

An introduction to Hrushovski's theorem  
on the elementary theory of the Frobenius.

J.-B. Bost (Essay)

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I Geometric formulations of Hrushovski's Main Theorem HMTG

II Comments on HMTG

1 HMTG for curves

2 Finiteness

3 Reduction to the "smooth surjective case"

III Concerning  $\infty$ -schemes

1 Spec  $\infty$  R and  $\infty$ -schemes

2 Un fin d'histoire: perfect ideals and the theorem of R.R.

3 Correspondences and  $\infty$ -schemes

4 Transformal dimension, total dimension, ...

IV Hrusovski's reduction functors, and the second form of the Main Theorem HMT<sub>0</sub>  
 $\stackrel{1}{\equiv}$  The reduction functors  
 $\stackrel{2}{\equiv}$  HMT<sub>0</sub>

V Concerning the proof.

In accordance with the basic principles of the "Arbeitsgemeinschaft" style workshops, the speaker has no serious enfeeblement on the subject.

Caveat emptor!

important or new notation

explanations / comments

$\hookrightarrow$  closed inclusion / immersion

$\hookrightarrow$  open inclusion / immersion

"integral subscheme" = "subvariety"

└ reduced and irreducible

Notation

•  $p$  prime number ;  $q = p^m$  "primary" number

•  $A$   $\mathbb{F}_p$ -algebra  $\text{Fr}_q^{(\infty)} = \text{Fr}_p^m : x \mapsto x^q =: x^{\text{Fr}_q}$

$\text{Fr}_q : \text{Spec } A \rightarrow \text{Spec } A$

more generally, for any scheme  $X$  such that  $\dots$  over  $X$

$\text{Fr}_q : X \rightarrow X : \text{identity on } |X|, \text{Fr}_q^* a = a^q$

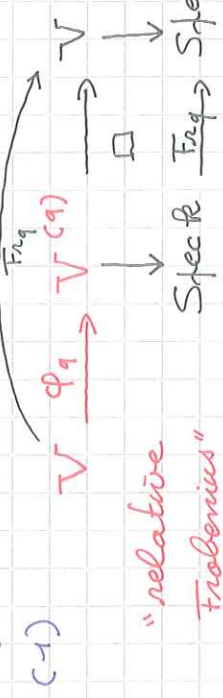
"absolute Frobenius"

$k$  field of characteristic  $p, V$   $k$ -scheme / variety

$\text{im} \rightarrow V^{(q)} := V \times_{k, \text{Fr}_q} k, \varphi_{V,q} : V \rightarrow V^{(q)} \quad k$ -morphism

$\Phi_{V,q} := \text{Graph of } \varphi_{V,q} \hookrightarrow V \times_k V^{(q)}$

may be described as follows:

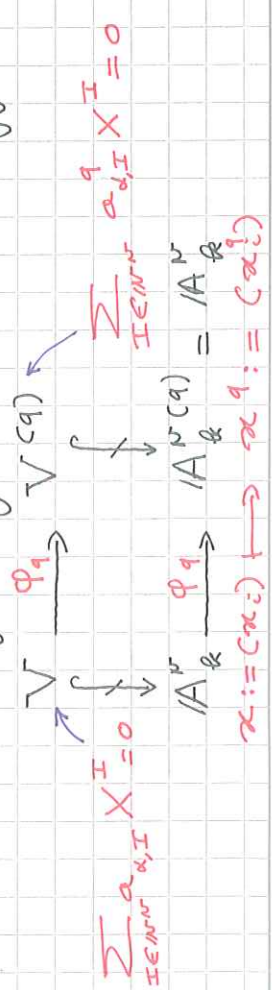


"relative Frobenius"

"geometric construction":

- functorial on  $k$ -schemes  $(\dots)$ ,
- compatible with arbitrary extensions of the base field  $k$ .

(2) locally may assume  $V$  affine



N.B.:  $K$  perfect extension of  $k$

$\Rightarrow \varphi_q : (V(K) \xrightarrow{\sim} V(K))$

$\Phi_{V,q} = (K = x^q)$

IV - Geometric Formulations of Hrushovski's Main Theorem

HMTG

Theorem HMTG "basic"

For any  $(N, d, S) \in W^3$ , there exist  $C_1(N, d, S)$  and  $C_2(N, d, S)$

such that :

- for any prime number  $p$  and any  $p$ -primary number  $q$ ,
- for any algebraically closed field  $K$  of characteristic  $p$ ,
- for any closed integral  $K$ - subscheme

$V \hookrightarrow \mathbb{P}_K^N$ ,  
 of closure  $\overline{V} \hookrightarrow \mathbb{P}_K^N$  of degree  $\leq d$ ,

- for any closed integral subscheme

of closure  $\overline{S} \hookrightarrow \mathbb{P}_K^N \times \mathbb{P}_K^N$  of degree  $\leq S$ , such that both

projections  $pr_0 : S \rightarrow V$  and  $pr_1 : S \rightarrow V^{(q)}$

are dominant and one of them is quasi-finite,

if  $q > C_1(N, d, S)$  and if  $d := \dim V = \dim S$ , then

$$\left| \# S \cap \Phi_{V, q}(K) - \frac{[S:V]}{[S:V^{(q)}]_{\text{inv}}} q^d \right| \leq C_2(N, d, S) q^{d-1/2}$$

$$\left( \{ \alpha \in A^N(K) \mid (\alpha, \alpha^q) \in S(K) \} \right)_{q \mid \alpha}$$

## Corollary:

$$S \cap \mathbb{I}_{V, q}(\mathbb{R}) \neq \emptyset \text{ if } q \gg_{N, d, S} 0$$

More generally, if  $S' \subsetneq S$  closed (reduced) subscheme of closure  $\overline{S'} \hookrightarrow \mathbb{P}_{\mathbb{R}}^N \times \mathbb{P}_{\mathbb{R}}^M$  of degree at most  $S'$

$$\#(S \setminus S') \cap \mathbb{I}_{V, q}(\mathbb{R}) \neq \emptyset \text{ if } q \gg_{N, d, S, S'} 0.$$

Variants I: choice of the base fields  $K$

for any pair  $(p, q)$  ( $p$  prime,  $q$   $p$ -primary), choose some algebraically closed field of characteristic  $p$ ,  $K_q$

$$\text{e.g. } K_q = \overline{\mathbb{F}_p}$$

Observation: for any family  $(K_q)$  of algebraically closed fields as above,

$$\text{HTMG "naive"} \iff \text{HTMG "naive" with } K = K_q$$

This follows from either:

- basic algebraic geometry

see for instance EGA IV § 8-9

in Publ. Math. IHES, 28 (1966),  
Chapitre III, troisième partie

Beilinson, Bernstein, Deligne - Faisceaux pervers  
Astérisque 100 (1982)

§ 6. De  $\mathbb{F}$  à  $\mathbb{C}$

"Note but dans ce paragraphe est d'expliquer  
et de justifier des recettes pour déduire d'énoncés  
sur la clôture algébrique d'un corps fini  
(...) des énoncés sur  $\mathbb{C}$ ."



already occurs in S. Noether  
and implicitly in Kronecker

or  
• basic model theory

- cf. A. Macintyre, Model theory in Modern Logic - Axsey 1980, 45-48

compactness;  $\uparrow$  LS

"... On the basis of this fact, and the model theory above one can

prove: Any injective morphism of an algebraic variety to itself is surjective.

In the special case of  $\mathbb{C}$ , the field of complex numbers, the result  
can be proved by appeal to the theory of analytic functions of  
several complex variables. (...).

For the general case, one can get pleasure by putting the question  
to an algebraic geometer and watching him squirm.  $Ax \rightarrow$

One should note that the application was not observed until 1968, by [2].

An algebraic proof may be found (or may not be found) in  
Grothendieck [18]. "

↳ EGA IV.

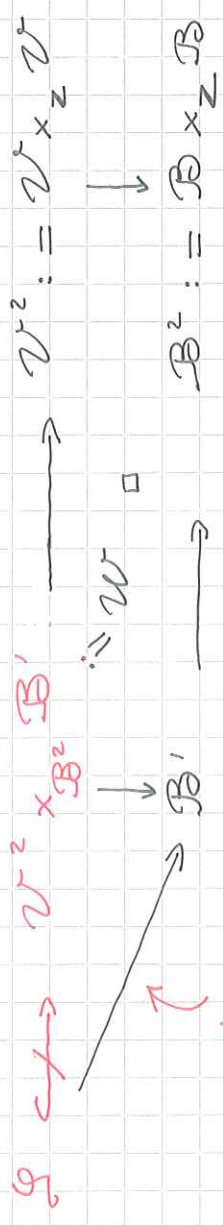
Variants II : "uniform data"  $(\mathcal{B}, \mathcal{B}', \mathcal{V}, \mathcal{G})$

Definition of "uniform data":

- $\mathcal{B}, \mathcal{B}'$  affine : integral schemes of finite type over  $\mathbb{Z}$   
 $\mathcal{B} = \text{Spec } \mathbb{Z}[X_1, \dots, X_n] / I \hookrightarrow \mathbb{A}_{\mathbb{Z}}^n$ ,  $I$  prime ideal in  $\mathbb{Z}[X_1, \dots, X_n]$   
 w. g. o. g., may replace  $\mathbb{Z}$  by  $\mathbb{Z}[1/m]$  or some finite field  $\mathbb{F}$

- $\mathcal{V} \rightarrow \mathcal{B}$  quasi-projective  $\mathcal{B}$ -scheme, with integral geometric fibers

$$\mathcal{B}' \stackrel{\beta = (\beta_0, \beta_1)}{\hookrightarrow} \mathcal{B}^2 := \mathcal{B} \times_{\mathbb{Z}} \mathcal{B} \quad \beta_{0,1} : \mathcal{B}' \rightarrow \mathcal{B}$$



integral geometric fibers

Otherwise:  $k$  field;  $b' \in \mathcal{B}'(k)$   $\mathcal{G}_{b'} \hookrightarrow \mathcal{W}_{b'} = \mathcal{V}_{\beta_0(b')} \times_k \mathcal{V}_{\beta_1(b')}$

such that: the two projections  $p_{i,i} : \mathcal{G} \rightarrow \mathcal{V} \times_{\beta_i} \mathcal{B}'$ ,  $i=0,1$ , are dominant on the fibers over  $b'$ ;  
 the first one is quasi-finite.

Claim: if  $h \in \mathcal{B}'(K_q)$  satisfies  $\beta(h) \in \Phi_{q, \mathcal{B}'}(K_q)$

$$\left\{ \beta_0(h) = \varphi_q(\beta_0(h)) \right\}$$

then

$$W_h = \mathcal{V}_{\beta(h)}^2 = \mathcal{V}_{\beta_0(h)} \times_{K_q} \mathcal{V}_{\varphi_q(\beta_0(h))} = \mathcal{V}_{\beta_0(h)} \times_{K_q} \mathcal{V}_{\beta_0(h)}^{(q)}$$

Bounds for uniform data:

$(\mathcal{B}, \mathcal{B}', \mathcal{V}, \mathcal{S})$  uniform data;  $(K_q)_q, \dots$   
 $\delta := \dim \mathcal{V} / \mathcal{B} = \dim \mathcal{S} / \mathcal{B}'$

There exists  $c_1$  and  $c_2$  in  $\mathbb{N}$  such that:

for any primary number  $q > c_1$ , and any  $h \in \mathcal{B}'(K_q)$  s.t.  $\beta(h) \in \Phi_{q, \mathcal{B}'}(K_q)$

$$(*) \quad \left| \# \mathcal{S}_h \cap \Phi_{q, \mathcal{V}_{\beta_0(h)}} - \frac{[\mathcal{S}_h: \mathcal{V}_{\beta_0(h)}]}{[\mathcal{S}_h: \mathcal{V}_{\beta_0(h)}]_{\text{link}}} \right|_q \leq c_2 q^{\delta - 1/2}$$

Weak bounds for uniform data:

- there exists  $\mathcal{B}'' \subset \mathcal{B}$  open dense s.t.  $\dots (*)$  holds for  $h \in \mathcal{B}''(K_q)$ .  
 $\neq \emptyset$

## Observations:

• for any uniform data  $(\mathcal{B}, \mathcal{B}', \mathcal{V}, \mathcal{G})$ , the validity of Hruslovski's bounds (\*) does not depend on the choice of the  $K_q$ 's.

• noetherian induction

$\implies$

weak Hr. bounds for any n. d.  $\implies$  Hr. bounds for any n. d.

• equivalence of the following 6 forms of HMTG

Scholium: "basic" HMTG for arbitrary  $k$   $\iff$  "basic" HMTG for  $k = K_q$   
 $\iff$  (weak) Hr. bounds for any uniform data and any choice of the  $K_q$ 's  
 $\iff$  (weak) Hr. bounds for any uniform data and a fixed choice of the  $K_q$ 's.