III. 3 Difference schemes and algebraic correspondences

- " σ - envelopes "

  D well - mixed σ - ring

  V D - scheme

__Proposition:__ The functor $\sigma\text{-schemes}/D \rightsquigarrow \mathrm{Sets}$

$$X \rightsquigarrow \mathrm{Hom}_{D\text{-scheme}}(X, V)$$

is representable by some σ - scheme / D $[\sigma]_D V$.

$$\mathrm{Hom}_{D\text{-schm}}(X, V) \xrightarrow{\sim} \mathrm{Hom}_{D\text{-}\sigma\text{-scheme}}(X, [\sigma]_D V)$$

universal morphism $[\sigma]_D V \longrightarrow V$
(of D - schemes)

__Examples:__ 1) Y affine of finite type $= \mathrm{Spec}\, D[X_1, \dots, X_n]/I$

$$[\sigma]_D Y = \mathrm{Spec}^\sigma D[X_1, \dots, X_n]_\sigma / (I)_{w}$$

well - mixed ideal generated by I

2) $\mathcal{D} = k$ σ-field $\qquad K$ σ-extension of k

V separated scheme of finite type over k "algebraic k-scheme"

$\rightsquigarrow [\sigma]_k V$ $\qquad$ σ-scheme of finite type over k

$$V(K) := \mathrm{Hom}_{k\text{-sch.}}(\operatorname{Spec} K, V) \simeq \mathrm{Hom}_{k\sigma\text{-sch.}}(\operatorname{Spec}^\sigma K, [\sigma]_k V) =: [\sigma]_k V(K)$$

$$\begin{array}{r} \big\downarrow \wr \Leftarrow \text{if } k = K \text{ ACFA} \\ \text{if } K \text{ ACFA} \longrightarrow \\ \{\text{closed points of } [\sigma]_k V\} \end{array}$$

⚠ k ACFA

the "σ-topology" on $[\sigma]_k V(k)$ is (in general strictly) finer than the Zariski topology on $V(K)$.

- σ-envelopes and projective systems.

k σ-field, V k-scheme, $X \hookrightarrow [\sigma]_k V$ σ-subscheme, σ-reduced

$\leadsto$ $V[n] := \prod_{0 \le i \le n} V^{\sigma^i}$, for any $n \in \mathbb{N}$.

canonical morphisms of k-ringed spaces:

$$p_n := \prod_{0 \le i \le n} \pi_i : [\sigma]_k V \longrightarrow V[n]$$

$$\begin{tikzcd}
{[\sigma]_k V} \arrow[r, "\sigma^n"] \arrow[d, "\pi_n"] & {[\sigma]_k V} \arrow[d, "\pi_0"] \\
V^{\sigma^n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\
\operatorname{Spec} k \arrow[r, "\text{“}\sigma^n\text{”}"] & \operatorname{Spec} k
\end{tikzcd}$$

σ^n is σ^n-linear

$\leftarrow$ universal k-morphism

<u>Example</u>: $V = \mathbb{A}^N_k$ $\qquad [\sigma]_k V \simeq \mathbb{A}^N_{k,\sigma} \xrightarrow{p_n} (\mathbb{A}^N_k)^{n+1}$

$$k[X_i]^{\sigma}_{1 \le i \le N} \longleftarrow k[X_{i,\alpha}]_{\substack{1 \le i \le N \\ 0 \le \alpha \le n}}$$

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "p_n"'] & {[\sigma]_k V} \arrow[d, "p_n"] \\
X[n] \arrow[r, hook] & V[n]
\end{tikzcd}$$

defined by I perfect ideal in $k[X_i]^{\sigma}_{1 \le i \le N}$

defined by $I \cap k[X_{i,\alpha}]_{\substack{1 \le i \le N \\ 0 \le \alpha \le n}}$.

$\mathcal{J}_{X[n]} := \ker p_n^* : \mathcal{O}_{V[n]} \to p_{n*} \mathcal{O}_{[\sigma]_k V}$

$\rightsquigarrow$ projective system of closed subschemes

$$\begin{tikzcd}
X[0] \arrow[d, hook] & X[1] \arrow[l] \arrow[d, hook] & X[2] \arrow[l] \arrow[d, hook] & \cdots \arrow[l] & X[n] & X[n+1] \arrow[l] & \arrow[l] \\
V[0] & V[1] \arrow[l] & V[2] \arrow[l] & \arrow[l]
\end{tikzcd}$$

actually :

- $\left\{ \text{closed } \sigma\text{-subschemes of } [\sigma]_k V \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{certain sub (pro) schemes} \\ \text{of } V[\infty] = \prod_{n \in \mathbb{N}} V^{\sigma^n} \end{array} \right\}$

  reduced ?

- $X(K) \simeq \varprojlim_n X[n](K)$

  $X(K) \wr \; \mathrm{Hom}_{\sigma\text{-sch}/k}(\mathrm{Spec}\, K, X)$ $\qquad$ $X[n](K) \wr \; \mathrm{Hom}_{\mathrm{sch}/k}(\mathrm{Spec}\, K, X[n])$