

### III. 3 Difference schemes and algebraic correspondences

" $\mathfrak{o}$ -envelopes"

$\mathbb{D}$  well-mixed  $\mathfrak{o}$ -ring

$\mathcal{V}$   $\mathbb{D}$ -scheme

Proposition: The functor  $\mathfrak{o}$ -schemes /  $\mathbb{D}$  maps sets

$$X \mapsto \text{Hom}_{\mathbb{D}\text{-scheme}}(X, \mathcal{V})$$

is representable by some  $\mathfrak{o}$ -scheme /  $\mathbb{D}$   $[\mathfrak{o}]_{\mathbb{D}} \mathcal{V}$ .

$$\text{Hom}_{\mathbb{D}\text{-schm}}(X, \mathcal{V}) \xrightarrow{\sim} \text{Hom}_{\mathbb{D}\text{-}\mathfrak{o}\text{-schm}}(X, [\mathfrak{o}]_{\mathbb{D}} \mathcal{V})$$

universal morphism  $[\mathfrak{o}]_{\mathbb{D}} \mathcal{V} \rightarrow \mathcal{V}$

(of  $\mathbb{D}$ -schemes)

Examples: 1)  $X$  affine of finite type =  $\text{Spec } \mathbb{D}[X_1, \dots, X_n] / I$

$$[\mathfrak{o}]_{\mathbb{D}} X = \text{Spec } \mathfrak{o}[\mathbb{D}[X_1, \dots, X_n]_{\mathfrak{o}} / (I)_{\mathfrak{o}}$$

well-mixed ideal  
generated by  $I$

2)  $D = \mathbb{R}$   $\omega$ -field

$K \omega$  - extension of  $K$

$V$  separated scheme of finite type over  $\mathbb{R}$  "algebraic  $\mathbb{R}$ -scheme"

$\omega$  - scheme of finite type over  $\mathbb{R}$

$\rightsquigarrow [V]_{\mathbb{R}}^V$

$$V(K) := \text{Hom}_{\mathbb{R}\text{-sch.}}(\text{Spec } K, V) \cong \text{Hom}_{\mathbb{R}\text{-sch.}}(\text{Spec } K, [V]_{\mathbb{R}}^V) =: [V]_{\mathbb{R}}^V(K)$$



$\mathbb{R}$  ACFA

{closed points of  $[V]_{\mathbb{R}}^V$ }

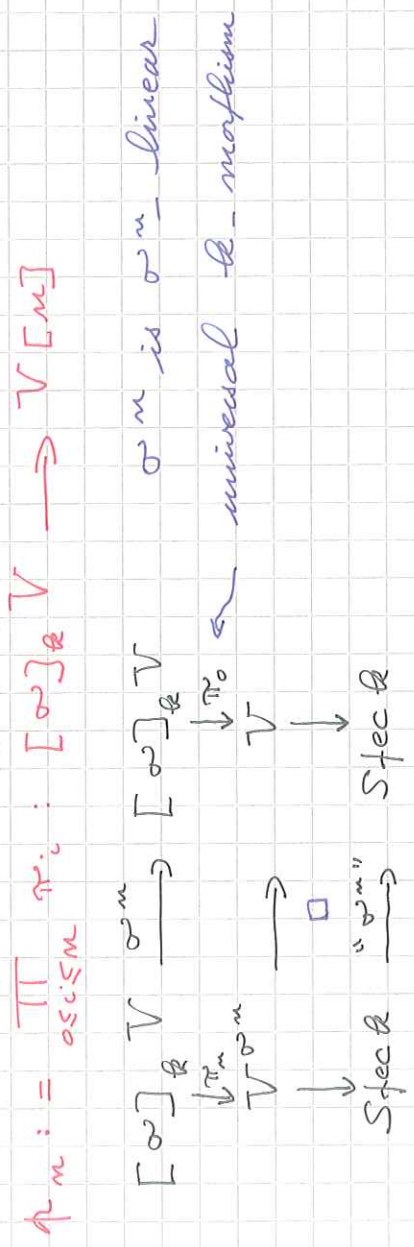
the " $\omega$ -topology" on  $[V]_{\mathbb{R}}^V(K)$  is (in general strictly) finer than the Zariski topology on  $V(K)$ .

$\sigma$ -envelopes and projective systems.

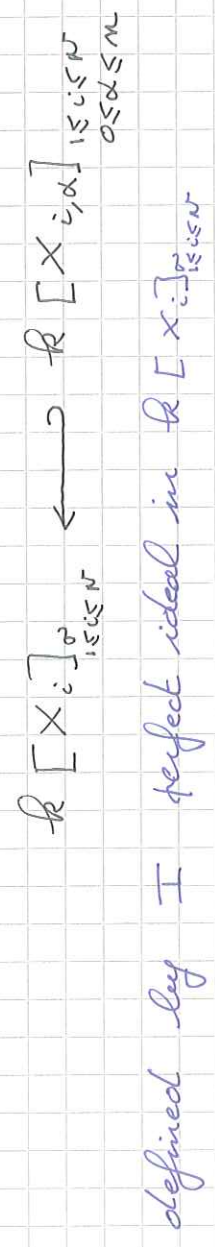
$k$   $\sigma$ -field,  $V$   $k$ -scheme,  $X \hookrightarrow [\sigma]_k V$   
 $\sigma$ -scheme,  $\sigma$ -reduced

$V[m] := \prod_{0 \leq i \leq m} V_{\sigma^i}$ , for any  $m \in \mathbb{N}$ .

canonical morphisms of  $k$ -ringed spaces:

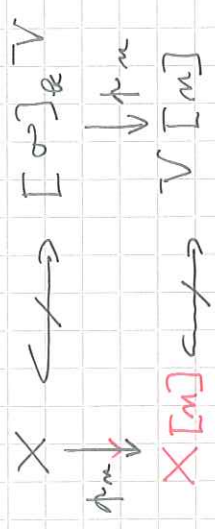


Example:  $V = A_k^N$   $\xrightarrow{\rho_m} A_{k, \sigma}^N \xrightarrow{\rho_{m+1}} (A_k^N)^{m+1}$



defined by  $I$  perfect ideal in  $\mathbb{R}[X_i]_{0 \leq i \leq N}$

defined by  $I \cap \mathbb{R}[X_{i, \alpha}]_{0 \leq i \leq N, 0 \leq \alpha \leq m}$ .



$\rho_m^* : \mathcal{O}_{V[m]} \rightarrow \rho_{m*} \mathcal{O}_{[\sigma]_k V}$

$n \rightarrow$  projective system of closed subschemes

$$\begin{array}{c}
 X[0] \leftarrow X[1] \leftarrow X[2] \leftarrow \dots \leftarrow X[n] \leftarrow X[n+1] \leftarrow \\
 \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
 V[0] \leftarrow V[1] \leftarrow V[2] \leftarrow \dots \leftarrow V[n] \leftarrow V[n+1] \leftarrow \\
 \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
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 \end{array}$$

actually:

$$\left\{ \begin{array}{l} \text{closed } \sigma\text{-subschemes of } [\sigma]_{\mathbb{R}}^V \\ \text{reduced?} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{certain sub (pro)schemes} \\ \text{of } V[\infty] = \prod_{m \in \mathbb{N}} V_{\sigma m} \end{array} \right\}$$

$$\cdot X(K) \simeq \varprojlim_m X[n](K)$$

$\text{Hom}_{\sigma\text{-sub}/\mathbb{R}}(\text{Spec } K, X)$

$\text{Hom}_{\text{sub}/\mathbb{R}}(\text{Spec } K, X[n])$