomorphism of G-sets between $A(e, c)$ and $A(e, c')$.

Following the notation we used in the previous example, if $c_1, c_2, \ldots, c_n$ is an n-tuple from $C(M)$, we denote by $j(e, c_1, \ldots, c_n)$ the map from $A(e, c_n)$ to $A(e, c_1)$, $(f_{ec_2c_1} \circ \cdots \circ f_{ec_nc_{n-1}})$. For any n-tuple $c_0, c_1, \ldots, c_{n-1}$ from $C(M)$, then $j(e, c_0, c_1, \ldots, c_{n-1}, c_0)$ will be a permutation of $A(e, c_0)$. Now we link the action of these permutations to the action of $G(M)$. We say that :

11. for each $e \in E(M)$ and each $c_0, c_1, \ldots, c_{n-1} \in C(M)$, the permutation $j(e, c_0, c_1, \ldots, c_{n-1}, c_0)$ acts on $A(e, c_0)$ like the translation by the following element of $G(M)$:
$i_{n,e}(h(e, c_0, c_1) + h(e, c_1, c_2) + \cdots + h(e, c_{n-1}, c_0))$.

12. For each $n \geq 0$, $G_n(M)$ is a subgroup of $G(M)$ of index 2 in $G(M)$, such that the $G_n(M)$'s are independent, i.e. such that every finite boolean combination of the $G_n(M)$'s is non empty.
This gives us in $(L^2)^{eq}$ for each $n \geq 0$ the projection π_n from $G(M)$ onto $G/G_n(M)$. We will denote by π the map $(\pi_0, \pi_1, \ldots, \pi_n, \ldots)$ from $G(M)$ onto $\prod_{n \geq 0} G/G_n$ which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\omega$.

Now we work with the language L^2 together with the sorts G/G_n and the maps π_n, for $n \geq 0$.
We need to increase the language once more to L^3 , where we add , for each $n \geq 0$, $\gamma^n \subset E \times C \times A \times (G/G_n)$.
We add that :

13. for each $(e, c) \in E(M) \times C(M)$, γ^n induces a map denoted $\gamma^n_{e,c}$ from $A(e, c)$ onto $G/G_n(M)$ such that:
for all $a \in A(e, c)$, for all $g \in G(M)$, $\gamma^n_{e,c}(g + a) = \pi_n(g) + \gamma_{e,c}(a)$

Now it remains only to describe the way in which the maps $f_{ecc'}$ compose with the maps $\gamma^n_{e,c}$ and $\gamma^n_{e,c'}$.
As in Section 1, we make this dependent only on the type of (e, c, c') in the graph structure induced by E on C. We introduce the same notation as before : let us fix some e in E. For any $c, c' \in C$, let $d(c, c')$ denote the distance between c and c' in the graph structure on C, if c and c' are in the same component, and infinity otherwise. If $d(c, c') < \infty$, let $d_e(c, c')$ denote the e-distance between c and c', that is, the number of edges with label e on the path between between c and c'. Now let $\delta_e(c, c')$ be equal to

$2d(c,c') - d_e(c,c')$ if c and c' are in the same component.
We say that:

14. for all $e \in E(M)$, for all $c, c' \in C(M)$, for all $a \in A(e,c)$, if $a' = f_{ecc'}(a)$, then

    (a) if $i + 1 < d(c,c')$, then $\gamma^i_{e,c'}(a') = \gamma^i_{e,c}(a)$

    (b) if $d(c,c')$ is finite, then

    - for $i < (\delta_e(c,c') - 1)$, $\gamma^i_{e,c'}(a') = \gamma^i_{e,c}(a)$
    - for $i \geq (\delta_e(c,c') - 1)$, $\gamma^i_{e,c'}(a') = \gamma^i_{e,c}(a) + 1$.

Note that this implies that if c and c' are not in the same component, then $\gamma^i_{e,c'}(a') = \gamma^i_{e,c}(a)$ for all $i \geq 0$. We denote by $\gamma_{e,c}$ the map $(\gamma^0_{e,c}, \ldots, \gamma^n_{e,c}, \ldots)$ from $A(e,c)$ onto $(\mathbb{Z}/2\mathbb{Z})^\omega$.
For $e, c, c' \in E(M) \times C(M) \times C(M)$, with c, c' in the same component , we define $s[e,c,c'] \in (\mathbb{Z}/2\mathbb{Z})^\omega$, by :
- $s[e,c,c'](i) = 0$ if $i < (\delta_e(c,c') - 1)$
- $s[e,c,c'](i) = 1$ if $i \geq (\delta_e(c,c') - 1)$.
Then the conditions above, for $d(c,c')$ finite can be denoted as

$$\gamma_{e,c'}(f_{ecc'}(a)) = \gamma_{e,c}(a) + s[e,c,c'].$$

It follows that if $a, a' \in A(e, c_0)$, and $a' = j(e, c_0, c_1, ..., c_{n-1}, c_0)(a)$, then

$$\gamma_{e,c_0}(a') = \gamma_{e,c_0}(a_0) + (s[e,c_0,c_1] + s[e,c_1,c_2] + \cdots + s[e,c_{n-1},c_0]).$$

We say that an L^3-structure M is a model of T_0 if M satisfies all the above conditions 1) to 13).

We leave the actual construction of a model of T_0 until a little later (Prop. 3.15).
For the moment, we will check that T_0 is superstable NOTOP non multi-dimensional by describing the invariants which characterize models up to isomorphism.

3.2.2 Structure of models of T_0

Let M be any model of T_0.
Let $\mathrm{Dim}(C(M))$ denote the number of R-components of $C(M)$.

The group $F(M)$ is Abelian of exponent 2 and by the map h, contains a copy $\hat{F}(M)$ of the free Abelian group of exponent 2 generated by $E(M) \times C^{[2]}(M)$. Hence $F(M) = \hat{F}(M) \oplus F'(M)$, for some subgroup $F'(M)$ and, given the cardinalities of $E(M)$ and $C(M)$, the isomorphism type of $F(M)$ is given by the dimension of $F'(M)$.
The group $G(M)$ contains an isomorphic copy of $F_\omega(M)$, by the maps $i_{n,e}$, which we denote by $G_\omega(M)$, $G(M) = G_\omega(M) \oplus G'(M)$, for some subgroup $G'(M)$. The dimension of $G(M)$ is given by the dimension of $G'(M)$, given $F_\omega(M)$, which again is determined by the cardinalities of $E(M)$ and $C(M)$.

Claim 3.12

- *(i) Let us fix some $c_0 \in C(M)$, and for each $e \in E(M)$, some $a(e) \in A(e, c_0)$. Then for each $c \in C(M)$, and each $a \in A(e, c)$, $\gamma_{e,c}(a)$ is determined by $\pi(G)$, $d(c, c_0), d_e(c, c_0)$ and $\gamma_{e,c_0}(a(e))$.*

- *(ii) For each $g \in G_\omega$, for each $n \geq 0$, if $g = i_{k,e}(h(e, c'_0, c'_1) + \cdots + h(e, c'_{k-1}, c'_0))$, then $\pi_n(g) = s[e, c'_0, c'_1] + \cdots + s[e, c'_{k-1}, c'_0]$.*

Proof of the Claim : By the axioms we have given, if $a \in A(e, c_0)$, there is $g \in G$ such that $a = g + a(e)$ and we must have, for each n,

$$\gamma^n_{e,c_0}(g + a(e)) = \pi_n(g) + \gamma^n_{e,c_0}(a(e)).$$

Now if $a \in A(e, c)$ for some $c \in C, c \neq c_0$, there is some $a' \in A(e, c_0)$ such that $a = f_{ec_0c}(a') = f_{ec_0c}(g + a(e))$ for some $g \in G$. It follows that

$$\gamma^n_{e,c}(a) = \pi_n(g) + \gamma^n_{e,c_0}(a(e)) + s[e, c_0, c](n).$$

This proves (i).
If $g \in i_{k,e}(F(k, e))$, i.e. $g = i_{k,e}(h(e, c'_0, c'_1) + \cdots + h(e, c'_{k-1}, c'_0))$, then for any $a \in A(e, c'_0)$, we must have both :

$$\gamma^n_{e,c'_0}(g + a) = \gamma^n_{e,c'_0}(j(e, c'_0, \ldots, c'_{k-1}, c'_0)(a))$$

$$= \gamma^n_{e,c'_0}(a) + (s[e, c'_0, c'_1](n) + \cdots + s[e, c'_{k-1}, c'_0](n))$$

and

$$\gamma^n_{e,c'_0}(g + a)) = \pi_n(g) + \gamma^n_{e,c'_0}(a).$$

Hence $\pi(i_{k,e}(h(e, c'_0, c'_1) + \cdots + h(e, c'_{k-1}, c'_0))) = s[e, c'_0, c'_1] + \cdots + s[e, c'_{k-1}, c'_0]$.
This proves (ii). $\square$

Proposition 3.13 *Let M and N be two models of T_0. If $c_0 \in C(M), d_0 \in C(N)$, for each $e \in E(M)$, $a(e) \in A(e, c_0)$, for each $e' \in E(N)$, $a(e') \in A(e', d_0)$ are such that*

- $|E(M)| = |E(N)|$, *Dim($C(M)$)= Dim($C(N)$), Dim($F'(M)$) = Dim($F'(N)$),*
- $(G'(M), (G_n \cap G'(M))_{1<n<\omega} \cong (G'(N), (G_n \cap G'(N))_{1<n<\omega}$, *(as groups)*
- *for each* $s \in (\mathbb{Z}/2\mathbb{Z})^\omega$,
  $|\{e \in E(M); \gamma_{e,c_0}(a(e)) = s\}| = |\{e' \in E(N); \gamma_{e,d_0}(a(e)) = s\}|$,

then M and N are isomorphic.

Proof: This follows in a straightforward fashion from the previous claim. □

It follows that, in cardinality $\aleph_\alpha$, the number of non-isomorphic models of T_0 is bounded by $|\omega + \alpha|^{2^{2^\omega}}$.
The theory T_0 is superstable non-ω-stable and by the above bound on the number of models must be *NOTOP* non-multidimensional.

3.2.3 One-dimensional models of T_0

We now construct a certain type of model of T_0 which we are going to need. First some notation, as in Section 1: let M be a model of T_0 with C-dimension one.
Let Γ denote the subgroup of $(\mathbb{Z}/2\mathbb{Z})^\omega$ generated by $\{s[e, c, c'];$ for $c, c' \in C(M)\}$. Let Γ_0 denote the subgroup of Γ generated by

$$\{s[e, c_0, c_1]+s[e, c_1, c_2]+\cdots+s[e, c_k, c_0];\ \text{for all } c_0, c_1, \ldots, c_k \in C(M),\ k \geq 2\}.$$

As in Section 1 (Claim 3.2), these do not depend on e nor on M as long as $C(M)$ is one-dimensional, Γ_0 is a subgroup of index 2 of Γ, and, if $\delta_e(c, c')$= 1, then $s[e, c, c'] \notin \Gamma_0$. Note also that Γ_0 is dense in $(\mathbb{Z}/2\mathbb{Z})^\omega$.
By Claim 3.12, $\pi(G_\omega) = \Gamma_0$.

Definition 3.14 *Let M be a model of T_0. We say that M is one-dimensional if it satisfies the following:*

- *$E(M)$ has cardinality $\aleph_1$*

- *$C(M)$ has dimension one , i.e. is just one component*
- *$F(M)$ is the free Abelian group of exponent 2 on $E \times C^{[2]}$*
- *$G(M) = F_\omega$*
- *for each $e \in E(M)$, for each $c \in C(M)$, $\gamma_{e,c}(A(e,c)) = \Gamma_0$ or $\gamma_{e,c}(A(e,c)) = s_1 + \Gamma_0$, for $s_1 \in \Gamma \setminus \Gamma_0$.*

Proposition 3.15

- *The theory T_0 has one-dimensional models.*
- *If M and N are two one-dimensional models of T_0, then M and N are isomorphic iff there are $c_0 \in C(M)$ and $d_0 \in C(N)$ such that*
  $|\{e \in E(M); \gamma_{e,c_0}(A(e,c_0)) = \Gamma_0\}| = |\{e' \in E(N); \gamma_{e',d_0}(A(e',d_0)) = \Gamma_0\}|$
  and
  $|\{e \in E(M); \gamma_{e,c_0}(A(e,c_0)) = s_1+\Gamma_0\}| = |\{e' \in E(N); \gamma_{e',d_0}(A(e',d_0)) = s_1 + \Gamma_0\}|$.
- *If M is a one-dimensional model of T_0, if $c, c' \in C(M)$, then* $|\{e \in E(M); \gamma_{e,c}(A(e,c)) \neq \gamma_{e,c'}(A(e,c'))\}|$ *is finite.*

Proof : We will just prove the first statement, that is construct a one-dimensional model of T_0.
The second statement then follows by Claim 3.13, and the third one is straightforward.
Take E an infinite set of cardinality $\aleph_1$, C a set on which the free group on E with relations $\{e^2 = 1; e \in E\}$ acts regularly. Let $F = \hat{F}$ i.e. the free Abelian group of exponent 2 generated by $E \times C^{[2]}$. Take $G = F_\omega = G_\omega$, hence $G = \{(e, c_0, c_1) + \cdots + (e, c_{n-1}, c_0); e \in E, c_0, \ldots, c_{n-1} \in C\}$.
For $g \in G$, of the form above, let $\pi_i(g) = s[e, c_0, c_1](i) + \cdots + s[e, c_{n-1}, c_0](i)$ and let $G_i = ker(\pi_i)$. It is straightforward to check that:

Claim 3.16 *For each $i \geq 0$, $[G/G_i] = 2$, and the G_i's are independent.*

Now choose some $c_0 \in C$. For each $e \in E$, let G act regularly on $A(e, c_0)$, pick some $a(e) \in A(e, c_0)$, hence $A(e, c_0) = G + a(e)$. Let $\{c_i; i < \aleph_1\}$ be an enumeration of C. Choose for $f_{ec_0c_i}$ any bijection from $A(e, c_0)$ into $A(e, c_i)$.

Let G act on $A(e, c_i)$ by the action induced by this bijection.
For $i < j$, the permutation $j(e, c_0, c_i, c_j, c_0)$ must be the translation by $(e, c_0, c_i) + (e, c_i, c_j) + (e, c_j, c_0)$ in G. This determines the G-sets isomorphism $f_{ec_ic_j}$ for all $i < j$,

$$f_{ec_ic_j} = f_{ec_0c_j} \circ j(e, c_0, c_i, c_j, c_0) \circ f_{ec_ic_0}.$$

Now choose for each $e \in E$, $\gamma_{e,c_0}(a(e))$ in the group Γ. Then extend γ_{e,c_0} and define γ_{e,c_i} as required by the axioms in T_0 (Claim 3.12). It is straightforward to check that $M = (E, C, F, G)$ as described above is a model of T_0.

Let s_1 be some element of Γ not in Γ_0. The proof of the next claim is exactly similar to that of Claim3.3.

Claim 3.17

- *For any $e \in E(M)$, for any $c \in C(M)$, either $\gamma_{e,c}(A(e,c)) = \Gamma_0$ or $\gamma_{e,c}(A(e,c)) = s_1 + \Gamma_0$.*
- *If $c, c' \in C(M)$ and $\delta_e(c, c')$ is odd, then $\gamma_{e,c}(A(e,c)) = \Gamma_0$ iff $\gamma_{e,c'}(A(e,c')) = s_1 + \Gamma_0$.*
  If $\delta_e(c, c')$ is even, then $\gamma_{e,c}(A(e,c)) = \gamma_{e,c'}(A(e,c'))$.

This finishes the proof of the proposition. □

3.2.4 The example with infinitely many sorts

Now, again as in the first example, we pass to a language with infinitely many sorts:

$$E, (C_i)_{i<\omega}, (A_i)_{i<\omega}, (F_i)_{i<\omega}, (G_i)_{i<\omega}.$$

For every $i < \omega$ the restriction of our models to the sorts E, C_i, A_i, F_i, G_i is a model of T_0, and there is no other link between any of the sorts. This theory, which we denote T_1 is again superstable $NOTOP$ non multidimensional.

We say that a model M of T_1 is *one-dimensional* if its restriction to the sorts E, C_i, A_i, F_i, G_i is a *one-dimensional* model of T_0 for every $i < \omega$.

To each *one-dimensional* model M of T_0 and to each choice $(c_i)_{i<\omega}$ with $c_i \in C_i(M)$ for each i, we associate a map $\Delta[M, (c_i)_{i<\omega}]$ from $E(M)$ to $\mathbf{2}^\omega$ in the following way:

$$\Delta[M, (c_i)_{i<\omega}](e)(i) = 0 \text{ iff}$$

$$\gamma_{e,c_i}(a) \in \Gamma_0 \text{ for some (any) } a \in A_i(e, c_i).$$

Now the proof that T_1 satisfies the right properties with respect to the language $L_{\infty,\epsilon}(d.q.)$ is similar to the first example and we leave it to the reader.

Proposition 3.18

- *Let M, N be one-dimensional models of T_1 such that $\Delta(E(M))$ and $\Delta(E(N))$ are $\aleph_1$-dense in $\mathbf{2}^\omega$.*
  1. *If M and N are isomorphic, then the symmetric difference between $\Delta(E(M))$ and $\Delta(E(N))$ must be countable.*
  2. *The models M and N are $\equiv_\infty^\epsilon$-equivalent.*
- *Given any $\aleph_1$-dense subset of $\mathbf{2}^\omega$ of cardinality $\aleph_1$, D, there is a one-dimensional model M of T_1 such that if Δ denotes the associated map, then $\Delta(E(M)) = D$.*

This finishes the second counterexample.

4 The depth 2 case

The ω-stable example we constructed in the first section has depth 3. Here we show that this is minimal possible. Recall the definition of $\equiv_\infty^\epsilon$-equivalent from section 1.1

Proposition 4.1 *Let T be a countable ω-stable theory, NDOP, shallow of depth 2. Let M, N be two models of T. Then M and N are isomorphic if and only if they are $\equiv_\infty^\epsilon$-equivalent.*

If T has depth 1 (T is then said to be non-multidimensional), it is easy to check that the above proposition holds.

We always suppose that $T = T^{eq}$ and in particular types over algebraically closed sets are stationary. As usual, we are working inside a "monster" saturated model for T, such that all models we consider are elementary submodels of this monster model .
Notation : If $p \in S(A)$ is stationary, and if $C \subseteq A \subseteq B$, p^B denotes the

unique non-forking extension of p over B and $p_{\restriction C}$ the restriction of p to C.

We will assume that the reader is familiar with notions like strong regularity, orthogonality, dimension and basic consequences of NDOP which can be found for example in [Ba], [La 87] or [Sh 90].
Nevertheless we recall briefly the specific basic facts about strongly regular types and depth 2 theories which we will use constantly:

Facts 4.2 *Let T be any ω-stable theory.*

1. *Let M be a model of T, $A \subset M$, A algebraically closed, $C, C' \subset M$, such that $t(C/A) = t(C'/A)$. Let $q \in S(C)$ be a strongly regular type, and let q' denote the conjugate of q over C'.*
   If $q \not\perp A$, then $q \not\perp q'$ and $dim(q, M) = dim(q', M)$.

2. *Let $M \preceq N$ be models of T, $A \subset M$, $p \in S(A)$ strongly regular. Then if I is a Morley sequence for p in N, $I \underset{p(M)A}{\downarrow} M$, where $p(M)$ denotes the set of realizations of p in M.*
   Moreover, $dim(p, N) = dim(p, M) + dim(p^M, N)$.

Facts 4.3 *Let T be countable ω-stable, NDOP, shallow of depth 2.*

1. *If M is a model of T, and N is a prime model over $M\bar{a}$, if p is a strongly regular type over N, $p \perp M$, then $p \not\perp M\bar{a}$.*

2. *Let $M \preceq N$ be models of T, let $E \subset N$ be a maximal independent set of realizations of strongly regular types over M. For each $e \in E$, let $M(e) \preceq N$ be a prime model over Me and let B_e be a maximal independent set of realizations of strongly regular types over $M(e)$, each orthogonal to M.*
   Then N is prime over $(\bigcup_{e \in E} M(e)) \cup (\bigcup_{e \in E} B_e)$.

From now on T is a countable ω-stable theory.

We need to introduce some definitions:

Definition: Let $p \in S(C)$ be strongly regular.

1. We say that p is *persistently isolated* if p is isolated and for all finite D, p^{CD} remains isolated.

2. We say that $p \in S(C)$ is *good* if p is either persistently isolated or not isolated.

3. Let $B \subset A \subseteq C$, C atomic over A. We say that p is *good for* (A, B) if $p \not\perp A$, $p \perp B$ and p is good.

Lemma 4.4 *1). Let M be a model of T. Let $q \in S(M)$ be strongly regular. If $A \subset M$ is such that q does not fork over A and is stationary over A, then there exists some finite $F \subset M$ such that $q_{\restriction A \cup F}$ is good.*
2). Let $q \in S(A)$ be strongly regular, let M be prime over A, then q is persistently isolated iff the dimension of q in M is countably infinite.

Proof : 1). We can suppose that M is prime over A and that $q_{\restriction A}$ is isolated but not persistently isolated.
We show that in this case the dimension of $q_{\restriction A}$ in M is finite. We can then take F to be a basis (i.e. maximal independent set of realizations) for $q_{\restriction A}$ in M.
So suppose the dimension of $p = q_{\restriction A}$ is infinite in M. Let D finite in some extension of M be such that $p^D \in S(AD)$ is not isolated. Let N be a model prime over AD, without loss of generality $M \preceq N$. Let I be an infinite basis for p in M ; then by finite weight, there must be some element in I, e such that e and D are independent over A, but then e realizes p^D, contradicting the fact that p^D is not isolated over AD.
Note that this also gives the second statement of the lemma. □

Lemma 4.5 *Let A be countable, let N be a model of T containing A. Then there is a prime model over A, $M \preceq N$ satisfying the following condition: let $q \in S(M)$ be strongly regular not orthogonal to A and let B, $A \subseteq B \subseteq M$, $B \setminus A$ finite be such that q does not fork over B and $q_{\restriction B}$ is stationary. If the dimension of $q_{\restriction B}$ in N is infinite countable then M contains a basis for $q_{\restriction B}$, i.e. the dimension of q in N is zero.*

Proof : We first check that we can suppose that the model N is countable: consider all the strongly regular types over N which are not orthogonal to A, there are only countably many up to pairwise orthogonality. For each class, choose a representative q and some set D_q, $A \subseteq D_q \subseteq N$, $D_q \setminus A$ finite, such that q does not fork over D_q and $q_{\restriction D_q}$ is stationary. If the dimension of $q_{\restriction D_q}$ is countable in N, choose J_q a basis for it in N. Note that the countability of this dimension does not depend on the actual choice of D_q. Take a prime model N' over A, the D_q''s and the J_q's, $N' \preceq N$; N' is countable. Now suppose we have proved the lemma above for N' instead of N. Then the prime model $M \preceq N'$ satisfying the conditions will also work for N : if $p \in S(M)$ is not orthogonal to A, it must be not orthogonal to one of the chosen representatives q and then the dimensions of $p^{N'}$ and $q^{N'}$ in N must be equal.

Hence we suppose that N is countable. Consider M maximal atomic over A in N, and suppose it does not satisfy the conditions in the lemma. Then let e be a realization of q not in M, we will show that Me is still atomic over A, contradicting the maximality of M. Let $B \subset M$ satisfy the conditions in the lemma and let c be any finite subset of M. Then $q_{\restriction Bc}$ still has infinite countable dimension in N and is still persistently isolated. Let J be an infinite basis for $q_{\restriction Bc}$ in M by lemma 4.4. Then for any a in J, aBc is atomic over A. Now for almost all $a \in J$, a and e have the same type over Bc, because q is the average of J. Hence for all finite $c \subset M$, ec is isolated over A, i.e. Me is still atomic over A. □

Lemma 4.6 *If $B \subset A \subseteq C \subseteq D$, $p \in S(C)$ is strongly regular good, $D \setminus C$ finite and D is atomic over C, then :*

1. *$p^{acl(D)}$ is good*
2. *if p is good for (A,B), then $p^{acl(D)}$ is good for (A,B)*
3. *if p is not isolated , if M is a model of T, $M \supset acl(D)$, and if I is a Morley sequence for p in M, then I remains a Morley sequence for $p^{acl(D)}$.*
4. *if p is isolated, then $p^{acl(D)}$ is isolated, and if M is any model of T containing $acl(D)$, then $Dim(p^{acl(D)}; M)$ is infinite.*

Proof : 1) and 2) are clear : suppose that p is isolated, then by definition of good, $p^{acl(D)}$ is still persistently isolated. If p is non isolated, then $p^{acl(D)}$ remains not isolated by non-forking.
By non-forking also, it is clear that if p is good for (A, B), then $p^{acl(D)}$ remains good for (A, B).

3) Let $M_0 \preceq M$ be a model atomic over C, containing D. As p is not isolated, it is not realized in M_0. Let $I \subset M$ be a Morley sequence in p. Then by 4.2, I remains a Morley sequence for p^M, hence also for $p^{acl(D)}$.
4) As p is isolated, it is realized in M. Now let J be any finite Morley sequence for p, in M. As p is good, p^J is still isolated, hence realized in M, and J cannot be maximal. $\square$

Recall that we say that A is ϵ-closed if for some finite $A_0 \subseteq A$, $A = acl(A_0)$. For convenience, we define some equivalence relations $\equiv_1$ and $\equiv_2$ which isolate the parts of $\equiv^{\epsilon}_{\infty}$ relevant to depth two theories.

Definition: Let M, N be models of T

1. Let $D \subseteq M$, $D' \subseteq N$ range over enumerated ϵ-closed sets which are atomic (over $\emptyset$). Let $e \in M$, with $t(e/D)$ strongly regular, and $e' \in N$. We say that $\begin{pmatrix} e \\ D \end{pmatrix}, M \Big) \equiv_1 \Big(\begin{pmatrix} e' \\ D' \end{pmatrix}, N \Big)$ if

   - $t(acl(eD)) = t(acl(e'D'))$, that is, there is an elementary isomorphism f, from D to D', and an elementary isomorphism f', from $acl(eD)$ onto $acl(e'D')$, such that f' extends f and $f'(e) = e'$.
   - for all $C \subseteq M$, $C \supseteq acl(eD)$, C ϵ-closed, C atomic over $acl(eD)$, there is $C' \subseteq N$ such that :
     (a) $t(Cacl(eD)) = t(C'acl(e'D'))$, that is, there is an elementary isomorphism g from C onto C', $g \restriction acl(eD) = f'$
     (b) for every strongly regular $q \in S(C)$, q good for $(acl(eD), D)$, if $q' = g(q)$ is the conjugate of q over C', $dim(q, M) = dim(q', N)$.
   - for all $C' \subseteq N$, $C' \supseteq acl(e'D')$, C' ϵ-closed, C' atomic over $D'e'$, there is $C \subseteq M$ such that:
     (a) $t(Cacl(eD)) = t(C'acl(e'D'))$
     (b) for every strongly regular $q' \in S(C')$, q' good for $(acl(e'D'), D')$, if q denotes the conjugate of q' over C, $dim(q, M) = dim(q', N)$.

2. Let $C \subseteq M$ and $C' \subseteq N$ range over enumerated ϵ-closed and atomic (over $\emptyset$) sets.
   We say that $(C, M) \equiv_2 (C', N)$ if,

(a) $t(C) = t(C')$

(b) for all $D \supseteq C$, $D \subseteq M$, D ϵ-closed and atomic over C, there is $D' \subseteq N$, $D' \supseteq C'$, such that $t(DC) = t(D'C')$, and, for all $p \in S(D)$, p strongly regular, good, there is I_p in M, maximal Morley sequence for p, and $I_{p'}$ in N, maximal Morley sequence for the conjugate of p over C' and a one-to-one correspondence h between I_p and I'_p such that for all $e \in I_p$,

$$\begin{pmatrix} e \\ D \end{pmatrix}, M \Bigg) \equiv_1 \Bigg(\begin{matrix} h(e) \\ D' \end{matrix}, N \Bigg).$$

(c) for all $D' \supseteq C'$, $D' \subseteq N$, D' ϵ-closed and atomic over C'', there is $D \subseteq M$, $D \supseteq C$, such that $t(DC) = t(D'C')$, and, for all $p \in S(D)$, p strongly regular, good, there is I_p in M, maximal Morley sequence for p, and $I_{p'}$ in N, maximal Morley sequence for the conjugate of p over C' and a one-to-one correspondence h between I_p and I'_p such that for all $e \in I_p$,

$$\Bigg(\begin{matrix} e \\ D \end{matrix}, M \Bigg) \equiv_1 \Bigg(\begin{matrix} h(e) \\ D' \end{matrix}, N \Bigg).$$

The next lemma follows easily from the definition of $\equiv^{\epsilon}_{\infty}$ and Lemma 1.1.

Lemma 4.7 *Let M and N be two models of T. Suppose that $C \subseteq M$, C ϵ-closed and atomic, and $C' \subseteq N$, C' ϵ-closed and atomic.*
If $(M, C) \equiv^{\epsilon}_{\infty} (N, C')$, then $(M, C) \equiv_2 (N, C')$.

Proof : This follows easily from Lemma 1.1. □

Now we will establish some properties of these equivalence relations.

Proposition 4.8 *Let $D \subseteq M$ $e \in M$, with D ϵ-closed and atomic (over $\emptyset$) and $t(e/D)$ strongly regular, good. Let $D' \subseteq N$ and $e' \in N$.*

1. *If $\Bigg(\begin{matrix} e \\ D \end{matrix}, M \Bigg) \equiv_1 \Bigg(\begin{matrix} e' \\ D' \end{matrix}, N \Bigg)$, and $C \supset acl(eD)$, C ϵ-closed and atomic over $acl(eD)$, then, for all $C' \supset acl(e'D')$, with $t(C'acl(e'D')) = t(Cacl(eD))$, if $q \in S(C)$ is good and $\not\perp (acl(eD)$ then $dim(q, M) = dim(q', N)$, where q' is the conjugate of q over C'.*

2. Let $C \supset D$, C ϵ-closed and atomic, $e \underset{D}{\downarrow} C$, and $C' \supset D'$ with $t(C'acl(e'D')) = t(Cacl(eD))$. Then $\begin{pmatrix} e \\ C \end{pmatrix}, M \Big) \equiv_1 \Big(\begin{pmatrix} e' \\ C' \end{pmatrix}, N \Big)$ iff $\Big(\begin{pmatrix} e \\ D \end{pmatrix}, M \Big) \equiv_1 \Big(\begin{pmatrix} e' \\ D' \end{pmatrix}, N \Big)$

Proof : 1) This is immediate : by definition there is some C'' in N with $t(C''acl(e'D')) = t(Cacl(eD))$ and such that q and q'' the conjugate of q over C" have the same dimension. But $t(C'/acl(e'D')) = t(C''/acl(e'D'))$, $q'' \not\perp acl(D'e')$, hence q'' and q' have the same dimension (by 4.2).

2) Suppose first that $\Big(\begin{matrix} e \\ D \end{matrix}, M \Big) \equiv_1 \Big(\begin{matrix} e' \\ D' \end{matrix}, N \Big)$.

Let $F \supseteq acl(Ce)$ be ϵ-closed and atomic over $acl(Ce)$. Let $F' \subseteq N$ be such that $t(Facl(CDe)) = t(F'acl(C'D'e'))$. Let $q \in S(F)$ be good for $(acl(Ce), C)$, and let q' denote the conjugate of q over F'. We want to show that $Dim(q) = Dim(q')$. We claim that in fact q is good for $(acl(eD), D)$. By our assumption and 1), this is enough. Let us check the claim: as e and C are independent over D, and $q \perp C$, by NDOP, $q \not\perp acl(eD)$ and $q \perp D$ because $q \perp C$. It remains to show that F is atomic over $acl(eD)$: if $t(e/D)$ is isolated, then as $t(e/D)$ is good, $t(e/C)$ is still isolated, so as F is atomic over Ce, and C is atomic over D, then F is atomic over De. If $t(e/D)$ is not isolated, then we know that $t(e/D) \vdash t(e/C)$ by 4.2; hence C remains atomic over De and F is also atomic over De.

Suppose now that $\Big(\begin{matrix} e \\ C \end{matrix}, M \Big) \equiv_1 \Big(\begin{matrix} e' \\ C' \end{matrix}, N \Big)$.

Let $Z \supseteq acl(eD)$ be ϵ-closed and atomic over $acl(eD)$, and let $Z' \subseteq N$ be such that $(Zacl(eD)) = t(Z'acl(e'D'))$. We want to show that for every strongly regular type q over Z, good for $(acl(eD), D)$, if q' denotes the conjugate of q over Z', the dimension of q in M is equal to the dimension of q' in N.

Note first that we may assume without loss of generality that $acl(CZ)$ is atomic over $acl(eD)$: it is easy to see by the same arguments as above that $acl(Ce)$ itself is atomic over $acl(eD)$. Now let $Z_0 \subseteq M$ be such that $t(Z/acl(eD)) = t(Z_0/acl(eD))$ and $acl(CZ_0)$ is atomic over $acl(eD)$. Consider q_0 the conjugate of q over Z_0. Let $Z_0' \subset N$ be such that $t(Z_0acl(eC)) = t(Z_0'acl(e'C'))$ and let q_0' denote the conjugate of q_0 over Z_0'. Then, as $q \not\perp acl(eD)$, q and q_0 have same dimension in M, and similarly, q' and q_0' have same dimension in N.

Hence we assume that $acl(CZ)$ is atomic over $acl(eD)$, and as all sets are

algebraic closures of finite sets, also over $acl(Ce)$, and also over Z. By 1), we can also suppose that $t(Z'/acl(C'D'e')) = t(Z/acl(CDe))$.
Let $q_1 = q^{acl(CZ)}$, and $q_1' = q'^{acl(C'Z')}$. Then, q_1 is good for $(acl(Ce), C)$: as C and De are independent over D, and $q \not\perp acl(eD)$ but $q \perp D$, it follows that $q_1 \perp C$. On the other hand, it is clear by non-forking that $q_1 \not\perp acl(Ce)$.
So by our assumption, $dim(q_1, M) = dim(q_1', N)$. If q is not isolated, as $acl(ZC)$ is atomic over Z, $q \vdash q_1$, in particular, $dim(q, M) = dim(q_1, M) = dim(q_1', N) = dim(q', N)$. If q is isolated, then as q is good, $dim(q, M)$ is infinite. But as C is contained in the algebraic closure of a finite set, $dim(q, M) - dim(q_1, M)$ is finite, and similarly $dim(q', N) - dim(q_1', N)$ is finite. It follows that $dim(q, M) = dim(q', N)$. □

We are now going to prove Proposition 4.1.

From now on, $T = T^{eq}$ is a countable ω-stable theory, NDOP, shallow of depth 2.

Consider M and N, such that $(M, acl(\emptyset)) \equiv_\infty^\epsilon (N, acl(\emptyset))$. We are going to construct isomorphic maximal independent trees of models in M and N, with three levels. By depth 2, M and N will be prime over these trees, hence isomorphic (by Facts 4.3).
We leave it to the reader to check that the next two propositions (4.9 and 4.10) are sufficient to construct the isomorphic trees.

Proposition 4.9 *There are models $M_0 \preceq M$, $N_0 \preceq N$, an elementary isomorphism f_0 between M_0 and N_0 and a set of pairwise orthogonal strongly regular types over M_0, $R(M_0)$, such that*

- *M_0, N_0 are prime models (over $\emptyset$)*
- *for every strongly regular type q, if $q \not\perp M_0$, then there is some $p \in R(M_0)$ such that $p \not\perp q$*
- *for each $p \in R(M_0)$, there are $I_p \subset M$, maximal Morley sequence for p, $J_p \subset N$, maximal Morley sequence for the conjugate of p, $f_0(p)$, and a one-to-one correspondence h_p between I_p and J_p such that:*
  for every $e \in I_p$, for every sufficiently large $C \subseteq M_0$, ϵ-closed, such

that $e \underset{C}{\perp} M_0$ *and* $t(e/C)$ *is good, then*

$$\begin{pmatrix} e \\ C \end{pmatrix}, M \Big) \equiv_1 \begin{pmatrix} h_p(e) \\ f_0(C) \end{pmatrix}, N \Big).$$

Proof: We are going to construct $M_0, N_0, f_0, R(M_0)$ by induction, each step of induction will itself be broken into four substeps.
First we fix an enumeration $\{\phi_k(\bar{v}) : k < \omega\}$ of all formulas such that each formula is repeated infinitely many times. We also fix a bijection from ω to $\omega \times \omega$, denoted π.

For each $n \geq 0$, by this induction process, we will construct $A_n, B_n, g_n, R(A_n)$ such that

(1) A_n, B_n are atomic and ϵ-closed, $A_n \subset M, B_n \subset N$, g_n is an elementary isomorphism from A_n onto B_n; if $n \geq 1$, $A_{n-1} \subseteq A_n$, $g_{n-1} \subseteq g_n$.

(2) We also have for each $n \geq 1$, $d_n \subseteq A_n$ finite such that $A_n = acl(d_n)$, and for $n \geq 1$, $d_{n-1} \subseteq d_n$ and increasing enumerations of the finite sequences in the d_n's, $(s_k)_{k \in FS(d_n)}$, where $FS(d_n)$ denotes the cardinality of the set of ordered finite subsets of d_n. For $n = 0$, $d_0 = \emptyset$.

(3) $R(A_n)$ is a set of pairwise orthogonal representatives for all good strongly regular types over A_n, $R(A_{n-1}) \subseteq R(A_n)$ by which we mean that if $p \in R(A_{n-1})$, then $p^{A_n} \in R(A_n)$ (p^{A_n} is good also by 4.6).

(4) For each $p \in R(A_n)$, we have $I_{p,n}$ and $J_{p,n}$, Morley sequences respectively of p in M and of the conjugate of p in N over B_n, $I_{p,n+1} \subseteq I_{p,n}$ and $J_{p,n+1} \subseteq J_{p,n}$.

(5) If $p \in R(A_n)$, and $p \notin R(A_{n-1})$ then $I_{p,n}$ is a maximal Morley sequence for p in M over A_n, and $J_{p,n}$ is a maximal Morley sequence for the conjugate of p over B_n in N. We have also a one-to-one map, h_p, from $I_{p,n}$ onto $J_{p,n}$ such that, for all $e \in I_{p,n}$,

$$\begin{pmatrix} e \\ A_n \end{pmatrix}, M \Big) \equiv_1 \begin{pmatrix} h_p(e) \\ B_n \end{pmatrix}, N \Big).$$

(6) If $n \equiv 1, 0(4)$, for all $p \in R(A_n)$, h_p restricted to $I_{p,n}$ is one-to-one onto $J_{p,n}$.

(7) If $n \equiv 2(4)$, if $p \in R(A_n)$ and if $k_0 < n$ is the first integer such that $p \in R(A_{k_0})$, then, for all $e \in I_{p,k_0} \setminus I_{p,n}$, $e \underset{A_{k_0}}{\not\downarrow} A_n$.

(8) If $n \equiv 3(4)$, if $p \in R(A_n)$ and if $k_0 < n$ is the first integer such that $p \in R(A_{k_0})$, then, for all $e' \in J_{p,k_0} \setminus J_{p,n}$, $e' \underset{B_{k_0}}{\not\downarrow} B_n$.

(9) if $n = 4k+1$, if $\pi(k) = (k_1, k_2)$, if $M \models \exists x \phi_{k_1}(x, s_{k_2})$, where s_{k_2} is a finite sequence from d_{n-1}, then there is $a \in A_n$ such that $M \models \phi_{k_1}(a, s_{k_2})$ and such that every strong type extending $t(a/d_{n-1})$ is also realized in A_n.

Then we let $M_0 = \bigcup_{n<\omega} A_n$, $N_0 = \bigcup_{n<\omega} B_n$, $f_0 = \bigcup_{n<\omega} g_n$.
Let $R(M_0) = \{p^{M_0}; p \in \bigcup_{n<\omega} R(A_n)\}$.
For each $q \in R(M_0)$, let k be minimal such that q is based on A_k and $p = q_{\restriction A_k}$ is good, hence such that $p \in R(A_k)$. Then we let $I_q = \bigcap_{k \leq n < \omega} I_{p,n}$, $J_q = \bigcap_{k \leq n < \omega} J_{p,n}$.
We must now check that the conditions in the proposition are satisfied :

- $M_0 \preceq M$ by (9), $N_0 \preceq N$ because by (1) f_0 is an elementary isomorphism.

- M_0 and N_0 are prime models over $\emptyset$ by (1).

- Let q be a strongly regular type over M_0. Then there is some m such that q does not fork over A_m, and $q_{\restriction A_m}$ is good. Then by definition of $R(A_m)$ in (3), there is some $p \in R(A_m)$ such that $q \not\perp p^{M_0}$.

- By (6), for $p \in R(M_0)$, h_p induces a one-to-one correspondence between I_p and J_p. Furthermore, if k_0 is the first integer such that p does not fork over A_{k_0} and $p_{\restriction A_{k_0}}$ is good, then by (5), for all $e \in I_p$,

$$\begin{pmatrix} e \\ A_{k_0} \end{pmatrix}, M \Bigg) \equiv_1 \begin{pmatrix} h_p(e) \\ B_{k_0} \end{pmatrix}, N \Bigg).$$

It follows by Prop.4.8 that for all ϵ-closed C, $A_{k_0} \subseteq C \subseteq M_0$, we have that

$$\left(\begin{matrix} e \\ C \end{matrix}, M \right) \equiv_1 \left(\begin{matrix} h_p(e) \\ f_0(C) \end{matrix}, N \right).$$

- For $p \in R(M_0)$, I_p is a Morley sequence in p, and J_p is a Morley sequence in the conjugate of p, $f_0(p)$, by (4). Furthermore, I_p and J_p must be maximal : if not let for example $a \in M$ be such that $I_p a$ is still a Morley sequence in p. But by (5), for some k_0, (the first such that p does not fork over A_{k_0}, and $p_{\restriction A_{k_0}}$ is good), I_{p,k_0} is maximal independent in M over A_{k_0}.
  Hence we have that $aI_p \not\downarrow_{A_{k_0}} (I_{p,k_0} \setminus I_p)$. But it follows from (7) that, for each $e \in I_{p,k_0} \setminus I_p$, $e \not\downarrow_{A_{k_0}} M_0$. Hence by strong regularity, as each element in $I_{p,k_0} \setminus I_p$ realizes a forking extension of the restriction of p to A_{k_0}, aI_p and $I_{p,k_0} \setminus I_p$ are independent over M_0, hence also independent over A_{k_0}, contradiction. Similarly, by (8), J_p must be maximal.

We can now begin the induction. Technical difficulties arise when we try to fulfill conditions (6), (7) and (8) and force us to construct some auxiliary subsets. In addition to the above conditions (1) to (9), we will have the following conditions which enable us to proceed with the induction:

(1*) $(M, A_n) \equiv^{\epsilon}_{\infty} (N, B_n)$.

(2*) For each n, for each $p \in R(A_n)$, if k_0 is the first integer such that p appears in $R(A_{k_0})$, we have a finite set $C(p,n) \subseteq I_{p,k_0} \setminus I_{p,n}$ such that $I_{p,n} \cup C(p,n)$ is a Morley sequence for p over A_n in M.

(3*) For each n, for all $p \in R(A_n)$ except a finite number, $C(p,n) = \emptyset$. If $C(p,n) \neq \emptyset$, then p must be isolated.

(4*) If $n \equiv 2, 3(4)$, then for all $p \in R(A_n)$, $C(p,n) = \emptyset$.

(5*) For each n, for each $p \in R(A_n)$, if k_0 is the first integer such that p appears in $R(A_{k_0})$, we have a finite set $D(p,n) \subseteq J_{p,k_0} \setminus J_{p,n}$ such that $J_{p,n} \cup D(p,n)$ is a Morley sequence for the conjugate of p over B_n in N.

(6*) For each n, for all $p \in R(A_n)$ except a finite number, $D(p,n) = \emptyset$. If $D(p,n) \neq \emptyset$, then p must be isolated.

(7*) If $n \equiv 3(4)$, then for all $p \in R(A_n)$, $D(p,n) = \emptyset$.

Case $n = 0$
Let $A_0 = acl(\emptyset)$ in M, and $B_0 = acl(\emptyset)$ in N. By our hypothesis, we have that

$(M, A_0) \equiv^{\epsilon}_{\infty} (N, B_0)$. Let g_0 be the corresponding elementary isomorphism. Let $R(A_0)$ be a set of pairwise orthogonal representatives for all strongly regular good types over A_0.
By 4.7, for each $p \in R(A_0)$ there are $I_{p,0}$ maximal Morley sequence for p in M and $J_{p,0}$, maximal Morley sequence in N for $g_0(p)$ the conjugate of p over B_0, and a one-to-one correspondence h_p between $I_{p,0}$ and $J_{p,0}$ such that for all $e \in I_{p,0}$,

$$\begin{pmatrix} e \\ A_0 \end{pmatrix}, M \Big) \equiv_1 \begin{pmatrix} h_p(e) \\ B_0 \end{pmatrix}, N \Big).$$

Let $C(p, 0) = D(p, 0) = \emptyset$.

Case $n = 4k + 1$, with $\pi(k) = (k_1, k_2)$
If $M \models \phi_{k_1}(m, s_{k_2})$ for some m, with $s_{k_2} \subseteq d_{n-1}$, choose one such m with isolated type over d_{n-1}. Then there are a finite number of strong types extending $t(m/d_{n-1})$, i.e. a finite number of (isolated) extensions over $A_{n-1} = acl(d_{n-1})$. Find $d \subset M$, finite, which contains a realization of each of these strong types over A_{n-1} and such that d is atomic over A_{n-1}. Let $A_n = acl(A_{n-1}d)$, then A_n is atomic over $\emptyset$ and also over A_{n-1}. By our induction assumptions, $(M, A_{n-1}) \equiv^{\epsilon}_{\infty} (N, B_{n-1})$, hence by Lemma 1.1 we can find $B_n \subseteq N$, such that $(M, A_n) \equiv^{\epsilon}_{\infty} (N, B_n)$.
Now define $R(A_n)$ to be the union of the set $\{p^{A_n}; p \in R(A_{n-1})\}$ and of a maximal set of pairwise orthogonal new strongly regular good types over A_n.
If $p \in R(A_n) \setminus R(A_{n-1})$, let $I_{p,n}$ be a maximal independent set of realizations of p in M. Then by 4.7, there is $J_{p,n}$, maximal independent set of realizations in N for the conjugate p' of p over B_n, and a one-to-one correspondence h_p such that :

$$\begin{pmatrix} e \\ A_n \end{pmatrix}, M \Big) \equiv_1 \begin{pmatrix} h_p(e) \\ B_n \end{pmatrix}, N \Big).$$

Let also $C(p, n) = D(p, n) = \emptyset$.
For all $p \in R(A_{n-1})$, except a finite number, $I_{p,n-1} \cup C(p, n-1)$ remains independent over A_n, and $J_{p,n-1} \cup D(p, n-1)$ also remains independent over B_n. In this case, let $I_{p,n} = I_{p,n-1}$ and $J_{p,n} = J_{p,n-1}$, and $C(p, n) = C(p, n-1)$, $D(p, n) = D(p, n-1)$.
For a finite number of $p's$ from $R(A_{n-1})$, $I_{p,n-1} \cup C(p, n-1)$ is no longer independent over A_n, or $J_{p,n-1} \cup D(p, n-1)$ is no longer independent over B_n. Note that such a p must be isolated over A_{n-1}, by 4.6. In this case, let $F(p, n)$ be a minimal finite subset of $I_{p,n-1} \cup C(p, n-1)$ such that $(I_{p,n-1} \cup C(p, n -$

$1))\setminus F(p,n)$ is independent over A_n, and let $G(p,n)$ be a minimal finite subset of $J_{p,n-1}\cup D(p,n-1)$ such that $(J_{p,n-1}\cup D(p,n-1))\setminus G(p,n)$ is independent over B_n. Let also $C_0(p,n)$ be a minimal subset in $(I_{p,n-1}\cup C(p,n-1))\setminus F(p,n)$ such that for all $e\in F(p,n)$, $e \underset{A_{n-1}}{\not\downarrow} C_0(p,n)A_n$. It follows by our induction assumption that, if k_0 is the first integer such that $p\in R(A_{k_0})$, then for all $e\in I_{p,k_0}\setminus I_{p,n-1}$, $e \underset{A_{k_0}}{\not\downarrow} C_0(p,n)A_n$.

Similarly, let $D_0(p,n)$ be a minimal finite subset of $(J_{p,n-1}\cup D(p,n-1))\setminus G(p,n)$ such that that for all $e'\in G(p,n)$, $e' \underset{B_{n-1}}{\not\downarrow} D_0(p,n)B_n$. Again it follows that for all $e'\in J_{p,k_0}\setminus J_{p,n-1}$, $e' \underset{B_{k_0}}{\not\downarrow} D_0(p,n)B_n$.

Now let $C(p,n)$ and $D(p,n)$ be finite sets such that :

- $C(p,n)\subseteq (I_{p,n-1}\cup C(p,n-1))\setminus F(p,n)$
- for every $e\in(C_0(p,n)\cup C(p,n-1))$, if $e\notin F(p,n)$, then $e\in C(p,n)$
- $D(p,n)\subseteq (J_{p,n-1}\cup D(p,n_1))\setminus G(p,n)$
- for every $e'\in(D_0(p,n)\cup D(p,n-1))$, if $e'\notin G(p,n)$, then $e'\in D(p,n)$
- h_p induces a one-to-one correspondence between $(I_{p,n-1}\cup C(p,n-1))\setminus(F(p,n)\cup C(p,n))$ and $(J_{p,n-1}\cup D(p,n-1))\setminus(G(p,n)\cup D(p,n))$.

Now let

$$I_{p,n}=(I_{p,n-1}\cup C(p,n-1))\setminus(F(p,n)\cup C(p,n))\ \subseteq I_{p,n-1}$$

and

$$J_{p,n}=(J_{p,n-1}\cup D(p,n-1))\setminus(G(p,n)\cup D(p,n))\ \subseteq J_{p,n-1}.$$

Note that now, for all $p\in R(A_n)$ we have :

- $I_{p,n}\cup C(p,n)$ is an independent set of realizations of p over A_n
- $J_{p,n}\cup D(p,n)$ is an independent set of realizations of the conjugate of p over B_n
- if $C(p,n)\neq\emptyset$ or $D(p,n)\neq\emptyset$, then p is isolated over A_n.
- for all p except a finite number, $C(p,n)$ and $D(p,n)$ are both empty.

- h_p induces a one-to-one correspondence between $I_{p,n}$ and $J_{p,n}$.
- if k_0 is the first integer such that $p \in R(A_{k_0})$, then for all $e \in I_{p,k_0} \setminus I_{p,n}$, $e \underset{A_{k_0}}{\not\downarrow} C(p,n)A_n$. Similarly, for all $e' \in J_{p,k_0} \setminus J_{p,n}$, $e' \underset{B_{k_0}}{\not\downarrow} D(p,n)B_n$.

Now define g_n and d_n and the enumeration of finite sequences of d_n in the obvious way.

Case $n = 4k+2$

Let $D = \bigcup C(p, n-1)$, for all $p \in R(A_{n-1})$; D must be finite. Let $A_n = acl(A_{n-1}D)$. Then because D consists of independent realizations of good isolated types in $R(A_{n-1})$, and D comes from some previous $I_{p,m}$'s, we must have that :

- $I_{p,n-1}$ is a Morley sequence in p^{A_n}
- if k_0 is the first integer m such that $p \in R(A_m)$, then for all $e \in I_{p,k_0} \setminus I_{p,n-1}$, $e \underset{A_{k_0}}{\not\downarrow} A_n$.
- A_n is atomic over A_{n-1} , hence over $\emptyset$.

By 1.1 we can find $B_n \subseteq N$ such that $(A_n, M) \equiv^{\epsilon}_{\infty} (B_n, N)$. Let NO_n be a maximal set of new pairwise orthogonal good strongly regular types over A_n. Let

$$R(A_n) = \{p^{A_n}; p \in R(A_{n-1}) \ \cup \ NO_n.$$

If $p \in R(A_n)$, p already appearing in $R(A_{n-1})$, let $I_{p,n} = I_{p,n-1}$ and let $C(p,n) = \emptyset$. Let $G(p,n)$ be finite minimal such that $(J_{p,n-1} \cup D(p,n-1)) \setminus G(p,n)$ remains independent over B_n. Note that if $G(p,n)$ is not empty, p must be isolated, and also that for all p except a finite number , $G(p,n)$ is empty.

Let $J_{p,n} = J_{p,n-1} \setminus (G(p,n) \cap J_{p,n-1})$. Let $D(p,n) = D(p,n-1) \setminus (G(p,n) \cap D(p,n-1))$. Note that $J_{p,n} \cup D(p,n)$ is an independent set of realizations of the conjugate of p over B_n. Note also that at this stage, h_p does not induce a one-to-one correspondence anymore between $I_{p,n}$ and $J_{p,n}$, for the finite number of p's such that $G(p,n)$ is not empty. If $p \in R(A_n)$ did not appear in $R(A_{n-1})$, define $I_{p,n}$, $J_{p,n}$, h_p a one-to-one correspondence, given by the fact that $(A_n, M) \equiv^{\epsilon}_{\infty} (B_n, N)$, such that for all $e \in I_{p,n}$,

$$\left(\begin{matrix} e \\ A_n \end{matrix} \ , \ M \right) \equiv_1 \left(\begin{matrix} h_p(e) \\ B_n \end{matrix} \ , \ N \right).$$

Let in this case $C(p,n) = D(p,n) = \emptyset$. Now define g_n, d_n and the numeration $(s_k)_{k \in FS(d_n)}$ in the obvious way.

Case $n = 4k+3$.
For $p \in R(A_{n-1})$, there are two cases :
- $G(p,n-1) \neq \emptyset$. In this case, let $D_0(p,n) \subset (J_{p,n-1} \cup D(p,n-1))$ be finite minimal containing $D(p,n-1)$ such that for all $e' \in G(p,n-1)$, $e' \underset{B_{n-2}}{\not\downarrow} B_{n-1}D_0(p,n)$.
- $G(p,n-1) = \emptyset$. In this case, let $D_0(p,n) = D(p,n-1)$.
Let $E = \bigcup D_0(p,n)$, for all $p \in R(A_{n-1})$. Then as above, E is finite and $B_{n-1} \cup E$ is atomic over B_{n-1}. Let $B_n = acl(B_{n-1}E)$ and let $J_{p,n} = J_{p,n-1} \setminus (D_0(p,n) \cap J_{p,n-1})$.
Then we have the following :

- for all $p \in R(A_{n-1})$, $J_{p,n}$ is a Morley sequence in N for the conjugate of p over B_n
- for all $p \in R(A_{n-1})$, if k_0 is the first integer such that p appears in $R(A_{k_0})$, for all $e' \in J_{p,k_0} \setminus J_{p,n}$, $e' \underset{B_{k_0}}{\not\downarrow} B_n$

Now let $A_n \subseteq M$ be such that $(A_n, M) \equiv_2 (B_n, N)$. For $p \in R(A_{n-1})$, let $F(p,n)$ be finite minimal such that $I_{p,n-1} \setminus F(p,n)$ remains independent over A_n. Again note that if $F(p,n) \neq \emptyset$, p must be isolated , and that for all p except a finite number, $F(p,n) = \emptyset$. Let $I_{p,n} = I_{p,n-1} \setminus F(p,n)$. Now as above, define $R(A_n) = \{p^{A_n}; p \in R(A_{n-1})\} \cup$
{a maximal set of pairwise orthogonal new good strongly regular types over A_n }.
For these new types in $R(A_n) \setminus R(A_{n-1})$, choose $I_{p,n}$, $J_{p,n}$, and the one-to-one correspondence h_p as usual. Now for all $p \in R(A_n)$, let $C(p,n) = D(p,n) = \emptyset$. Again define, d_n, g_n and $(s_k)_{k \in FS(d_n)}$, in the obvious way.

Case $n = 4k+4$.
For $p \in R(A_{n-1})$, there are two cases :
- $F(p,n-1) \neq \emptyset$. In this case, let $C_0(p,n) \subseteq I_{p,n-1}$ be finite minimal such that for all $e \in F(p,n-1)$, $e \underset{A_{n-2}}{\not\downarrow} A_{n-1}C_0(p,n)$.
- $F(p,n-1) = \emptyset$. In this case, let $C_0(p,n) = \emptyset$.
Now let $C(p,n)$ and $D(p,n)$ be finite sets such that :

- $C(p,n) \subseteq I_{p,n-1}$, $C_0(p,n) \subseteq C(p,n)$
- $D(p,n) \subseteq J_{p,n-1}$
- h_p induces a one-to-one correspondence between $I_{p,n-1} \setminus C(p,n)$ and $J_{p,n-1} \setminus D(p,n)$

Let $I_{p,n} = I_{p,n-1} \setminus C(p,n)$ and $J_{p,n} = J_{p,n-1} \setminus D(p,n)$, $A_n = A_{n-1}$ and $B_n = B_{n-1}$. Note that the following hold :

- for all $p \in R(A_n)$, $I_{p,n} \cup C(p,n)$ is a Morley sequence in M for p over A_n
- for all $p \in R(A_n)$, $J_{p,n} \cup D(p,n)$ is a Morley sequence in N for the conjugate of p over B_n
- for all $p \in R(A_n)$, if k_0 is the first integer such that p appears in $R(A_{k_0})$, for all $e \in I_{p,k_0} \setminus I_{p,n}$, $e \underset{A_{k_0}}{\not\perp} A_n C(p,n)$
- for all $p \in R(A_n)$, if k_0 is the first integer such that p appears in $R(A_{k_0})$, for all $e' \in J_{p,k_0} \setminus J_{p,n}$, $e' \underset{B_{k_0}}{\not\perp} B_n D(p,n)$
- for all $p \in R(A_n)$ except a finite number, $C(p,n) = D(p,n) = \emptyset$
- if $C(p,n) \neq \emptyset$, or $D(p,n) \neq \emptyset$, p is isolated
- for each $p \in R(A_n)$, h_p induces a one-to-one correspondence between $I_{p,n}$ and $J_{p,n}$.

We let of course $g_n = g_{n-1}$ and $d_n = d_{n-1}$, and this finishes the induction step. $\square$

The second (and final) step of the construction is much shorter:

Proposition 4.10 *Suppose we have $M_0 \preceq M$, $N_0 \preceq N$, M_0 and N_0 both prime over $\emptyset$, and an elementary isomorphism f_0 between M_0 and N_0. Let $e \in M$ realize a strongly regular type over M_0, let $e' \in N$ realize the conjugate by f_0 of $t(e/M_0)$ over N_0. Suppose that for all sufficiently large ϵ-closed $D \subseteq M_0$ such that $e \underset{D}{\perp} M_0$ and $t(e/D)$ is good, we have $\begin{pmatrix} e \\ D \end{pmatrix}, M \Bigg) \equiv_1 \Bigg(\begin{pmatrix} e' \\ f_0(D) \end{pmatrix}, N$. Then there are $M_1 \preceq M$, $N_1 \preceq N$ and an elementary isomorphism f_1 from M_1 onto N_1, such that*

- *M_1 is prime over M_0e*
- *f_1 extends f_0, $f_1(e) = e'$, hence N_1 is prime over N_0e'*
- *for all $q \in S(M_1)$, strongly regular, such that $q \perp M_0$, $dim(q, M) = dim(f_1(q), N)$.*

Proof : Let $M_1 \preceq M$ be a model prime over M_0e such that for all $q \in S(M_1)$, $q \perp M_0$, strongly regular, if there is $C_q \subseteq M_1$ such that

- C_q is finite
- q does not fork over M_0eC_q and $q_{\restriction M_0eC_q}$ is stationary
- $q_{\restriction M_0eC_q}$ is persistently isolated
- the dimension in M of $q_{\restriction M_0eC_q}$ is countable (it must then be infinite countable)

then, $dim(q, M) = 0$. Such a model exists by 4.5. Similarly, let $N_1 \preceq N$ be prime over N_0e', with the same property. By uniqueness of prime models, there is an elementary isomorphism f_1 from M_1 onto N_1, extending f_0 and taking e to e'.
Now let $q \in S(M_1)$ be strongly regular orthogonal to M_0, and let q' denote the conjugate of q over N_1. First, by depth 2 (Facts 4.3), as M_1 is atomic over M_0e, $q \not\perp M_0e$. By assumption we can find $D \subseteq M_0$, and $C \subseteq M_1$ ϵ-closed such that

- q does not fork over $acl(DeC)$
- $eC \underset{D}{\downarrow} M_0$, $q \not\perp acl(De)$
- $q_{\restriction acl(DeC)}$ is good for $(acl(De), D)$.
- $t(e/D)$ is good.
- $\left(\begin{matrix} e \\ D \end{matrix}, M\right) \equiv_1 \left(\begin{matrix} e' \\ D' \end{matrix}, N\right)$, where D' denotes $f_0(D)$.

Then the dimension of $q_{\restriction acl(DeC)}$ in M is equal to the dimension of the conjugate $q'_{\restriction acl(D'e'C')}$ in N. Now, as $q_{\restriction acl(DeC)} \perp D$, and $eC \underset{D}{\downarrow} M_0$, any independent set of realizations of $q_{\restriction acl(DeC)}$ remains an independent set of realizations of $q_{\restriction acl(M_0eC)}$. Similarly for q'. Hence the dimensions of $q_{\restriction acl(M_0eC)}$ and $q'_{\restriction acl(N_0e'C')}$ remain equal.
If $q_{\restriction acl(M_0eC)}$ is not isolated, then as M_1 is atomic over $acl(M_0eC)$, by Facts 4.2, any Morley sequence for $q_{\restriction acl(M_0eC)}$ remains a Morley sequence for q, and similarly for q'. Hence it follows that q and q' have same dimension .
If $q_{\restriction acl(M_0eC)}$ is isolated : if it has uncountable dimension , as M_1 is countable, then its dimension remains the same over M_1, and similarly for q'. If it has countable dimension, then our assumption on M_1 and N_1 ensures that $dim(q, M) = dim(q', N) = 0$. □

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