Energy Flow in Extended Gradient Partial Differential Equations

Th. Gallay
Université de Paris-Sud
Mathématique, UMR 8628
Bâtiment 425
F-91405 Orsay, France
Thierry.Gallay@math.u-psud.fr

S. Slijepčević
Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb, Croatia
slijepce@math.hr

Abstract. *As an example of an extended, formally gradient dynamical system, we consider the damped hyperbolic equation $u_{tt} + u_t = \Delta u + F(x,u)$ in $\mathbf{R}^N$, where F is a locally Lipschitz nonlinearity. Using local energy estimates, we study the semiflow defined by this equation in the uniformly local energy space $\mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$. If $N \le 2$, we show in particular that there exist no periodic orbits, except for equilibria, and we give a lower bound on the time needed for a bounded trajectory to return in a small neighborhood of the initial point. We also prove that any nonequilibrium point has a neighborhood which is never visited on average by the trajectories of the system, and we deduce that the only uniformly recurrent points are equilibria. Counter-examples are given which show that these results cannot be extended to higher space dimensions.*

1. Introduction

Describing the dynamics of nonlinear partial differential equations on unbounded domains is a challenging problem, which remains widely open so far. We are interested here in dissipative systems that have a *formal gradient structure* in the following sense: if we restrict the corresponding equation to a bounded domain $\Omega \subset \mathbf{R}^N$ and impose appropriate boundary conditions, the resulting system has a strict Lyapunov function. Typical examples are the semilinear parabolic equation $u_t = \Delta u + F(x,u)$ and the damped hyperbolic equation $u_{tt} + u_t = \Delta u + F(x,u)$, which define gradient dynamical systems in any bounded domain Ω. In this case, under standard assumptions on the nonlinearity F, it can be shown that the systems above have a compact global attractor $\mathcal{A}$ which captures the long-time evolution of any bounded set of initial data [Ha], [Te]. Due to the gradient structure, the attractor can be characterized as the unstable manifold of the set $\mathcal{E}$ of all equilibria. Moreover, by LaSalle's principle, the ω-limit set of any single trajectory is contained in $\mathcal{E}$.

In the case of unbounded domains, the study of attractors has been initiated about ten years ago by Babin and Vishik [BV]. More recently, the existence of a global attractor has been proved by Feireisl [Fe] for the damped wave equation and by Mielke and Schneider [MS] for the reaction-diffusion equation, using the uniformly local Sobolev spaces introduced by Kato [Ka]. These spaces are large enough to contain various solutions of physical interest, such as traveling waves, periodic or quasi-periodic patterns, and self-similar diffusive profiles. However, the resulting attractor is compact and attracting in a weak, localized topology only, and no characterization of its content has been obtained yet, even in very simple situations [HN], [M2]. Moreover, examples have been constructed which show that solutions to such extended gradient systems do not necessarily converge to the set of equilibria as $t \to \infty$, even in a localized topology [ER]. It is thus interesting to understand which kind of restrictions are imposed on the behavior of the solutions by the underlying gradient structure.

Recently, this question has been investigated by the second author for a class of abstract dynamical systems, called "extended gradient semiflows" [Sl]. These systems are characterized by the existence of an infinite sequence (H_n, A_n) of approximate Lyapunov functions H_n and of "correction functions" A_n, where A_n measures by how much H_n fails to be an exact Lyapunov function. In the applications to extended dissipative systems, H_n is typically the energy contained in a ball of radius n in physical space, while A_n represents the energy flux entering this ball through the boundary. As a consequence of a few simple axioms, interesting conclusions can be drawn about the dynamics of such systems. For instance, nontrivial periodic orbits cannot exist, and any nonequilibrium point has a neighborhood which is never visited on average by the trajectories of the system. It then follows from Birkhoff's ergodic theorem that all invariant measures are supported on the set $\mathcal{E}$ of equilibria.

Although they were motivated mainly by the desire of understanding the dynamics of discrete dynamical systems, such as the Frenkel-Kontorova model, the abstract results in [Sl] also apply to some nonlinear partial differential equations in one dimension, like the Ginzburg-Landau equation on the real line. Higher-dimensional equations, however, do not fit directly into the abstract setting. Thus, an important question is whether the underlying gradient structure still imposes restrictions on the dynamics of partial differential equations on $\mathbf{R}^N$ if $N \geq 2$. Following the same lines of thought as in [Sl], we show in this paper, for a typical example, that the answer is positive at least if $N = 2$.

To be specific, we consider the damped hyperbolic equation

$$\alpha u_{tt} + u_t \;=\; \Delta u + F(x,u)\;, \quad x \in \mathbf{R}^N\;, \quad t \in \mathbf{R}\;, \tag{1.1}$$

where α is a positive parameter. The parabolic case $\alpha = 0$ is included, and will be treated simultaneously. We assume that the nonlinearity $F : \mathbf{R}^{N+1} \to \mathbf{R}$ satisfies

$$\begin{aligned} |F(x,u) - F(y,u)| \;&\le\; C_0\omega(|x-y|)(1+|u|^{\beta+1})\;, \qquad && x,y \in \mathbf{R}^N\;, \\ |F(x,u) - F(x,v)| \;&\le\; C_0(1+|u|^\beta + |v|^\beta)|u-v|\;, \qquad && u,v \in \mathbf{R}\;, \end{aligned} \tag{1.2}$$

where $C_0 > 0$, $\beta > 0$, and $\omega : \mathbf{R}_+ \to [0,1]$ is such that $\omega(r) \to 0$ as $r \to 0+$. For instance, the conditions (1.2) are fulfilled if $F(x,u)$ is a polynomial in u of degree $d \le \beta + 1$, the coefficients of which are bounded, uniformly continuous functions of $x \in \mathbf{R}^N$. For simplicity, we also assume that $(N-2)\beta \le 2$, but we believe that most of the arguments below extend to the whole subcritical range $(N-2)\beta < 4$.

Following [Fe] and [MS], we study the solutions of (1.1) in the uniformly local Sobolev spaces $\mathrm{H}^k_{\mathrm{ul}}(\mathbf{R}^N)$, which provide a natural framework for the use of localized energy estimates. For easy reference, the definition and the basic properties of these Banach spaces are recalled in the Appendix below (Section 7), which includes a short discussion of the Cauchy problem for Eq.(1.1). The results in Section 7 can be summarized as follows. In the hyperbolic case $\alpha > 0$, we set $w = (u, u_t)$ and we look for solutions w of (1.1) in the "energy" space $X = \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$. Then, for all $w_0 \in X$, there exists $T > 0$ such that (1.1) has a unique (mild) solution $w \in \mathcal{C}^0([0,T], X)$ satisfying $w(0) = w_0$. This solution $w(t)$ depends continuously on the initial data w_0 in the topology of X, uniformly in $t \in [0,T]$. Moreover, if $w_0 \in Y = \mathrm{H}^2_{\mathrm{ul}}(\mathbf{R}^N) \times \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$ and if F satisfies additional regularity assumptions, then $w \in \mathcal{C}^1([0,T], X) \cap \mathcal{C}^0([0,T], Y)$ is a classical solution of (1.1). In the parabolic case $\alpha = 0$, we set $w = u$, $X = \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$, and $Y = \mathrm{H}^2_{\mathrm{ul}}(\mathbf{R}^N)$. Then, for all $w_0 \in X$, there exists $T > 0$ such that (1.1) has a unique solution $w \in \mathcal{C}^0([0,T], X) \cap \mathcal{C}^1((0,T], X) \cap \mathcal{C}^0((0,T], Y)$ satisfying $w(0) = w_0$. Again, this solution depends continuously on the initial data $w_0 \in X$. Remark that, unlike in the hyperbolic case, any solution of (1.1) with $\alpha = 0$ is classical for $t > 0$.

Given any $w \in X$, we define the energy density

$$h(w) \;=\; \frac{\alpha u_t^2}{2} + \frac{|\nabla u|^2}{2} + V(x,u)\;, \tag{1.3}$$

where $V(x,u) = -\int_0^u F(x,v)\,\mathrm{d}v$ is the potential associated to the nonlinearity F. The assumptions (1.2) imply that $h(w) \in \mathrm{L}^1_{\mathrm{ul}}(\mathbf{R}^N) \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^N)$, and that the mapping $h : X \to \mathrm{L}^1_{\mathrm{ul}}(\mathbf{R}^N)$ is $\mathcal{C}^1$, see Section 7.2. The starting point of all developments in this paper is the following elementary observation: if $w \in \mathcal{C}^1([0,T],X) \cap \mathcal{C}^0([0,T],Y)$ is any classical solution of (1.1), then the energy density $h(w)$ satisfies the *dissipation law*

$$\frac{\mathrm{d}}{\mathrm{d}t} h(w) + \mathrm{div}\, j(w) \;=\; -\lambda(w)\;, \quad t \in [0,T]\;, \tag{1.4}$$

where $j(w) = -u_t \nabla u$ is the energy flux (density) and $\lambda(w) = u_t^2$ the energy dissipation (density). As we shall show, the dissipation law (1.4) imposes serious restrictions on the behavior of the solutions of (1.1) in X, at least if $N \le 2$.

In the sequel, we mainly use the integrated version of (1.4). For any $w \in Y$ and all $R > 0$, we define

$$H_R(w) \;=\; \int_{B_R} h(w)\,\mathrm{d}x\;, \quad J_R(w) \;=\; -\int_{\partial B_R} (j(w)\cdot\nu)\,\mathrm{d}\sigma\;, \quad \Lambda_R(w) \;=\; \int_{B_R} \lambda(w)\,\mathrm{d}x\;,$$

where B_R denotes the ball of radius R centered at zero in $\mathbf{R}^N$, $\nu = x/|x|$ is the exterior unit normal to the sphere ∂B_R, and σ is the Euclidean surface measure. Clearly, $H_R(w)$ can be interpreted as the amount of energy contained in the ball B_R, $J_R(w)$ is the energy flux entering this ball from outside, and $\Lambda_R(w)$ is the energy dissipation rate inside B_R. If $w \in \mathcal{C}^1([0,T],X) \cap \mathcal{C}^0([0,T],Y)$ is a classical solution of (1.1), we deduce from (1.4) that the following *balance equation* is satisfied

$$\frac{\mathrm{d}}{\mathrm{d}t} H_R(w) \;=\; J_R(w) - \Lambda_R(w)\;, \quad R > 0\;, \quad t \in [0,T]\;. \tag{1.5}$$

Of course, the relation (1.5) for a single $R > 0$ does not restrict the dynamics of Eq.(1.1) very much, because the boundary term J_R prevents H_R of being a Lyapunov function. However, as was observed in [Sl], the study of the whole family $(H_R)_{R>0}$ provides valuable information on the behavior of the solutions. A simple, somewhat naive result in this direction is the following. Assume that $\alpha > 0$, and let

$$\begin{aligned} \overline{h}(w) &= \limsup_{R\to+\infty} \frac{1}{R^N} H_R(w)\;, \quad & \overline{\lambda}(w) &= \limsup_{R\to+\infty} \frac{1}{R^N} \Lambda_R(w)\;, \\ \underline{h}(w) &= \liminf_{R\to+\infty} \frac{1}{R^N} H_R(w)\;, \quad & \underline{\lambda}(w) &= \liminf_{R\to+\infty} \frac{1}{R^N} \Lambda_R(w)\;, \end{aligned} \tag{1.6}$$

for all $w \in X$. As is easily verified (see Section 7.2), the limits in (1.6) are finite, and define continuous functions of $w \in X$.

Proposition 1.1. *Assume $\alpha > 0$ and that $w \in \mathcal{C}^0([0,T],X)$ is a solution of (1.1). Then*

$$\int_0^T \underline{\lambda}(w(t))\,\mathrm{d}t \;\le\; \overline{h}(w(0)) - \overline{h}(w(T)) \;\le\; \int_0^T \overline{\lambda}(w(t))\,\mathrm{d}t\;, \tag{1.7}$$

and the same inequalities hold for $\underline{h}(w)$ instead of $\overline{h}(w)$.

Proof. By density, it is sufficient to prove (1.7) for solutions w of (1.1) such that $w(0) \in Y$. In this case, although the solution w is in general not classical, we know that $w \in \mathcal{C}^1([0,T], \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^N))$ and that the norm $\|w(t)\|_Y$ stays bounded for $t \in [0,T]$, see Section 7.2. In particular, the dissipation law (1.4) is valid in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^N)$, so that the balance equation (1.5) holds for all $R > 0$. Integrating (1.5) with respect to time, we obtain

$$\frac{1}{R^N} H_R(w(T)) \;=\; \frac{1}{R^N} H_R(w(0)) + \frac{1}{R^N}\int_0^T \bigl(J_R(w(t)) - \Lambda_R(w(t))\bigr)\,\mathrm{d}t\;, \tag{1.8}$$

for all $R > 0$. Now, as is mentioned at the end of Section 7.2, there exists $C > 0$ such that $|J_R(w)| \le C R^{N-1} \|\nabla u\|_{\mathrm{H}^1_{\mathrm{ul}}} \|u_t\|_{\mathrm{H}^1_{\mathrm{ul}}} \le C R^{N-1} \|w\|_Y^2$ for all $R \ge 1$, hence

$$\lim_{R\to+\infty} \frac{1}{R^N}\int_0^T J_R(w(t))\,\mathrm{d}t \;=\; 0\;.$$

Therefore, we deduce from (1.8) that

$$\begin{aligned}\liminf_{R\to+\infty} \frac{1}{R^N}\int_0^T \Lambda_R(w(t))\,\mathrm{d}t \;&\le\; \overline{h}(w(0)) - \overline{h}(w(T)) \\ &\le\; \limsup_{R\to+\infty} \frac{1}{R^N}\int_0^T \Lambda_R(w(t))\,\mathrm{d}t\;,\end{aligned} \tag{1.9}$$

and (1.7) follows by Fatou's Lemma. The proof for $\underline{h}(w)$ is similar. $\square$

Remark. In the parabolic case $\alpha = 0$, the quantities $\overline{\lambda}(w)$ and $\underline{\lambda}(w)$ are well defined only if $w \in Y = \mathrm{H}^2_{\mathrm{ul}}(\mathbf{R}^N)$. If $w(0) \in Y$, the solution of (1.1) satisfies $w \in \mathcal{C}^0([0,T],Y)$, hence (1.7) holds, with a similar proof. In the general case where $w(0) \in X$ only, Eq.(1.7) has to be replaced with Eq.(1.9) above, which is proved by density, see Section 7.3.

Proposition 1.1 shows in particular that the quantities $\overline{h}$ and $\underline{h}$ defined in (1.6) are Lyapunov functions for the system (1.1) in X. However, these Lyapunov functions are highly non strict, since many $w \in X$ satisfy $\underline{\lambda}(w) = \overline{\lambda}(w) = 0$ without being equilibria of (1.1). This is the case for instance whenever $u_t(x,t)$ converges to zero as $|x| \to \infty$. For such solutions, $\overline{h}(w)$ and $\underline{h}(w)$ are time-independent, and (1.7) is trivially satisfied.

In the rest of this paper, we derive other consequences of the energy balance equation (1.5) when $N \le 2$. In Section 2, we prove that there exists no nontrivial solution $w \in \mathcal{C}^0([0,T],X)$ of (1.1) satisfying $H_R(w(T)) \ge H_R(w(0))$ for all sufficiently large $R > 0$. This in particular precludes the existence of periodic orbits. In Section 3, we give counter-examples showing that such a result does not hold in higher dimensions. Section 4 is devoted to studying the recurrence properties of general trajectories of (1.1). In particular, we give an estimate on the time needed for a bounded trajectory to return in a neighborhood of the initial point, and we discuss the existence of homoclinic or heteroclinic orbits. In Section 5, we prove that, if $w \in \mathcal{C}^0([0,+\infty),X)$ is a solution of (1.1) which stays bounded in X, then the time average $T^{-1}\int_0^T \Lambda_R(w(t))\,\mathrm{d}t$ converges to zero as $T \to \infty$ for any $R > 0$. This implies that any nonequilibrium point has a neighborhood in X which is never visited on average by the trajectories of the system. We deduce that uniformly recurrent orbits cannot exist, except for equilibria, and that the ω-limit set (with respect to some localized topology) of any bounded trajectory contains an equilibrium. This last statement is a weaker version of LaSalle's principle. Finally, open problems and possible generalizations are briefly mentioned in Section 6. Section 7 is an appendix which contains the definitions and some basic properties of the uniformly local Sobolev spaces $\mathrm{H}^k_{\mathrm{ul}}(\mathbf{R}^N)$, together with a discussion of the Cauchy problem for Eq.(1.1) in these spaces.

Acknowledgements. Part of this work was done when the second author (S.S) was a PhD student at the University of Cambridge, Department of Applied Mathematics and Theoretical Physics.

2. Nonexistence of energy increasing solutions

To formulate our results, it is convenient to equip the space $X = \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$ (respectively, $X = \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$ if $\alpha = 0$) with the following "preorder" relation:

Definition. Let $w_1, w_2 \in X$. We say that $w_1 \succeq w_2$ (resp. $w_1 \succ w_2$) if there exists $R_0 > 0$ such that

$$H_R(w_1) \ge H_R(w_2) \quad \text{(resp. } H_R(w_1) > H_R(w_2)\text{)} \quad \text{for all } R \ge R_0\ .$$

We recall that $H_R(w)$ is the integral of the energy density (1.3) over the ball $B_R = \{x \in \mathbf{R}^N \,|\, |x| \le R\}$. The relation $\succeq$ is reflexive, transitive, but not antisymmetric; it thus induces a partial order on the quotient space $X/{\sim}$, where $\sim$ is the equivalence relation defined by $w_1 \sim w_2$ if $H_R(w_1) = H_R(w_2)$ for all sufficiently large $R > 0$.

The main result of this section is:

Proposition 2.1. *Assume that $N \le 2$, and that $w \in \mathcal{C}^0([0,T],X)$ is a solution of (1.1) satisfying $w(T) \succeq w(0)$. Then $w(t) = w(0)$ for all $t \in [0,T]$.*

Remark. In particular, if $N \le 2$, Proposition 2.1 precludes the existence of (nontrivial) periodic orbits of Eq.(1.1) in the space X.

Proof. Assume first that $w \in \mathcal{C}^1([0,T],X) \cap \mathcal{C}^0([0,T],Y)$ is a classical solution of (1.1) satisfying $H_R(w(T)) \ge H_R(w(0))$ for all $R \ge R_0 > 0$. Integrating the balance equation (1.5) over $t \in [0,T]$, we obtain, for $R \ge R_0$,

$$\int_0^T J_R(w(t))\,\mathrm{d}t - \int_0^T \Lambda_R(w(t))\,\mathrm{d}t \;=\; H_R(w(T)) - H_R(w(0)) \;\ge\; 0 \;, \qquad (2.1)$$

hence

$$\int_0^T \Lambda_R(w(t))\,\mathrm{d}t \;\le\; \int_0^T |J_R(w(t))|\,\mathrm{d}t \;, \quad R \ge R_0 \;. \qquad (2.2)$$

To simplify the notations, we define, for all $R > 0$,

$$f(R) \;=\; \int_0^T \Lambda_R(w(t))\,\mathrm{d}t \;=\; \int_0^T \int_{B_R} u_t^2\,\mathrm{d}x\,\mathrm{d}t \;, \quad g(R) \;=\; \int_0^T \int_{\partial B_R} |\nabla u|^2\,\mathrm{d}\sigma\,\mathrm{d}t \;.$$

Since $|\nabla u| \in \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$, the function $g : \mathbf{R}_+ \to \mathbf{R}$ is Hölder continuous, and there exists $C > 0$ such that $g(R) \le CR^{N-1}$ for all $R \ge 1$, see Section 7.1. Similarly, $f : \mathbf{R}_+ \to \mathbf{R}$ is continuously differentiable, and

$$f'(R) \;=\; \int_0^T \int_{\partial B_R} u_t^2\,\mathrm{d}\sigma\,\mathrm{d}t \;, \quad R > 0 \;. \qquad (2.3)$$

Using the definition of $J_R(w)$ and Hölder's inequality, we thus find

$$\int_0^T |J_R(w(t))|\,\mathrm{d}t \;\le\; \int_0^T \int_{\partial B_R} |u_t \nabla u|\,\mathrm{d}\sigma\,\mathrm{d}t \;\le\; \bigl(f'(R)g(R)\bigr)^{1/2} \;, \quad R \ge R_0 \;,$$

hence (2.2) implies the following differential inequality for f:

$$f(R) \le \left(f'(R)g(R)\right)^{1/2} , \quad R \ge R_0 . \tag{2.4}$$

Integrating this inequality, we obtain the bound

$$f(R) \le \left(\int_R^\infty \frac{1}{g(r)}\,\mathrm{d}r\right)^{-1} , \quad R \ge R_0 . \tag{2.5}$$

Indeed, if $f(R) > 0$ for some $R \ge R_0$, then $f(r) \ge f(R) > 0$ for all $r \ge R$. By (2.4), it follows that $g(r) > 0$ and $f'(r) \ge f(r)^2/g(r)$ for all $r \ge R$, hence

$$\int_R^{R_1} \frac{1}{g(r)}\,\mathrm{d}r \le \frac{1}{f(R)} - \frac{1}{f(R_1)} \le \frac{1}{f(R)} , \quad R_1 \ge R .$$

Taking the limit $R_1 \to +\infty$, we arrive at (2.5). Now, if $N \le 2$, the bound $g(R) \le CR^{N-1}$ implies that $\int_R^\infty g(r)^{-1}\,\mathrm{d}r = \infty$ for any $R > 0$. By (2.5), it follows that $f(R) = 0$ for all $R \ge R_0$, which means that $w(t) = w(0)$ for all $t \in [0,T]$. This proves the claim for classical solutions.

We now consider the general case where $w \in \mathcal{C}^0([0,T],X)$ is a mild solution only. Using (2.1) and a density argument (see the proof of Proposition 1.1 if $\alpha > 0$, or Section 7.3 if $\alpha = 0$), it is easy to show that

$$\int_0^T \int_R^{R+1} \Lambda_r(w(t))\,\mathrm{d}r\,\mathrm{d}t \le \int_0^T \int_R^{R+1} J_r(w(t))\,\mathrm{d}r\,\mathrm{d}t , \tag{2.6}$$

for all $R \ge R_0$. Proceeding as in the previous case, we introduce the averaged quantities

$$\widetilde{f}(R) = \int_R^{R+1} f(r)\,\mathrm{d}r , \quad \widetilde{g}(R) = \int_R^{R+1} g(r)\,\mathrm{d}r , \quad R > 0 . \tag{2.7}$$

Then $\widetilde{f} \in \mathcal{C}^1(\mathbf{R}_+)$, $\widetilde{g} \in \mathcal{C}^0(\mathbf{R}_+)$, and it follows from (2.6) and Hölder's inequality that $\widetilde{f}(R) \le (\widetilde{f}'(R)\widetilde{g}(R))^{1/2}$ for all $R \ge R_0$. Moreover, there exists $C > 0$ such that $\widetilde{g}(R) \le CR^{N-1}$ for all $R \ge 1$. Therefore, if $N \le 2$, we conclude that $\int_R^\infty \widetilde{g}(r)^{-1}\,\mathrm{d}r = \infty$, hence $\widetilde{f}(R) = 0 = f(R)$ for all $R \ge R_0$. The proof of Proposition 2.1 is now complete. $\square$

Remarks.

1. It is important to realize that Eq.(1.1) is not fully used in the proof of Proposition 2.1: all that we need is the dissipation law (1.4), together with the formulas for the current

j and the dissipation λ. Surprisingly enough, the energy density h does not play any role, provided it is locally integrable over $x \in \mathbf{R}^N$. As a consequence, the conclusion of Proposition 2.1 remains valid in more general situations. For instance, if $\alpha > 0$, we may assume that the constant C_0 in (1.2) is replaced by a locally bounded function of x. Regarding the solution $w = (u, u_t)$, it is then sufficient to suppose that $w \in \mathcal{C}^0([0,T], \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^N))$, provided $\nabla u \in \mathcal{C}^0([0,T], \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N))$. Alternatively, we may drop the assumptions (1.2) completely if F is smooth and if we consider only regular solutions, say $w \in \mathrm{H}^{k+1}_{\mathrm{loc}}(\mathbf{R}^N) \times \mathrm{H}^k_{\mathrm{loc}}(\mathbf{R}^N)$ for $k > N/2$ and $\nabla u \in \mathrm{L}^\infty(\mathbf{R}^N)$.

2. The assumption $\nabla u \in \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$ or $\nabla u \in \mathrm{L}^\infty(\mathbf{R}^N)$, which implies the bound $\widetilde{g}(R) \le CR^{N-1}$, is of course essential in our argument. If it is relaxed, then energy increasing solutions can exist in any space dimension. For instance, the function $u(x,t) = x^2 + 2t$ is a solution of the one-dimensional heat equation $u_t = u_{xx}$, is not time-independent, and satisfies $H_R(w(t)) = H_R(w(0))$ for all $t \in \mathbf{R}$ and all $R > 0$. In the next section, further examples will be presented which show that the condition $\nabla u \in \mathrm{L}^\infty$ is responsible for the difference between the dimensions $N \le 2$ and $N > 2$.

3. If $N > 2$ and if $w \in \mathcal{C}^1([0,T], X) \cap \mathcal{C}^0([0,T], Y)$ is a classical solution of (1.1) satisfying $w(T) \succeq w(0)$, then by (2.5) there exists $C > 0$ such that

$$f(R) \;=\; \int_0^T \int_{B_R} u_t^2 \,\mathrm{d}x\,\mathrm{d}t \;\le\; CR^{N-2}\;,$$

for all sufficiently large $R > 0$. The same result holds for $\widetilde{f}(R)$ in the general case. This estimate shows that the motion of any energy increasing solution is in some sense "localized" in space.

4. The reader may wonder why the particular case of classical solutions is treated separately in the proof of Proposition 2.1, whereas the argument we give in the general case is completely independent. The reason is that expressions such as J_R, Λ_R, or $f(R)$ have a direct interpretation in terms of energy flux or energy dissipation, while the physical meaning of the averaged quantities (2.7) is much less transparent. From our point of view, the essence of the argument is better understood for classical solutions, averaging over R being just a technical trick to obtain the result in the general case.

In the case where $N = 1$ and w is a classical solution, we can strengthen Proposition 2.1 as follows:

Proposition 2.2. *Assume that $N = 1$ and that $w \in \mathcal{C}^1([0,T], X) \cap \mathcal{C}^0([0,T], Y)$ is a* *nonequilibrium classical solution of (1.1). Then* $w(0) \succ w(T)$.

Proof. Assume on the contrary that there exists a sequence $R_k \to +\infty$ such that $H_{R_k}(w(T)) \ge H_{R_k}(w(0))$ for all $k \in \mathbf{N}$. In particular, the inequalities (2.2), (2.4) hold with $R = R_k$ for all $k \in \mathbf{N}$. Since $N = 1$, there exists $C > 0$ such that, for all $R > 0$,

$$\int_0^T |J_R(w(t))| \,\mathrm{d}t \le C\ , \quad \text{and} \quad g(R) = \int_0^T \left(|u_x(R,t)|^2 + |u_x(-R,t)|^2\right) \mathrm{d}t \le C\ ,$$

see Section 7. Therefore, (2.2) implies that $f(R_k) \le C$ for all $k \in \mathbf{N}$, hence $f(R) \le C$ for all $R > 0$ because f is nondecreasing. On the other hand, since w is a nonequilibrium solution, we can find $R_0 > 0$ and $\varepsilon > 0$ such that $f(R) \ge \varepsilon$ for all $R \ge R_0$. In view of (2.4), we thus have

$$\varepsilon^2 \le f^2(R_k) \le C f'(R_k)\ , \tag{2.8}$$

for all $k \in \mathbf{N}$ such that $R_k \ge R_0$. Now, since $N = 1$ and $u_t \in \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R})$, the function f' defined by (2.3) belongs to $\mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}_+)$, hence is uniformly Hölder continuous on $\mathbf{R}_+$. Therefore, there exists $\delta > 0$ such that

$$f'(R) \ge \frac{\varepsilon^2}{2C}\ , \quad R \in [R_k - \delta, R_k + \delta]\ , \tag{2.9}$$

for all $k \in \mathbf{N}$ such that $R_k \ge R_0$. This of course contradicts the fact that f is nondecreasing and uniformly bounded. Since the sequence (R_k) was arbitrary, we conclude that $H_R(w(0)) > H_R(w(T))$ for all sufficiently large $R > 0$. $\square$

3. Counter-examples in higher dimensions

In this section, we give two counter-examples showing that Proposition 2.1, or rather its generalizations suggested in Remark 1 above, do not hold in higher dimensions. For simplicity, we assume in both examples that the function $F(x,u)$ in (1.1) is linear in u and radially symmetric in x, and we restrict ourselves to radially symmetric solutions.

3.1. A trajectory with strictly increasing energy

Let p, q be two real numbers satisfying $0 \le p \le q$ and $p+q < N$. We define the functions

$$\psi(r) \;=\; \frac{1}{(1+r^2)^{q/2}}\;, \quad \varphi(r) \;=\; \int_0^r \frac{s}{(1+s^2)^{p/2}}\,\mathrm{d}s\;, \quad r \ge 0\;.$$

Then ψ solves the differential equation

$$\psi''(r) + \frac{N-1}{r}\psi'(r) + V(r)\psi(r) \;=\; 0\;, \quad r > 0\;,$$

where

$$V(r) \;=\; \frac{q(q+2)}{(1+r^2)^2} + \frac{q(N-q-2)}{1+r^2}\;.$$

Next, fix any $\gamma > 0$, and let $\varepsilon > 0$ be sufficiently small so that $\varepsilon q < 1$ and $\varepsilon\gamma \le N-p-q$. We consider the function $u : \mathbf{R}^N \times \mathbf{R}_+ \to \mathbf{R}$ defined by

$$u(x,t) \;=\; \varphi(|x|) + \delta(t)\psi(|x|)\;, \quad x \in \mathbf{R}^N\;, \quad t \ge 0\;,$$

where $\delta(t) = \varepsilon(1 - \mathrm{e}^{-\gamma t})$. It is easy to verify that u is a solution of the linear diffusion equation

$$u_t(x,t) \;=\; \Delta u(x,t) + (V(|x|) - \gamma)u(x,t) + W(|x|)\;, \tag{3.1}$$

where

$$W(r) \;=\; \gamma(\varphi(r) + \varepsilon\psi(r)) - \Big(\varphi''(r) + \frac{N-1}{r}\varphi'(r) + V(r)\varphi(r)\Big)\;, \quad r \ge 0\;.$$

We now study the energy density of this solution, namely

$$h(x,t) \;=\; \frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{2}(\gamma - V(|x|))u(x,t)^2 - W(|x|)u(x,t)\;.$$

A direct calculation shows that

$$\frac{\partial h}{\partial t}(x,t) \;=\; \delta'(t)\big(A(|x|) + \delta(t)B(|x|)\big)\;,$$

where

$$\begin{aligned} A(r) \;&=\; \varphi'(r)\psi'(r) + (\gamma - V(r))\varphi(r)\psi(r) - W(r)\psi(r) \\ &=\; \varphi'(r)\psi'(r) + \psi(r)\Big(\varphi''(r) + \frac{N-1}{r}\varphi'(r)\Big) - \varepsilon\gamma\psi(r)^2 \\ &=\; \frac{1}{(1+r^2)^{(p+q)/2}}\Big(N - (p+q)\frac{r^2}{1+r^2}\Big) - \frac{\varepsilon\gamma}{(1+r^2)^q}\;, \end{aligned}$$

and

$$B(r) \;=\; \psi'(r)^2 + (\gamma - V(r))\psi(r)^2 \;=\; \frac{\gamma}{(1+r^2)^q} - \frac{q}{(1+r^2)^{q+1}}\Big(N - 2(q+1)\frac{r^2}{1+r^2}\Big)\ .$$

Since $q \ge p$ and $p+q+\varepsilon\gamma \le N$, it is clear that $A(r) > 0$ for all $r \ge 0$. On the other hand, using the inequalities $\varepsilon q < 1$ and $2(q+1) \ge p+q$, we obtain

$$\begin{aligned} A(r) + \varepsilon B(r) \;&= \\ &= \frac{1}{(1+r^2)^{(p+q)/2}}\Big(N - (p+q)\frac{r^2}{1+r^2}\Big) - \frac{\varepsilon q}{(1+r^2)^{q+1}}\Big(N - 2(q+1)\frac{r^2}{1+r^2}\Big) \\ &\ge \frac{1}{(1+r^2)^{(p+q)/2}}\Big(N - (p+q)\frac{r^2}{1+r^2}\Big)(1-\varepsilon q) > 0\ , \quad r \ge 0\ . \end{aligned}$$

Since $\delta'(t) > 0$ and $0 \le \delta(t) < \varepsilon$ for all $t \ge 0$, we conclude that the energy density h is strictly increasing with time:

$$\frac{\partial h}{\partial t}(x,t) \;>\; 0\ , \quad x \in \mathbf{R}^N\ , \quad t \ge 0\ .$$

In particular, the solution $u(x,t)$ of (3.1) satisfies $u(\cdot,t) \succ u(\cdot,t')$ for all $t > t' \ge 0$.

Now, if $N > 2$, we can choose $1 \le p \le q < N/2$, so that $\varphi' \in \mathrm{L}^\infty(\mathbf{R}_+)$, hence $\nabla u \in \mathrm{L}^\infty(\mathbf{R}^N)$. This shows that the energy arguments used in the proof of Proposition 2.1 cannot be extended to higher dimensions. In particular, the generalizations suggested in Remark 1 (Section 2) fail as soon as $N > 2$. If $N > 4$, we can even assume that $2 < p \le q < N/2$, so that $\varphi \in \mathrm{L}^\infty(\mathbf{R}_+)$ and $W \in \mathrm{L}^\infty(\mathbf{R}_+)$. In this case, the example constructed here fits exactly into the setting of Proposition 2.1, and shows that the conclusion fails in high dimensions.

3.2. A periodic orbit

Let $\rho, \varphi : [0,+\infty) \to \mathbf{R}$ be smooth functions satisfying $\rho(r) > 0$ for all $r \ge 0$, $\rho'(0) = 0$, $\varphi(r) > 0$ for all $r > 0$, and $\varphi(0) = 0$. We set $\Phi(r) = \int_0^r \varphi(s)\,\mathrm{d}s$, $r \ge 0$. Given any $\omega \neq 0$, we consider the function $u : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ defined by

$$u(x,t) \;=\; \rho(|x|)\sin\big(\omega(\Phi(|x|) + t)\big)\ , \quad x \in \mathbf{R}^N\ , \quad t \in \mathbf{R}\ . \tag{3.2}$$

It is straightforward to verify that u solves the linear diffusion equation

$$u_t(x,t) \;=\; \Delta u(x,t) + W(|x|)u(x,t)\ , \quad x \in \mathbf{R}^N, \quad t \in \mathbf{R}\ , \tag{3.3}$$

if and only if the following relations hold for all $r > 0$:

$$\begin{aligned} 0 \;&=\; \rho''(r) - \omega^2\rho(r)\varphi(r)^2 + \frac{N-1}{r}\rho'(r) + W(r)\rho(r)\;, \\ \rho(r) \;&=\; 2\rho'(r)\varphi(r) + \rho(r)\varphi'(r) + \frac{N-1}{r}\rho(r)\varphi(r)\;. \end{aligned} \tag{3.4}$$

The first equation in (3.4) defines the potential W in terms of the functions ρ and φ:

$$W(r) \;=\; \omega^2\varphi(r)^2 - \left(\frac{\rho''(r)}{\rho(r)} + \frac{N-1}{r}\frac{\rho'(r)}{\rho(r)}\right)\;, \quad r > 0\;. \tag{3.5}$$

The second relation is a differential equation which allows to determine ρ in terms of φ (or the converse). After elementary calculations, we find

$$\rho(r)^2 \;=\; \frac{C_0}{\varphi(r)r^{N-1}}\exp\left(\int_1^r \frac{1}{\varphi(s)}\,\mathrm{d}s\right)\;, \quad r > 0\;, \tag{3.6}$$

where C_0 is an arbitrary positive constant. As is easily verified, the function ρ defined by (3.6) satisfies $\rho(0) > 0$ and $\rho'(0) = 0$ if and only if $\varphi'(0) = 1/N$ and $\varphi''(0) = 0$.

As was explained in the remarks after Proposition 2.1, we are interested in solutions u of (3.3) satisfying $\nabla u \in \mathrm{L}^\infty(\mathbf{R}^N)$. In view of (3.2), this is equivalent to $\rho' \in \mathrm{L}^\infty(\mathbf{R}_+)$ and $\rho\varphi \in \mathrm{L}^\infty(\mathbf{R}_+)$. If ρ and φ are related through (3.6), the second condition becomes

$$\varphi(r)\exp\left(\int_1^r \frac{1}{\varphi(s)}\,\mathrm{d}s\right) \;\le\; C_1 r^{N-1}\;, \quad r > 0\;, \tag{3.7}$$

for some $C_1 > 0$. Setting $G(r) = \exp(\int_1^r \varphi(s)^{-1}\,\mathrm{d}s)$, we see that G satisfies the differential inequality

$$G(r)^2 \;\le\; C_1 r^{N-1} G'(r)\;, \quad r > 0\;,$$

which has no nontrivial solution if $N \le 2$. Therefore, the condition (3.7) cannot be fulfilled if $N \le 2$. On the other hand, it is easy to construct solutions φ of (3.7) if $N > 2$. An example is

$$\varphi(r) \;=\; \frac{r(1+r^2)}{N + (N-2p)r^2}\;, \quad r \ge 0\;,$$

where $1 < p < N/2$. In this case, the function ρ defined by (3.6) (with $C_0 = 2^{-p}$) becomes

$$\rho(r) \;=\; \frac{1}{(1+r^2)^{p/2}}\left(N - 2p\frac{r^2}{1+r^2}\right)^{1/2}\;, \quad r \ge 0\;.$$

With this choice of ρ and φ, the function u defined by (3.2) is bounded and converges to zero as $|x| \to \infty$. The same properties hold for the gradient ∇u, so that $u \in \mathcal{C}^0(\mathbf{R}, \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N))$, and u is periodic in time. This example shows that extended gradient partial differential equations can have nontrivial recurrence properties if $N > 2$. Note, however, that the potential $W(|x|)$ in (3.3) has a quadratic growth as $|x| \to \infty$, so that the conditions (1.2) are not satisfied in this particular case.

4. Recurrent and nonrecurrent trajectories

The proof of Proposition 2.1 shows that, if $N \le 2$ and if $w \in \mathcal{C}^0([0,T], X)$ is a nonstationary solution of (1.1), then the energy Λ_R that is dissipated in a sufficiently large ball B_R cannot be compensated by the flux J_R coming from outside, so that the total energy H_R contained in B_R decreases with time. In this section, we extend this argument and obtain a lower bound for the energy loss $H_R(w(0)) - H_R(w(T))$. This will allow us to draw some conclusions about recurrent trajectories or homoclinic orbits of (1.1).

As in Section 2, it is sufficient to work with classical solutions, the extension to the general case being straightforward. We thus assume that $w \in \mathcal{C}^1([0,T], X) \cap \mathcal{C}^0([0,T], Y)$ is a classical solution of (1.1), and we define

$$f(R,T) \;=\; \int_0^T \int_{B_R} u_t^2 \,\mathrm{d}x \,\mathrm{d}t \;, \quad g(R,T) \;=\; \int_0^T \int_{\partial B_R} |\nabla u|^2 \,\mathrm{d}\sigma \,\mathrm{d}t \;, \quad R > 0 \;. \tag{4.1}$$

Proceeding as in the proof of Proposition 2.1, it is easy to verify that

$$f(R,T) \;\le\; (f'(R,T) g(R,T))^{1/2} + \Delta(R,T) \;, \quad R > 0 \;, \tag{4.2}$$

where $f'(R,T) = \partial_R f(R,T)$ and $\Delta(R,T) = H_R(w(0)) - H_R(w(T))$. If $N \le 2$, we shall show that the differential inequality (4.2) for f has nontrivial solutions only if the functions g and Δ obey some compatibility conditions. To keep the analysis as simple as possible, we replace g and Δ in (4.2) by the following upper bounds, which can be derived from the definitions of the spaces X, Y (see Section 7). There exist $\gamma(T) > 0$, $\delta(T) > 0$ and $R_0 > 0$ such that

$$g(R,T) \;\le\; \frac{1}{4}\gamma(T) R^{N-1} \;, \quad \Delta(R,T) \;\le\; \frac{1}{2}\delta(T) R^N \;, \quad \text{for all } R \ge R_0 \;. \tag{4.3}$$

We set $R_*(T) = (N\gamma(T)/\delta(T))^{1/2}$, and we introduce the functions $\mathcal{X}_1, \mathcal{X}_2 : \mathbf{R}_+ \to (0,1]$ defined by

$$\mathcal{X}_1(s) \;=\; \begin{cases} s(2-s) & \text{if } 0 < s < 1 \;, \\ 1 & \text{if } s \ge 1 \;, \end{cases} \qquad \mathcal{X}_2(s) \;=\; \begin{cases} s^2(1 - 2\log s) & \text{if } 0 < s < 1 \;, \\ 1 & \text{if } s \ge 1 \;. \end{cases}$$

Remark that $\mathcal{X}_N(s) \leq 1$ $(N = 1, 2)$ for all $s > 0$.

Lemma 4.1. *Assume that $N \leq 2$, and that f is a solution of the differential inequality (4.2), where g and Δ satisfy the bounds (4.3). Then*

$$\delta(T) \;\geq\; \sup_{R \geq R_0} \left\{ \mathcal{X}_N\Big(\frac{R}{R_*(T)}\Big) \frac{f(R,T)}{R^N} \right\} . \tag{4.4}$$

Proof. Since $T > 0$ is fixed, we shall simply write $f(R)$ instead of $f(R,T)$, and similarly for g, Δ, γ, δ and R_*. If (4.4) fails, there exists $R_1 \geq R_0$ such that $f(R_1)\mathcal{X}_N(R_1/R_*) > \delta R_1^N$. In particular, since $\mathcal{X}_N \leq 1$, this implies $f(R_1) > \delta R_1^N$. Consider the interval

$$I \;=\; \big\{ R_2 \geq R_1 \,|\, f(R) \geq \delta R^N \text{ for all } R \in [R_1, R_2] \big\} \;.$$

If $R \in I$, then $\Delta(R) \leq \frac{1}{2}\delta R^N \leq \frac{1}{2} f(R)$, hence $f(R) \leq 2(f'(R)g(R))^{1/2}$ by (4.2). Integrating this differential inequality, we find

$$\frac{1}{f(R)} \;\leq\; \frac{1}{f(R_1)} - \frac{1}{4}\int_{R_1}^{R} \frac{1}{g(r)}\,\mathrm{d}r \;\leq\; \frac{1}{f(R_1)} - \frac{1}{\gamma}\int_{R_1}^{R} \frac{1}{r^{N-1}}\,\mathrm{d}r \;, \quad R \in I \;. \tag{4.5}$$

Now, consider the function $\Phi : \mathbf{R}_+ \to \mathbf{R}$ defined by

$$\Phi(R) \;=\; \frac{1}{\delta R^N} + \frac{1}{\gamma}\int_{R_1}^{R} \frac{1}{r^{N-1}}\,\mathrm{d}r - \frac{1}{f(R_1)} \;, \quad R > 0 \;.$$

As is easily verified, $\Phi'(R) < 0$ if $0 < R < R_*$ and $\Phi'(R) > 0$ if $R > R_*$. Thus, if $R_1 \geq R_*$, then $\Phi(R) \geq \Phi(R_1) > 0$ for all $R \geq R_1$. If $R_0 \leq R_1 < R_*$, then $\Phi(R) \geq \Phi(R_*)$ for all $R \geq R_1$, and a short calculation shows that

$$\Phi(R_*) \;=\; \frac{\mathcal{X}_N(R_1/R_*)}{\delta R_1^N} - \frac{1}{f(R_1)} \;>\; 0 \;.$$

In both cases, we conclude that $\Phi(R) > 0$ for all $R \in I$. Using (4.5), we deduce that

$$\frac{1}{f(R)} \;\leq\; \frac{1}{f(R_1)} - \frac{1}{\gamma}\int_{R_1}^{R} \frac{1}{r^{N-1}}\,\mathrm{d}r \;<\; \frac{1}{\delta R^N} \;, \quad R \in I \;,$$

which means that $I = [R_1, +\infty)$. This is of course impossible, since the right-hand side of (4.5) becomes negative when R is sufficiently large. $\square$

Remark. If $\delta(T) = 0$, then (4.4) implies that $f(R,T) = 0$ all $R > 0$, hence $w(t) = w(0)$ for all $t \in [0,T]$. Thus, Proposition 2.1 can be recovered as a particular case of Lemma 4.1.

Assume now that $w \in \mathcal{C}^0([0,+\infty),X)$ is a solution of (1.1) which stays bounded in X, and that $w_0 = w(0)$ is not an equilibrium point. If $N \le 2$ and if $w_1 \in X$ satisfies $w_1 \succeq w_0$, we know from Proposition 2.1 that $w(t) \neq w_1$ for all $t > 0$. It is thus natural to ask whether the trajectory $w(t)$ can come arbitrarily close to w_1 as $t \to +\infty$, or whether there exists a neighborhood $\mathcal{V}$ of w_1 such that $w(t) \notin \mathcal{V}$ for all $t \ge 1$. In particular, if $w_1 = w_0$, the question we address is whether the extended gradient system (1.1) can have nontrivial *recurrent* points if $N \le 2$. Although we do not know the answer to this question, we think that our energy arguments do not preclude the existence of a nontrivial trajectory w containing in its ω-limit set some function $w_1 \in X$ with $w_1 \succeq w_0$. However, if such a solution exists, it follows from Lemma 4.1 that the time needed for $w(t)$ to enter a small neighborhood of w_1 is very long. A precise formulation of this property is:

Proposition 4.2. *Assume that $N \le 2$, and that $w \in \mathcal{C}^0([0,+\infty),X)$ is a nonstationary solution of (1.1) which stays bounded in X. Let $w_1 \in X$ such that $w_1 \succeq w_0 = w(0)$. Then there exist positive constants C_1 and ε_0 such that, if $\|w(T_\varepsilon) - w_1\|_X \le \varepsilon$ for some $T_\varepsilon \ge 1$ and some $\varepsilon \le \varepsilon_0$, then*

$$T_\varepsilon \;\ge\; \begin{cases} C_1/\varepsilon & \text{if } N = 1\ , \\ C_1 \log(1/\varepsilon) & \text{if } N = 2\ . \end{cases} \tag{4.6}$$

Proof. We first assume that $w \in \mathcal{C}^1([0,+\infty),X) \cap \mathcal{C}^0([0,+\infty),Y)$ is a classical solution of (1.1) which stays bounded in Y. For all $T > 0$, we define $f(R,T)$ and $g(R,T)$ by (4.1). Since w_0 is not an equilibrium point, there exists $R_1 > 0$ such that $f(R_1,1) > 0$. In addition, since w is bounded in Y, there exists $\gamma_0 > 0$ such that the bound (4.3) holds for g with $\gamma(T) = \gamma_0 T$. On the other hand, in view of Eq.(7.7) below, there exists $\delta_0 > 0$ such that

$$|H_R(w_1) - H_R(w_2)| \;\le\; \frac{1}{2}\delta_0 \varepsilon R^N\ , \quad R \ge 1\ , \tag{4.7}$$

for all $\varepsilon \le 1$ and all $w_2 \in X$ such that $\|w_1 - w_2\|_X \le \varepsilon$. In particular, if $T_\varepsilon \ge 1$ is such that $\|w(T_\varepsilon) - w_1\|_X \le \varepsilon \le 1$, then (4.7) holds with $w_2 = w(T_\varepsilon)$. Since $w_1 \succeq w_0$ by assumption, il follows that there exists $R_2 \ge 1$ such that

$$\Delta(R,T_\varepsilon) \;=\; H_R(w_0) - H_R(w(T_\varepsilon)) \;\le\; \frac{1}{2}\delta_0 \varepsilon R^N\ , \quad R \ge R_2\ ,$$

hence the bound (4.3) holds for Δ with $\delta(T_\varepsilon) = \delta_0\varepsilon$. Now, we set $R_0 = \max(R_1, R_2)$, $\varepsilon_0 = \min(1, N\gamma_0\delta_0^{-1}R_0^{-2})$, and we apply (4.4) with $R = R_0 \le R_*(T_\varepsilon) = (N\gamma_0 T_\varepsilon/\delta_0\varepsilon)^{1/2}$. If $N = 1$, we obtain

$$\delta_0\varepsilon \;\ge\; \frac{1}{R_*(T_\varepsilon)}\left(2 - \frac{R_0}{R_*(T_\varepsilon)}\right) f(R_0, T_\varepsilon) \;\ge\; \left(\frac{\delta_0\varepsilon}{\gamma_0 T_\varepsilon}\right)^{1/2} f(R_1, 1)\ ,$$

which is the desired result. If $N = 2$, we find

$$\delta_0\varepsilon \;\ge\; \frac{1}{R_*(T_\varepsilon)^2}\left(1 + 2\log\frac{R_*(T_\varepsilon)}{R_0}\right) f(R, T_\varepsilon) \;\ge\; \frac{\delta_0\varepsilon}{2\gamma_0 T_\varepsilon}\big(C + \log(T_\varepsilon/\varepsilon)\big) f(R_1, 1)\ ,$$

where $C = 1 + \log(2\gamma_0/\delta_0 R_0^2)$, and (4.6) follows since $T_\varepsilon \ge 1$. This proves the claim if w is a classical solution. The general case is proved by applying the same argument to the averaged quantities

$$\begin{aligned}
\widetilde{f}(R,T) \;&=\; \int_R^{R+1} f(r,T)\,\mathrm{d}r\ , \quad \widetilde{g}(R,T) \;=\; \int_R^{R+1} g(r,T)\,\mathrm{d}r\ ,\\
\widetilde{\Delta}(R,T) \;&=\; \int_R^{R+1} \Delta(r,T)\,\mathrm{d}r\ .
\end{aligned}$$

This concludes the proof of Proposition 4.2. □

In the same spirit, Lemma 4.1 implies the following estimates for homoclinic or heteroclinic trajectories:

Proposition 4.3. *Assume that $N \le 2$, and that $w \in \mathcal{C}^0(\mathbf{R}, X)$ is a nonstationary solution of (1.1) satisfying*

$$\lim_{t\to-\infty} \|w(t) - w_-\|_X \;=\; 0\ , \quad \lim_{t\to+\infty} \|w(t) - w_+\|_X \;=\; 0\ ,$$

for some $w_-, w_+ \in X$ such that $w_+ \succeq w_-$. Then there exists a constant $C_2 > 0$ such that, for all $T \ge 1$,

$$\|w(-T) - w_-\|_X + \|w(T) - w_+\|_X \;\ge\; \begin{cases} C_2/T & \text{if } N = 1\ ,\\ C_2\exp(-T/C_2) & \text{if } N = 2\ .\end{cases}$$

Proof. The proof of Proposition 4.3 is very similar to that of Proposition 4.2, and thus may be omitted.

Remark. In particular, if $N = 1$ and if w_-, w_+ are *hyperbolic* equilibria of (1.1) satisfying $w_+ \succeq w_-$, Proposition 4.3 shows that w_- cannot be connected to w_+ by a heteroclinic orbit in X. However, such a heteroclinic connection is possible if we endow X with a weaker topology, such as the "localized" topology discussed in Section 5.2 below. An example is provided by the annihilation of a pair of "kinks" in the real Ginzburg-Landau equation $u_t = u_{xx} + u - u^3$ [ER].

We conclude this section with a simple sufficient condition for $w_0 \in X$ to be a nonrecurrent point of the system (1.1) in X.

Proposition 4.4. *Assume that $\alpha > 0$. If $w_0 \in X$ satisfies $\overline{\lambda}(w_0) > 0$ (in the sense of (1.6)), then w_0 is a nonrecurrent point of the system (1.1).*

Proof. If $\overline{\lambda}(w_0) = \varepsilon > 0$, there exists a sequence $R_k \to +\infty$ such that

$$\lim_{k\to\infty} \frac{1}{R_k^N} \Lambda_{R_k}(w_0) \;=\; \varepsilon > 0 \;.$$

In analogy with (1.6), we define

$$\tilde{h}(w) \;=\; \limsup_{k\to\infty} \frac{1}{R_k^N} H_{R_k}(w) \;, \quad \tilde{\lambda}(w) \;=\; \liminf_{k\to\infty} \frac{1}{R_k^N} \Lambda_{R_k}(w) \;, \quad w \in X \;.$$

As was already observed, $\tilde{h}$ and $\tilde{\lambda}$ are continuous functions of $w \in X$. Thus, if $w \in \mathcal{C}^0([0, T_*), X)$ is the maximal solution of (1.1) with initial data w_0, there exists $T \in (0, T_*)$ such that $\tilde{\lambda}(w(t)) \geq \varepsilon/2$ for all $t \in [0, T]$. Using the analogue of (1.7), we find, for all $t \in [T, T_*)$,

$$\tilde{h}(w(t)) \;\leq\; \tilde{h}(w_0) - \int_0^t \tilde{\lambda}(w(s))\,\mathrm{d}s \;\leq\; \tilde{h}(w_0) - \frac{\varepsilon T}{2} \;,$$

hence $w(t) \in X \setminus \mathcal{V}$ for all $t \in [T, T_*)$, where $\mathcal{V} = \{w_1 \in X \,|\, |\tilde{h}(w_1) - \tilde{h}(w_0)| < \varepsilon T/2\}$. This shows that w_0 is nonrecurrent. $\square$

Remark. In the parabolic case $\alpha = 0$, Proposition 4.4 holds for all $w_0 \in Y$ such that $\overline{\lambda}(w_0) > 0$.

5. Convergence on average and uniform recurrence

A well-known property of gradient dynamical systems is that all trajectories approach the set $\mathcal{E}$ of equilibria as $t \to +\infty$. This is not true in general for extended gradient systems such as Eq.(1.1), even if the convergence is understood in a weak sense, for instance in the topology of $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^N)$ (respectively, $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N)$ if $\alpha = 0$). An example of a trajectory exhibiting a more complicated behavior is the multikink solution constructed by Eckmann and Rougemont [ER], [R] for the real Ginzburg-Landau equation. However, assuming that $N \le 2$ and using the same energy arguments as in the previous sections, it is possible to show that all solutions of (1.1) converge locally to the set of equilibria *on average*, in the sense of ergodic theory. We deduce from this result that uniformly recurrent orbits cannot exist, except for equilibria, and we obtain a weaker version of LaSalle's principle.

5.1. Convergence on average to the set of equilibria

The main result of this section is the following:

Proposition 5.1. *Assume that $N \le 2$, and that $w \in \mathcal{C}^0([0,+\infty), X)$ is a trajectory of (1.1) which stays bounded in X. Then, for all $R > 0$,*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \Lambda_R(w(t))\,\mathrm{d}t \;=\; 0 \;. \tag{5.1}$$

In particular, any nonequilibrium point $w_1 \in X \setminus \mathcal{E}$ has a neighborhood $\mathcal{V}$ in X such that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{V}}(w(t))\,\mathrm{d}t \;=\; 0 \;. \tag{5.2}$$

Remark. Eq.(5.2) means that the average time spent by the trajectory $w(t)$ in a small neighborhood of any nonequilibrium point w_1 is equal to zero. In particular, in the case where $w_1 = w_0 \equiv w(0)$, we see that the nonequilibrium point w_0 is nonrecurrent *on average*.

Proof. Assume first that $w \in \mathcal{C}^1([0,+\infty), X) \cap \mathcal{C}^0([0,+\infty), Y)$ is a classical solution of (1.1) which stays bounded in Y. For any sequence $T_k \to +\infty$, we define, for all $R > 0$,

$$f(R) \;=\; \liminf_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} \Lambda_R(w(t))\,\mathrm{d}t \;=\; \liminf_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} \int_{B_R} u_t^2\,\mathrm{d}x\,\mathrm{d}t \;,$$

$$\varphi(R) \;=\; \liminf_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} \int_{\partial B_R} u_t^2\,\mathrm{d}\sigma\,\mathrm{d}t \;, \quad g(R) \;=\; \limsup_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} \int_{\partial B_R} |\nabla u|^2\,\mathrm{d}\sigma\,\mathrm{d}t \;,$$

together with $\Phi(R) = \int_0^R \varphi(r)\,\mathrm{d}r$. Using the results of Section 7.1 and the assumption that w is bounded in Y, it is not difficult to verify that the functions f, φ, g are (Hölder) continuous on $\mathbf{R}_+$, and that $g(R) \le CR^{N-1}$ for some $C > 0$ and all $R \ge 1$. In addition, $\Phi(R) \le f(R)$ for all $R > 0$ by Fatou's Lemma. On the other hand, dividing the first equality in (2.1) by $T = T_k$ and using the fact that $H_R(w(t))$ stays bounded (as a function of t) for all $R > 0$, we obtain

$$\liminf_{k\to\infty} \frac{1}{T_k}\int_0^{T_k} \Lambda_R(w(t))\,\mathrm{d}t \;\le\; \liminf_{k\to\infty} \frac{1}{T_k}\int_0^{T_k} |J_R(w(t))|\,\mathrm{d}t\;, \quad R > 0\;.$$

Moreover, the right-hand side is bounded by $(\varphi(R)g(R))^{1/2}$ as a consequence of Hölder's inequality. Summarizing, we have shown that

$$\Phi(R) \;\le\; f(R) \;\le\; (\Phi'(R)g(R))^{1/2}\;, \quad R > 0\;. \tag{5.3}$$

Now, if $N \le 2$, the differential inequality (5.3) implies that $\Phi(R) = 0$ for all $R > 0$, because $\int_R^\infty g(r)^{-1}\,\mathrm{d}r = +\infty$. Using (5.3) again, we conclude that $f(R) = 0$ for all $R > 0$. Since the sequence $T_k \to +\infty$ was arbitrary, this proves (5.1) in the case of classical solutions. In particular, the proof of (5.1) is complete in the parabolic case $\alpha = 0$, since any bounded solution $w \in \mathcal{C}^0([0,+\infty), X)$ is classical and bounded in Y for all $t \ge 1$.

We now consider the general case where $w \in \mathcal{C}^0([0,+\infty), X)$ is a mild solution only. In analogy with (2.7), we introduce the averaged quantities

$$\begin{aligned}
\widetilde{f}(R) \;&=\; \liminf_{k\to\infty} \frac{1}{T_k}\int_0^{T_k}\!\!\int_R^{R+1} \Lambda_r(w(t))\,\mathrm{d}r\,\mathrm{d}t\;, \quad \widetilde{\Phi}(R) \;=\; \widetilde{f}(0) + \int_0^R \widetilde{\varphi}(r)\,\mathrm{d}r\;,\\
\widetilde{\varphi}(R) \;&=\; \liminf_{k\to\infty} \frac{1}{T_k}\int_0^{T_k}\!\!\int_{A_R} u_t^2\,\mathrm{d}\sigma\,\mathrm{d}t\;, \quad \widetilde{g}(R) \;=\; \limsup_{k\to\infty} \frac{1}{T_k}\int_0^{T_k}\!\!\int_{A_R} |\nabla u|^2\,\mathrm{d}\sigma\,\mathrm{d}t\;,
\end{aligned}$$

where $A_R = B_{R+1} \setminus B_R = \{x \in \mathbf{R}^N \,|\, R < |x| \le R+1\}$. Since w is bounded in X, it is again straightforward to verify that the function $\widetilde{f}$ is locally Lipschitz on $\mathbf{R}_+$, and that $f, \widetilde{\varphi}, \widetilde{g}$ are locally bounded. In addition, there exists $C > 0$ such that $\widetilde{g}(R) \le CR^{N-1}$ for all $R \ge 1$. Proceeding as in the previous case, we obtain the inequality

$$\widetilde{\Phi}(R) \;\le\; \widetilde{f}(R) \;\le\; (\widetilde{\Phi}'(R)\widetilde{g}(R))^{1/2}\;, \quad R > 0\;,$$

which implies that $\widetilde{\Phi}(R) = \widetilde{f}(R) = 0$ for all $R > 0$ if $N \le 2$. Finally, since f is nondecreasing and since

$$\int_R^{R+1} f(r)\,\mathrm{d}r \;\le\; \widetilde{f}(R) \;=\; 0\;, \quad \text{for all } R > 0\;,$$

by Fatou's lemma, we conclude that $f(R) = 0$ for all $R > 0$, and (5.1) follows.

To prove (5.2), we first assume that $\alpha > 0$. If $w_1 \in X \setminus \mathcal{E}$, there exist $R > 0$, $\varepsilon > 0$ and a neighborhood $\mathcal{V}$ of w_1 in X such that, for all $w = (u, u_t) \in \mathcal{V}$,

$$\Lambda_R(w) \;=\; \int_{B_R} u_t^2 \,\mathrm{d}x \;\geq\; \varepsilon\ . \tag{5.4}$$

As a consequence, we have for any $T > 0$

$$\frac{1}{T}\int_0^T \Lambda_R(w(t))\,\mathrm{d}t \;\geq\; \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{V}}(w(t))\Lambda_R(w(t))\,\mathrm{d}t \;\geq\; \frac{\varepsilon}{T}\int_0^T \mathbf{1}_{\mathcal{V}}(w(t))\,\mathrm{d}t\ ,$$

hence (5.2) follows from (5.1). In the parabolic case $\alpha = 0$, this argument has to be modified as follows. If $w_1 \in X \setminus \mathcal{E}$, there exist $\tau > 0$, $R > 0$, $\varepsilon > 0$ and a neighborhood $\mathcal{V}$ of w_1 in $X = \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$ such that, for all $\bar{w}_0 \in \mathcal{V}$, the solution $\bar{w}(t)$ of (1.1) with $\bar{w}(0) = \bar{w}_0$ exists for $t \in [0, \tau]$ and satisfies $\int_0^\tau \Lambda_R(\bar{w}(t))\,\mathrm{d}t \geq \varepsilon$, see Section 7.3. Thus, for any $T > 0$,

$$\begin{aligned}\frac{1}{T}\int_0^{T+\tau} \Lambda_R(w(t))\,\mathrm{d}t \;&\geq\; \frac{1}{\tau T}\int_0^T \Big(\int_t^{t+\tau} \Lambda_R(w(s))\,\mathrm{d}s\Big)\,\mathrm{d}t \\ &\geq\; \frac{1}{\tau T}\int_0^T \mathbf{1}_{\mathcal{V}}(w(t))\Big(\int_t^{t+\tau} \Lambda_R(w(s))\,\mathrm{d}s\Big)\,\mathrm{d}t \;\geq\; \frac{\varepsilon}{\tau T}\int_0^T \mathbf{1}_{\mathcal{V}}(w(t))\,\mathrm{d}t\ .\end{aligned}$$

This concludes the proof of Proposition 5.1. □

Remarks.

1. As is clear from the proof, we can even find a neighborhood $\mathcal{V}$ of w_1 in the topology of $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^N)$ (resp. $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N)$ if $\alpha = 0$) such that Eq.(5.2) holds. This generalization will be used in the next section.

2. The example of a periodic solution given in Section 3.2 shows that the conclusion of Proposition 5.1 fails if $N > 2$.

5.2. Uniform recurrence

We avoided so far the discussion of different topologies on X, and assumed always the "uniformly local" topology $\mathcal{T}_{\mathrm{ul}}$ defined by the norm $\|\cdot\|_X$. However, since $X \subset X_{\mathrm{loc}} = \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^N)$ (resp. $X_{\mathrm{loc}} = \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^N)$ if $\alpha = 0$), we may also use the weaker topology $\mathcal{T}_{\mathrm{loc}}$ on X that is induced by the (usual) topology of X_{loc}. This "localized" topology is important, because if the trajectory $w \in C^0([0,\infty), X)$ stays bounded in X (in the sense of the norm $\|\cdot\|_X$), then the set $\{w(t)\,|\,t \ge 0\}$ is relatively compact with respect to $\mathcal{T}_{\mathrm{loc}}$. Therefore, in the localized topology, we can study the dynamical properties of (1.1) using standard concepts such as ω-limit sets and attractors, which rely on compactness. Note that the solutions of (1.1) in X depend continuously on the initial data in the localized topology $\mathcal{T}_{\mathrm{loc}}$. This property can be proved using the finite speed of propagation if $\alpha > 0$, or the spatial decay of the heat kernel if $\alpha = 0$.

In the rest of this section, we denote by $\mathcal{B}$ the set of all $w_0 \in X$ such that the solution $w(t)$ of (1.1) with initial data w_0 exists for all $t \ge 0$ and remains bounded (in the sense of the norm $\|\cdot\|_X$). We write $w(t) = S_t w_0$ for all $t \ge 0$. We say that $w_0 \in \mathcal{B}$ is *uniformly recurrent* in a topology $\mathcal{T}$ if, for all neighborhood $\mathcal{V}$ of w_0 in $\mathcal{T}$, the set $A = \{t \ge 0\,|\,S_t w_0 \in \mathcal{V}\} \subset \mathbf{R}_+$ has the following property:

(P) There exist $n \ge 1$ and $t_1, \dots, t_n \ge 0$ such that $[0,+\infty) = \bigcup_{i=1}^{n} \{t \ge 0\,|\,t + t_i \in A\}$

(see Furstenberg [Fu] for a similar definition in the discrete-time case, where such sets are called "syndetic"). A direct consequence of (P) is:

Lemma 5.2. *Assume that $A \subset [0,+\infty)$ is measurable (with respect to Lebesgue measure μ) and has the property (P). Then*

$$\liminf_{T\to+\infty} \frac{1}{T}\,\mu\big(A \cap [0,T]\big) \;\ge\; \frac{1}{n} \;>\; 0\;.$$

Proof. For $T > 0$ and $i \in \{1,\dots,n\}$, let $A_i^T = \{t \in [0,T]\,|\,t + t_i \in A\}$. Then A_i^T is measurable, and

$$[0,T] \;=\; \bigcup_{i=1}^{n} A_i^T\;, \qquad \text{hence} \quad T \le \sum_{i=1}^{n} \mu(A_i^T)\;. \tag{5.5}$$

On the other hand,

$$\mu(A_i^T) \;=\; \int_0^T \mathbf{1}_A(t+t_i)\,\mathrm{d}t \;\le\; \int_0^{T+m} \mathbf{1}_A(t)\,\mathrm{d}t\;, \tag{5.6}$$

where $m = \max\{t_1, \ldots, t_n\}$. Using (5.5) and (5.6), we obtain

$$T \leq n \int_0^{T+m} \mathbf{1}_A(t)\,\mathrm{d}t \ , \qquad \text{hence} \quad \liminf_{T\to+\infty} \frac{1}{T} \int_0^T \mathbf{1}_A(t)\,\mathrm{d}t \geq \frac{1}{n} \ .$$

□

Combining Proposition 5.1 and Lemma 5.2, we obtain the following characterization of uniformly recurrent points:

Proposition 5.3. *Assume that $N \leq 2$ and that $w_0 \in \mathcal{B}$ is uniformly recurrent in the localized topology $\mathcal{T}_{\mathrm{loc}}$. Then $w_0 \in \mathcal{E}$.*

Remark. Of course, if w_0 is uniformly recurrent in $\mathcal{T}_{\mathrm{ul}}$, then *a fortiori* w_0 is uniformly recurrent in $\mathcal{T}_{\mathrm{loc}}$ and Proposition 5.3 applies.

Proof. As was already observed, if $w_0 \in \mathcal{B}$ is not an equilibrium point and if $w(t) = S_t w_0$, there exists a neighborhood $\mathcal{V}$ of w_0 in $\mathcal{T}_{\mathrm{loc}}$ such that (5.2) holds. Using Lemma 5.2, we deduce that the (measurable) set $A = \{t \geq 0 \,|\, S_t w_0 \in \mathcal{V}\}$ has not the property (P), which means that w_0 is not uniformly recurrent. □

We say that a subset $\mathcal{U}$ of $\mathcal{B}$ is *invariant* for (1.1) if $S_t w_0 \in \mathcal{U}$ for all $w_0 \in \mathcal{U}$ and all $t \geq 0$. Given a topology $\mathcal{T}$, we say that a closed invariant set $\mathcal{U} \subset \mathcal{B}$ is *minimal* if it has no nonempty proper closed invariant subset. Note that, if $\mathcal{U} \subset \mathcal{B}$ is a closed invariant set for $\mathcal{T}_{\mathrm{ul}}$ or $\mathcal{T}_{\mathrm{loc}}$, then (1.1) defines a continuous semiflow $(S_t)_{t\geq 0}$ on $\mathcal{U}$. As is easily verified, the set $\mathcal{U}$ is then minimal if and only if, for all $w_0 \in \mathcal{U}$, the orbit $\{S_t w_0 \,|\, t \geq 0\}$ is dense in $\mathcal{U}$.

The following result is an elementary fact from ergodic theory. A proof can be found in [Fu], Lemma 1.14 and Theorem 1.15.

Theorem 5.4. *Given a continuous semiflow on a compact metrizable space X, every minimal closed invariant set consists of uniformly recurrent points.*

Corollary 5.5. *Assume that $N \leq 2$, and that $\mathcal{U} \subset \mathcal{B}$ is a nonempty minimal compact invariant set with respect to the localized topology $\mathcal{T}_{\mathrm{loc}}$. Then $\mathcal{U}$ is a single equilibrium.*

Remark. If $\mathcal{U} \subset \mathcal{B}$ is a (nonempty) minimal compact invariant set with respect to $\mathcal{T}_{\mathrm{ul}}$, then *a fortiori* $\mathcal{U}$ is compact and minimal with respect to $\mathcal{T}_{\mathrm{loc}}$, and Corollary 5.5 applies.

Proof. Equation (1.1) induces a continuous semiflow on $\mathcal{U}$. Theorem 5.4 and Proposition 5.3 then imply that every point of $\mathcal{U}$ is an equilibrium. Thus, by definition of a minimal invariant set, $\mathcal{U}$ must be a single equilibrium. □

We finally give our weaker version of LaSalle's principle for Eq.(1.1).

Corollary 5.6. *Assume that $N \le 2$, and let $w_0 \in \mathcal{B}$. Then the ω-limit set of w_0 with respect to the localized topology $\mathcal{T}_{\mathrm{loc}}$ is nonempty and contains an equilibrium.*

Proof. Since $w_0 \in \mathcal{B}$, the ω-limit set of w_0 with respect to $\mathcal{T}_{\mathrm{loc}}$ is a nonempty compact invariant set, hence contains a nonempty minimal compact invariant set (by Zorn's lemma). By Corollary 5.5, this minimal compact invariant set is an equilibrium. □

Remark. An alternative, more constructive proof of Corollary 5.6 is the following: using (5.1), we can find sequences $t_n \to \infty$, $R_n \to \infty$ such that $\Lambda_{R_n}(w(t_n)) \le 1/n$. Any limit point (in the localized topology $\mathcal{T}_{\mathrm{loc}}$) of the sequence $w(t_n)$ is an equilibrium of (1.1), which is contained in the ω-limit set of w_0.

6. Conclusions and perspectives

In this paper, we investigated the constraints that are imposed to the solutions of Eq.(1.1) by the local energy dissipation law (1.4). The damped hyperbolic equation (1.1) has been chosen as a representative example of an extended, formally gradient dynamical system, but the arguments we used are very general, and it is clear that they apply in other contexts as well. Among possible generalizations, we mention the hyperbolic system

$$\alpha U^i_{tt} + U^i_t \;=\; \Delta U^i + F^i(x,U) \;, \quad i \in \{1,\dots,M\} \;, \tag{6.1}$$

where $U = (U^1,\dots,U^M) \in \mathbf{R}^M$ and $F = (F^1,\dots,F^M) : \mathbf{R}^{N+M} \to \mathbf{R}^M$. However, we have to assume that there exists a potential $V(x,U)$ such that $F = -\nabla_U V$, a condition that is quite restrictive if $M \ge 2$. Under this assumption, the dissipation law (1.4) holds with

$$h(W) \;=\; \frac{1}{2}\sum_{i=1}^{M}\big(\alpha(U^i_t)^2 + |\nabla U^i|^2\big) + V(x,U) \;.$$

We may also replace the Laplacian Δ in (1.1), or in (6.1), by the diffusion operator $\mathrm{div}(a(x)\nabla)$, where $a \in \mathcal{C}_b^1(\mathbf{R}^N, \mathbf{R}_+)$ is bounded away from zero. Then the energy density reads, in the scalar case,

$$h(w) \;=\; \frac{\alpha u_t^2}{2} + \frac{1}{2}a(x)|\nabla u|^2 + V(x,u)\ .$$

Another interesting possibility is to study Eq.(1.1) in a smooth connected unbounded domain $\Omega \subset \mathbf{R}^N$, with homogeneous Neumann condition on $\partial\Omega$. This no-flux condition ensures that no energy can enter the system through the boundary. Thus, replacing B_R with $B_R \cap \Omega$ in the definitions of H_R, J_R and Λ_R, we again obtain the balance equation (1.5) for all $R > 0$. Assume moreover that there exist $C > 0$ and $d > 0$ such that

$$\sigma\big(\{x \in \Omega \,|\, |x| = R\}\big) \;\sim\; CR^{d-1}\ , \quad R \to +\infty\ ,$$

where σ is the Euclidean surface measure. This is the case, for instance, if Ω is an exterior domain ($d = N$), a sector ($d = N$), or a cylinder ($d = 1$). Then it is easy to verify that all results in this paper remain valid, with the important difference that the now familiar assumption $N \le 2$ can be replaced with the weaker condition $d \le 2$. Remark that d need not be an integer.

In some particular cases, Eq.(1.1) has other remarkable properties, in addition to the gradient structure. If $\alpha = 0$, the reaction-diffusion equation (1.1) is an *order preserving* system, due to the parabolic maximum principle. This monotonicity property is a strong constraint, and it would be interesting to combine it with the gradient structure to possibly strengthen the results of this paper. Note, however, that all counter-examples in Section 3 are constructed for linear parabolic equations satisfying the maximum principle. On the other hand, if the nonlinearity F does not depend explicitly on x, Eq.(1.1) is *translation invariant*, a property that plays an important role in extended dynamical systems. For instance, this symmetry is used in [MS] to show that the global attractor of the Ginzburg-Landau equation on the real line is attracting not only in the localized topology (see Section 5.2), but also in a stronger sense. In an abstract context, the dynamics of extended gradient semiflows is studied in [Sl] at the level of probability measures on the phase space. Assuming translation invariance, it is shown that the induced semiflow on the space of translationaly invariant probability measures is strictly gradient, and that the stationary measures for this semiflow are supported on the set of equilibria of the original system. These results suggest that one could obtain more information on the solutions of Eq.(1.1) in the translation invariant case.

7. Appendix

We first recall the definitions of the uniformly local Sobolev spaces $\mathrm{H}^k_{\mathrm{ul}}(\mathbf{R}^N)$, together with some basic properties that are used throughout this work. We then briefly discuss the Cauchy problem for Eq.(1.1) in these spaces, both in the hyperbolic case $\alpha > 0$ and in the parabolic limit $\alpha = 0$. For a more detailed discussion, including some proofs, we refer to [Ka], [MS], [M1], [Fe], and [GV].

7.1. The uniformly local Sobolev spaces

Let $\rho : \mathbf{R}^N \to (0, +\infty)$ be a smooth, rapidly decreasing function, for instance $\rho(x) = 1/\cosh(|x|)$, where $|x| = (x_1^2 + \ldots + x_N^2)^{1/2}$ is the Euclidean norm in $\mathbf{R}^N$. For any $y \in \mathbf{R}^N$, we denote by T_y the translation operator defined by $(T_y u)(x) = u(x-y)$ for all $u : \mathbf{R}^N \to \mathbf{R}$. For any $p \in [1, +\infty)$, the uniformly local Lebesgue space $\mathrm{L}^p_{\mathrm{ul}}(\mathbf{R}^N)$ is defined by

$$\mathrm{L}^p_{\mathrm{ul}}(\mathbf{R}^N) = \left\{u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^N) \,|\, \|u\|_{\mathrm{L}^p_{\mathrm{ul}}} < \infty\,,\ \lim_{y\to 0} \|T_y u - u\|_{\mathrm{L}^p_{\mathrm{ul}}} = 0\right\}\,,$$

where

$$\|u\|_{\mathrm{L}^p_{\mathrm{ul}}} = \sup_{y\in\mathbf{R}^N} \Big(\int_{\mathbf{R}^N} \rho(x+y)|u(x)|^p \,\mathrm{d}x\Big)^{1/p}\,.$$

Similarly, for any $k \in \mathbf{N}$, we define the uniformly local Sobolev space

$$\mathrm{H}^k_{\mathrm{ul}}(\mathbf{R}^N) = \left\{u \in \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N) \,|\, \partial^\alpha u \in \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N) \text{ for } |\alpha| \le k\right\}\,,$$

equipped with the norm

$$\|u\|_{\mathrm{H}^k_{\mathrm{ul}}} = \Big(\sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{\mathrm{L}^2_{\mathrm{ul}}}\Big)^{1/2}\,.$$

Here, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}^N$ is a multiple index, and $|\alpha| = |\alpha_1| + \ldots + |\alpha_N|$. It is not difficult to verify that $\mathrm{H}^k_{\mathrm{ul}}(\mathbf{R}^N)$ is a Banach space, and that $\mathcal{C}^\infty_b(\mathbf{R}^N)$ (the space of all $u \in \mathcal{C}^\infty(\mathbf{R}^N)$ that are bounded together with all their derivatives) is dense in $\mathrm{H}^k_{\mathrm{ul}}(\mathbf{R}^N)$. Also, different choices of the weight ρ give rise to the same family of spaces $\mathrm{H}^k_{\mathrm{ul}}(\mathbf{R}^N)$, with equivalent norms.

The usual Sobolev embeddings have natural counterparts in the framework of uniformly local spaces. For instance, assume that $\beta > 0$ satisfies $(N-2)\beta \le 2$. Then, for all $u, v \in \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$, the product $|u|^\beta v$ belongs to $\mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$, and there exists a constant $C > 0$ (independent of u, v) such that

$$\| |u|^\beta v \|_{\mathrm{L}^2_{\mathrm{ul}}} \;\le\; C \|u\|^\beta_{\mathrm{H}^1_{\mathrm{ul}}} \|v\|_{\mathrm{H}^1_{\mathrm{ul}}} \;.$$

As a consequence, if $F : \mathbf{R}^{N+1} \to \mathbf{R}$ satisfies (1.2) and if $\mathcal{F}$ is the substitution operator defined by $(\mathcal{F}(u))(x) = F(x, u(x))$, then $\mathcal{F}$ maps $\mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$ into $\mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$ and there exists $K_0 > 0$ such that

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{\mathrm{L}^2_{\mathrm{ul}}} \;\le\; K_0 \big(1 + \|u_1\|^\beta_{\mathrm{H}^1_{\mathrm{ul}}} + \|u_2\|^\beta_{\mathrm{H}^1_{\mathrm{ul}}}\big) \|u_1 - u_2\|_{\mathrm{H}^1_{\mathrm{ul}}} \;, \tag{7.1}$$

for all $u_1, u_2 \in \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$. Similarly, if $\mathbf{V}$ is the substitution operator associated to the primitive $V(x,u) = -\int_0^u F(x,v)\,\mathrm{d}v$, then $\mathbf{V}$ maps $\mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$ into $\mathrm{L}^1_{\mathrm{ul}}(\mathbf{R}^N)$ and

$$\|\mathbf{V}(u_1) - \mathbf{V}(u_2)\|_{\mathrm{L}^1_{\mathrm{ul}}} \;\le\; K_1 \big(1 + \|u_1\|^{\beta+1}_{\mathrm{H}^1_{\mathrm{ul}}} + \|u_2\|^{\beta+1}_{\mathrm{H}^1_{\mathrm{ul}}}\big) \|u_1 - u_2\|_{\mathrm{H}^1_{\mathrm{ul}}} \;, \tag{7.2}$$

for all $u_1, u_2 \in \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$. In addition, $\mathbf{V}(0) = 0$ and $\mathbf{V}$ is a $\mathcal{C}^1$ mapping whose Fréchet derivative is $-\mathcal{F}$.

For any $R > 0$, let $B_R = \{x \in \mathbf{R}^N \,|\, |x| \le R\}$ and $\partial B_R = \{x \in \mathbf{R}^N \,|\, |x| = R\}$. If $u \in \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$, then the function $f : \mathbf{R}_+ \to \mathbf{R}$ defined by

$$f(R) \;=\; \int_{B_R} |u(x)|^2 \,\mathrm{d}x \;, \quad R > 0 \;, \tag{7.3}$$

is continuous, and there exists $C > 0$ (independent of u) such that $|f(R)| \le C R^N \|u\|^2_{\mathrm{L}^2_{\mathrm{ul}}}$ for all $R \ge 1$. If $u \in \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$, one can show that $f \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}_+)$, where $p = N/(N-1)$ if $N \ge 3$, $p < 2$ if $N = 2$ and $p = 2$ if $N = 1$. In addition,

$$f'(R) \;=\; \int_{\partial B_R} |u(x)|^2 \,\mathrm{d}\sigma \;, \quad R > 0 \;, \tag{7.4}$$

where σ is the Euclidean surface measure on ∂B_R. The proof of this result also shows that $|f'(R)| \le C R^{N-1} \|u\|^2_{\mathrm{H}^1_{\mathrm{ul}}}$ for all $R \ge 1$.

7.2. The Cauchy problem for the wave equation

Fix $\alpha > 0$, and let $X = \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N) \times \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$, $Y = \mathrm{H}^2_{\mathrm{ul}}(\mathbf{R}^N) \times \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$. Setting $w = (u, u_t)$, we rewrite Eq.(1.1) into the abstract form

$$\frac{\mathrm{d}w}{\mathrm{d}t} = Aw + G(w) \ , \tag{7.5}$$

where

$$A = \begin{pmatrix} 0 & \mathbf{1} \\ \alpha^{-1}\Delta & 0 \end{pmatrix} \ , \quad G(w) = \frac{1}{\alpha}\begin{pmatrix} 0 \\ \mathcal{F}(u) - u_t \end{pmatrix} \ .$$

Using the *finite speed of propagation* together with classical energy estimates for the wave equation, it is not difficult to show that the solutions of the linear equation (7.5) with $G = 0$ define a C_0-group $S(t) = \exp(tA)$ of bounded linear operators in X, the domain of the generator A being precisely $Y \subset X$. On the other hand, it follows from (7.1) that the mapping $G : X \to X$ is locally Lipschitz, uniformly on bounded sets. Therefore, for all $w_0 \in X$, there exists $T > 0$ such that Eq.(7.5) has a unique mild solution $w \in \mathcal{C}^0([0,T], X)$ satisfying $w(0) = w_0$ [Pa, Chap. 6]. This solution $w(t)$ depends continuously on the initial data w_0 in the topology of X, uniformly in $t \in [0,T]$. In addition, if $w_0 \in Y$ and if $G \in \mathcal{C}^1(X)$, then $w \in \mathcal{C}^0([0,T], Y) \cap \mathcal{C}^1([0,T], X)$ and Eq.(7.5) is satisfied in the classical sense.

Unfortunately, if $w_0 \in Y$ and if G is only locally Lipschitz, we cannot prove that the solution w is classical, because the function space X is not reflexive. However, using again the finite speed of propagation and the fact that the space $\mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)$ is reflexive for any bounded $\Omega \subset \mathbf{R}^N$, we can show that $w \in \mathcal{C}^1([0,T], \mathrm{H}^1_{\mathrm{loc}} \times \mathrm{L}^2_{\mathrm{loc}})$ and that $\|Aw(t)\|_X$ is uniformly bounded for $t \in [0,T]$. These properties are sufficient to justify all calculations in this paper. For instance, if h is the energy density (1.3), then $t \mapsto h(w(t)) \in \mathcal{C}^1([0,T], \mathrm{L}^1_{\mathrm{loc}})$ and the dissipation law (1.4) holds in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^N)$. In particular, the balance equation (1.5) is satisfied for any $R > 0$.

Using (7.2), we note that the energy density h maps X into $\mathrm{L}^1_{\mathrm{ul}}(\mathbf{R}^N)$ and that

$$\|h(w_1) - h(w_2)\|_{\mathrm{L}^1_{\mathrm{ul}}} \le K_2\big(1 + \|w_1\|_X^{\beta+1} + \|w_2\|_X^{\beta+1}\big)\|w_1 - w_2\|_X \ , \tag{7.6}$$

for all $w_1, w_2 \in X$. In addition, $h : X \to \mathrm{L}^1_{\mathrm{ul}}(\mathbf{R}^N)$ is a $\mathcal{C}^1$ mapping, whose Fréchet derivative at $w \in X$ is the linear map $\hat{w} \mapsto (D_w h)\hat{w}$ defined by

$$(D_w h)\hat{w} = \alpha u_t \hat{u}_t + \nabla u \cdot \nabla \hat{u} - \mathcal{F}(u)\hat{u} \ ,$$

where $w = (u, u_t)$ and $\hat{w} = (\hat{u}, \hat{u}_t)$. On the other hand, integrating $h(w)$ over the ball B_R and using (7.6), we find

$$|H_R(w_1) - H_R(w_2)| \le K_3 R^N \bigl(1 + \|w_1\|_X^{\beta+1} + \|w_2\|_X^{\beta+1}\bigr)\|w_1 - w_2\|_X \ , \tag{7.7}$$

for all $R \ge 1$. This inequality implies that the limits $\overline{h}(w)$ and $\underline{h}(w)$ in (1.6) exist, and define continuous functions of $w \in X$. Finally, proceeding as in (7.4), it is easy to verify that the energy flux J_R in (1.5) satisfies $|J_R(w)| \le CR^{N-1}\|\nabla u\|_{\mathrm{H}^1_{\mathrm{ul}}}\|u_t\|_{\mathrm{H}^1_{\mathrm{ul}}} \le CR^{N-1}\|w\|_Y^2$ for all $w \in Y$ and all $R \ge 1$.

7.3. The Cauchy problem for the parabolic equation

Let $\alpha = 0$, $X = \mathrm{H}^1_{\mathrm{ul}}(\mathbf{R}^N)$, and $Y = \mathrm{H}^2_{\mathrm{ul}}(\mathbf{R}^N)$. Setting $w = u$, we rewrite Eq.(1.1) into the form

$$\frac{\mathrm{d}w}{\mathrm{d}t} = Aw + \mathcal{F}(w) \ , \tag{7.8}$$

where $A : Y \to \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$ is the Laplace operator. Using the explicit representation given by the Poisson kernel, it is straightforward to verify that A generates an analytic semi-group $S(t) = \exp(tA)$ of bounded linear operators in $\mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$. On the other hand, the nonlinearity $\mathcal{F} : X \to \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$ is locally Lipschitz, uniformly on bounded sets. Therefore, for any $w_0 \in X$, there exists $T > 0$ such that Eq.(7.8) has a unique solution $w \in \mathcal{C}^0([0,T], X) \cap \mathcal{C}^1((0,T], X) \cap \mathcal{C}^0((0,T], Y)$ with $w(0) = w_0$ [He, Chap. 3]. This solution $w(t)$ depends continuously on the initial data w_0 in the topology of X, uniformly in $t \in [0,T]$. Note that all solutions are classical for $t > 0$. In particular, we conclude that any solution $w \in \mathcal{C}^0([0,T], X)$ can be uniformly approximated by classical solutions.

The following observation is used several times in the preceding sections. If $w \in \mathcal{C}^0([0,T], X)$ is a solution of (1.1) and if

$$v(x) = \left(\int_0^T u_t(x,t)^2 \, \mathrm{d}t\right)^{1/2} \ , \quad x \in \mathbf{R}^N \ ,$$

then $v \in \mathrm{L}^2_{\mathrm{ul}}(\mathbf{R}^N)$, and v is a locally Lipschitz function of $w_0 = w(0) \in X$. Since

$$\int_0^T \Lambda_R(w(t)) \, \mathrm{d}t = \int_{B_R} v(x)^2 \, \mathrm{d}x \ , \tag{7.9}$$

it follows from this remark and from Eq.(7.3) that the first and the last members in (1.9) are well-defined and depend continuously on w_0. Thus, since the inequality (1.9) holds for classical solutions, it remains valid by density in the general case. Eq.(2.6) in the proof of Proposition 2.1 is justified by the same argument, and it follows that the function $\widetilde{f}$ defined by (2.7), which is continuously differentiable due to (7.9), satisfies the inequality $\widetilde{f}(R) \le (\widetilde{f}'(R)\widetilde{g}(R))^{1/2}$ for all $R \ge R_0$.

References

[BV] A.V. Babin and M.I. Vishik: Attractors of partial differential equations in an unbounded domain, Proc. Royal Soc. Edinburgh **116A** (1990), 221–243.

[ER] J.-P. Eckmann and J. Rougemont: Coarsening by Ginzgurg-Landau dynamics, Comm. Math. Phys. **199** (1998), 441–470.

[Fe] E. Feireisl: Bounded, locally compact global attractors for semilinear damped wave equations on $\mathbf{R}^n$, J. Diff. Integral Eqns. **9** (1996), 1147–1156.

[Fu] H. Furstenberg: *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[GV] J. Ginibre, G. Velo: The Cauchy problem in local spaces for the complex Ginzburg-Landau equation, II. Contraction methods, Comm. Math. Phys. **187** (1997), 45–79.

[Ha] J.K. Hale: *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and monographs, Vol. 25, AMS, Providence, 1988.

[HN] F. Hamel and N. Nadirashvili: Entire solutions of the KPP equation, Comm. Pure Appl. Math. **52** (1999), 1255–1276.

[He] D. Henry: *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, 1981.

[Ka] T. Kato: The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Rat. Mech. Anal. **58** (1975), 181–205.

[M1] A. Mielke: The complex Ginzburg-Landau equation on large and unbounded domains: sharper bounds and attractors, Nonlinearity **10** (1997) 199–222

[M2] A. Mielke: The Ginzburg-Landau equation in its role as a modulation equation, in *Handbook for Dynamical Systems III: Towards Applications*, B. Fiedler, G. Iooss, N. Koppell, F. Takens (eds), Springer-Verlag, to appear.

[MS] A. Mielke and G. Schneider: Attractors for modulation equations on unbounded domains — existence and comparison, Nonlinearity **8** (1995), 743–768.

[Pa] A. Pazy: *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. **44**, Springer-Verlag, New-York (1983).

[R] J. Rougemont: Dynamics of kinks in the Ginzburg-Landau equation: approach to metastable shape and collapse of embedded pair of kinks, Nonlinearity **12** (1999), 539–554.

[Sl] S. Slijepčević: Extended gradient systems: dimension one, to appear in Discr. Cont. Dynam. Systems.

[Te] R. Temam: *Infinite–Dimensional Systems in Mechanics and Physics*, Springer, Berlin, 1988.