Intertwining and supercuspidal types for p-adic classical groups

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Abstract : Let F be a non-archimedean local field, of residual characteristic different from 2, and let G be a unitary, symplectic or orthogonal group, considered as the fixed point subgroup inf $\tilde{G} = \mathrm{GL}(N,F)$ of an involution σ. We generalize the notion of a *simple character* of $\tilde{G}$ ([**3**, §3.2]) to define semisimple characters. Given a semisimple character θ of $\tilde{G}$ fixed by σ, we transfer it to a character θ_- of G and calculate its intertwining. If the torus associated to θ_- is maximal compact, we obtain supercuspidal representations of G.

Résumé : Soient F un corps local non-archimédien, de caractéristique résiduelle différente de 2, et G un groupe unitaire, symplectique ou orthogonal, considéré comme le sousegroupe des points fixes d'une involution σ de $\tilde{G} = \mathrm{GL}(N,F)$. Nous généralisons la notion de *caractère simple* de $\tilde{G}$ ([**3**, §3.2]) pour définir des caractères semisimples. Pour θ un caractère semisimple de $\tilde{G}$ fixé par σ, nous le transférons en un caractère θ_- de G et calculons son entrelacement. Si le tore associé à θ_- est compact maximal, nous obtenons des représentations supercuspidales de G.

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Introduction

Let F be a non-archimedean local field of residual characteristic $p \neq 2$, with a (possibly trivial) galois involution $\bar{\ }$ with fixed field F_0. Let V be an N-dimensional F-vector space, equipped with a nondegenerate ϵ-hermitian form h and let σ be the involution on $\tilde{G} = \mathrm{Aut}_F V \simeq \mathrm{GL}(N, F)$ induced by h. We put $\Sigma = \{1, \sigma\} \subset \mathrm{Aut}\tilde{G}$ and set $G = \tilde{G}^{\Sigma}$, a unitary group defined over F_0 (symplectic or orthogonal if $F = F_0$).

Let π be an irreducible supercuspidal representation of G. Following the strategy of Bushnell and Kutzko [**4**] for the classification of the representations of G, we would like to construct a $[G, \pi]_G$-type, that is a pair (J, λ) consisting of a compact subgroup J of G and an irreducible representation λ of J such that π is the unique irreducible representation (upto equivalence) of G which contains λ by restriction.

We construct such types starting from types for $\tilde{G}$ which are fixed by Σ and the main tool for transferring from $\tilde{G}$ to G is Glauberman's correspondence ([**7**]). This gives us a correspondence $\boldsymbol{g}$ between the equivalence classes of irreducible representations ρ of a pro-p subgroup K of $\tilde{G}$ which are fixed by Σ and the equivalence classes of irreducible representations of K^{Σ}. The lemma (2.4) which allows us to make use of this correspondence relates to the intertwining: for $g \in G$, the intertwining space $\mathrm{Hom}_{\,^gK \cap K}(^g\rho, \rho)$ has dimension coprime to p if and only if the intertwining space $\mathrm{Hom}_{\,^gK^{\Sigma} \cap K^{\Sigma}}(^g\boldsymbol{g}(\rho), \boldsymbol{g}(\rho))$ has dimension coprime to p. In particular, if $\rho = \chi$ is a character then the intertwining of $\boldsymbol{g}(\chi)$ is $I_G(\boldsymbol{g}(\chi)) = I_{\tilde{G}}(\chi) \cap G$.

However, we cannot use Glauberman's correspondence directly for types (J, λ) for $\tilde{G}$, since J is not a pro-p subgroup, so we must delve into the construction process for such types ([**3**, §§2,3,5]). The first step is a *simple stratum* and the transfer for these (including their intertwining) is described in [**15**]. In particular, there is a field extension E/F associated to such a stratum.

The crucial step is the construction of *simple characters* ([**3**, §3.2]). These are certain arithmetically defined characters θ of a pro-p subgroup H^1 of $\tilde{G}$. If the simple stratum used in the construction of θ is *skew* ("contained" in $\mathrm{Lie}\, G$) then H^1 is fixed by Σ and we may consider the (non-empty) set of simple characters fixed by Σ. Then we define the simple characters for G to be the Glauberman transfers of these characters (in fact, just their restrictions). Then, using the intertwining lemma and a theorem on the decomposition of double cosets ([**15**, (1.1)]), we may calculate the intertwining of these characters (3.7).

There is another pro-p group $J^1 \supset H^1$ and the next step is to show that there is a unique irreducible representation η of J^1 containing θ. Moreover, if θ is fixed by Σ then so is η so we may consider the Glauberman transfer $\boldsymbol{g}(\eta)$. In fact this is the unique irreducible representation of $J^1 \cap G$ containing $\boldsymbol{g}(\theta)$. Lemma (2.4) applies again and thus we calculate the intertwining of $\boldsymbol{g}(\eta)$.

The final stage in the construction is only considered in a special case, which we describe below.

While in $\tilde{G}$ the simple strata are sufficient to give all supercuspidal representations, this is certainly no longer the case for the group G. Indeed, there are compact tori in G of the form $N_1(E_1) \times N_1(E_2)$, where, for $i = 1, 2$, E_i is a field extension of F

with a galois involution which restricts to $^-$ on F and has fixed field $E_{i,0}$, and N_1 denotes the elements e of norm $N_{E_i/E_{i,0}}e = 1$.

To take account of this, in [**15**, §3.2] the notion of a *semisimple* stratum is introduced – this is a sum of simple strata which are "sufficiently different" from each other. Here we extend this to define *semisimple characters* for $\tilde{G}$, which are in bijection with products of simple characters for smaller $\mathrm{GL}(N_i, F)$, $\sum N_i = N$. The process described above for simple characters works equally well in this semisimple case and we obtain a representation $\boldsymbol{g}(\eta)$ of $J^1 \cap G$, together with its intertwining.

Finally, we suppose that the field extension E_i associated to each simple stratum from which our semisimple stratum is built is maximal – that is, $\sum[E_i : F] = N$. In this case the intertwining of $\eta_- = \boldsymbol{g}(\eta)$ is contained in $J_- = J \cap G$ and, since $J_-/J^1 \cap G$ is a product of cyclic groups, η_- admits a finite number of extensions to a representation κ_- of J_-. Then the induced representation $\pi = \mathrm{Ind}_{J_-}^{G}\,\kappa_-$ is irreducible supercuspidal and (J_-, κ_-) is a $[G, \pi]_G$-type.

We now give a brief summary of the contents of each chapter. In §1 we introduce the notations and give some definitions. The intertwining lemma for Glauberman's correspondence is given in §2. In §3 we define simple and semisimple characters for G and calculate their intertwining and in §4 we construct the representation η and its transfer to G. In §5 we construct types and supercuspidal representations of G in the case of a maximal compact torus. We finish with a few remarks concerning possible further work. Finally, in §6, we show that any simple character fixed by Σ does in fact come from a skew simple stratum.

Although the methods used here are somewhat different, I was partly inspired by the work of L.Morris (see [**13**], [**12**], for example). Supercuspidal representations have also been constructed in the *tame* case in [**1**], [**10**], [**16**], and in the general case in [**9**]. This paper generalizes the constructions of [**9**].

The results here for simple characters formed part of my PhD thesis. I would like to thank my supervisor, Colin Bushnell, for starting me on the project and for his support and encouragement. The use of Glauberman's correspondence, in particular with regard to intertwining, was suggested to me by Guy Henniart.

1. *Preliminaries*

Let F be a non-archimedean local field, $\mathfrak{o}_F$ its ring of integers, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue class field and $q_F = p^{f_F} = \mathrm{card}\, k_F$. We assume throughout that the residual characteristic p is not 2.

Suppose that F comes equipped with a galois involution $^-$, with fixed field F_0; we allow the possibility that $F = F_0$. Then we denote by $\mathfrak{o}_0$, $\mathfrak{p}_0$, k_0, $q_0 = p^{f_0}$ the objects for F_0 analogous to those above for F. We also fix a uniformizer π_F of F such that $\overline{\pi_F} = \pm\pi_F$ (the sign depending on whether F/F_0 is ramified or not).

Let ψ_0 be a character of the additive group of F_0, with conductor $\mathfrak{p}_0$. Then we put $\psi_F = \psi_0 \circ \mathrm{tr}_{F/F_0}$; since $p \neq 2$, F/F_0 is at worst tamely ramified so ψ_F is a character of the additive group of F with conductor $\mathfrak{p}_F$.

Let V be an N-dimensional F-vector space and put $A = \mathrm{End}_F(V) \simeq \mathbb{M}(N,F)$ so that $\tilde{G} = \mathrm{Aut}_F(V)$ may be identified with $\mathrm{GL}(N,F)$. Let ψ_A be the character of A given by $\psi_A = \psi_F \circ \mathrm{tr}_{A/F}$. Let h be a nondegenerate ϵ-hermitian form on V and let $^-$ be the adjoint involution on A associated to h; this extends the involution on F (for F embedded diagonally in A). We also denote by σ the involution on $\tilde{G}$ given by $x \mapsto \overline{x}^{-1}$ and by Σ the subgroup of $\mathrm{Aut}\,\tilde{G}$ consisting of σ and the identity. Note that the action of σ on $\mathrm{Lie}\,\tilde{G} \simeq A$, via the differential, is given by $x \mapsto -\overline{x}$.

We put $G = \tilde{G}^{\Sigma} = \{g \in \tilde{G} : h(gv, gw) = h(v,w)\}$ for all $v, w \in V\}$, a unitary group over F_0 (possibly symplectic or orthogonal). We also put $A_- = A^{\Sigma} = \{x \in A : x + \overline{x} = 0\} \simeq \mathrm{Lie}\,G$ and $A_+ = \{x \in A : x = \overline{x}\}$; since F is not of characteristic 2 we have $A = A_- \oplus A_+$ and, moreover, this decomposition is orthogonal with respect to the pairing induced by $\mathrm{tr}_0 = \mathrm{tr}_{F/F_0} \circ \mathrm{tr}_{A/F}$ since, for $x \in A_-$, $y \in A_+$, we have

$$\mathrm{tr}_0(xy) = \overline{\mathrm{tr}_0(xy)} = \mathrm{tr}_0(\overline{xy}) = \mathrm{tr}_0(-yx) = -\mathrm{tr}_0(xy).$$

For S any subset of A, we write S_- (or sometimes S^-) for $S \cap A_-$ and S_+ for $S \cap A_+$. If S is an $\mathfrak{o}_F$-lattice fixed by the involution then we have $S = S_- \oplus S_+$, since the residual characteristic of F is not 2.

Recall from [**5**, (2.1)] that an $\mathfrak{o}_F$-*lattice sequence* in V is a function Λ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V such that

(i) $n \geq m$ implies $\Lambda(n) \subset \Lambda(m)$;

(ii) there exists a positive integer $e = e(\Lambda)$ (the $\mathfrak{o}_F$-*period* of Λ) such that $\Lambda(n+e) = \mathfrak{p}_F\Lambda(n)$, for all $n \in \mathbb{Z}$.

An $\mathfrak{o}_F$-lattice sequence Λ is called *strict* if $\Lambda(n) \neq \Lambda(n+1)$, for all $n \in \mathbb{Z}$.

A lattice sequence Λ gives rise to a filtration on A by

$$\mathfrak{a}_n = \mathfrak{a}_n(\Lambda) = \{x \in A : x\Lambda(m) \subset \Lambda(m+n), m \in \mathbb{Z}\}, \quad n \in \mathbb{Z}.$$

This then gives rise to a "valuation" ν_Λ on A by

$$\nu_\Lambda(x) = \sup\{n \in \mathbb{Z} : x \in \mathfrak{a}_n\}, \quad \text{for } x \in A,$$

with the understanding that $\nu_\Lambda(0) = \infty$.

From a lattice sequence Λ we obtain a compact open subgroup $U = U(\Lambda) = \mathfrak{a}_0(\Lambda)^\times$ of $\tilde{G}$, equipped with a filtration

$$U_n = U_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda), \quad n \in \mathbb{Z}, n > 0.$$

This is also the Moy-Prasad filtration associated to a certain rational point in the building of $\mathrm{GL}(N,F)$. We define the normalizer of the filtration to be

$$\mathfrak{K}(\Lambda) = \bigcap_{r \geq 0} N_{\tilde{G}}(U_r),$$

where $N_{\tilde{G}}$ denotes normalizer.

For L an $\mathfrak{o}_F$-lattice in V, we define the dual lattice $L^\#$ by

$$L^\# = \{v \in V : h(v, L) \subset \mathfrak{p}_F\}.$$

Then $L^\#$ can be identified with $\mathrm{Hom}_{\mathfrak{o}_F}(L, \mathfrak{p}_F)$ by the non-degeneracy of h and we have $L^{\#\#} = L$. For Λ an $\mathfrak{o}_F$-lattice sequence, define the dual sequence $\Lambda^\#$ by

$$\Lambda^\#(n) = \Lambda(-n)^\#, \quad n \in \mathbb{Z}.$$

We say that Λ is self-dual if there exists $d \in \mathbb{Z}$ such that $\Lambda^\#(n) = \Lambda(n+d)$, for all $n \in \mathbb{Z}$. In this case, the filtration $\mathfrak{a}_n$ on A induced by Λ satisfies $\overline{\mathfrak{a}}_n = \mathfrak{a}_n$, for $n \in \mathbb{Z}$. In particular, the groups U, U_n, $n \geq 1$, are fixed by Σ and we put $P = U^\Sigma$, a compact open subgroup of G, and $P_n = U_n^\Sigma$, for $n \geq 1$, a filtration on P. Further, by [**13**, (2.13)(c)] we have a bijection $\mathfrak{a}_n^- \to P_n$ given by the Cayley map $x \mapsto (1+\frac{x}{2})(1-\frac{x}{2})^{-1}$.

(1.1) LEMMA. *Let Λ be an $\mathfrak{o}_F$-lattice sequence in V and let $m, n \in \mathbb{Z}$ satisfy $2n \geq m > n \geq 1$.*
(i) The map $x \mapsto 1+x$ induces an isomorphism of abelian groups $\mathfrak{a}_n/\mathfrak{a}_m \tilde{\to} U_n/U_m$.
(ii) If Λ is self-dual then the map $x \mapsto 1+x$ induces an isomorphism of abelian groups $\mathfrak{a}_n^-/\mathfrak{a}_m^- \tilde{\to} P_n/P_m$.

Let S be an $\mathfrak{o}_F$-lattice in A, hence an $\mathfrak{o}_0$-lattice in A. We define the $\mathfrak{o}_F$-lattice

$$\begin{aligned} S^* &= \{a \in A : \mathrm{tr}_0(aS) \subset \mathfrak{p}_0\} \\ &= \{a \in A : \mathrm{tr}_{A/F}(aS) \subset \mathfrak{p}_F\}, \end{aligned}$$

since F is at worst tamely ramified over F_0. If S is also stable under the involution, we can define

$$(S_-)^* = \{a \in A_- : \mathrm{tr}_0(aS_-) \subset \mathfrak{p}_0\} = (S^*)_-$$

since the direct sum $S = S_- \oplus S_+$ is orthogonal with respect to tr_0.
We recall from [**5**, (2.10)] that, if Λ is an $\mathfrak{o}_F$-lattice sequence in V with associated filtration $\mathfrak{a}_n$, then we have $\mathfrak{a}_n^* = \mathfrak{a}_{1-n}$.
Let "hat" ^ denote the Pontrjagin dual. Then we have the following:

(1.2) LEMMA. *Let Λ be an $\mathfrak{o}_F$-lattice sequence in V and let $m, n \in \mathbb{Z}$ satisfy $2n \geq m > n \geq 1$.*
(i) There is a $\mathfrak{K}(\Lambda)$-equivariant isomorphism of abelian groups

$$\begin{array}{ccc} \mathfrak{a}_{1-m}/\mathfrak{a}_{1-n} & \tilde{\to} & (U_n/U_m)\hat{} \\ b + \mathfrak{a}_{1-n} & \mapsto & \psi_b \end{array}$$

where $\psi_b(u) = \psi_F(\mathrm{tr}_{A/F}(b(u-1)))$ for $u \in U_n$.
(ii) If Λ is self-dual then there is a P-equivariant isomorphism of abelian groups

$$\begin{array}{ccc} (\mathfrak{a}_{1-m}^-)/(\mathfrak{a}_{1-n}^-) & \tilde{\to} & (P_n/P_m)\hat{} \\ b + (\mathfrak{a}_{1-n}^-) & \mapsto & \psi_b^- \end{array}$$

where $\psi_b^-(p) = \psi_0(\mathrm{tr}_0(b(p-1)))$ *for* $p \in P_n$. *Moreover, for* $b \in (\mathfrak{a}_{1-m}^-)$, ψ_b^- *is the restriction to* P_n *of* ψ_b.

We now recall some definitions from [**3**], keeping the language of [**5**]

(1.3) Definition ([**3**, (1.5)], [**5**, (3.1)]). A *stratum* in A is a 4-tuple $[\Lambda, n, r, b]$ consisting of a lattice sequence Λ in V, $n \in \mathbb{Z}$, $r \in \mathbb{R}$ with $r < n$, and an element $b \in \mathfrak{a}_{-n}(\Lambda)$. We say that two strata $[\Lambda, n, r, b_i]$, $i = 1, 2$, are *equivalent* if $b_1 \equiv b_2 \pmod{\mathfrak{a}_{-r}(\Lambda)}$.

Let $[\Lambda, n, r, b]$ be a stratum in A and suppose that the integers r, n satisfy

$$n > r \geq [\tfrac{n}{2}] \geq 0, \tag{1.4}$$

where $[x]$ is the greatest integer less than or equal to x. By(1.2)(i), an equivalence class of strata $[\Lambda, n, r, b]$ corresponds to the character ψ_b of $U_{r+1}(\Lambda)/U_{n+1}(\Lambda)$.

(1.5) Definition ([**3**, (1.5.5)]). Let $[\Lambda, n, r, \beta]$ be a stratum in A. It is *pure* if
(*i*) the algebra $E = F[\beta]$ is a field;
(*ii*) Λ is an $\mathfrak{o}_E$-lattice sequence;
(*iii*) $\nu_\Lambda(\beta) = -n$.

If $[\Lambda, n, r, \beta]$ is a pure stratum, we put, for $k \in \mathbb{Z}$, $\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{a}_0(\Lambda) : \beta x - x\beta \in \mathfrak{a}_k\}$, an $\mathfrak{o}_F$-lattice in A. Then we define $k_0(\beta, \Lambda)$ to be the least integer k such that $\mathfrak{n}_{k+1}(\beta, \Lambda)$ is contained in $B \cap \mathfrak{a}_0 + \mathfrak{a}_1$, where B is the A-centralizer of β; we understand that if $F[\beta] = F$ then $k_0(\beta, \Lambda) = -\infty$. If $F[\beta] \neq F$ then $k_0(\beta, \Lambda)$ is an integer greater than or equal to $-n$, with equality if and only if β is *minimal* (see [**3**, (1.4.14)]).

(1.6) Definition ([**3**, (1.5.5)]). A pure stratum $[\Lambda, n, r, \beta]$ is called *simple* if $r < -k_0(\beta, \Lambda)$.

We now consider the situation for our group G.

(1.7) Definition. A stratum $[\Lambda, n, r, b]$ in A is called *skew* if $b + \overline{b} = 0$ and Λ is self-dual.

Again, if (1.4) is satisfied then, by (1.2)(ii), an equivalence class of skew strata $[\Lambda, n, r, b]$ corresponds to the character ψ_b^- of $P_{r+1}(\Lambda)/P_{n+1}(\Lambda)$.

We finish this section with some technical results concerning simple strata.

(1.8) Lemma. *Let* Λ *be an* $\mathfrak{o}_F$*-lattice sequence in* V *and let* $\{\gamma_t : t \in \mathbb{Z}\}$ *be a sequence of elements of* $\mathfrak{K}(\Lambda)$ *which converges to a non-zero element* γ *in* A. *Then* $\gamma \in \mathfrak{K}(\Lambda)$.

Proof. Let $\nu = \nu_\Lambda(\gamma) < +\infty$ and let $T \in \mathbb{Z}$ be such that $\gamma_T \equiv \gamma \pmod{\mathfrak{a}_{\nu+1}}$. Let $x = \gamma - \gamma_T \in \mathfrak{a}_{\nu+1}$; then $\gamma = \gamma_T(1 + \gamma_T^{-1}x) \in \mathfrak{K}(\Lambda)U_1(\Lambda) = \mathfrak{K}(\Lambda)$. ■

(1.9) Proposition. *Let* $\{[\Lambda, n, r, \gamma_t] : t \in \mathbb{N}\}$ *be a sequence of equivalent simple strata such that* γ_t *converges to some* $\gamma \in A$. *Then* $[\Lambda, n, r, \gamma]$ *is a simple stratum.*

Proof. Let $\Phi_t(X) \in F[X]$ be the characteristic polynomial of γ_t and let $P_t(X)$ be its minimal polynomial, which is irreducible. Let $\Phi(X) \in F[X]$ be the characteristic polynomial of γ; then we certainly have $\lim_{t\to\infty} \Phi_t(X) = \Phi(X)$.

Let $\overline{F}$ be an algebraic closure of F. Then the set of roots of $\Phi_t(X)$ in $\overline{F}$, for all $t \in \mathbb{Z}$, is bounded. Hence the set of coefficients (in F) of $P_t(X)$, $t \in \mathbb{Z}$, is also bounded and $\{P_t(X) : t \in \mathbb{Z}\}$ has a convergent subsequence. Let $P(X) \in F[X]$ be the limit of this subsequence; in particular we have $P(\gamma) = 0$.

If $P(X)$ factorizes into coprime factors then, by Hensel's Lemma, $P_t(X)$ factorizes for $P_t(X)$ sufficiently close to $P(X)$, which is absurd. Hence $P(X) = \Pi(X)^m$, for some $\Pi(X) \in F[X]$ irreducible, $m \in \mathbb{N}$.

Now we show $\Pi(\gamma) = 0$. For suppose not, then $\Pi(\gamma)$ is nilpotent. But $\Pi(\gamma_t) \in \mathfrak{K}(\Lambda)$ converges to $\Pi(\gamma)$ so, by(1.8), $\Pi(\gamma)$ is a nilpotent element of $\mathfrak{K}(\Lambda)$, which is absurd.

In particular, $F[\gamma]$ is a field, whose non-zero elements normalize Λ by (1.8) so $[\Lambda, n, r, \gamma]$ is a pure stratum. Then, since $F[\gamma]$ is of degree less than or equal to that of $F[\gamma_1]$, $[\Lambda, n, r, \gamma]$ is in fact simple, by [**3**, (2.4.1)(i)]. ■

We now give the skew analogue of [**3**, (2.4.1)]. In particular, this will allow us to conclude that, for a skew simple stratum, the groups determined by the stratum are invariant under Σ.

(1.10) PROPOSITION. *Let $[\Lambda, n, r, \beta]$ be a pure stratum with Λ self-dual and $\beta+\overline{\beta} \in \mathfrak{a}_{-r}$. Then there exists a skew simple stratum $[\Lambda, n, r, \gamma]$ equivalent to $[\Lambda, n, r, \beta]$.*

Proof. By [**3**, (2.4.1)(i)], there exists a simple stratum $[\Lambda, n, r, \gamma_0]$ equivalent to $[\Lambda, n, r, \beta]$. Then $\gamma_0+\overline{\gamma}_0 \in \mathfrak{a}_{-r}$ and $[\Lambda, n, r, -\overline{\gamma}_0]$ is also a simple stratum equivalent to $[\Lambda, n, r, \beta]$.

We find, by induction, simple strata $[\Lambda, n, r, \gamma_t]$ equivalent to $[\Lambda, n, r, \beta]$ such that $\gamma_t + \overline{\gamma}_t \in \mathfrak{a}_{t-r}$ and $\gamma_t - \gamma_{t+1} \in \mathfrak{a}_{t-r}$. Granting this, we let γ be the limit of the γ_t, as $t \to \infty$, $\gamma \in \beta + \mathfrak{a}_{-r}$. Then $[\Lambda, n, r, \gamma]$ is simple, by(1.9), equivalent to $[\Lambda, n, r, \beta]$ and skew, as required.

We have found γ_0 so assume we have γ_t as required, for some $t \geq 0$. Let $E = F[\gamma_t]$, $B = C_A(E)$, $\mathfrak{b}_n = \mathfrak{a}_n \cap B$, for $n \in \mathbb{Z}$, and let s be a tame corestriction relative to E/F (see [**3**, (1.3.3)]). We will also write $\Lambda_{\mathfrak{o}_E}$ when we think of Λ as an $\mathfrak{o}_E$-lattice sequence. The simple strata $[\Lambda, n, r-t, \gamma_t]$ and $[\Lambda, n, r-t, -\overline{\gamma}_t]$ are equivalent so, by [**3**, (2.4.1)(ii)] and since $2 \in \mathfrak{o}_F^\times$, there exists $\delta_t \in E$ such that $s(\gamma_t + \overline{\gamma}_t) \equiv -2\delta_t \pmod{\mathfrak{a}_{1-r+t}}$.

We put $b_t = -\frac{1}{2}(\gamma_t + \overline{\gamma}_t)$; then the stratum $[\Lambda_{\mathfrak{o}_E}, r-t, r-t-1, s(b_t)]$ in B is equivalent to the stratum $[\Lambda_{\mathfrak{o}_E}, r-t, r-t-1, \delta_t]$, which is either simple or equivalent to the null stratum $[\Lambda_{\mathfrak{o}_E}, r-t, r-t-1, 0]$. In the latter case, by [**3**, (2.2.1)], there exists $u \in U_1(\Lambda)$ such that the skew stratum $[\Lambda, n, r-t-1, \gamma_t+b_t]$ is equivalent to the simple stratum $[\Lambda, n, r-t-1, u\gamma_t u^{-1}]$ so we put $\gamma_{t+1} = u\gamma_t u^{-1}$, with $k_0(\gamma_{t+1}, \Lambda) = k_0(\gamma_t, \Lambda)$. In the former case, by [**3**, (2.2.8)], the skew stratum $[\Lambda, n, r-t-1, \gamma_t + b_t]$ is equivalent to a simple stratum $[\Lambda, n, r-t-1, \gamma_{t+1}]$; moreover, $k_0(\gamma_{t+1}, \Lambda) = k_0(\gamma_t, \Lambda)$, since $k_0(\delta_t, \Lambda_{\mathfrak{o}_E}) = -\infty$. In both cases we have that $[\Lambda, n, r, \gamma_{t+1}]$ is simple, as required. ■

(1.11) COROLLARY (cf. [**3**, (2.2.8)]). *Let $[\Lambda, n, r, \beta]$ be a skew simple stratum* *in A. Let B be the A-centralizer of $E = F[\beta]$ and $\mathfrak{b}_n = \mathfrak{a}_n \cap B$. Let $b \in \mathfrak{a}_{-r}^-$ and let s be a tame corestriction on A relative to E/F. Suppose that the stratum $[\Lambda_{\mathfrak{o}_E}, r, r-1, s(b)]$ is equivalent to a simple stratum in B. Then $[\Lambda, n, r-1, \beta+b]$ is equivalent to a skew simple stratum $[\Lambda, n, r-1, \beta_1]$ and, moreover, $k_0(\beta_1, \Lambda) = \max\{k_0(\beta, \Lambda), k_0(c, \Lambda_{\mathfrak{o}_E}\}$.*

Proof. By [**3**, (2.2.8)], the skew stratum $[\Lambda, n, r-1, \beta+b]$ is equivalent to a simple stratum $[\Lambda, n, r-1, \beta_0]$. Hence $\beta_1 + \overline{\beta}_1 \in \mathfrak{a}_{1-r}$ and(1.10) implies that $[\Lambda, n, r-1, \beta_0]$ is equivalent to a skew simple stratum $[\Lambda, n, r-1, \beta_1]$. The final assertion follows from [**3**, (2.2.8)]. ∎

We remark that(1.10) (and hence also(1.11)) is easily generalizable to the situation of a group Γ of automorphisms of $\tilde{G}$ of order coprime to p and $G = \tilde{G}^{\Gamma}$.

2. *The principal lemma*

We now take a digression and consider Glauberman's correspondence of characters (see [**7**]). The notation in the first part of this section is independent of that in the remainder of this paper. For the exposition of Glauberman's results, we follow [**2**].

Let H be a finite group and Γ a soluble subgroup of $\mathrm{Aut}H$ such that $|H|$, $|\Gamma|$ are relatively prime. We denote the centralizer of Γ in H by H^{Γ}.

We write $\mathrm{Irr}\,(H)$ for the set of equivalence classes of irreducible representations of H and use a similar notation for other groups. The group Γ acts on $\mathrm{Irr}\,(H)$; we denote the set of fixed points by $\mathrm{Irr}\,(H)^{\Gamma}$.

We have the following result of Glauberman, whose formulation is taken from [**11**]:

(2.1) Theorem. *There is a uniquely determined bijection*

$$\boldsymbol{g} = \boldsymbol{g}_{\Gamma,H} : \mathrm{Irr}\,(H)^{\Gamma} \xrightarrow{\simeq} \mathrm{Irr}\,(H^{\Gamma})$$

with the following properties:
(i) if Γ is an l-group, for some prime number l, and $\rho \in \mathrm{Irr}\,(H)^{\Gamma}$ then $\boldsymbol{g}(\rho)$ occurs in $\rho|_{H^{\Gamma}}$ with multiplicity incongruent to 0 modulo l;
(ii) if Δ is a normal subgroup of Γ then

$$\boldsymbol{g}_{\Gamma,H} = \boldsymbol{g}_{\Gamma/\Delta,H^{\Delta}} \circ \boldsymbol{g}_{\Delta,H}.$$

In fact, in case (i), one has that $\boldsymbol{g}(\rho)$ occurs in $\rho|_{H^{\Gamma}}$ with multiplicity congruent to $\pm 1 \pmod{l}$, by [**7**, Corollary 6].

In this situation we also have the following results, from [**8**]:

(2.2) Theorem ([**8**, Theorem A]). *Let K be a Γ-stable subgroup of H. Let $\rho \in \mathrm{Irr}\,(H)^{\Gamma}$, $\sigma \in \mathrm{Irr}\,(K)^{\Gamma}$.*
(i) If $\rho \simeq \mathrm{Ind}\,_K^H \sigma$ then $\boldsymbol{g}(\rho) \simeq \mathrm{Ind}\,_{K^{\Gamma}}^{H^{\Gamma}} \boldsymbol{g}(\sigma)$.
(ii) If $\rho|_K \simeq \sigma$ then $\boldsymbol{g}(\rho)|_{K^{\Gamma}} \simeq \boldsymbol{g}(\sigma)$.

(2.3) LEMMA ([**8**, (2.3)]). *Suppose Γ is an l-group, for l a prime number. Let K be a Γ-stable subgroup of H. Let $\rho \in \operatorname{Irr}(H)^\Gamma$, $\sigma \in \operatorname{Irr}(K)^\Gamma$. Then σ occurs in $\rho|_K$ with multiplicity incongruent to* $0 \pmod{l}$ *if and only if $\boldsymbol{g}(\sigma)$ occurs in $\boldsymbol{g}(\rho)|_{K^\Gamma}$ with multiplicity incongruent to* $0 \pmod{l}$.

In the situation of(2.3), suppose that $\boldsymbol{g}(\rho)$ occurs in ρ with multiplicity congruent to $\epsilon \pmod{l}$ and $\boldsymbol{g}(\sigma)$ occurs in σ with multiplicity congruent to $\eta \pmod{l}$. Then, if σ occurs in $\rho|_K$ with multiplicity $r \pmod{l}$, the proof of [**8**, (2.3)] in fact shows that $\boldsymbol{g}(\sigma)$ occurs in $\boldsymbol{g}(\rho)|_{K^\Gamma}$ with multiplicity $\epsilon\eta r \pmod{l}$.

We now return to the situation of §1 and apply the results concerning Glauberman's correspondence (in particular (2.3)) to the intertwining of representations of pro-p subgroups of $\tilde{G} = \mathrm{GL}(N,F)$. Firstly though observe that Glauberman's correspondence can be applied to representations of pro-p subgroups of $\tilde{G}$ by quotienting by a (small enough) normal compact open subgroup. We will apply all the above results to this situation without further comment.

For $i = 1,2$, let ρ_i be a representation of a subgroup H_i of $\tilde{G}$. For $g \in \tilde{G}$, the intertwining space $I_g(\rho_1,\rho_2)$ is defined to be

$$I_g(\rho_1,\rho_2) = \operatorname{Hom}_{{}^gH_1\cap H_2}({}^g\rho_1,\rho_2),$$

where ${}^gH_1 = gH_1g^{-1}$ and ${}^g\rho_1$ is the representation $x \mapsto \rho_1(gxg^{-1})$ of ${}^gH_1$. We put $I_{\tilde{G}}(\rho_1,\rho_2) = \{g \in \tilde{G} : I_g(\rho_1,\rho_2) \neq 0\}$ and say that g *intertwines* ρ_1 with ρ_2 if $g \in I_{\tilde{G}}(\rho_1,\rho_2)$.

(2.4) PRINCIPAL LEMMA. *Let Γ be a finite soluble subgroup of* $\mathrm{Aut}\tilde{G}$. *Suppose also that Γ is an l-group, $l \neq p$ a prime number. For $i = 1,2$, let H_i be Γ-stable pro-p subgroups of $\tilde{G}$ and let $\rho_i \in \operatorname{Irr}(H_i)^\Gamma$. Let $g \in G := \tilde{G}^\Gamma$. Then*

$$\dim_{\mathbb{C}}(I_g(\rho_1,\rho_2)) \not\equiv 0 \pmod{l} \iff \dim_{\mathbb{C}}(I_g(\boldsymbol{g}(\rho_1),\boldsymbol{g}(\rho_2))) \not\equiv 0 \pmod{l}.$$

Proof. Let S be the set of triples (σ,m_1,m_2) consisting of an (equivalence class of) irreducible representation σ of ${}^gH_1\cap H_2$ and the multiplicity m_1 (respectively m_2) with which σ occurs in (the restriction of) ${}^g\rho_1$ (respectively ρ_2). The contribution of the triple (σ,m_1,m_2) to the intertwining space $I_g(\rho_1,\rho_2)$ has dimension m_1m_2. Let S_l be the subset of S consisting of those triples with $m_1m_2 \not\equiv 0 \pmod{l}$.

The group Γ acts on S_l since ${}^gH_1\cap H_2$, ${}^g\rho_1$ and ρ_2 are each fixed by Γ. Let O be an orbit for this action; if O is not just one triple then it has the form $\{(\sigma,m_1,m_2),(\sigma^\gamma,m_1,m_2),\ldots,(\sigma^{\gamma^{l-1}},m_1,m_2)\}$, where γ is some generator of Γ. In particular, the contribution of O to the intertwining space has dimension $lm_1m_2 \equiv 0 \pmod{l}$.

Let $S_l^\Gamma = \{(\sigma_j,m_1^j,m_2^j)\}$ be the fixed points of S_l for the action of Γ. Then we have

$$\dim_{\mathbb{C}}(I_g(\rho_1,\rho_2)) \equiv \sum_j m_1^jm_2^j \pmod{l}.$$

Similarly, let T be the set of triples (τ,n_1,n_2) consisting of an (equivalence class of) irreducible representation τ of ${}^gH_1^\Gamma\cap H_2^\Gamma$ and the multiplicity n_1 (respectively

n_2) with which τ occurs in (the restriction of) ${}^g\boldsymbol{g}(\rho_1)$ (respectively $\boldsymbol{g}(\rho_2)$). Let $T_l = \{(\tau_k, n_1^k, n_2^k)\}$ be the subset consisting of those triples with $n_1 n_2 \not\equiv 0 \pmod{l}$. Then, as above, we have

$$\dim_{\mathbb{C}}(I_g(\boldsymbol{g}(\rho_1), \boldsymbol{g}(\rho_2))) \equiv \sum_k n_1^k n_2^k \pmod{l}.$$

By (2.3), Glauberman's correspondence gives a bijection $S_l^\Gamma \leftrightarrow T_l$ and, using the remark following(2.3), we can examine how this bijection affects the multiplicities. For $i = 1, 2$, let $\epsilon_i = \pm 1$ be such that $\boldsymbol{g}(\rho_i)$ occurs in ρ_i with multiplicity congruent to $\epsilon_i \pmod{l}$. For $(\sigma_j, m_1^j, m_2^j) \in S_l^\Gamma$, let $\eta_j = \pm 1$ be such that $\boldsymbol{g}(\sigma_j)$ occurs in σ_j with multiplicity congruent to $\eta_j \pmod{l}$. Then, if $(\boldsymbol{g}(\sigma_j), n_1^j, n_2^j) \in T_l$, we have $n_i^j \equiv \epsilon_i \eta_j m_i^j \pmod{l}$, for $i = 1, 2$. In particular,

$$\sum_j n_1^j n_2^j \equiv \sum_j \epsilon_1 \eta_j m_1^j \epsilon_2 \eta_j m_2^j \equiv \epsilon_1 \epsilon_2 \sum_j m_1^j m_2^j \pmod{l},$$

and the lemma follows. ■

(2.5) Corollary. *With notation as in (2.4), suppose $\rho_i = \chi_i$ are characters, $i = 1, 2$. Then $I_g(\chi_1, \chi_2) \neq 0$ if and only if $I_g(\boldsymbol{g}(\chi_1), \boldsymbol{g}(\chi_2)) \neq 0$.*

We will, in particular, apply(2.4) and(2.5) to the case $\Gamma = \Sigma$, where Σ is as in §2.

3. *Characters and intertwining*

3.1. *Simple characters in $\tilde{G}$*

We begin by recalling the definitions of the orders $\mathfrak{H}$, $\mathfrak{J}$ from [**3**, (3.1)]. Throughout this section $[\Lambda, n, 0, \beta]$ will be a simple stratum in A. We also assume that Λ is a strict lattice sequence; this restriction is due to the fact that the results of [**3**, §3] are only available in this case, although it seems likely that they will remain valid in the general case. We set $r = -k_0(\beta, \Lambda)$ and let $\{\mathfrak{a}_t : t \in \mathbb{Z}\}$ be the filtration on A associated to Λ. We write B_β for the centralizer of β in A, dropping the subscript when the meaning is clear, and put $\mathfrak{b}_t = \mathfrak{b}_{\beta,t} = \mathfrak{a}_t \cap B_\beta$.

We define $\mathfrak{H}$ and $\mathfrak{J}$ inductively on r. If β is minimal over F we put

$$(3.1) \qquad \begin{cases} \mathfrak{H}(\beta) & = \mathfrak{H}(\beta, \Lambda) = \mathfrak{b}_\beta + \mathfrak{a}_{[\frac{n}{2}]+1}; \\ \mathfrak{J}(\beta) & = \mathfrak{J}(\beta, \Lambda) = \mathfrak{b}_\beta + \mathfrak{a}_{[\frac{n+1}{2}]}. \end{cases}$$

Otherwise, suppose that $r < n$ and let $[\Lambda, n, r, \gamma]$ be a simple stratum equivalent to $[\Lambda, n, r, \beta]$, which exists by [**3**, (2.4.1)]; then we put

$$(3.2) \qquad \begin{cases} \mathfrak{H}(\beta) & = \mathfrak{H}(\beta, \Lambda) = \mathfrak{b}_\beta + \mathfrak{H}(\gamma, \Lambda) \cap \mathfrak{a}_{[\frac{r}{2}]+1}; \\ \mathfrak{J}(\beta) & = \mathfrak{J}(\beta, \Lambda) = \mathfrak{b}_\beta + \mathfrak{J}(\gamma, \Lambda) \cap \mathfrak{a}_{[\frac{r+1}{2}]}. \end{cases}$$

Note that this inductive definition is independent of the choice of γ such that $[\Lambda, n, r, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, r, \beta]$. We also have filtrations on $\mathfrak{H}(\beta)$ and $\mathfrak{J}(\beta)$ given by

$$\left.\begin{aligned} \mathfrak{H}^t(\beta) &= \mathfrak{H}(\beta) \cap \mathfrak{a}_t \\ \mathfrak{J}^t(\beta) &= \mathfrak{J}(\beta) \cap \mathfrak{a}_t \end{aligned}\right\} \qquad \text{for } t \geq 0.$$

We define two families of compact open pro-p subgroups of $\tilde{G}$ by

$$\left.\begin{aligned} H^t(\beta) = H^t(\beta,\Lambda) &= \mathfrak{H}(\beta,\Lambda) \cap U_t(\Lambda) \\ J^t(\beta) = J^t(\beta,\Lambda) &= \mathfrak{J}(\beta,\Lambda) \cap U_t(\Lambda) \end{aligned}\right\} \qquad \text{for } t \geq 0.$$

We may now recall the definition of the sets $\mathcal{C}(\Lambda, m, \beta)$ of simple characters associated to the simple stratum $[\Lambda, n, 0, \beta]$.

(3.3) Definition ([**3**, (3.2.1)]). Let β be minimal over F, $E = F[\beta]$. For $0 \leq m \leq n-1$, let $\mathcal{C}(\Lambda, m, \beta)$ denote the set of characters θ of $H^{m+1}(\beta)$ such that
(*i*) $\theta|_{H^{m+1}(\beta)\cap U_{[\frac{n}{2}]+1}(\Lambda)} = \psi_\beta$;
(*ii*) $\theta|_{H^{m+1}(\beta)\cap B_\beta^\times}$ factors through $\det{}_{B_\beta} : B_\beta^\times \to E^\times$.

(3.4) Definition ([**3**, (3.2.3)]). Suppose $r < n$ and let $[\Lambda, n, r, \gamma]$ be a simple stratum equivalent to the pure stratum $[\Lambda, n, r, \beta]$. Then, for $0 \leq m \leq r-1$, let $\mathcal{C}(\Lambda, m, \beta)$ denote the set of characters θ of $H^{m+1}(\beta)$ such that
(*i*) $\theta|_{H^{m+1}(\beta)\cap B_\beta^\times}$ factors through $\det{}_{B_\beta}$;
(*ii*) θ is normalized by $\mathfrak{K}(\Lambda) \cap B_\beta^\times$;
(*iii*) if $m' = \max\{m, [\frac{r}{2}]\}$, the restriction $\theta|_{H^{m'+1}(\beta)}$ is of the form $\theta_0\psi_c$ for some $\theta_0 \in \mathcal{C}(\Lambda, m', \gamma)$, $c = \beta - \gamma$.

In the latter case, for $m \geq r$ we set $\mathcal{C}(\Lambda, m, \beta) = \mathcal{C}(\Lambda, m, \gamma)$.

For $\theta \in \mathcal{C}(\Lambda, m, \beta)$, we write $I_{\tilde{G}}(\theta)$ for the intertwining set $I_{\tilde{G}}(\theta, \theta)$. For $0 \leq m \leq r-1$, we also set $\mathfrak{m}_m = \mathfrak{m}_m(\beta, \Lambda) = \mathfrak{a}_{r-m} \cap \mathfrak{n}_{-m} + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)$.

(3.5) Theorem ([**3**, (3.3.2)]). *Let $[\Lambda, n, 0, \beta]$ be a simple stratum in A, $r = -k_0(\beta, \Lambda)$. Let $0 \leq m \leq r-1$ and $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Then*

$$I_{\tilde{G}}(\theta) = (1 + \mathfrak{m}_m) B_\beta^\times (1 + \mathfrak{m}_m).$$

3.2. *Simple characters in G*

Recall that σ is the involution of $\tilde{G}$ given by $x \mapsto x^{-1}$ and $\Sigma = \{1, \sigma\} \subset \mathrm{Aut}\tilde{G}$. We now look at the situation in our unitary group $G = \tilde{G}^\Sigma$. We consider again a simple stratum $[\Lambda, n, 0, \beta]$ with Λ a strict lattice sequence and continue with the notation of the previous section.

(3.6) LEMMA. *Let $[\Lambda, n, 0, \beta]$ be a skew simple stratum in A. Then the groups $H^t(\beta, \Lambda)$ and $J^t(\beta, \Lambda)$, $t \geq 0$, are fixed by Σ.*

Proof. This follows by induction along $r = -k_0(\beta, \Lambda)$, since in the definitions of $\mathfrak{H}(\beta, \Lambda)$, $\mathfrak{J}(\beta, \Lambda)$ ((3.1), (3.2)) we may choose the simple stratum $[\Lambda, n, r, \gamma]$ to be skew, by (1.10). ■

From now on, we suppose that the stratum $[\Lambda, n, 0, \beta]$ is skew. Hence Σ acts on the set of equivalence classes of irreducible representations of $H^t(\beta, \Lambda)$. For $0 \leq m \leq n-1$, we put $\mathcal{C}^{\Sigma}(\Lambda, m, \beta) = \{\theta \in \mathcal{C}(\Lambda, m, \beta) : \theta^\sigma = \theta\}$. Note that this set is non-empty since card $\mathcal{C}(\Lambda, m, \beta)$ is a power of p, by [**3**, (3.3.21)].

We define two families of compact open subgroups of G by

$$\left.\begin{aligned} H^t_-(\beta, \Lambda) &= H^t(\beta, \Lambda) \cap G \\ J^t_-(\beta, \Lambda) &= J^t(\beta, \Lambda) \cap G \end{aligned}\right\} \qquad \text{for } t \geq 0.$$

Then, since $H^t_-(\beta, \Lambda) = H^t(\beta, \Lambda)^{\Sigma}$, $H^t(\beta, \Lambda)$ is a pro-p subgroup of $\tilde{G}$ and $p \neq 2 = \operatorname{card}\Sigma$, we have Glauberman's correspondence $\boldsymbol{g}$ (see (2.1)) between the set of equivalence classes of irreducible representations of $H^t(\beta, \Lambda)$ fixed by Σ and the set of equivalence classes of irreducible representations of $H^t_-(\beta, \Lambda)$.

We put $\mathcal{C}_-(\Lambda, m, \beta) = \{\boldsymbol{g}(\theta) : \theta \in \mathcal{C}^{\Sigma}(\Lambda, m, \beta)\}$ and call an element of $\mathcal{C}_-(\Lambda, m, \beta)$ a simple character for G. Note here that, since θ is a character, $\boldsymbol{g}(\theta)$ is just the restriction of θ.

REMARK. We could also have defined the simple characters for G directly, analogously to the definitions (3.3), (3.4) for $\tilde{G}$. These two definitions coincide (see [**14**, §6.2]).

For $0 \leq m \leq r-1$, let $Q_m = Q_m(\beta, \Lambda)$ denote the group $(1 + \mathfrak{m}_m(\beta, \Lambda)) \cap G$, where $\mathfrak{m}_m = \mathfrak{m}_m(\beta, \Lambda) = \mathfrak{a}_{r-m} \cap \mathfrak{n}_{-m} + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)$ as in (3.5).

(3.7) THEOREM. *Let $[\Lambda, n, 0, \beta]$ be a skew simple stratum in A, $0 \leq m \leq r-1$ and $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$. Then*

$$I_G(\theta_-) = Q_m \cdot B_\beta \cap G \cdot Q_m.$$

Proof. Let $\theta \in \mathcal{C}^{\Sigma}(\Lambda, m, \beta)$ be such that $\theta_- = \boldsymbol{g}(\theta)$. Then, by (2.5), $I_G(\theta_-) = I_{\tilde{G}}(\theta) \cap G = (1+\mathfrak{m}_m)B_\beta^\times(1+\mathfrak{m}_m) \cap G$, by(3.5), and this will decompose as required if we can show that $(1+\mathfrak{m}_m)$, $B_\beta^\times$, Σ satisfy the conditions of [**15**, (1.3)]. The only condition which is not automatic is that, for $b \in B_\beta^\times$, we have

$$(1+\mathfrak{m}_m)b(1+\mathfrak{m}_m) \cap B_\beta^\times = ((1+\mathfrak{m}_m) \cap B_\beta^\times)b((1+\mathfrak{m}_m) \cap B_\beta^\times).$$

However, we have $\mathfrak{b}_{\beta,m} \subset \mathfrak{m}_m \subset \mathfrak{a}_m$ so, for $b \in B_\beta^\times$, we have

$$\begin{aligned}(1+\mathfrak{b}_{\beta,m})b(1+\mathfrak{b}_{\beta,m}) &\subset (1+\mathfrak{m}_m)b(1+\mathfrak{m}_m) \cap B \\ &\subset (1+\mathfrak{a}_m)b(1+\mathfrak{a}_m) \cap B = (1+\mathfrak{b}_{\beta,m})b(1+\mathfrak{b}_{\beta,m}),\end{aligned}$$

by [**3**, (1.6.1)]. ∎

3.3. *Semisimple characters in* $\tilde{G}$

We now extend the results of the previous sections to *semisimple* strata, whose definition we now recall (see [**15**, §3.2]).

Let $[\Lambda, n, r, \beta]$ be a stratum in A and put $e = e(\Lambda)$, the $\mathfrak{o}_F$-period of Λ. Set $g = (n, e)$ and consider $y_\beta = \pi_F^{n/g}\beta^{e/g}$. We define the characteristic polynomial $\varphi_\beta(X)$ of the stratum to be the reduction modulo $\mathfrak{p}_F$ of the characteristic polynomial of $y_\beta \in A$ (which lies in $\mathfrak{o}_F[X]$).

Let V_i, $i = 1, 2$, be subspaces of V such that $V = V_1 \oplus V_2$. Let $\mathbf{1}^i$ denote the projection $V \to V^i$ with kernel V^j, $j \neq i$, and put $A^{ij} = \mathbf{1}^i \cdot A \cdot \mathbf{1}^j$, for $i, j = 1, 2$. We identify $A^{ij} = \mathrm{Hom}_F(V^j, V^i)$ and abbreviate $A^{ii} = A^i$. We use the notation

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}.$$

If S is an $\mathfrak{o}_F$-lattice in A, we set $S^{ij} = L \cap A^{ij}$. We also put $\mathcal{M} = A^1 \oplus A^2$, $M = \mathcal{M}^\times$, a Levi subgroup of $\tilde{G}$, $N_u = 1 + A^{12}$, $N_l = 1 + A^{21}$ and $P_u = MN_u$, $P_l = MN_l$.

For $i = 1, 2$, let Λ^i be a lattice sequence in V_i and put $\Lambda = \Lambda^1 \oplus \Lambda^2$, a lattice sequence in V of period $e = \mathrm{lcm}\,(e_1, e_2)$. Let $\beta_i \in A^i$ and put $n_i = -\nu_{\Lambda^i}(\beta_i)$. Then we put $\beta = \beta_1 \oplus \beta_2$ and $n = e \cdot \max\{n_1/e_1, n_2/e_2\}$ so that $\nu_\Lambda(\beta) = -n$. Thus we obtain a stratum $[\Lambda, n, r, \beta]$ in A, for any $0 \leq r \leq n-1$.

(3.8) Definition. A stratum $[\Lambda, n, r, \beta]$ as above is called *split* if
(i) $\beta_1 \in \mathfrak{K}(\Lambda^1)$;
(ii) either $n_1/e_1 > n_2/e_2$ or else all the following conditions hold:
 (a) $n_1/e_1 = n_2/e_2$,
 (b) $\beta_2 \in \mathfrak{K}(\Lambda^2)$,
 (c) $\gcd\,(\varphi_{\beta_1}, \varphi_{\beta_2}) = 1$.

(3.9) Definition. (Inductive on the dimension of V.) A stratum $[\Lambda, n, r, \beta]$ is called *semisimple* if either it is a simple stratum and Λ is strict or it is split as above and $[\Lambda^i, n_i, r_i, \beta_i]$ is a semisimple stratum, for $i = 1, 2$, where $r_i = [re_i/e]$.

Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum as above and let $\{\mathfrak{a}_i : i \in \mathbb{Z}\}$ be the filtration on A associated to Λ. We define the $\mathfrak{o}_F$-orders $\mathfrak{H} = \mathfrak{H}(\beta, \Lambda)$ and $\mathfrak{J} = \mathfrak{J}(\beta, \Lambda)$ inductively on the dimension of V. If $[\Lambda, n, 0, \beta]$ is a simple stratum then $\mathfrak{H}$ and $\mathfrak{J}$ are as defined in (3.1), (3.2). Otherwise, we put

$$\mathfrak{H}(\beta,\Lambda) = \begin{pmatrix} \mathfrak{H}(\beta_1,\Lambda^1) & \mathfrak{a}^{12}_{[\frac{n}{2}]+1} \\ \mathfrak{a}^{21}_{[\frac{n}{2}]+1} & \mathfrak{H}(\beta_2,\Lambda^2) \end{pmatrix}, \qquad \mathfrak{J}(\beta,\Lambda) = \begin{pmatrix} \mathfrak{J}(\beta_1,\Lambda^1) & \mathfrak{a}^{12}_{[\frac{n+1}{2}]} \\ \mathfrak{a}^{21}_{[\frac{n+1}{2}]} & \mathfrak{J}(\beta_2,\Lambda^2) \end{pmatrix}.$$

For $t \geq 0$, we put $\mathfrak{H}^t(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda) \cap \mathfrak{a}_t$ and $\mathfrak{J}^t(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda) \cap \mathfrak{a}_t$ and also define the groups $H^t(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda) \cap U_t(\Lambda)$, $J^t(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda) \cap U_t(\Lambda)$. We often write $J(\beta, \Lambda)$ in place of $J^0(\beta, \Lambda)$. Note that, since $\mathfrak{a}_{n+1}(\Lambda) \cap A^i = \mathfrak{a}_{[ne_i/e]+1}(\Lambda^i)$, we have $\mathfrak{H}^{t+1}(\beta,\Lambda)\cap A^i = \mathfrak{H}^{[\frac{te_i}{e}]+1}(\beta_i,\Lambda^i)$, for $i=1,2$, and likewise for $\mathfrak{J}$, so that $H^{t+1}(\beta,\Lambda)\cap M = H^{[\frac{te_1}{e}]+1}(\beta_1,\Lambda^1)\times H^{[\frac{te_2}{e}]+1}(\beta_2,\Lambda^2)$.

(3.10) Lemma. *For $[\frac{n}{2}]\le t\le n-1$, $J(\beta,\Lambda)$ normalizes ψ_β on $U_{t+1}(\Lambda)$.*

Proof. The case of a simple stratum is a special case of [**3**, (3.3.1)] and the general case follows by induction since, for $x\in\mathfrak{a}_{t+1}$ and putting $x=\sum_{i,j=1}^2 x_{ij}$ with $x_{ij}=\mathbf{1}^i x\mathbf{1}^j\in A^{ij}$, we have $\psi_\beta(1+x)=\psi_{\beta_1}(1+x_{11})\psi_{\beta_2}(1+x_{22})$. ∎

For $0\le m\le n-1$, we would now like to define (inductively) a set of *semisimple* characters $\mathcal{C}(\Lambda,m,\beta)$ of the group $H^{m+1}(\beta,\Lambda)$. If $[\Lambda,n,0,\beta]$ is a simple stratum then this is just the set of simple characters ((3.3), (3.4)). Otherwise, we put $m_i=[\frac{me_i}{e}]$ and assume we have defined the sets $\mathcal{C}(\Lambda^i,m_i,\beta_i)$, for $i=1,2$.

(3.11) Definition. Let $[\Lambda,n,0,\beta]$ be a semisimple stratum as above. For $0\le m\le n-1$, let $\mathcal{C}(\Lambda,m,\beta)$ denote the set of characters θ of $H=H^{m+1}(\beta,\Lambda)$ such that
(*i*) $\theta|_{H\cap U_{[\frac{n}{2}]+1}(\Lambda)}=\psi_\beta$;
(*ii*) Writing $\theta|_{H\cap M}=\theta_1\otimes\theta_2$, we have $\theta_i\in\mathcal{C}(\Lambda^i,[\frac{me_i}{e}],\beta_i)$, for $i=1,2$.

(3.12) Lemma. *Let $[\Lambda,n,0,\beta]$ be a semisimple stratum as above and fix $m\in\mathbb{Z}$, $0\le m\le n-1$. Let $\theta_i\in\mathcal{C}(\Lambda^i,[\frac{me_i}{e}],\beta_i)$, for $i=1,2$, and put $m'=\max\{m,[\frac{n}{2}]\}$. Then there exists a unique character θ of $H=H^{m+1}(\beta,\Lambda)$ such that $\theta|_{H\cap M}=\theta_1\otimes\theta_2$ and $\theta|_{U_{m'+1}(\Lambda)}=\psi_\beta$. Moreover, θ is normalized by $J(\beta,\Lambda)$.*

Proof. We proceed by induction on the dimension of V. In the case of a simple stratum, the first statement is empty while the second is [**3**, (3.3.1)]. So assume $[\Lambda,n,0,\beta]$ is split semisimple; in particular, we assume that θ_i is normalized by $J(\beta_i,\Lambda^i)$, for $i=1,2$. As $H=H\cap M\cdot U_{m'+1}(\Lambda)$, uniqueness is clear so we need only prove existence of such a character θ.

Put $\theta_M=\theta_1\otimes\theta_2$, a character of $H\cap M$. Now $H\cap M\subset J(\beta,\Lambda)$, so $H\cap M$ normalizes the character ψ_β of $U_{m'+1}(\Lambda)$, by (3.10). Moreover, θ_M and ψ_β agree on $U_{m'+1}\cap M$ so we define θ by

$$\theta(uh)=\psi_\beta(u)\theta_M(h),\qquad \text{for } u\in U_{m'+1}(\Lambda),\ h\in H\cap M.$$

For $j\in J(\beta,\Lambda)$, $h\in U_{m'+1}$, we have $\theta(jhj^{-1})=\theta(h)$ by (3.10). Similarly, we have that H normalizes ψ_β on $J^{[\frac{n+1}{2}]}(\beta,\Lambda)$ so that, for $h\in H$, $j\in J^{[\frac{n+1}{2}]}(\beta,\Lambda)$, we have $\theta(jhj^{-1})=\psi_\beta(jhj^{-1}h^{-1})\theta(h)=\theta(h)$. Hence, to prove that $J(\beta,\Lambda)$ normalizes θ, we need only show that $J(\beta,\Lambda)\cap M$ normalizes $\theta|_{H\cap M}=\theta_M$, which is true by induction. ∎

Note that we also have

$$\theta(h_l h_M h_u)=\theta_M(h_M),\qquad \text{for } h_l\in H\cap N_l,\ h_M\in H\cap M,\ h_u\in H\cap N_u,$$

since $H\cap N_l\subset\ker\psi_\beta$ and likewise for $H\cap N_u$.

Lemma (3.12) says that we have a bijection

$$\mathcal{C}(\Lambda, m, \beta) \longleftrightarrow \mathcal{C}(\Lambda^1, [\tfrac{me_1}{e}], \beta_1) \times \mathcal{C}(\Lambda^2, [\tfrac{me_2}{e}], \beta_2). \tag{3.13}$$

(3.14) THEOREM. *Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum in A and let $\theta \in \mathcal{C}(\Lambda, 0, \beta)$. Then*

$$I_{\tilde{G}}(\theta) = J^1 B^\times J^1,$$

where $B \subset M$ is the centralizer of β in A.

Proof. We proceed by induction on the dimension of V, the simple case being (3.5). We have $I_{\tilde{G}}(\theta) \subset I_{\tilde{G}}(\psi_\beta|U_{[\frac{n}{2}]+1}(\Lambda)) \subset U_{[\frac{n+1}{2}]} M U_{[\frac{n+1}{2}]}$, by [**15**, (3.9)]. Now, by (3.12), J^1 normalizes θ and $U_{[\frac{n+1}{2}]} \subset J^1$ so we have $I_{\tilde{G}}(\theta) = J^1 I_M(\theta) J^1$. But $I_M(\theta) \subset J^1 B^\times J^1$, by induction, so the result follows. ∎

3.4. *Semisimple characters in G*

We return once more to the group $G = \widetilde{G}^\Sigma$, with the notation of the previous sections.

(3.15) DEFINITION. A semisimple stratum $[\Lambda, n, r, \beta]$ is called *skew* if (inductive definition, on the dimension) either $[\Lambda, n, r, \beta]$ is a skew simple stratum in the sense of(1.7) or $V = V_1 \perp V_2$ and each $[\Lambda^i, n_i, r_i, \beta_i]$ is a skew semisimple stratum, $i = 1, 2$, where $r_i = [re_i/e]$.

From now on, we suppose $[\Lambda, n, 0, \beta]$ a skew semisimple stratum. The orders $\mathfrak{H}(\beta, \Lambda)$, $\mathfrak{J}(\beta, \Lambda)$ are fixed by the involution $^-$, since this is true in the skew simple case and $\overline{\mathfrak{a}_t^{12}} = \mathfrak{a}_t^{21}$, for $t \in \mathbb{Z}$. Hence the groups $H^t(\beta, \Lambda)$ and $J^t(\beta, \Lambda)$, $t \geq 0$, are fixed by Σ and, for $0 \leq m \leq n-1$, we put $\mathcal{C}^\Sigma(\Lambda, m, \beta) = \{\theta \in \mathcal{C}(\Lambda, m, \beta) : \theta^\sigma = \theta\}$. From (3.13) we have a bijection

$$\mathcal{C}^\Sigma(\Lambda, m, \beta) \longleftrightarrow \mathcal{C}^\Sigma(\Lambda^1, [\tfrac{me_1}{e}], \beta_1) \times \mathcal{C}^\Sigma(\Lambda^2, [\tfrac{me_2}{e}], \beta_2).$$

We define compact open subgroups of G by $H^t_-(\beta, \Lambda) = H^t(\beta, \Lambda) \cap G$, $J^t_-(\beta, \Lambda) = J^t(\beta, \Lambda) \cap G$, for $t \geq 0$. As before, we have Glauberman's correspondence $\boldsymbol{g}$ between the set of equivalence classes of irreducible representations of $H^t(\beta, \Lambda)$ fixed by Σ and the set of equivalence classes of irreducible representations of $H^t_-(\beta, \Lambda)$.
We put $\mathcal{C}_-(\Lambda, m, \beta) = \{\boldsymbol{g}(\theta) : \theta \in \mathcal{C}^\Sigma(\Lambda, m, \beta)\}$. Then we have a bijection

$$\mathcal{C}_-(\Lambda, m, \beta) \longleftrightarrow \mathcal{C}_-(\Lambda^1, [\tfrac{me_1}{e}], \beta_1) \times \mathcal{C}_-(\Lambda^2, [\tfrac{me_2}{e}], \beta_2).$$

As in the simple case, we also have $\mathcal{C}_-(\Lambda, m, \beta) = \{\theta|_{H^{m+1}_-} : \theta \in \mathcal{C}(\Lambda, m, \beta)\}$.

(3.16) THEOREM. *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A and let $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$. Then*

$$I_G(\theta_-) = J^1_- \cdot B \cap G \cdot J^1_-,$$

where $B \subset M$ is the centralizer of β in A.

Proof. Let $\theta \in \mathcal{C}^{\Sigma}(\Lambda, 0, \beta)$ be such that $\theta_- = \boldsymbol{g}(\theta)$. We proceed by induction on the dimension of V, the simple case being (3.7). We have $I_G(\theta_-) \subset I_G(\psi_\beta^- | P_{[\frac{n}{2}]+1}(\Lambda)) \subset P_{[\frac{n+1}{2}]}(M \cap G) P_{[\frac{n+1}{2}]}$, by [**15**, (3.15)]. Now, by (3.12), J^1 normalizes θ so that J_-^1 normalizes θ_-. Then $P_{[\frac{n+1}{2}]} \subset J_-^1$ so we have $I_G(\theta) = J_-^1 I_{M\cap G}(\theta_-) J_-^1$. But $I_{M\cap G}(\theta_-) \subset J_-^1 \cdot B \cap G \cdot J_-^1$, by induction, so the result follows. ∎

4. *Heisenberg representations*

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A and let $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$. We would now like to obtain all irreducible representations of $J_-^1(\beta, \Lambda)$ containing the character θ_- of $H_-^1(\beta, \Lambda)$. Since $J^1(\beta, \Lambda)$ is a pro-p subgroup of $\tilde{G}$, we are able to proceed as above via Glauberman's correspondence. We remark the the results for $\tilde{G}$ hold regardless of the skewness of the stratum.

(4.1) Proposition. *Let $\theta \in \mathcal{C}(\Lambda, 0, \beta)$. Then the pairing*

$$\boldsymbol{k}_\theta : (j, j') \mapsto \theta([j, j']), \qquad j, j' \in J^1(\beta, \Lambda),$$

induces a nondegenerate alternating bilinear form $J^1/H^1 \times J^1/H^1 \to k_F$.

Proof. We proceed by induction on $\dim V$, the simple case being given by [**3**, (3.4.1)]. As in the proof of [**3**, (3.4.1)], we need only show that, for $x \in \mathfrak{J}^1$,

$$\theta([1+x, 1+y]) = 1 \quad \forall y \in \mathfrak{J}^1 \qquad \Longleftrightarrow \qquad x \in \mathfrak{H}^1,$$

the implication $\Leftarrow$ being clear, since J^1 normalizes θ.
We certainly have the implication $\Rightarrow$ for $x \in \mathfrak{J}^1 \cap \mathcal{M}$, by the induction hypothesis, so we suppose that $x \in \mathfrak{J}^1 \cap A^{21} = \mathfrak{a}^{21}_{[\frac{n+1}{2}]}$ satisfies the left hand side (the case $x \in \mathfrak{J}^1 \cap A^{12}$ will follow symmetrically). For $y \in \mathfrak{J}^1 \cap A^{12}$, we have $[1+x, 1+y] \subset U_n(\Lambda)$ so

$$\begin{aligned} 1 = \theta([1+x, 1+y]) = \psi_\beta(1 + xy - yx) &= \psi_F \circ \mathrm{tr}_{A/F}(\beta xy - \beta yx) \\ &= \psi_F \circ \mathrm{tr}_{A/F}(x(y\beta - \beta y)). \end{aligned}$$

Now the map $y \mapsto y\beta - \beta y = y\beta_2 - \beta_1 y$ sends $\mathfrak{a}^{12}_{[\frac{n+1}{2}]}$ onto $\mathfrak{a}^{12}_{-[\frac{n}{2}]}$, by [**5**, (3.7) Lemma 1], so we have $\psi_A(x\mathfrak{a}^{12}_{-[\frac{n}{2}]}) \equiv 1$, that is $x \in (\mathfrak{a}^{12}_{-[\frac{n}{2}]})^* = \mathfrak{a}^{21}_{[\frac{n}{2}]+1} = \mathfrak{H}^1 \cap A^{21}$. ∎

As usual, we write B for the centralizer of β in A. Note that we have $B \subset M$.

(4.2) Corollary (cf. [**3**, (5.1.1)]). *Let $\theta \in \mathcal{C}(\Lambda, 0, \beta)$. Then there exists a unique irreducible representation η of $J^1(\beta, \Lambda)$ which contains θ. Moreover, $\eta|_{H^1(\beta,\Lambda)}$ is a multiple of θ, $\dim \eta = (J^1(\beta, \Lambda) : H^1(\beta, \Lambda))^{\frac{1}{2}}$ and $I_{\tilde{G}}(\eta) = J^1 B^\times J^1$.*

Now let $\theta \in \mathcal{C}^{\Sigma}(\Lambda, 0, \beta)$ and let η be the unique irreducible representation of $J^1(\beta, \Lambda)$ containing θ. Then η^σ is an irreducible representation of $J^1(\beta, \Lambda)$ containing $\theta^\sigma = \theta$ so $\eta^\sigma \simeq \eta$ by uniqueness. Let $\eta_- = \boldsymbol{g}(\eta)$ be the irreducible representation of $J_-^1(\beta, \Lambda)$ corresponding to η via Glauberman's correspondence.

It contains $\theta_- = \boldsymbol{g}(\theta)$ by (2.3), since η contains θ with multiplicity $\dim \eta$, which is odd. Moreover, $\eta_-|_{H^1_-}$ is a multiple of θ_-, since $J^1_-(\beta, \Lambda) \subset I_G(\theta_-)$.

REMARK. For J^1/H^1 considered as a k_F-vector space, σ is a linear map which preserves the bilinear form $\boldsymbol{k}_\theta$. The space J^1/H^1 then decomposes into orthogonal eigenspaces and the +1-eigenspace is precisely J^1_-/H^1_-. In particular, the restriction of $\boldsymbol{k}_\theta$ to J^1_-/H^1_- is nondegenerate and we deduce that η_- is the unique irreducible representation of J^1_- containing θ_-.

In order to obtain information on the intertwining of the representation η_-, we need more precise information on the intertwining spaces for η.
Recall that for ρ a representation of a subgroup K of $\tilde{G}$, the intertwining space $I_g(\rho|K) = I_g(\rho, \rho)$ is defined to be

$$I_g(\rho|K) = \mathrm{Hom}_{\,{}^gK\cap K}({}^g\rho, \rho),$$

with the notations of §2.

(4.3) PROPOSITION (cf. [**3**, (5.1.8)]). *Let $\theta \in \mathcal{C}(\Lambda, 0, \beta)$ and let η be the representation of $J^1(\beta, \Lambda)$ given by (4.2). Then, for $g \in \tilde{G}$, we have*

$$\dim_{\mathbb{C}}(I_g(\eta|J^1)) = \begin{cases} 1 & \textit{if } g \in J^1B^\times J^1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. As in the proof of [**3**, (5.1.8)], and since $I_{\tilde{G}}(\theta) = J^1B^\times J^1$, the proposition will follow if we can show that, for $y \in B^\times$, J^1yJ^1 is the union of $(J^1 : H^1)$ distinct (H^1, H^1) double cosets (where $(J^1 : H^1)$ is the group index). This will, in turn, follow from:

(4.4) LEMMA (cf. [**3**, (5.1.10)]). *For $y \in B^\times$ we have $(J^1 : J^1 \cap (J^1)^y) = (H^1 : H^1 \cap (H^1)^y)$.*

Proof. We write $J^1 = 1 + \mathfrak{j}$, $H^1 = 1 + \mathfrak{h}$. In the simple case, by [**3**, (3.1.16)], we have an exact sequence

$$0 \longrightarrow \mathfrak{b}_1 \longrightarrow \mathfrak{j} \xrightarrow{a_\beta} \mathfrak{h}^* \xrightarrow{s} \mathfrak{b}_0 \longrightarrow 0,$$

where $\mathfrak{b}_i = \mathfrak{a}_i \cap B$, for $i = 0, 1$, a_β is the map $A \to A$ given by $x \mapsto \beta x - x\beta$ and s is a *tame corestriction* on A (see [**3**, (1.3)]).
Suppose now we are in the semisimple case. Then we have

$$\mathfrak{h}^* = \begin{pmatrix} \mathfrak{H}(\beta_1, \Lambda^1)^* & \mathfrak{a}^{12}_{-[\frac{n}{2}]} \\ \mathfrak{a}^{21}_{-[\frac{n}{2}]} & \mathfrak{H}(\beta_2, \Lambda^2)^* \end{pmatrix}.$$

Moreover, by [**5**, (3.7) Lemma 1] a_β induces a bijection $\mathfrak{a}^{12}_{[\frac{n+1}{2}]} \to \mathfrak{a}^{12}_{-[\frac{n}{2}]}$, that is between $\mathfrak{j}^{12}$ and $\mathfrak{h}^{*12}$ (and likewise for $\mathfrak{a}^{21}$). Hence, by induction, we have an exact sequence

$$0 \longrightarrow \mathfrak{b}_1 \longrightarrow \mathfrak{j} \xrightarrow{a_\beta} \mathfrak{h}^* \xrightarrow{s} \mathfrak{b}_0 \longrightarrow 0,$$

where s is a *tame corestriction* on A given by

$$s\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} s_1(a_{11}) & 0 \\ 0 & s_2(a_{22}) \end{pmatrix},$$

for s_1, s_2 tame corestrictions on A^1, A^2 respectively.

For $y \in B^\times$, the map a_β also induces a bijection between $\mathfrak{j}^{12} \cap (\mathfrak{j}^{12})^y$ and $\mathfrak{h}^{*12} \cap (\mathfrak{h}^{*12})^y$. Hence, as above (and as in the proof of [**3**, (5.1.10)] in the simple case) we have an exact sequence

$$0 \longrightarrow \mathfrak{b}_1 + \mathfrak{b}_1^y \longrightarrow \mathfrak{j} + \mathfrak{j}^y \xrightarrow{a_\beta} \mathfrak{h}^* + (\mathfrak{h}^*)^y \xrightarrow{s} \mathfrak{b}_0 + \mathfrak{b}_0^y \longrightarrow 0.$$

The lemma now follows exactly as in [**3**, (5.1.10)]. ∎

This also completes the proof of (4.3). ∎

We may now apply (2.4) to obtain the intertwining of η_-.

(4.5) Proposition. *With notation as above, we have $I_G(\eta_-) = J^1_- \cdot B \cap G \cdot J^1_-$.*

Proof. Let $g \in J^1 B^\times J^1 \cap G$. Since $\dim_{\mathbb{C}}(I_g(\eta|J^1)) = 1$, by (2.4) we have $g \in I_G(\eta_-)$ so $I_G(\eta_-) \supset J^1 B^\times J^1 \cap G \supset J^1_- \cdot B \cap G \cdot J^1_-$. But, since $\eta_-|_{H^1_-}$ is a multiple of θ_-, we also have $I_G(\eta_-) \subset I_G(\theta_-) = J^1_- \cdot B \cap G \cdot J^1_-$, by (3.7), so there is equality, as required. ∎

5. *Maximal tori*

We now consider the extensions of the representation η_- of J^1_- to a representation κ_- of J_-. For any such extension, the pair (J_-, κ_-) will be a type and $\operatorname{Ind}_{J_-}^G \kappa_-$ will be an irreducible supercuspidal representation (see (5.2)). Observe that we cannot use Glauberman's correspondence for this stage since J is not a pro-p group (more precisely, because 2 divides $(J : J^1)$).

We continue with the notation of the previous section; in particular, $[\Lambda, n, 0, \beta]$ is a skew semisimple stratum.

(5.1) Definition. A simple stratum $[\Lambda, n, r, \beta]$ is called *maximal* if $F[\beta]$ is a maximal subfield of A. We extend this definition to semisimple strata inductively, as usual.

From now on we suppose that $[\Lambda, n, 0, \beta]$ is a maximal skew semisimple stratum. This implies that the centralizer $B_\beta \cap G$ of β in G is a compact maximal torus in G: if we write $B_\beta^\times = \prod_i E_i^\times$ with E_i an extension of F, then each E_i is stable under the involution $\bar{\ }$, with fixed field $E_{i,0} \neq E_i$, and we have $B_\beta \cap G = \prod_i N_1(E_i)$ where $N_1(E_i) = \{e \in E_i^\times : e\bar{e} = 1\}$ is the norm-1 group of the extension $E_i/E_{i,0}$. We write k_i, $k_{i,0}$ for the residue fields of E_i, $E_{i,0}$ respectively.

We have $J_- = (B_\beta \cap G)J^1_-$ so that $J_-/J^1_- = \prod_i N_1(k_i)$, where $N_1(k_i)$ is the norm-1 group of the extension $k_i/k_{i,0}$. Each group $N_1(k_i)$ is cyclic, of order 2 if $E_i/E_{i,0}$ is ramified, order $\operatorname{card} k_{i,0} + 1$ otherwise.

Let $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$ and let η_- be the (unique) irreducible representation of J^1_- containing θ_-, given by(4.2). The group J_- normalizes η_- and J_-/J^1_- is a product of cyclic groups so there exists an extension of η_- to a representation κ_- of J_-. Moreover, every extension of η_- is of the form $\kappa_- \otimes \chi$, for χ a character of J_- obtained by inflation from J_-/J^1_-.

(5.2) Theorem. *Let $[\Lambda, n, 0, \beta]$ be a maximal skew semisimple stratum in A, $\theta_- \in \mathcal{C}_-(\Lambda, 0, \beta)$ and η_- the unique irreducible representation of J^1_- containing θ_-. Let κ_- be any extension of η_- to a representation of J^1_-. Then $\pi = \mathrm{Ind}_{J_-}^{G} \kappa_-$ is an irreducible supercuspidal representation of G and (J_-, κ_-) is a $[G, \pi]_G$-type.*

Proof. We have $I_G(\kappa_-) \subset I_G(\eta_-) = J^1_- \cdot B_\beta \cap G \cdot J^1_- = J_-$. The first assertion now follows from e.g. [**6**, (1.5)], while the second is a direct consequence, by [**4**, (5.4)]. ■

Remarks. 1. The method described to construct supercuspidal representations essentially starts from a sequence of simple strata $[\Lambda_i, n_i, 0, \beta_i]$ of decreasing level. It should be straightforward to allow the final strata to be of level 0, though we do not pursue the matter here.

2. We have constructed supercuspidal representations starting from split strata. In [**5**, §6], there is also the notion of "relatively split" strata, where a derived stratum $[\mathfrak{B}, n, n-1, s(b)]$ is split. It should be possible to use the results there to repeat the above constructions in this case also.

3. It is possible to do much of the above construction in the more general setting of §2, that is $\Gamma \subset \mathrm{Aut}\tilde{G}$ is an l-group, $l \neq p$, and $G = \tilde{G}^\Gamma$. Indeed, the essential tools(2.4) and [**15**, (1.3)] are given in this generality (see also the remark following (1.11)).

6. *Skew simple characters in $\tilde{G}$*

In this technical section we examine the simple characters θ in $\tilde{G}$ which are fixed by the involution σ and show that we may choose the stratum defining θ to be skew. In particular, this implies that we have considered all such characters in §3. The techniques used here rely heavily on the results from [**3**, §3.5].

Let $[\Lambda, n, m, \beta]$ be a simple stratum, with Λ a strict lattice sequence, and let $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Let $E = F[\beta]$, $B = C_A(E)$ and $\mathfrak{b}_n = \mathfrak{a}_n(\Lambda) \cap B$, as usual, and write $\Lambda_{\mathfrak{o}_E}$ when we are thinking of Λ as an $\mathfrak{o}_E$-lattice sequence. By [**3**, (3.3.17)], the $\tilde{G}$-normalizer of θ is

$$N_{\tilde{G}}(\theta) = \mathfrak{K}(\Lambda_{\mathfrak{o}_E})(1 + \mathfrak{m}_m),$$

where $\mathfrak{m}_m = \mathfrak{a}_{r-m} \cap \mathfrak{n}_{-m} + \mathfrak{J}^{[\frac{r+1}{2}]}(\beta)$. In particular, the unique maximal compact subgroup of $N_{\tilde{G}}(\theta)$ is ${}^0N_{\tilde{G}}(\theta) = U(\Lambda_{\mathfrak{o}_E})(1 + \mathfrak{m}_m)$.

Let $L \subset V$ be an $\mathfrak{o}_F$-lattice stabilized by ${}^0N_{\tilde{G}}(\theta)$. Then L is an $\mathfrak{o}_E$-lattice since $\mathfrak{o}_E^\times \subset {}^0N_{\tilde{G}}(\theta)$. Consider the strict $\mathfrak{o}_E$-lattice sequence $\Lambda_L(n) = \mathfrak{p}_E^n L$ and put $\mathfrak{a}_L = \mathfrak{a}_0(\Lambda_L)$. Then $\mathfrak{a}_L \cap B$ is a maximal $\mathfrak{o}_E$-order in B and $\mathfrak{b}_0 \subset \mathfrak{a}_L \cap B$ since $U(\Lambda_{\mathfrak{o}_E})$ normalizes Λ_L. Hence $\mathfrak{a}_0 \subset \mathfrak{a}_L$. In particular, $\mathfrak{a}_0 \subset \mathrm{Stab}\, L$.

This is true for all lattices stabilized by ${}^0N_{\tilde{G}}(\theta)$ so we have

$$\mathfrak{a}_0 = \bigcap_L \operatorname{Stab} L,$$

where the intersection is taken over all lattices stabilized by ${}^0N_{\tilde{G}}(\theta)$. In particular, $\mathfrak{a}_0$ is determined by ${}^0N_{\tilde{G}}(\theta)$, hence by θ. Then, since Λ is strict, Λ is also determined by θ. Moreover, the integers n, m are also determined by θ, since $n = \min\{k \in \mathbb{N} : U_k(\Lambda) \subset \ker\theta\}$ and $m = \max\{k \in \mathbb{Z}, k \geq 0 : U_{k+1}(\Lambda) \supset H\}$, where H is the subgroup on which θ is defined.

(6.1) LEMMA. *Let $[\Lambda, n, m, \beta]$ be a simple stratum, with Λ a strict lattice sequence, and let $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Suppose that θ is fixed by σ. Then Λ is a self-dual.*

Proof. We also have that $[\Lambda^{\#}, n, m, -\overline{\beta}]$ is a simple stratum and $\theta = \theta^\sigma \in \mathcal{C}(\Lambda^{\#}, m, -\overline{\beta})$. But θ determines Λ so we have $\Lambda = \Lambda^{\#}$, as required. ■

We require one more preliminary lemma.

(6.2) LEMMA. *Let $[\Lambda, n, m, \beta]$ be a skew simple stratum in A and let θ be a simple character in $\mathcal{C}^{\Sigma}(\Lambda, m, \beta)$. Then there exists a simple character $\theta_0 \in \mathcal{C}^{\Sigma}(\Lambda, m-1, \beta)$ which restricts to θ on $H^{m+1}(\beta, \Lambda)$.*

Proof. Consider the set of $\theta' \in \mathcal{C}(\Lambda, m-1, \beta)$ which restricts to θ. By [**3**, (3.3.21)], this set has cardinality a power of p. But Σ acts on it and, since $p \neq 2$, there is a fixed point. ■

We may now state the main result of this section.

(6.3) THEOREM. *Let $[\Lambda, n, m, \beta]$ be a simple stratum, with Λ a strict lattice sequence, and let $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Suppose that θ is fixed by σ. Then there exists a skew simple stratum $[\Lambda, n, m, \gamma]$ such that $\theta \in \mathcal{C}(\Lambda, m, \gamma)$.*

Proof. By (6.1), we already have that Λ is self-dual. We proceed by induction along $k_0(\beta, \Lambda)$ so we assume first that β is minimal.

If $m \geq [\frac{n}{2}]$ then $\psi_\beta = \theta = \theta^\sigma = \psi_{-\overline{\beta}}$ so we have $\beta + \overline{\beta} \in \mathfrak{a}_{-m}$. Then, by(1.10), there exists a skew stratum $[\Lambda, n, m, \gamma]$ equivalent to $[\Lambda, n, m, \beta]$ and $\theta = \psi_\gamma \in \mathcal{C}(\Lambda, m, \gamma)$ as required.

Assume now $m < [\frac{n}{2}]$. We have $H^{[\frac{n}{2}]+1}(\beta, \Lambda) = U_{[\frac{n}{2}]+1}(\Lambda)$; the restriction of θ to this group is ψ_β, while the restriction of θ^σ is $\psi_{-\overline{\beta}}$. Then, since θ is fixed by σ, we have $\beta + \overline{\beta} \in \mathfrak{a}_{-[\frac{n}{2}]}$. By (1.10), there exists a skew simple stratum $[\Lambda, n, [\frac{n}{2}], \gamma]$ equivalent to $[\Lambda, n, [\frac{n}{2}], \beta]$ and γ is minimal by [**3**, (2.1.4)]. By [**3**, (3.1.9)], we have $H^{m+1}(\gamma, \Lambda) = H^{m+1}(\beta, \Lambda)$ and, furthermore, we have that $\theta | U_{[\frac{n}{2}]+1}(\Lambda) = \psi_\beta = \psi_\gamma$.

To conclude that $\theta \in \mathcal{C}(\Lambda, m, \gamma)$ we need only show that $\theta | U^{m+1}(\Lambda_{\mathfrak{o}_{F[\gamma]}})$ factors through the determinant, or, equivalently by [**3**, (2.4.11)], that it is intertwined by all of $B_\gamma^\times$. But the intertwining of the equivalent simple strata $[\Lambda, n, [\frac{n}{2}], \gamma]$ and $[\Lambda, n, [\frac{n}{2}], \beta]$ is

$$U_{[\frac{n+1}{2}]}(\lambda) B_\gamma^\times U_{[\frac{n+1}{2}]}(\lambda) = U_{[\frac{n+1}{2}]}(\beta) B_\gamma^\times U_{[\frac{n+1}{2}]}(\beta),$$

by [**3**, (1.5.8)]. But this is precisely the intertwining of θ, by [**3**, (3.3.2)], so $\theta|U^{m+1}(\Lambda_{\mathfrak{o}_{F[\gamma]}})$ is indeed intertwined by all of $B_\gamma^\times$, as required.

We now assume $k_0(\beta,\Lambda)=-r>-n$. We will first reduce to the case $m=r-1$ so we suppose $m<r-1$ and that we have the result for $m'>m$ In particular we have that $[\Lambda,n,m+1,\beta]$ is a simple stratum. The restriction $\widetilde{\theta}=\theta|_{H^{m+2}(\beta)}$ is a simple character in $\mathcal{C}(\Lambda,m+1,\beta)$ fixed by Σ so, by induction, we have a simple stratum $[\Lambda,n,m+1,\gamma']$ with $\widetilde{\theta}\in\mathcal{C}(\Lambda,m+1,\gamma')$. By [**3**, (3.5.9)], we have $H^{m+1}(\beta)=H^{m+1}(\gamma')$

Let $\theta_1\in\mathcal{C}^\Sigma(\Lambda,m,\gamma')$ be such that $\theta_1|_{H^{m+2}}=\theta|_{H^{m+2}}$. In particular, we have $\theta=\theta_1\psi_b$ for some $b\in\mathfrak{a}_{-1-m}$ and, since θ, θ_1 are both fixed by Σ, $\psi_b=\psi_{-\bar{b}}$ on H^{m+1}. Hence $\psi_{\frac{b-\bar{b}}{2}}=\psi_b$ on H^{m+1} and we may assume $b\in A_-$. Let s be a tame corestriction on A relaive to $F[\gamma']/F$; then, ss in the proof of [**3**, (3.5.9)], we have $s(b)\in F[\gamma']+\mathfrak{b}_{\gamma',-m}$. Hence $[\Lambda_{\mathfrak{o}_{F[\gamma']}},m+1,m,s(b)]$ is simple. Then by (1.11), there exists a skew simple stratum $[\Lambda,n,m,\gamma]$ equivalent to $[\Lambda,n,m,\gamma'+b]$. By [**3**, (3.3.20)], we have $\theta=\theta_1\psi_b\in\mathcal{C}(\Lambda,m,\gamma)$ as required.

Thus we assume $m=r-1$. Let $[\Lambda,n,m+1,\xi]$ be a simple stratum equivalent to $[\Lambda,n,m+1,\beta]$. Then $\widetilde{\theta}=\theta|_{H^{m+2}}\in\mathcal{C}^\Sigma(\Lambda,m+1,\xi)$ so, by induction, there exists a skew simple stratum $[\Lambda,n,r,\xi']$ with $\widetilde{\theta}\in\mathcal{C}(\Lambda,m+1,\xi')$. By [**3**, (3.5.9)], we have $H^{m+1}(\beta)=H^{m+1}(\xi)=H^{m+1}(\xi')$ and we may alter ξ to assume that $\mathcal{C}(\Lambda,m,\xi)=\mathcal{C}(\Lambda,m,\xi')$.

Let $\phi\in\mathcal{C}^\Sigma(\Lambda,m,\xi')$ be such that $\phi|_{H^{m+2}}=\widetilde{\theta}$. Let $b\in\mathfrak{a}_{-m-1}$ be such that $\theta=\phi\psi_b$; as in the previous case, since θ, ϕ are both fixed by Σ, we may assume $b\in A_-$. Let s, s' be tame corestrictions on A relative to $F[\xi]/F$, $F[\xi']/F$ respectively. By [**3**, (3.5.13)], $[\Lambda_{\mathfrak{o}_{F[\xi]}},m+1,m,s(b)]$ is equivalent to a simple stratum. Let $\mathcal{I}$ be the G-intertwining of θ and put $\mathcal{R}=(\mathcal{I}\cap\mathfrak{a}_0)+\mathfrak{a}_1/\mathfrak{a}_1$. Then, as in [**3**, (3.5.14)], we have

$$\begin{aligned}\mathcal{R}&=\{x\in\mathfrak{b}_{\xi,0}/\mathfrak{b}_{\xi,1}:xs(b)\equiv s(b)x \pmod{\mathfrak{b}_{\xi,-m}}\}\\&=\{x\in\mathfrak{b}_{\xi',0}/\mathfrak{b}_{\xi',1}:xs'(b)\equiv s'(b)x \pmod{\mathfrak{b}_{\xi',-m}}\}.\end{aligned}$$

Since $[\Lambda_{\mathfrak{o}_{F[\xi]}},m+1,m,s(b)]$ is equivalent to a simple stratum, $\mathcal{R}$ is a semisimple k_F-algebra, by [**3**, (2.4.13)], and so $[\Lambda_{\mathfrak{o}_{F[\xi']}},m+1,m,s'(b)]$ is also equivalent to a simple stratum. Then, by (1.11), $[\Lambda,n,m,\xi'+b]$ is equivalent to a skew simple stratum $[\Lambda,n,m,\gamma]$ and $\theta\in\mathcal{C}(\Lambda,m,\gamma)$ by [**3**, (3.3.20)]. ■

Since here we rely only on the results of [**3**, §3.5], together with (1.10) and (1.11), the above results will remain valid in the situation of an l-group Γ acting on $\tilde{G}$, $G=\tilde{G}^\Gamma$ (see the remark following (1.11)).

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