An introduction to Hrushovski's theorem on the elementary theory of the Frobenius.

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I Geometric formulations of Hrushovski's Main Theorem HMTG

II Comments on HMTG

1. HMTG for curves
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IV Hrushovski's reduction functors, and the second form of the Manin Theorem HMT_σ

1. The reduction functors

2. HMT_σ

V Concerning the proof.

In accordance with the basic principles of the "Arbeitsgemeinschaft" style workshops, the speaker has no serious expertise on the subject.

Caveat emptor!

important or new notation   explanations / comments

$\hookrightarrow$ closed inclusion / immersion

$\overset{\circ}{\hookrightarrow}$ open inclusion / immersion

"integral subscheme" = "subvariety"

reduced and irreducible

Notation

- p prime number ; $q = p^m$ "p-primary" number
- A $\mathbb{F}_p$-algebra $\quad \mathrm{Fr}_q^{(*)} = \mathrm{Fr}_p^m : x \longmapsto x^q =: x^{\mathrm{Fr}_q}$

  $\mathrm{Fr}_q : \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, A$

  more generally, for any scheme X such that over X

  $\mathrm{Fr}_q : X \longrightarrow X$: identity on $|X|$, $\mathrm{Fr}_q^* a = a^q$

  "absolute Frobenius"

- k field of characteristic p, V k-scheme/variety

  $\rightsquigarrow$ $V^{(q)} := V \times_{k, \mathrm{Fr}_q} k$, $\varphi_{V,q} : V \longrightarrow V^{(q)}$ k-morphism

  $\Phi_{V,q} :=$ Graph of $\varphi_{V,q} \hookrightarrow V \times_k V^{(q)}$

may be described as follows:

(1)
$$\begin{tikzcd} V \arrow[r, "\varphi_q"] \arrow[rr, bend left, "\mathrm{Fr}_q"] & V^{(q)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ & \mathrm{Spec}\, k \arrow[r, "\mathrm{Fr}_q"] & \mathrm{Spec}\, k \end{tikzcd}$$

"relative Frobenius"

"geometric construction":

- functorial on k-schemes (...),
- compatible with arbitrary extensions of the base field k.

(2) locally may assume V affine

$$\begin{tikzcd} V \arrow[r, "\varphi_q"] \arrow[d, hook] & V^{(q)} \arrow[d, hook] \\ \mathbb{A}^N_k \arrow[r, "\varphi_q"] & \mathbb{A}^{N(q)}_k = \mathbb{A}^N_k \end{tikzcd}$$

$\sum_{I \in \mathbb{N}^N} a_{\alpha,I} X^I = 0$ $\qquad$ $\sum_{I \in \mathbb{N}^N} a_{\alpha,I}^q X^I = 0$

$x := (x_i) \longmapsto x^q := (x_i^q)$

N.B.: . K perfect extension of k

$\Rightarrow \varphi_q : (V(K) \xrightarrow{\sim} V(K)$

. $\Phi_{V,q} =$ "$(y = x^q)$"

I – Geometric formulations of Hrushovski's Main Theorem

HMTG

Theorem HMTG "basic"

For any $(N, d, \delta) \in \mathbb{N}^3$, there exist $c_1(N, d, \delta)$ and $c_2(N, d, \delta)$ such that :

- for any prime number p and any p-primary number q,
- for any algebraically closed field k of characteristic p,
- for any closed integral k-subscheme

$$V \hookrightarrow \mathbb{A}^N_k,$$

of closure $\overline{V} \hookrightarrow \mathbb{P}^N_k$ of degree $\leq d$,

- for any closed integral subscheme

$$S \hookrightarrow V \times_k V^{(q)}$$

of closure $\overline{S} \hookrightarrow \mathbb{P}^N_k \times \mathbb{P}^N_k$ of degree $\leq \delta$, such that both projections

$$pr_0 : S \longrightarrow V \quad \text{and} \quad pr_1 : S \longrightarrow V^{(q)}$$

are dominant and one of them is quasi-finite,

if $q > c_1(N, d, \delta)$ and if $d := \dim V = \dim S$, then

$$\left| \# S \cap \Phi_{V,q}(k) - \frac{[S:V]}{[S:V(q)]_{insep}} q^d \right| \leq c_2(N, d, \delta)\, q^{d - 1/2}.$$

$$\Big(\{x \in \mathbb{A}^N(k) \mid (x, \underbrace{x^q}_{\varphi_q(x)}) \in S(k)\}$$

<u>Corollary</u> : $S \cap \Phi_{V,q}(k) \neq \emptyset$ if $q \gg_{N,d,\delta} 0$

More generally, if $S' \subsetneq S$ closed (reduced) subscheme of closure $\overline{S'} \hookrightarrow \mathbb{P}_k^N \times \mathbb{P}_k^N$ of degree at most δ'

$$\#(S \setminus S') \cap \Phi_{V,q}(k) \neq \emptyset \quad \text{if } q \gg_{N,d,\delta,\delta'} 0.$$

<u>Variants I</u> : choice of the base fields K

for any pair (p, q) (p prime, q p primary), choose some algebraically closed field of characteristic p, K_q

e.g. $K_q = \overline{\mathbb{F}_p}$

<u>Observation</u> : for any family (K_q) of algebraically closed fields as above,

HTMG "naive" $\Longleftrightarrow$ HTMG "naive" with $K = K_q$

This follows from either :

- basic algebraic geometry

see for instance EGA IV § 8 – 9

in Publ. Math. IHES, 28 (1966), Chapitre IV, troisième partie

Beilinson, Bernstein, Deligne – Faisceaux pervers Astérisque 100 (1982)

§ 6 . De $\mathbb{F}$ à $\mathbb{C}$

"Notre but dans ce paragraphe est d'expliquer et de justifier des recettes pour déduire d'énoncés vrais sur la clôture algébrique d'un corps fini (...) des énoncés sur $\mathbb{C}$."

already occurs in E. Noether
and implicitly in Kronecker

or

- basic model theory

cf. A. Macintyre, Model theory in Modern Logic - A survey 1980, 45-63

Steinitz; compactness; ↑LS

" ... On the basis of this fact, and the model theory above one can prove: Any injective morphism of an algebraic variety to itself is surjective.

In the special case of $\mathbb{C}$, the field of complex numbers, the result can be proved by appeal to the theory of analytic functions of several complex variables. (...).

For the general case, one can get pleasure by putting the question to an algebraic geometer and watching him squirm.

Ax

One should note that the application was not observed until 1968, by [2]. An algebraic proof may be found (or may not be found) in Grothendieck [18]. "

EGA IV.

Variants II : "uniform data" $(\mathcal{B}, \mathcal{B}'; \mathcal{V}, \mathcal{S})$

Definition of "uniform data":

. $\mathcal{B}, \mathcal{B}'$ affine integral schemes of finite type over $\mathbb{Z}$

$\mathcal{B} = \operatorname{Spec} \mathbb{Z}[X_1, \dots, X_N]/I \hookrightarrow \mathbb{A}^N_{\mathbb{Z}}$, I prime ideal in $\mathbb{Z}[X_1, \dots, X_N]$

w. g. o. g., may replace $\mathbb{Z}$ by $\mathbb{Z}[1/m]$ or some finite field $\mathbb{F}$

. $\mathcal{V} \longrightarrow \mathcal{B}$ quasi-projective $\mathcal{B}$-scheme, with integral geometric fibers

$\mathcal{B}' \xrightarrow{\beta = (\beta_0, \beta_1)} \mathcal{B}^2 := \mathcal{B} \times_{\mathbb{Z}} \mathcal{B}$ $\qquad$ $\beta_{0,1} : \mathcal{B}' \longrightarrow \mathcal{B}$

.
$$\begin{array}{ccccc} \mathcal{S} & \hookrightarrow & \mathcal{V}^2 \times_{\mathcal{B}^2} \mathcal{B}' =: \mathcal{W} & \longrightarrow & \mathcal{V}^2 := \mathcal{V} \times_{\mathbb{Z}} \mathcal{V} \\ & \searrow & \downarrow & \square & \downarrow \\ & & \mathcal{B}' & \longrightarrow & \mathcal{B}^2 := \mathcal{B} \times_{\mathbb{Z}} \mathcal{B} \end{array}$$

integral geometric fibers

Observe : k field ; $b' \in \mathcal{B}'(k)$ $\qquad$ $\mathcal{S}_{b'} \hookrightarrow \mathcal{W}_{b'} = \mathcal{V}_{\beta_0(b')} \times_k \mathcal{V}_{\beta_1(b')}$

such that : the two projections $p_i : \mathcal{S} \longrightarrow \mathcal{V} \times_{\beta_i} \mathcal{B}'$, $i = 0, 1$, are dominant on

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{V} \times_{\beta_i} \mathcal{B}' \\ & \searrow & \downarrow \\ & & \mathcal{B}' \end{array}$$

the fibers over the fibers ;

the first one is quasi-finite.

Observe : if $b \in \mathcal{B}'(K_q)$ satisfies $\beta(b) \in \Phi_{q,\mathcal{B}_{\mathbb{F}_p}}(K_q)$

"$\beta_1(b) = \varphi_q(\beta_0(b))$"

then $\mathcal{W}_b = \mathcal{V}^2_{\beta(b)} = \mathcal{V}_{\beta_0(b)} \times_{K_q} \mathcal{V}_{\varphi_q(\beta_0(b))} = \mathcal{V}_{\beta_0(b)} \times_{K_q} \mathcal{V}^{(q)}_{\beta_0(b)}$

Bounds for uniform data :

$(\mathcal{B}, \mathcal{B}', \mathcal{V}, \mathcal{E})$ uniform data ; $(K_q)_q$ ...

$\delta := \dim \mathcal{V}/\mathcal{B} = \dim \mathcal{E}/\mathcal{B}'$

There exists c_1 and c_2 in $\mathbb{N}$ such that :

for any primary number $q > c_1$ and any $b \in \mathcal{B}'(K_q)$ s.t. $\beta(b) \in \Phi_{q,\mathcal{B}_{\mathbb{F}_p}}(K_q)$,

$$(*) \qquad \left| \# \mathcal{E}_b \cap \Phi_{q,\mathcal{V}_{\beta_0(b)}} - \frac{[\mathcal{E}_b : \mathcal{V}_{\beta_0(b)}]}{[\mathcal{E}_b : \mathcal{V}^{(q)}_{\beta_0(b)}]_{insep}} q^\delta \right| \leq c_2 \, q^{\delta - 1/2}$$

Weak bounds for uniform data :

there exists $\mathcal{B}'' \hookrightarrow \mathcal{B}$ open dense s.t. ... $(*)$ holds for $b \in \mathcal{B}''(K_q)$.

$\neq \emptyset$

<u>Observations</u> :

- for any uniform data $(\mathcal{B}, \mathcal{B}', \mathcal{V}, \mathcal{E})$, the validity of Hrushovski's bounds $(*)$ does not depend on the choice of the K_q's.

- noetherian induction

  $\rightsquigarrow$ weak Hr. bounds for any u.d. $\Longrightarrow$ Hr. bounds for any u.d.

- equivalence of the following 6 forms of HMTG

<u>Scholium</u>:

$$\begin{array}{ccc}
\text{"basic" HMTG for arbitrary } k & \Longleftrightarrow & \text{"basic" HMTG for } k = K_q \\
\Updownarrow & & \Updownarrow \\
\text{(weak) Hr. bounds for any uniform data and any choice of the } K_q\text{'s} & \Longleftrightarrow & \text{(weak) Hr. bounds for any uniform data and a fixed choice of the } K_q\text{'s.}
\end{array}$$