IV Comments on HMTG

IV.1 HMTG for curves

$\delta = 1$

straightforward extension of the proof of Hasse – Weil based on the Hodge index theorem on projective surface "Castelnuovo – Severi inequality"

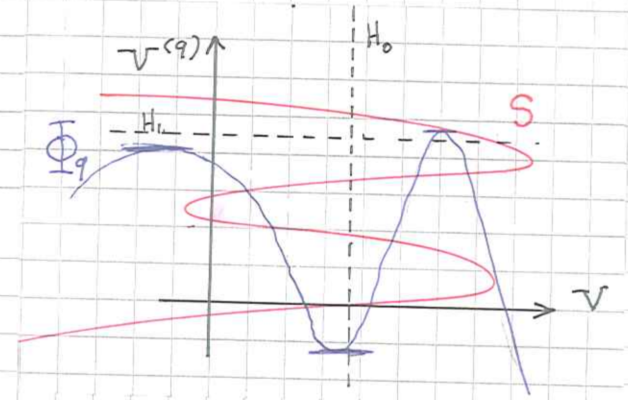
V smooth projective curve of genus g over k, field of characteristic p say algebraically closed ; $q = p^{\cdots}$

$S \hookrightarrow V \times V^{(q)}$ irreducible, with dominant projections

$\Phi_q :=$ graph of $\varphi_q : V \longrightarrow V^{(q)}$
$\hookrightarrow V \times V^{(q)}$

$V \times V^{(q)} \overset{pr_1}{\underset{pr_0}{\rightrightarrows}} \begin{matrix} V^{(q)} \\ V^{q} \end{matrix}$

$H_i := pr_i^{-1}(*) \qquad i = 0, 1$



Proposition :
The matrix of intersection numbers is :

	H_0	H_1	S	Φ_q
H_0	0	1	$[S:V]$	1
H_1	1	0	$[S:V^{(q)}]$	q
S	$[S:V]$	$[S:V^{(q)}]$	$S.S$	$\Phi_q . S$
Φ_q	1	q	$\Phi_q . S$	$q(2-2g)$

Hodge index theorem + $(H_1 + H_2)^2 > 0$ $\quad + \cdot \_ \cdot \_ \cdot \_ = \_$

$d_0 := \deg pr_0|S = [S:V]$

$d_1 := \text{"} \quad \text{"} pr_1 \ldots = [S:V^{(q)}]$

$$0 \geq \begin{vmatrix} 0 & 1 & d_0 & 1 \\ 1 & 0 & d_1 & q \\ d_0 & d_1 & S.S & \phi_q.S \\ 1 & q & \phi_q.S & q(2-2g) \end{vmatrix} = \begin{vmatrix} 0 & 1 & * & * \\ 1 & 0 & * & * \\ 0 & 0 & A & B \\ 0 & 0 & B & C \end{vmatrix} = AC - B^2$$

where $A := S.S - 2 d_0 d_1$

$B := \phi_q . S - d_1 - q d_0$

$C := -2qg$

$$\begin{cases} AC \geq 0 \\ |B| \leq \sqrt{AC} \end{cases}$$

$S.S \leq 2 d_0 d_1$
Castelnuovo–Severi for S

$$\left| \phi_q . S - [S:V^{(q)}] - q[S:V] \right| \leq 2 q^{1/2} \left[g\left([S:V][S:V^{(q)}] - \tfrac{1}{2} S.S \right) \right]^{1/2}$$

refined intersection theoretic version of HMTG for $\delta = 1$

Nota Bene :

(1) $V/\mathbb{F}_q$ $\quad V$ "=" $V^{(q)}$ $\quad S$

choose $S = \Delta_V$; then $\phi_q \cdot S = \# V(\mathbb{F}_q)$, and we recover

$$\left| \# V(\mathbb{F}_q) - (q+1) \right| \leq 2 g q^{1/2}$$

Hasse-Weil !

(2) $\deg_{H_0+H_1} \phi_{q,V} = q+1$

$\neq$ $[S:V] + [S:V^{(q)}] = \deg_{H_0+H_1} S$ if $q \geq [S:V] + [S:V^{(q)}]$.

Then $\Phi_{q,V} \cap S(k)$ is finite.

(3) Need also to know :

Intersection theoretic version $\Longrightarrow$ " actual " version.

of HMTG $\delta = 1$

Intersection theoretic version of HMTG $g=1$ $\Longrightarrow$ "actual" version

$$\left| \phi_{q,v} \cdot S - q\,[S:V] \right| \leq C_{geom}\, q^{1/2} \qquad \left| \# \phi_{q,v} \cap S(k) - \frac{[S:V]}{[S:V^{(q)}]_{ins}}\, q \right| \leq C'_{geom}\, q^{1/2}$$

① "separable case"

$$\begin{tikzcd} S \arrow[r, "pr_{1|S}"] & V^{(q)} \\ \tilde{S} \arrow[u, "\nu"] \arrow[ur, "pr_{1|S} \circ \nu"'] & \end{tikzcd}$$

normalization $\tilde{S}$ — separable (= generically étale)

$$S \cdot \phi_{q,v} = \deg_{\tilde{S}} \nu^* \mathcal{O}(\phi_{q,v})$$

$$= \sum_{\tilde{P} \in \tilde{S}(k)} \operatorname{mult}_{\tilde{P}} \nu^* \phi_q$$

+ Riemann – Hurwitz

$$\Longrightarrow \left| \phi_{q,v} \cdot S - \# \phi_{q,v} \cap S(k) \right| \leq C''_{geom}$$

$\operatorname{mult}_{\tilde{P}} \nu^* \phi_q \geq 0$, $\neq 0$ iff $\nu(\tilde{P}) \in S \cap \phi_{q,v}$

" " $\leq \lg_{\tilde{P}} \Omega^1_{\tilde{S}/V} + 1$

local computation : t local parameter at $\tilde{P}$ on $\tilde{S}$

$\nu(\tilde{P}) = P =: (Q_0, Q_1) \in \phi_{q,v}$ — x " " Q_0 on V

y " " Q_1 on $V^{(q)}$

ν : $x = \alpha(t)$, $y = \beta(t)$; $\phi_{q,v}$ "=" $(y = x^q)$

$\Rightarrow \operatorname{val}_0 (\beta - \alpha^q) \leq \operatorname{val}_0 (\beta - \alpha^q)' + 1 = \operatorname{val}_0 \beta' + 1$.

II.e

② "separable case" $\Longrightarrow$ general case

$$[\tilde{S} : V^{(q)}]_{insep} = [S : V^{(q)}]_{insep} = p^e \overset{!}{\leq} [S : V^{(q)}]$$ controlled by the geometry!

introduce the factorization:

$$\begin{tikzcd} \tilde{S} \arrow[r, "pr_1 \circ \nu"] \arrow[d, "\varphi_{p^e,\tilde{S}}"'] & V^{(q)} \\ \tilde{S}^{(p^e)} \arrow[ur, "\eta"'] & \end{tikzcd}$$

η = k-morphism generically étale

$$[\tilde{S}^{(p^e)} : V^{(q)}] = \deg \eta = \frac{[S : V^{(q)}]}{[S : V^{(q)}]_{insep}}$$

and get a commutative diagram:

$$\begin{tikzcd}[column sep=small] & \tilde{S} \arrow[rrr, "\varphi_{\tilde{S},p^e}"] \arrow[d, "\nu"] & & & \tilde{S}^{(p^e)} \arrow[d, "\nu'"] \arrow[dr, bend left, "\eta"] & \\ & S \arrow[dl, "pr_0"'] \arrow[dr, "pr_1"] & & & S' \arrow[dl, "pr_0'"'] \arrow[dr, "pr_1'"] & \\ V \arrow[rr, bend right, "\varphi_{p^e}"'] & & V^{(q)} \arrow[r] \arrow[rrr, bend right, "="'] & V^{(p^e)} & & V^{(q)} = (V^{(p^e)})^{(q')} \end{tikzcd}$$

$S' := \mathrm{im}\,(\underbrace{(pr_0 \circ \nu)^{(p^e)}}_{=:\nu'}, \eta)$

$q = p^m \geq p^e$

$\Leftarrow q \geq [S : V^{(q)}]$

$$q' := p^{m-e} = \frac{q}{[S : V^{(q)}]_{insep}}$$

$$s \in \tilde{S}(k) \overset{\varphi_{p^e}}{\longleftrightarrow} s' \in \tilde{S}^{(p^e)}$$

$$\{s \in \tilde{S}(k) \mid \nu(s) \in \Phi_{q,V}\} \simeq \{s' \in \tilde{S}^{(p^e)} \mid \nu'(s') \in \Phi_{q';V^{(p^e)}}\}$$

$$q \frac{[S : V]}{[S : V^{(q)}]_{insep}} = q' [\tilde{S}^{(p^e)} : V^{(p^e)}]$$

+ easy bounds on "ramified points".

II. 2 Finiteness and rough bounds

$$S \hookrightarrow V \times V^{(q)} \hookleftarrow \phi_{q,V}$$

$$\left\{ \begin{array}{l} pr_1 : S \longrightarrow V^{(q)} \text{ quasi-finite} \\ C \text{ subvariety of } C \end{array} \right. \Longrightarrow [C : p_1(C)] \leqslant [S : V^{(q)}]$$

$$C \text{ subvariety of } \Phi_{q,V} \Longrightarrow [C : p_1(C)] = q^{\dim C}$$

$$\rightsquigarrow \quad q > [S : V^{(q)}] \Longrightarrow S \cap \Phi_q \text{ finite}$$

[compare Deligne ?
Pink, Ann. Math $\underline{\underline{135}}$ (1992) 483–525 ; Lemma 7.1.2]

Once the finiteness is known, a crude form of Bézout's theorem establishes the following bound on the cardinality of $S \cap \phi_q$:

$$\# S \cap \phi_{q,V}(k) \leqslant C_{geom}\, q^d .$$

Indeed

$$\deg_* \Phi_{q,V} \leqslant C'_{geom}\, q^d .$$

III. 3 Reduction to the "smooth projective case"

$\left\{\begin{array}{l}\text{rough bounds} \\ \text{resolution of singularities, alterations}\end{array}\right.$ $\rightsquigarrow$ may assume $\mathcal{V} \hookrightarrow \overline{\mathcal{V}}$, with $\overline{\mathcal{V}}$ projective, smooth over $\mathbb{B}$

Hironaka, de Jong