$\pi_1^\infty \widetilde{M}^3 = 0$
A SHORT OUTLINE OF THE PROOF

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1 Introduction. We want to announce here the proof of the following theorem

THEOREM **A**. *Let M^3 be a closed 3-dimensional manifold and let $\widetilde{M}^3$ be its universal covering space. Then the $\widetilde{M}^3$ has an exhaustion by compact, simply connected 3-dimensional submanifolds, i.e. it is* ***simply-connected at infinity*** *(which we will write as $\pi_1^\infty \widetilde{M}^3 = 0$, like in the title.)*

In full generality if X is any topological space admitting a countable exhaustion by compacta, we will say that X is simply-connected at infinity if for every compact subset $k \subset X$ there exists a larger compact $k \subset K \subset X$, such that any closed loop $\lambda \subset X - K$ which is null-homomotopic in X is already null-homotopic in $X - k$. If this is the case, we write $\pi_1^\infty X = 0$ and if not, we write $\pi_1^\infty X \neq 0$, without attaching any other meaning to "π_1^∞", in this present paper. Our definition has the obvious virtue that for a compact X, automatically we have $\pi_1^\infty X = 0$. Another important feature is that, when X is an open 3-manifold, the $\pi_1^\infty X = 0$ is implied by the existence of an exhaustion by finite (i.e. compact) simply-connected subcomplexes. This criterion is not valid, generally speaking, outside the realm of 3-manifolds.

An equivalent formulation of theorem A is the following

THEOREM **B**. *1) Let M^3 be a closed irreducible 3-manifold with infinite $\pi_1 M^3$. Then $\widetilde{M}^3 = R^3$.*

2) More generally, let N^3 be a closed 3-manifold which does not contain any fake homotopy-ball $\Delta^3 \neq B^3$ and let $e(\pi_1 N^3)$ be the space of ends of $\pi_1 N^3$. Then

$$\widetilde{N}^3 = S^3 - e(\pi_1 N^3).$$

Here $e(\pi_1 N^3)$ can be $\emptyset$($\Leftrightarrow \pi_1 N^3$ finite), $\{p\}$ (= a single point), $\{p, q\}$, or a ***tame*** *Cantor set.*

This corresponds to a rather large chunck of the topological counterpart of Thurston's geometrization conjecture [**Th**].

All this also looks like a genuinely 3-dimensional result and even theorem A, as stated above, certainly has to do with dimension three ; remember also that M. Davis [**Da**] has proved that in any dimension $n \geq 4$ there exist closed n-manifolds $M^n = K(\pi_1 M^n, 1)$ such that $\pi_1^\infty M^n \neq 0$, while 1) in theorem B

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covers exactly the case of those $M^3 = K(\pi_1 M^3, 1)$ (and which are also irreducible).

On the other hand the proof of $\pi_1^\infty \widetilde{M}^3 = 0$ which is outlined in the present Note (and the complete details of which are to be found in [**Po14**], at present in hand-written form, hopefully to be soon available as a typed preprint) has hardly anything to do with the techniques of 3-dimensional topology, as the reader will soon find out. Also, at least in the very coarse sense used above, we can easily make sense of "$\pi_1^\infty G$", for any finitely presented group G, so that for a compact connected complex K we find that $\pi_1^\infty(\pi_1 K) = \pi_1^\infty \widetilde{K}$ [**Ta**]. [A priori $\pi_1^\infty G$ is defined in terms of a presentations of G, but via Tietze one shows that it is actually presentation independent. Also if $H \subset G$ and if the index of H in G is finite, then $\pi_1^\infty H = \pi_1^\infty G$. It is very likely that $\pi_1^\infty G$ is a "geometric" concept, in the precise technical sense of Margulis and Gromov, but I have not checked this.]

Personally, I prefer to think of theorem **A** as being a purely group-theoretical result, which happens to come in a topological disguise. In the same vein, remember at this point that both the present author and also A. Casson, M. Bestvina, G. Mess and others, have independently already proved special cases of theorem **A**, all taking the following form: if $\pi_1 M^3$ verifies some ***nice geometric condition***, in the sense of geometric group theory, then $\pi_1^\infty \widetilde{M}^3 = \pi_1^\infty(\pi_1 M^3) = 0$. Here the nice geometric conditions in question include Gromov hyperbolicity [**Gr**], [**G-H**], [**C-D-P**], J. Cannon's almost covexity [**Can**], Thurston's combability [**E-C-H-L-P-T**], Casson's condition [**S-G**] a.s.o. A far from exhaustive list of the papers where these special cases of theorem **A** are proved is [**Po3**],[**Po4**],[**S-G**],[**B-M**],[**PoTa1**] and [**PoTa2**].

In the present paper we ouline a proof of $\pi_1^\infty(\pi_1 M^3) = 0$ where no other restriction, beyond being the fundamental group of a closed 3-manifold, is imposed on $\pi_1 M^3$, and although some of the ideas from the papers just listed have been useful, what we do here requires radically new tecniques. It will be useful to have in mind, as a starting point on our road, the following result in the DIFF, high-dimensional category, which is of the same vintage as, let's say, [**Po3**], [**Po4**].

THEOREM **C**. *Let V^3 be an open simply-connected 3-manifold which is such that there exists an $n \in Z_+$ with the following property. The smooth non-compact manifold with non-empty boundary $V^3 \times B^n$ contains a* PROPER, *codimesion zero,* ***geometrically simply-connected*** *smooth submanifold X^{n+3}, such that*

$$V^3 \times (*) \subset X^{n+3} \subset V^3 \times B^n. \tag{1}$$

Then $\pi_1^\infty V^3 = 0$.

If should be immediately stressed here that, just like it was already the case for thorem **A**, if in the statement above V^3 is replaced by V^p with $p \geq 4$, then the corresponding statement becomes false. The reasons are connected with the simple fact that only in dimension three does the simple connectivity of a compact manifold imply the simple connectivity of its boundary.

[The reader is also reminded that a manifold M^p, which might be non-compact, with non-empty boundary, is said to be ***geometrically simply-connected*** if it has a PROPER handlebody decomposition without handles of index $\lambda = 1$.

There is also an equivalent Morse theoretical version, of this notion: there exists a PROPER smooth function $M^p \xrightarrow{F} R_+$ such that all the singularities of F are in $\text{int} M^n$, of Morse type and with index $\lambda \neq 1$; moreover $F | \partial M^p$ is also Morse and all its "meaningful" singularities (i.e. those for which crossing the corresponding critical value implies a change in topology), also have $\lambda \neq 1$.]

The special case $V^{n+3} = V^3 \times B^n$ of theorem **C** is proved already in [**Po1**], [**Po2**], but the techniques of those papers can be extended so as to prove our present, slightly more general statement too. Also L. Funar and L. Funar-T. Thickstun [**Fu**], [**Fu-T**] have vastly generalized [**Po2**], and their results also imply theorem **C**, as well as the more practically useful 2) in lemma 8 below.

The present proof of theorem **A** follows, in broad lines, the program which was already sketched in the last section of [**Po5**]; that paper should also be very useful to the reader for getting a good feeling for the mathematical surroundings of the present work. [A warning, nevertheless. Things became, eventualy, trickyer than I had thought at the time when I wrote [**Po5**]. In particular in [**Po5**] one finds CONJECTURE (C) which was supposed to be proved on the road to $\pi_1^\infty \widetilde{M}^3 = 0$. That particular conjecture is still unproved, as such; but enough of its mathematical content is, available anyway to allow us to get theorem **A** above.]

THE PLAN OF THE PAPER. For an arbitrary 3-manifold W^3, with $\partial W^3 = \emptyset$ we introduce the notion of S_u-structure and its assiciated S_b-structure. These come with some "underlying spaces", which are high-dimensional smooth manifolds $\|S_u W^3\|$ and $|S_b W^3|$. With $\dim \|S_u W^3\| = n+3$ the $\|S_u W^3\|$ always enters, as X^{n+3}, into a diagram similar to (1) but slightly more singular, while for W^3 open and simply connected $|S_b W^3|$ is always geometrically simply connected. Theorem 1 below and lemmas 2,3,4,6,7,8, concern the case of open simply-connected 3-manifolds, while (13) in lemma 4, lemma 5 and Theorem 9 are true for $\underbrace{\text{arbitrary}}$ 3-manifolds. This brings us to the end of section 5. By then, the proof $\pi_1^\infty \widetilde{M}^3 = 0$ is reduced to showing, for appropriate structures, that $\|S_u \widetilde{M}^3\| \overset{\text{DIFF}}{=\!=} |S_b \widetilde{M}^3|$. This is done in section 6. So, before section 6, the 3-manifolds considered are sometimes arbitrary but, mostly open simply-connected, like the $\underbrace{\text{Whitehead}}$ manifold Wh^3. It is only in section 6 that the discrete Galois symmetry of $\widetilde{M}^3$, with its compact fundamental domain is finally used. At that point we also rely heavily on our previous paper [**PoTa3**], which proves the existence of invariant S_u-structures.

This paper owes a lot to David Gabai who, over a long period of time helped me in more than one way, with the mathematics of the present work and also of [**Po5**].

Thanks are also due to Mathew Brin, Louis Funar, Steve Gersten, Etienne Ghys, Misha Gromov, Lucien Guillou, François Laudenbach, Alexis Marin,

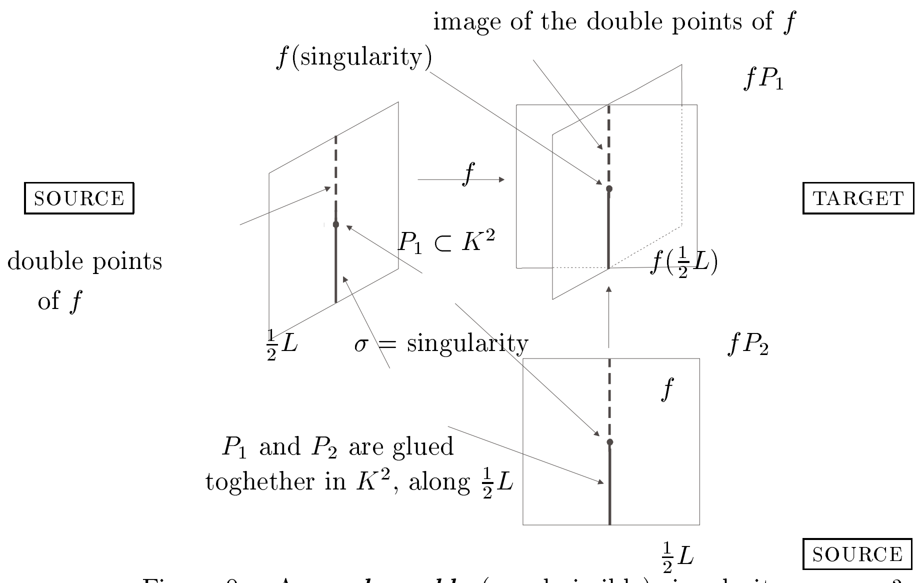

Figure 0 : An ***undrawable*** (or admissible) singularity..

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Barry Mazur, Frédéric Paulin, Larry Siebenmann, John Stallings, Dennis Sullivan and Corrado Tanasi for very helpful conversations and/or correspondance. I also wish to thank the I.H.E.S. where I was working at a time when some of the most crucial steps for this paper, took shape.

2. S_u**-structures.** It will be very convenient to replace the 3-manifolds W^3 which we will be considering (and which we suppose with $\partial W^3 = \emptyset$) by " W^3 with very many holes" which, up to homeomorphism is to be described as

$$W_h^3 \overset{\text{def}}{=} W^3 - \{\text{a } \textbf{\textit{tame}} \text{ totally discontinous closed subset}\}. \tag{2}$$

It is very easy to see that $\pi_1^\infty W^3 = \pi_1^\infty W_h^3$ and W_h^3 will turn out to be easier for us to handle than W^3.

We remind the reader, at this point, that if $K^2 \xrightarrow{f} W^3$ is a non-degenerate (let's say simplicial) map, a point $x \in K^2$ is an ***undrawable singularity*** for f if an open neighbourhood of x in K^2 consists of two planes P_1, P_2 such that:
i) $f|P_1, f|P_2$ inject and $fP_1 \cap fP_2$ is a line of transversal intersection, i.e. fP_1 and fP_2 go through each other transversally, at the target.

ii) At the source K^2, P_1 and P_2 are glued together along a half-line stopping at x (see fig. 1.1-h in [**Ga**] or fig. 1 in [**Po6**] or figure 0 below). Clearly f is not an immersion at x (which makes x a singularity) and a double line of $M^2(f) \subset K^2 \times K^2$ starts at (x,x). We will denote by $M_2(f) \subset K^2$ the image of $M^2(f)$ in K^2 and the (undrawable) singularities will be included, by definition, in the set of double points (which, otherwise, has no chance of being a closed subset.) The reason for the terminology "undrawable", suggested by Barry Mazur some years ago, should be pretty clear.

It is convenient at this point, to remind the reader of our COLLAPSIBLE PSEUDO-SPINE REPRESENTATION THEOREM for homotopy 3-spheres Σ^3 [**Po6**]. What this theorem says is the following. *For any Σ^3 there exists a non-degenerate simplicial map $K^2 \xrightarrow{f} \Sigma^3$, all the singularities of which are undrawable, such that*

a) K^2 is a finite collapsible 2-complex. [But it turns out to be enough here if we replace "collapsible" by "almost-collapsible", which means (collapsible)+(2-cells added along their boundaries).]

b) *The double points of f can be exhausted by **zipping moves** starting from the finite set Sing(f).* [The reader who wants more details concerning such moves, can find them in [**Po6**], [**Ga**], [**Po7**]; in [**Po1**] a very general and conceptual version of "zipability" is presented, in terms of the equality of two equivalence relations $\Psi(f) = \Phi(f)$ where $\Phi(f)$ is the simple-minded $(x,y) \in \Phi(f) \Longleftrightarrow fx = fy$ while $\Psi(f) \subset \Phi(f)$ is the much subtler "smallest equivalence relation compatible with f, which kills all the singularities" (i.e. the non-immersive points)].

c) *The open regular neighbourhood Nbd(fK^2) is*

$$\text{Nbd}(fK^2) = \Sigma^3 - \{\text{a finite set}\}. \tag{3}$$

[REMARK. Almost-collapsibility is, I think, a very useful concept. Here are some of its very nice features:

i) The 2-skeleton of a collapsible K^n is almost-collapsible (although not collapsible).

ii) If K^n has an almost collapsible 2-skeleton, this stays true after nice subdivision (barycentric, stellar, or Siebenmann's bissections [**Si**].)

iii) A compact smooth manifold is geometrically simply-connected if and only if it admits a smooth triangulation the 2-skeleton of which is almost-collaspible.]

This theorem is the motivation behind the definition which follows now. Let us say that if one wants to prove something like the collapsible pseudo-spine representation theorem, not for the compact Σ^3 but for an open simply-connected 3-manifold what one finds (with some work) is the definition below and the theorem 1 which follows. We start with a W^3 which is an arbitrary 3-manifold with $\partial W^3 = \emptyset$, possibly with the celebrated Whitehead manifold Wh^3 [**Wh2**] for which, as we know $\pi_1^\infty Wh^3 \neq 0$, although Wh^3 is contractible.

By definition, an S_u-***structure*** on W^3 is a non-degenerate map

$$X^2 \xrightarrow{f} W^3 \tag{4}$$

with the following list of properties.

(4.1) X^2 is a locally-finite simplicial complex and the singular set Sing$(f) \subset X^2$, which is discrete inside X^2, with $f|\text{Sing}(f)$ injective, but with $f(\text{Sing}(f)) \approx$

Sing(f) not necessarily discrete inside W^3, contains only undrawable singularities. Outside Sing(f), our f is a ***generic*** immersion

But we do ***not*** ask for f to be simplicial (for reasons to become clear shortly), the best we can do is to ask for some reasonable local conditions (not to be developed here) which make that the ***open*** regular neighbourhood Nbd$(fX^2) \subset W^3$ is well-defined. Our fX^2 will not be a locally finite space, hence not a simplicial sub-complex of W^3, for any kind of triangulation; this already signals the fact that f cannot be, in any way, simplicial. But it is still required to be non-degenerate, i.e. for a simplicial structure on X^2, which is sufficiently fine, f injects when restricted to a simplex.

[Before we go on with our definition, the reader might want to see some examples.

a) Take $W^3 = R^3$, take $x_1, x_2, \ldots, x_i \neq x_j, \ldots$ a sequence with $\lim_{n\to\infty} x_n = x_\infty$, $X^2 = \{$the disjoined union $\sum_1^\infty (x = x_n) + (y = 0)\}$ and f the obvious map of this infinite collection of planes into R^3.

The example above was non-singular. Here is a singular example. Start with

$$E = (x > 0, y > 0) \cap [(z = 0) \cup (\sum_1^\infty (x = \frac{1}{n}, z \geq 0)) \cup (\sum_1^\infty (y = \frac{1}{m}, z \geq 0))] \subset R^3$$

and then ***ungle***, for each $n, m \in Z_+$ the sheets $x = \frac{1}{n}, y = \frac{1}{m}$ of E along the $\frac{1}{2}$-lines $(x = \frac{1}{n}, y = \frac{1}{m}, z > 1)$. This gives an $X^3 \xrightarrow{f} R^3$.]

(4.2) Let us consider any ***tight*** precompact transversal $\Lambda \subset X^2$ to $M_2(f) \subset X^2$ (which, locally, looks like a large, generally speaking *infinite* collection of parallel lines, or a finite union of such collections.) Think here of $M_2(f) \subset X^2$ as being somehow similar to a ***lamination*** **[Th]** $\mathcal{L}$, except that, while for $\mathcal{L}$ we find that $\lim(\Lambda \cap \mathcal{L}) = \Lambda \cap \mathcal{L}$, now we are required to have $(\lim(\Lambda \cap M_2(f))) \cap M_2(f) = \emptyset$. Let us also say that a foliation $\mathcal{F}$, a lamination $\mathcal{L}$, or (the collections building up) our double point set $M_2(f)$ are transversally, respectively like R(=the real line), like a Cantor set, or like a countable set $S \subset R$ which is disjoined from its limit points.

We will ask for the following basic ***finiteness condition*** to be fulfilled

$$\text{card}(\lim(\Lambda \cap M_2(f))) < \infty \tag{$*$-1}$$

(where $\lim(\Lambda \cap M_2(f))$ means the set of points $x_\infty \in \Lambda$ for which there exists sequences $x_1, x_2, \ldots \in \Lambda \cap M_2(f)$ where all $x_i \neq x_j$, such that $x_\infty = \lim x_n$; actually, as already said, it will also be understood that $x_\infty \notin \Lambda \cap M_2(f)$.) With these conditions it will eventually turn out that the subset

$$\textsc{lim} M_2(f) \overset{\text{def}}{=} \overline{\bigcup_\Lambda (\Lambda \cap M_2(f))} \subset X^2 \tag{$*$-2}$$

is a bona fide locally finite simplicial subcomplex of X^2 of dimension one.

[*Explanation.* It will turn out that in the cases of interest for us, we will ***not*** be able to ask that $M_2(f) \subset X^2$ be a ***closed*** subset unlike what had (trivially) happened in the context of the collapsible pseudo-spine represenation theorem above. The next best thing which we might reaonably ask is (∗-1). Notice also that

$$\{M_2(f) \text{ closed}\} \Leftrightarrow \text{LIM} M_2(f) = \emptyset.$$

Once $M_2(f)$ is ***not*** closed, f cannot be simplicial either so we have to content ourselves with the ***open*** $\text{Nbd}(fX^2)$.]

(4-3) Like in the collapsible pseudo-spine representation theorem, $M^2(f)$ can be zipped, starting from $\text{Sing}(f)$, but infinitely many elementary zipping moves will be necessary now.

(4-4) Up to an isotopy which, generally speaking is ***non***-ambient, we have

$$\text{Nbd}(fX^2) = W_h^3,$$

a formula which should be compared to (3).

This finishes our definition (the only feature of which, is of a global, and not of a local caracter is (4-3)); at least for the time being one should also notice that no particular conditions have been imposed on X^2 in (4). In [**PoTa3**] one proves, among other things, the following result which should be regarded as a vast extension of the collapsible pseudo-spine representation theorem.

THEOREM 1. *1) Let W^3 be open and simply connected. Then W^3 admits an S_u-structure (4) where X^2 is **almost-arborescent** (i.e. it can be gotten from a point by an infinite (but* PROPER*) sequence of Whitehead dilatations of dimension ≤ 2 and additions of 2-cells; this is the correct extension of the almost-collapsibility discussed before, from the compact to the non compact case.)*

This S_u-structure has some additional features which I will list now.

2) The set

$$f\big(\text{LIM} M_2(f)\big) \subset f(X^2) \tag{5}$$

*is **closed**.*

3) Enough of the canonical retraction $\text{Nbd}(fX^2) \xrightarrow{r} fX^2$ is available, so that $r^{-1} f\big(\text{LIM} M_2(f)\big) \subset \text{Nbd} f(X^2)$ is well-defined. This

$$\Sigma(\infty) \overset{\text{def}}{=} r^{-1} f\big(\text{LIM} M_2(f)\big) \subset \text{Nbd} f(X^2) = W_h^3 \tag{6}$$

is a PROPER *smooth (stratified) 2-dimensional submanifold, the local models of which are R^2 and (open, 3-legged figure Y)$\times R$.*

In the rest of the paper, unless the contrary is explicitely said, W^3 will be taken to mean any open simply-connected 3-manifold. Moreover, any $S_u W^3$-structure considered for such an open simply-connected W^3 will ***always*** have an almost arborescent X^2, like in the theorem. [One might think of this as being another global condition concerning the definition of S_u-structures].

REMARK. Contrary to $\text{LIM} M_2(f)$ our $f(\text{LIM} M_2(f))$ is actually ***not locally finite***; this happen exactly in the neighbourhood of those points $p_{\infty\infty} \in f(\text{LIM} M_2(f))$ which are themselves limit points of $p_{\infty n} \in f\big(M_2(f|\text{LIM} M_2(f))\big)$.

We will call $\Sigma(\infty)$ the ***bad locus***. We have $\Sigma(\infty) = \emptyset$ iff $M_2(f)$ is closed and we have the following

LEMMA 2. *If W^3 admits an S_u-structure with $\Sigma(\infty) = \emptyset$, then $\pi_1^\infty W^3 = 0$.*

A relatively direct proof of this lemma is outlined in [**Po5**], but other ways of proving it will emerge below. Notice anyway, that this lemma 2 makes it hopeless to ask for anything stronger than the basic finiteness condition ($*$-1); the W^3 could be the Whitehead manifold, after all.

We want to add now as much boundary as possible to $\text{Nbd}(fX^2)$. Here is the idea how to go about it. One starts with a decomposition into reasonable pieces $X^2 = \bigcup_{i=0}^\infty L_i$ for which $f|L_i$ is injective too. One considers open thickenings, call them, $fL_i \times (-\varepsilon_i, \varepsilon_i) \subset W^3$ (with $\lim_{n=\infty} \varepsilon_n = 0$) and then, with a good choice of $\varepsilon_1 > \varepsilon_2 > \dots$ (going sufficiently fast to zero) we have, essentially,

$$\text{Nbd}(fX^2) = \bigcup_{i=0}^{\infty} fL_i \times (-\varepsilon_i, \varepsilon_i) = W_h^3 \text{ (see (4-4))}. \tag{7}$$

We get a manifold with (quite a large) boundary if, as much as possible, we replace the open intervals $(-\varepsilon_i, \varepsilon_i)$'s by the closed arcs $[-\varepsilon_i, \varepsilon_i]$'s in (7). This gives, roughtly speaking

$$[W_h^3] \stackrel{\text{def}}{=} \bigcup_{i=0}^{\infty} fL_i \times [-\varepsilon_i, \varepsilon_i] - \{\text{the points at infinity of } \Sigma(\infty), \text{think of them as } \bigcup_{i=0}^{\infty} (fL_i \cap f\text{LIM}M_2(f)) \times \{-\varepsilon_i, \varepsilon_i\}, \text{which are exactly those where } \bigcup_{i=0}^{\infty} fL_i \times [-\varepsilon_i, \varepsilon_i] \text{ fails to be a manifold }\}. \tag{8}$$

For this $[W_h^3]$ (a 3-manifold with very large boundary which really depends on the whole structure $S_u W^3$ and not just on W_h^3) we have $\text{int}[W_h^3] = W_h^3$ and it also comes naturally equipped with a PROPER embedding

$$\Sigma(\infty) \subset [W_h^3]. \tag{9}$$

Figure 1 is supposed to give an idea of (8), (9); in that figure, we have $L_0 = (y = 0), L_{n>0} = (x = x_n)$ and what we see is a section of $fX^2 \cap \{\text{coordinate neighbourhood } R^3\}$ and of its thickening, by the plane $z = 0$. We have $\lim x_n = x_\infty$, the dotted vertical line is in $\Sigma(\infty)$ the fat points (∞) are at the infinity of $\Sigma(\infty)$ (and deleted in (8), (9)). Everything is glued at the target along the doubly hached zones (which are images of double points.)

A HEURISTIC REMARK. When we deal with objects like fX^2, $\text{Nbd}(fX^2)$, $[W_h^3]$, $\Sigma(\infty)$ and their likes which will appear in the next sections, the word "at infinity" will take several, inter-related meanings. There is, certainly, the mythical and mysterious INFINITY of W^3 itself, the one to which W^3 being "simply-connected at infinity" refers to. But, for fX^2 and its likes there has to be infinity all over the place, under our steps wherever we might go, so as to be able to corral the bad accumulations which, otherwise, would jeopardize the various finiteness conditions like ($*$-1) or even the local finiteness of

X^2 itself. In more precise terms if $U \subset W^3$ (and here W^3 can be arbitrary) is any compact coordinate neighbourhood, when S_u-structures are being considered $fX^2 \cap U, \Sigma(\infty) \cap U$ or $\mathrm{Nbd}(fX^2) \cap U$ will never be *compact*. They will have their "local" infinities $\overline{fX^2 \cap U} - fX^2 \cap U, \overline{\Sigma(\infty) \cap U} - \Sigma(\infty) \cap U$ or $\overline{\mathrm{Nbd}(fX^2) \cap U} - \mathrm{Nbd}(fX^2) \cap U$ and putting together these local pieces, or "local infinities", which are quite concrete, one gets the global "infinity" of fX^2, or $\Sigma(\infty)$ or of $\mathrm{Nbd}(fX^2)$, as opposed to the "INFINITIES" of these objects (which are, essentially, the same as the INFINITY of W^3 itself.) All this is essentially heuristics which we do not have to make precise, and I will not try to formalize the distintion infinity versus INFINITY. At the time when that might have been otherwise necessary, in section 6) below, $W^3 = \widetilde{M}^3$ will have been replaced by the ***compact*** M^3 and there will be no INFINITY at all left to worry about. There will only plentiful of "infinity", precise, controllable and, by that time, itself compact.

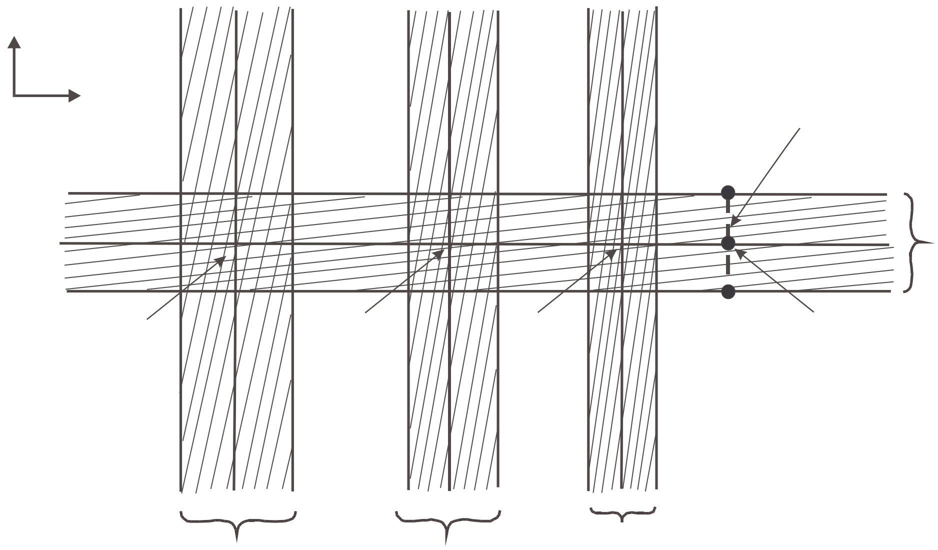

Figure 0 : Which?.

So far, we have taken the view that $\mathrm{int}[W_h^3] = W_h^3$ but we can change, sometimes, our viewpoint, as follows. Pull everything in fig.1 towards the interior, except at the fat points (∞). This allows us to get a PROPER embedding $[W_h^3] \subset W_h^3$ which continues to engulf $\Sigma(\infty)$ as a PROPER submanifold. It is this second viewpoint which will be used in the present section. With this, we introduce now the following provisional version for the ***underlying space*** of $S_u W^3$, which will anyway be a smooth manifold of high dimension (let us say $n + 3 \geq 6$). By definition, this is the following

$$(10) \qquad |S_u W^3| \overset{\text{def}}{=} [W_h^3] \times B^n.$$

Notice that, if $B^n(\varepsilon) \subset B^n$ is another, concentric ball of much smaller radius, then (just like in (1)) we have PROPER inclusion maps

$$W_h^3 \times (*) \subset |S_u W^3| \cup (W_h^3 \times B^n(\varepsilon)) \subset W_h^3 \times B^n. \tag{11}$$

Notice, also, that the middle term in (11) is gettable from $|S_u W^3|$ by a PROPER sequence of Whitehead dilatations It follows from these remarks, combined with theorem **C** from the introduction, that we have the following implication

$$\{|S_u W^3| \text{ geometrically simply connected}\} \Rightarrow \pi_1^\infty W^3 = 0 \tag{12}$$

Consider now a $S_u W^3$-structure $X^2 \xrightarrow{f} W^3$. There is a canonical way to thicken X^2 in $(n+3)$-dimensions, call this regular neighbourhood $\Theta^{n+3}(X^2)$. Since X^2 is almost arborescent, $\Theta^{n+3}(X^2)$ is geometrically simply-connected. Also, f can be zipped by an infinite sequence of "elementary zipping moves". From a strict ***homotopical*** view, whenever they are not plainly (simple) homotopy equivalences (in the sense of simple homotopy theory), these moves are additions of 2-cells along their boundary. What happens with these moves when we thicken, in the world of ***smooth*** $(n+3)$-manifolds, depends very critically on the range of the dimensions $(n+3)$ which are concerned, but for the high $(n+3)$'s which concern us here they thicken, canonically, into Whitehead dilatations or according to the case into smooth additions of handles of index $\lambda = 2$. [In my work on the Poincaré conjecture (see [**Ga**],[**Po6**], [**Po7**], [**Po8**], [**Po9**], [**Po10**], [**Po11**], [**Po12**], [**Po13**]), the dimension concerned is, exactly, $n+3=4$; then $\Theta^4(K^2)$ is not uniquely defined, it also depends on the choice of a ***desingularization*** of K^2. More importantly, only when a certain COHERENCE condition is verified do the zippings which are not homotopy equivalences smoothen into additions of 2-handles. The COHERENCE issue is the central theme of my Program for the Poincaré Conjecture, but the obstruction to COHERENCE vanishes as soon as one goes from dimension four to dimension five or more, by taking a product with $[0,1]$.] Closing this prentice and going back to $\Theta^{n+3}(X^2)$ and its smoothened zipping, it is not hard at all to prove the following

LEMMA 3. *If $\Sigma(\infty) = \infty$ then $|S_u W^3|$ is geometrically simply connected*

(which, of course, together with Theorem C and with (11) proves lemma 2).

But when $\Sigma(\infty) \neq \emptyset$, this can make havoc of the way in which the $(n+3)$-dimensional thickening of the zipping moves leading from X^2 to fX^2 are to be assembled together. Actually, since W^3 might as well be the Whitehead manifold, there ***has*** to be a $\Sigma(\infty)$ and, unless one can get rid of it when $W^3 = \widetilde{M}^3$, we are certainly not closer to $\pi_1^\infty \widetilde{M}^3$, after all this. There is actually something one can say about $|S_u W^3|$ even when $\Sigma(\infty) \neq \emptyset$, but this rather complicated result, to be found in [**PoTa4**] is irrelevant for $\widetilde{M}^3$; see also section 3 in [**Po5**] which gives a good overview of [**PoTa4**]. In point of fact, looking at things from the vantage point which we have reached by now, it can also be said that what those papers which showed that for nice $\pi_1 M^3$'s, the π_1^∞ is zero (like [**Po3**], [**Po4**], [**PoTa1**], [**PoTa2**]) did,was to construct special $S_u\widetilde{M}^3$'s (under reasonable geometric conditions for $\pi_1 M^3$ like, let's say, Gromov hyperbolic) for which $\Sigma(\infty) = \emptyset$. But our philosophical principle is that the general $\pi_1^\infty \widetilde{M}^3$ issue cannot be handled without dealing with an alive $\Sigma(\infty)$, in one disguise or another; we will come back to this issue in the last section. So, our strategy now is to accept that $\Sigma(\infty)$ is there and to try to come to grips with it.

3) S_b**-Structures**. Let us consider now an $S_u W^3$-structure $X^2 \xrightarrow{f} W^3$, as provided by theorem 1, let us say. Going from X^2 to fX^2, out of which $|S_u W^3|$ was eventually build, is an operation of taking a ***quotient*** (our zipping process) and the idea now is to ***add*** more material to X^2, rather than quotieting it, in a way which should imitate or mimick the zipping. So, we try to build up a diagram of the form

$$\begin{tikzcd} W^3 & & \overline{X}^2 = X^2 \cup Y^2 \arrow[ll, "\overline{f}"'] \\ & X^2 \arrow[ul, "f"] \arrow[ur] & \end{tikzcd} \tag{12}$$

This not yet explicit (12) will be called the $S_b W^3$-structure (attached to the $S_u W^3$-structure $X^2 \xrightarrow{f} W^3$.) But instead of attempting to embark into a formal definition, I will give, right away, the simplest, almost trivial, example which, I hope, will be quite illuminating for what we are trying to achieve. Let U^3 be some coordinate neighbourhood in R^3. Fig.1 presents an $X^2 \xrightarrow{f} U$ which is an $S_u U^3$-structure, without any infinity at all. The X^2 is the union of two sheets P_1, P_2 along the fat arcs $[a_1, s_1]$, $[a_2, s_2]$ and the fat dotted line is $fM_2(f)$. (Some readers might have recognized, by now, the local model of an O(3)-zipping move [**Ga**], [**Po6**], [**Po7**]. They should also be warned not to worry about COHERENCE, that issue never occurs in this paper.) Of course this X^2 is not simply-connected, but then no global conditions are imposed on X^2 in the framework of the definition (4), (4-1) to (4-4) either. Here is how an $S_b U^3$ will be attached to this particular $S_u U^3$. In figure 3-(A) we have redrawn X^2, source of f with a simple closed curve $\gamma \subset X^2$. Instead of closing in the double points of f (by a quotient projection) we leave them now open, but we add to X^2 a disk $Y^2 = D^2 \hookrightarrow U^3$ along γ.

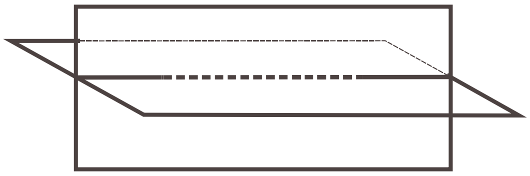

Figure 0 : Which page?.

In a section cutting through the double line, this $\overline{X}^2 = X^2 \cup Y^2$ is suggested in figure 3-(B), where now P_1, P_2 are no longer glued together at the point d. So, the general idea for building up (12) is to add a system Y^2 of gutters along $M_2(f)$, in the way which figure 3-(A), (B) might suggest, so as to imitate the zipping. We also want that $X^2 \Rightarrow \overline{X}^2 = X^2 \cup Y^2$ should consist of Whitehead dilatations and additions of 2-cells. Now, even in the case when X^2 is finite, implementing this program involves some technicalities, too long to be explained here in detail, occuring in the neighbourhood of the forking points of $M_2(f)$ and of the triple points $M_3(f)$.

Actually, exactly for a forking points of $M_2(f)$ the situation of Y^2 is represented in figure 4.1. When $X^2 \xrightarrow{f} W^3$ is the $S_u W^3$-structure from theorem 1, then it so happens that the detailed proof of the theorem in question comes with sufficient additional informations (like a canonical, coherent system of transversal orientations) which makes that the exact location of Y^2 (for which in fig.3 there are four possibilities) is actually ***canonically*** determined. Anyway, when $\Sigma(\infty) = \emptyset$ there is the following easy lemma.

LEMMA 4. *Let $S_u W^3$ be an S_u-structure with $\Sigma(\infty) = \emptyset$. Then the following things happen.*

1) $\Theta^{n+3}(X^2 \cup Y^2)$ is well-defined and geometrically simply-connected. (Apart from this last fact, our lemma applies to a general W^3).

2) We have a diffeomorphism

$$|S_u W^3| = \Theta^{n+3}(X^2 \cup Y^2). \tag{13}$$

This again implies lemma 2, of course. But the trouble is that, when $\Sigma(\infty) \neq \emptyset$ our $\overline{X}^2 = X^2 \cup Y^2$ is ***no*** longer ***locally-finite*** and $\Theta^{n+3}(X^2 \cup Y^2)$, even in an open version, is no longer well-defined. The following (admitedly not completely precise) statement is one of the important ingredients in our approach.

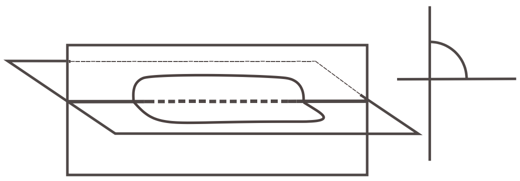

Figure 0 : which page.

LEMMA 5. *There is a **canonical** procedure for rendering (12) locally finite*

$$X^2 \cup Y^2 \Longrightarrow \Theta^3(X^2 \cup Y^2)_{\text{locally finite}} \tag{14}$$

*and, for the R.H.S. of (14), which is now a **singular 3-dimensional manifold**, with 3-dimensional undrawable singularities (see [**Ga**], [**Po7**] for more details concerning such things), the $\Theta^{n+3}(\Theta^3(X^2 \cup Y^2)_{\text{locally finite}})$ is well-defined.*

Moreover, with $X^2 \xrightarrow{f} W^3$ like in theorem 1 the smooth non-compact manifold with very large boundary (and which, by definition, is the ***underlying space*** *of our $S_b W^3$-structure)*

$$|S_b W^3| \overset{\text{def}}{=} \Theta^{n+3}\big(\Theta^3(X^2 \cup Y^2)_{\text{locally finite}}\big) \tag{15}$$

is geometrically simply-connected.

We can only give some hints about what is involved here (and this will be certainly necessary for understanding the rest of the paper.) But let me say, right away, that the word "canonical" used in the statement (and also before, in this section) is not just cosmetics. We will be, eventually, interested in $(X^2 \xrightarrow{f} W^3)$'s with specific equivariance property (with respect to some group-action $G \times W^3 \longrightarrow W^3$). We will then want our S_b-structures to be such that these equivariance properties are also verified for $\overline{X}^2 \xrightarrow{\overline{f}} W^3$ and then for $\Theta^3(X^2 \cup Y^2)_{\text{locally finite}}$ (which will be required to carry the group-action too.) The "canonicity" of the various constructions is the necessary ingredient to have the equivariance properties be transferable from one object to the next one.

I will give an idea about how the lack of local finiteness for $\overline{X}^2 = X^2 \cup Y^2 \xrightarrow{\overline{f}} W^3$ might come about, in the simplest possible set-up where this happens. We start with an $X^2 \xrightarrow{f} W^3$ which is almost like the one in figure 1 except that now $L_0 = (y = 0) \cup \big((z = 0) \cap (y \geq 0)\big)$(i.e. $M_2(f)$ has now a branching point, from which our troubles will stem). The fX^2 (thickened) is presented in fig. 4 which is a section through $z = 0$. This is to be compared with fig.1, with the following differences.

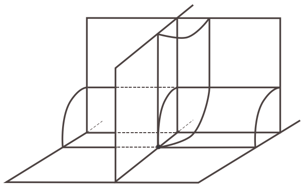

Figure 0 : which page?.

a) In fig.1 the double points had been zipped while here there are not; along the hatched zones, the contributions $(x = x_n)$ and $(y = 0)$ are not glued together. Actually these double points continue along $(x = x_n, y \geq 0)$, and so does the bad locus $\Sigma(\infty)$.

b) We add now the Y^2 (see (12)) contributions, which create an infinity of undrawable singularities $s_1, s_2, \dots$. From the viewpoint of $\big((z = 0) \cap (y \geq 0)\big) \times [-\varepsilon_0, \varepsilon_0]$, which is part of the (thickened) X^2, these ***accumulate at finite distance*** and this is something which we cannot cope with, at all.

c) In fig. 4, I have only suggest the system of gutters Y^2 which follow the double lines in the plane $y = 0$. But the system continues to follow the double lines inside $(z=0)\cap(y\geq 0)$ too, in a canonical fashion, of course; see figure 4.1, which might help undrstand figure 4. The gutters from figure 4 should be compared to Y_2 in figure 4.1 and the analogue of gutters Y_1 are not respresented in figure 4 at all. In order to keep this ouline at a reasonable length we will forget altogether about them in the present very sketchy and incomplete acount.

I'll give a very heuristic first idea of how to cope with the bad acumulation of the s_n's. Staying 2-dimensional, just very provisionally, thinck now of adding to $L_0=(y=0)\cup\big((z=0)\cap(y\geq 0)\big)$ a ***fin*** $(x=0)\cap(y^2+z^2\leq 1)\cap(y\geq 0)$ along the arc $(x=y=0,-1\leq z\leq 1)$, without glueing, otherwise, this fin to $(z=0,y\geq 0)$, and then slide the infinity of gutters on the surface of the fin. All this would destroy the s_n's and create a unique singularity of the fin. (Of course, there is now infinite accumulation inside the fin, towards the outher rim, let's say; also, do not worry about mapping things into W^3, any longer, at this point.)

To make this very rough idea really work, with local finiteness as the end-product, one has to go 3-dimensional, and work with a non-compact fin (so that the gutters which have produced the s_n's accumulate at the infinity of the fin (one sends the original outer rim to infinity, for instance.) This addition of the 3-dimensional fin is still a PROPER Whitehead dilatation, suggested by fig. 5. This figure develops a detail of fig.4 and it also suggests a (non-compact) singularity of the "undrawable" type, which gets created by the fin.

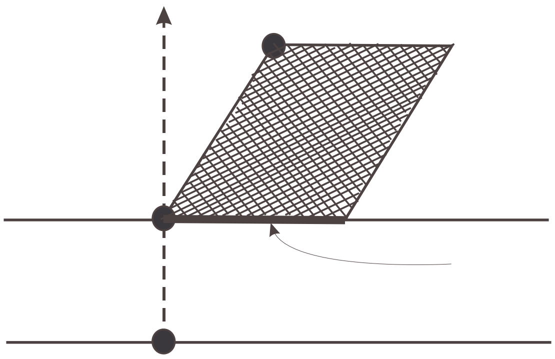

Figure 0 : which page.

We do not bother to consider maps $\Theta^3(X^2\cup Y^2)_{\text{locally finite}}\xrightarrow{f} W^3$, but under appropriate condition, group actions $G\times W^3\longrightarrow W^3$ will be transferable to actions $G\times\Theta^3(X^2\cup Y^2)_{\text{locally finite}}\longrightarrow\Theta^3(X^2\cup Y^2)_{\text{locally finite}}$, I will refrain from unfolding the whole complete story of the fins here, it is too long and technical.

Important Remark. The kind of fin which we have just discussed is clearly isolated, but this comes from the fact that $x_1,x_2,\ldots$ had only x_∞ as a limit point and this, eventually, can be traced to (∗-1) from (4.2), i.e. to our basic finiteness condition. If $\lim(\Lambda \cap M_2(f))$ could become a Cantor set, we would have to face accumulation of fins at finite distance, which would be exactly as bad as the accumulation of s_n's at finite distance, the disease which our fin was supposed to cure, to begin with.

Now there is another, much more complicated source of non local finiteness for (12) and that stems from the points $p_{\infty\infty} \in f(\text{LIM} M_2(f))$ which were introduced in the previous section. Infinitely many fins will be necessary for a given $p_{\infty\infty}$ but one can proceed in such a way that they do not accumulate at finite distance (but at the "local infinity" instead).

This is as much as I plan to say here concerning lemma 5 and the obvious idea now is to try to relate the manifolds $|S_u W^3|$ (which has the feature (11), like in theorem **C**) and $|S_b W^3|$ (which is geometrically simply connected). Actually they turn out to look quite different, their infinities look already hopelessly different. In order to take care of this last problem at least, one replaces, to begin with, the smooth $[W_h^3]$ by

(16)

$$[W_h^3(\text{singular})] \stackrel{\text{def}}{=} \{[W_h^3] \text{ with the (non-compact) fins of (14) already added}\},$$

and, after this, we introduce the real, fancifyed, underlying space for $S_u W^3$

$$\|S_u W^3\| \stackrel{\text{def}}{=} \Theta^{n+3}\big([W_h^3(\text{singular})]\big) \tag{17}$$

which has now the same infinity (see the HEURISTIC REMARK in the previuos section) as $|S_b W^3|$ (This is not an automatic fact, at all, since the zipping touches the bad locus which moreover, has its own internal double points; see the $\lim_{n=\infty} p_{\infty n} = p_{\infty\infty}$ from section 2.) Technically speacking, one gets $\|S_u W^3\|$ from $|S_u W^3|$ by a PROPER system of Whitehead dilatations, which send more boundary points to infinity (i.e. we create "punctures"). After a certain amount of such punctures, the infinities of $\|S_u W^3\|$ and $|S_b W^3|$ look alike. Our next statement is now the following

LEMMA 6. *Let W^3 be such that*

$$\|S_u W^3\| \xlongequal{\text{DIFF}} |S_b W^3|. \tag{18}$$

Then $\pi_1^\infty W^3 = 0$.

The proof will be sketched in the next section. But notice the following fact already.

α) One connot hope for (18) to be true in general, since $|S_b W^3|$ is already geometrically simply-connected and W^3 might well be the Whitehead manifold.

β) In order to prove lemma 6, the only thing one still needs is the following implication

$$\{\|S_u W^3\| \text{ geometrically simply-connected}\} \Longrightarrow \pi_1^\infty W^3 = 0\}. \tag{19}$$

4 Dehn-exhaustibility. Let X be a locally finite simplicial complex. We will say that X is ***Dehn-exhaustible*** if for every compact $K \subset X$ there exists a commutative diagram

$$
\begin{tikzcd}
K \arrow[rr, "j"] \arrow[dr] & & Y \arrow[dl, "g"] \\
& X &
\end{tikzcd} \tag{20}
$$

such that: a) Y is a finite simplicial complex with $\pi_0 Y = \pi_1 Y = 0$ and j is an injection.

b) The map g is a simplicial ***immersion*** such that

$$
jK \cap M_2(g) = \emptyset. \tag{21}
$$

The next lemma can be found, between the lines, in papers like [**Po2**], [**Po3**]; it is, essentially, another way to state Dehn's Lemma [**Po2**] and it is also about the only piece of real 3-dimensional topology to be found in the present paper.

LEMMA 7. *Let V^3 be an open simply-connected **3-manifold**. If V^3 is Dehn-exhaustible, then $\pi_1^\infty V^3 = 0$.*

We go back now to (11) and, with a similar change of viewpoint as for (11) one has the similar PROPER inclusion maps

$$
\begin{aligned}
\operatorname{int}[W_h^3(\text{singular})] \times (*) \subset \|S_u W^3\| \cup \big(\operatorname{int}[W_h^3(\text{singular})]) \times B^n(\varepsilon)\big) \subset \\
\subset \Theta^{n+3} \operatorname{int}\big([W_h^3(\text{singular})]\big).
\end{aligned} \tag{22}
$$

With all this, (19) will have been proved, once we have the following

LEMMA 8. *1) The middle term in (22) can be gotten from $\|S_u W^3\|$ by a* PROPER *system of Whitehead dilatations.*

*2) **If** the middle term in (22) is **geometrically simply-connected**, then* $\operatorname{int}[W_h^3(\text{singular})]$ *is Dehn exhaustible.*

3) If $\operatorname{int}[W_h^3(\text{singular})]$ *is Dehn-exhaustible, then so is* $\operatorname{int}[W_h^3] = W_h^3$. (And to W_h^3 one can apply now lemma 7).

Here 1) is elementary. Point 2) is a fancified version of theorem **C** from the introduction (theorem **C**, as such, is not part of the logic our proof, but it has been very instrumental in suggesting it.) The techniques from [**Po2**] (which are also the ones for the proof of theorem **C** can be extended so as to give what we want here. Also, 2) above is contained, as a special case, in the work of Funar and Funar-Thickstun, already quoted earlier. One should also stress the fact that the proof of 2) in our lemma (just like the proof of theorem **C** in the introduction) requires the DIFF category. The proof in question uses. among other things J.H.C. Whitehead's SMOOTH HAUPTVERMUTUNG THEOREM [**Wh1**] (which is true without dimensional or homotopical restrictions.) Applying the TOP Hauptvermutung theorems of D. Sullivan [**Su**] or R. Kirby and L. Siebenmann [**K-Si**] in our non-compact context with non-empty boundary, might be a very cumbersome if not straightforwardly haphazardous affair. There is, finally, a not too hard, direct argument for 3).

What we want now is to establish (18), when $W^3 = \widetilde{M}^3$.

5. Structures with Holes. Typically $fX^2 \subset W^3$ is a (non locally finite) stratified set, all the 2-dimensional strata of which are open 2-cells. When, in the proof of theorem 1 our $S_u W^3$-structure is constructed, one also endows W^3 with an ***ideal handlebody decomposition***, the 2-skeleton (= the union of the lateral surfaces of the various handles) of which is denoted by $\overline{\Sigma}(\infty)$ and, we have

$$f\,\mathrm{LIM}M_2(f) = fX^2 \cap \overline{\Sigma}(\infty), \Sigma(\infty) = W_h^3 \cap \overline{\Sigma}(\infty) = [W_h^3] \cap \overline{\Sigma}(\infty); \tag{23}$$

here $W_h^3 = \mathrm{Nbd}(fX^2)$, and $\overline{\Sigma}(\infty)$ should not be mixed up with the "closure of $\Sigma(\infty)$" (which is actually contained in $\overline{\Sigma}(\infty)$, but much smaller.) Actually, $\overline{\Sigma}(\infty) - \{$the closure of $\Sigma(\infty)\}$ is an infinite union of open 2-cells D^2_∞ (which have no common points with $fX^2, \mathrm{Nbd}(fX^2) = W_h^3$ or $[W_h^3]$) and for each of them there is an infinite collection of parallel 2-dimensional strata of fX^2 such that, inside W^3, we have $\lim_{n=\infty} D_n^2 = D^2_\infty$. (By now the word "ideal" used above, should be abundantly clear).

Outside everything interesting met so far (like $fM_2(f), Y^2$, a.s.o.) we will delete out of each D_n^2 an open 2-cell which I call a Hole (not to be mixed up with the holes apearing in (2)). So we go from fX^2 to $fX^2 - \mathrm{H}$ and, similarly, from $\overline{X}^2$ to $\overline{X}^2 - \mathrm{H}$ or from $\Theta^3(X^2 \cup Y^2)_{\mathrm{l.f.}}$ to $\Theta^3(X^2 \cup Y^2)_{\mathrm{l.f.}} - \mathrm{H}$.

The Holes make sense at the level of $\|S_u W^3\|, |S_b W^3|$ where they correspond to surgeryes: deleting the open tubular neighbourhood $\sum_{\mathrm{H}} (D^2 \times \mathrm{int} B^{n+1})(\mathrm{H})$ of an infinite PROPER and properly embedded system of disks. (Reminder: like usual, by "PROPER" I mean "inverse image of compact is compact" and by "proper" I mean "boundary goes to boundary and interior goes to interior"; so something can be PROPER but not proper or proper but not PROPER or both, or none, of course.) On the surged manifolds, this leaves as a trace, an infinite ***framed*** link which is PROPER

$$\begin{tikzcd}[column sep=large] & \partial(\|S_u W^3\| - \mathrm{H}) \\ \sum_{\mathrm{H}} S^1(\mathrm{H}) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & \partial(|S_b W^3| - \mathrm{H}). \end{tikzcd} \tag{24}$$

Whatever was said, so far, leaves a certain flexibility concerning both the size of the family $\mathcal{F} = \{\mathrm{H}\}$ of the Holes i.e. of how many strata carry Holes (of which there will never be more than one per stratum and which will never accumulate at finite distance inside fX^2 either) and also the metric size of the individual Holes H.

THEOREM 9.*1) For **any** W^3 and its corresponding S_uW^3, as soon as $\mathcal{F} = \{H\}$ is sufficiently large (in the sense that it contains, at last, a certain precise infinite, minimal family of Holes $\mathcal{F}_0$) there is a diffeomorphism*

$$|S_bW^3| - H \xrightarrow[\eta]{\qquad\qquad} \|S_uW^3\| - H. \tag{25}$$

*2) The following diagram commutes **up to homotopy***

$$\begin{tikzcd} \partial(|S_bW^3| - \mathrm{H}) \arrow[rr, "\eta", "\approx"'] & & \partial(\|S_uW^3\| - \mathrm{H}) \\ & \sum_H S^1(\mathrm{H}) \arrow[ul, "\beta"] \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{26}$$

*But it is **not** claimed that this homotopy is* PROPER.

3) On the other hand, for each $S^1(\mathrm{H})$ there is a particular point $p_{\mathrm{H}} \in S^1(\mathrm{H})$ such that $\eta\beta(p_{\mathrm{H}}) = \alpha(p_{\mathrm{H}})$, and the homotopy above respects this condition. Also the framings of the individual $S^1(\mathrm{H})$'s are respected in the context of (26).

The proof is an argument which has both global and local features (in particular, the globality of $M_2(f)$ has to be considered) and where one makes an explicit use of the fact that the infinities involved in both terms of (25) are the same; it will not be outlined here. But notice now that ***if*** the homotopy above was PROPER then, because of the high-dimensionality which is involved, (26) would commute up to ambient ***isotopy*** and, since the framings are correct too, we could then deduce that $\|S_uW^3\| \overset{\text{DIFF}}{=\!=} |S_bW^3|$ (see (18)) which, of course cannot be true, in general, since W^3 might be the Whitehead manifold as well.

On the other hand in $\partial(\|S_uW^3\| - \mathrm{H})$ we have the closed loops

$$\alpha(S^1(\mathrm{H})) \cdot \underbrace{(\eta\beta(S^1(\mathrm{H}))^{-1}}_{\alpha(p_H)},$$

call them C_{H}, which are such that $\lim_{\mathrm{H}\to\infty} C_{\mathrm{H}} = \infty$ (i.e only finitely many touch a given compact set) and which are all null-homototic in $\partial(\|S_u W^3\| - \mathrm{H})$. If the singular disks D^2_{H} which they bound would also be such that $\lim_{\mathrm{H}\to\infty} D^2_{\mathrm{H}} = \infty$, then the homotopy in 2) would be PROPER, and everything would be O.K.

This boils down to saying that, for $W^3 = \widetilde{M}^3$, if we could prove that $\pi_1^\infty(\|S_u W^3\| - \mathrm{H}) = 0$, then we could also show that $\|S_u W^3\| \overset{\mathrm{DIFF}}{=\!=} |S_b W^3|$, i.e. that $\|S_u W^3\|$ is geometrically simply-connected, which as we well know implies that $\pi_1^\infty W^3 = 0$. So, $\pi_1^\infty = 0$ requires geometric simple connectivity, which itself requires $\pi_1^\infty = 0$. Here our approach bites its own tail, and this really looks like a bad vicious circle. Breaking it will be the object of the next section.

6. The core of the proof. There is actually a reason why the S_u-structures or S_b-structures (where, by the way, "u" stands for "usual" and "b" for "bizarre") are called "structures", to begin with. The archetype to have in mind here is a foliation or a lamination, on a manifold N^3. This is to be thought of as a certain amount of 2-dimensional material piled up inside N^3, subject to certain mild (but useful) functorial properties, like push-forward or pull-back (under apropriate conditions, of course.) They also have an underlying space which is, trivially, N^3 itself. With this in mind let us consider a $S_u W^3$-structure $X^2 \xrightarrow{f} W^3$ (with arbitrary W^3.) If $W^3 \xrightarrow{g} W^3$ is a diffeomorphism, we define the push-forward structure

$$g_* S_u W^3 \overset{\mathrm{def}}{=} \{ \text{ the } S_u\text{-structure } X^2 \xrightarrow{g\circ f} W^3 \}. \tag{27}$$

If we have equality of subsets $gfX^2 = fX^2 \subset W^3$ then, one can also prove that g lifts uniquely to X^2 and $g_* S_u W^3 = S_u W^3$, which makes the whole $S_u W^3$ g-invariant.

If $G \times W^3 \longrightarrow W^3$ is a free action of the discrete group G, with

$$W^3 \xrightarrow{\pi} W^3/G \tag{28}$$

and if an $S_u W^3$-structure is G-invariant (i.e. if $g_* S_u W^3 = S_u W^3$ for all $g \in G$) then one can quotient $S_u W^3$ by G and define $\pi_* S_u W^3$ which is a $S_u(W^3/G)$-structure. If $Z^3 \xrightarrow{\pi} W^3$ is a covering space which, for the sake of simplicity I will assume to be Galois with automorphism group G, then any $S_u W^3$ lifts to a pull-back structure $\pi^* S_u W^3 = S_u Z^3$ which is G-invariant and $\pi_* \pi^* S_u W^3 = S_u W^3, \pi^* \pi_* S_u Z^3 = S_u Z^3$, a.s.o. All these things pass, automatically to the underlying spaces too. For instance, if the $Z^3 \xrightarrow{\pi} W^3$ is the universal covering space ($Z^3 = \widetilde{W}^3$) then

$$\|S_u \widetilde{W}^3\| \overset{\mathrm{def}}{=} \|\pi^* S_u W^3\| = \|S_u W^3\|\widetilde{\ }. \tag{29}$$

So, operations are defined at the level of the 3-manifolds and then transferred to the underlying spaces. Similar things hold for S_b. Moreover the two things hang together nicely, in the sense that the passage $S_u \Rightarrow S_b$ is canonical and, hence, we have something like

$$S_b(gS_uW^3) = gS_b(S_uW^3), \tag{30}$$

with all the formal consequences which this implies.

This is why the word "structure" was used and after this piece of abstract nonsense here is

THEOREM 10. *Let $\widetilde{M}^3 \xrightarrow{\pi} M^3$ be the universal covering space of a closed 3-manifold M^3, endowed with the free action $\pi_1 M^3 \times \widetilde{M}^3 \longrightarrow \widetilde{M}^3$. There is then a $S_u\widetilde{M}^3$-structure which is $\pi_1 M^3$-**invariant**.*

Of course, just like in theorem 1, with which theorem 10 should be compared, the $X^2 \xrightarrow{f} \widetilde{M}^3$ is here almost arborescent. In fact, both theorems 1 and 10 are proved in [**PoTa3**]. There is no room even for sketching the proof of this last, crucial result here, I will only make a few comments.

But let me notice first that, once we have a $\pi_1 M^3$-invariant $S_u\widetilde{M}^3$ we can project both it and also its associated $\pi_1 M^3$-invariant $S_b\widetilde{M}^3$ down to the compact M^3 and get

$$S_u M^3 \overset{\text{def}}{=} \pi_* S_u \widetilde{M}^3, \quad S_b M^3 \overset{\text{def}}{=} \pi_* S_b \widetilde{M}^3 \tag{31}$$

which will be our objects of investigation in the present section. Of course what we have gained in this process $\widetilde{M}^3 \Rightarrow M^3$ is that any INFINITY has vanishes and only the (locally defined) infinity of, let's say $\partial(\|S_uM^3\| - \mathrm{H})$ or $\|S_uM^3\| - \mathrm{H}$ (they are the same) is left. Notice, also, that, trivially, we have $\pi_1^\infty M_h^\infty \;=\; \pi_1^\infty M^3 \;=\; 0$. It is not hard to show that

$$\pi_1^\infty[M_h^3] = \pi_1^\infty \|S_uM^3\| = \pi_1^\infty \partial\|S_uM^3\| = 0 \tag{31-1}$$

too and our project now is to try to extend (31-1) to the fancier formula

$$\pi_1^\infty(\|S_uM^3\| - \mathrm{H}) = \pi_1^\infty(\partial(\|S_uM^3\| - \mathrm{H})) = 0,$$

which is the kind of ingredient which we were still missing, after we got theorem 9. This should not sound too unreasonable.

Here are now my comments concerning theorem 10.

I) The proof of theorem 10 is ***not*** the special case of the proof of theorem 1 when $W^3 = \widetilde{M}^3$. What the argument which proves the theorem 1 (for general open simply-connected W^3) can give us, is a $\pi_1 M^3$-invariant ideal handlebody decomposition (see the $\overline{\Sigma}(\infty)$ (23)), but not more than that. That proof (of theorem 1 alone) is still too close to the original collapsible pseudo-spine representation theorem for Σ^3. The way that particular proof proceeds, is to start with a representation $M^3 = \{$a 3-cell Δ with triangulated boundary and with the corresponding triagles identified in pairs$\}$. That produces both a surjection

$$F(= \text{free group}) \to \pi_1 M^3 \tag{32}$$

and a large tree-like object on which F acts freely. Specializing the proof of theorem 1 to the special case $W^3 = \widetilde{M}^3$ would then yield an X^2 (call it now X_1^2)

which is quite close to being the tree-like object in question and, in particular, which also carries a free action $F \times X_1^2 \longrightarrow X_1^2$.

There is no chance of this X_1^2 to be $X_{10}^2 = \{\text{the } X^2 \text{ from theorem } 10\}$, since $\pi_1 M^3$ does not act freely on trees (generally speaking), see [**Se**].

II) So, the very general idea is to go from X_1^2 to $X^2 = X_{10}^2$ by ***adding*** a lot of material without loosing any desirable property, but breaking the F-symmetry and forcing a $\pi_1 M^3$-symmetry for X_{10}^2, instead.

We move now back to (31) and we ***compactify*** $\partial(\|S_u M^3\| - \mathrm{H})$ by considering, for some high N the embedding

$$\partial(\|S_u M^3\| - \mathrm{H}) \subset \|S_u M^3\| - \mathrm{H} \subset \underbrace{M^3 \times B^N}_{\text{this is now compact}} , \tag{33}$$

and taking the closure

$$\begin{aligned} \overline{\partial(\|S_u M^3\| - \mathrm{H})} &= \partial(\|S_u M^3\| - \mathrm{H}) \cup \partial^\wedge \Sigma(\infty) \subset \\ &\subset \overline{\|S_u M^3\| - \mathrm{H}} = (\|S_u M^3\| - \mathrm{H}) \cup \partial^\wedge \Sigma(\infty) \end{aligned} \tag{34}$$

where $\partial^\wedge \Sigma(\infty)$ is, for the time being, just a notation: it is the infinity which one has to add so as to get the compact closure. It will certainly contain the bona fide boundary $\partial \Sigma(\infty)$ of the bad locus (hence the notation); it will also contain the union of the infinities of the non-compact fins, plus more stuff. It will be, of course, compact too. Now for the lemma which comes next to hold, some fine tuning is necessary: the $\mathcal{F} = \{\mathrm{H}\}$, i.e. the family Holes has to contain at least a second family $\mathcal{F}_1 \supset \mathcal{F}_0$, beyond $\mathcal{F}_0$ (see theorem 9), the sizes of the individual H's have to be chosen sufficiently large and some precautions have to be taken concerning the precise location of the fins. With this, we get the

LEMMA 11.*1) The one-dimensional compact set $\partial^\wedge \Sigma(\infty)$ is connected but not arcwise connected. It has finitely many "small" compact arc-wise connected components and one unique main large $\partial_0^\wedge \Sigma(\infty)$, non-compact arcwise connected component which contains $\partial \Sigma(\infty) \cup \{$the contribution of the fins$\}$. [Think here of the closure of a finite truncation of the graph of $y = sin \frac{1}{x}$, as being a toy-model for our $\partial_0^\wedge \Sigma(\infty)$.]*

2) (The crucial step.) The map

$$\pi_1(\partial^\wedge \Sigma(\infty)) \longrightarrow \pi_1(\overline{\partial(\|S_u M^3\| - H})) \tag{35}$$

injects. [As intermediary steps in the proof of the injectivity of (35), one has to show that

$$\pi_1(\Sigma(\infty)) \longrightarrow \pi_1([M_h^3] - H)$$

and

$$\pi_1(\partial_0^\wedge \Sigma(\infty)) \longrightarrow \pi_1(\Sigma(\infty) \cup \partial_0^\wedge \Sigma(\infty))$$

also inject.]

3) Moreover, the arguments which prove the various injectivities in 2) above also show that for any continuous closed curve mapped inside $\partial^\wedge\Sigma(\infty))$, call it $C_\infty \subset \partial^\wedge\Sigma(\infty))$, there is a neighbourhood, arbitrarily close to C_∞,

$$C_\infty \subset U \subset \overline{\partial(\|S_u M^3\| - H))}$$

such that

$$\pi_1(U - \partial^\wedge\Sigma(\infty))) \longrightarrow \pi_1(\partial(\|S_u M^3\| - H)) \tag{36}$$

also injects.

4) As a consequence of the various compactness properties which are involved (in particular of the fact that $\partial(\|S_u M^3\| - H) \cup \partial^\wedge\Sigma(\infty)$ is compact and $\partial^\wedge\Sigma(\infty)$ too), for any sequence of closed loops $C_1, C_2, \ldots$ in $\partial(\|S_u M^3\| - H)$, only finitely many of which meet any given compact subset of $\partial(\|S_u M^3\| - H)$, one can extract a subsequence which, inside $\partial(\|S_u M^3\| - H) \cup \partial^\wedge\Sigma(\infty) = \overline{\partial(\|S_u M^3\| - H)}$ converges uniformly to some closed curve $C_\infty \subset \partial^\wedge\Sigma(\infty)$.

Let us say that once the INFINITY is out of the picture (M^3 is compact), there is nothing else our sequence of curves can do.

[Notational remarks. This is to avoid confusions, if the readers looks at [**Po14**]. What we call here $\partial^\wedge\Sigma(\infty)$ is constructed in stages, and in [**Po14**] it is denoted by

$$\overset{\wedge\wedge\wedge}{\sigma_1(\infty)} \supset \overset{\wedge\wedge}{\sigma_1(\infty)} \supset \overset{\wedge}{\sigma_1(\infty)} \supset \sigma_1(\infty)$$

so as to keep track of the various stages in the construction. I thought it was too cumbersome to drag the three-hatted notation here.]

Finally, modulo everything already said, here is THE PROOF OF THEOREM **A**). We have the following steps.

A) As an easy consequence of lemma 11 one has now

$$\pi_1^\infty(\partial(\|S_u M^3\| - \mathrm{H})) = 0. \tag{37}$$

B) If one cobines (37) with theorem 9 and with the comments which follow it, one gets the diffeomorphism

$$\|S_u M^3\| \overset{\text{DIFF}}{=\!=} \|S_b M^3|, \tag{38}$$

downstairs. One should stress that the present $|S_b M^3|$ is neither simply-connected nor,of course, geometrically simply connected. The $\|S_u M^3\|$, on the other hand, continues to have property (22), with $[W_h^3(\text{singular})] \Rightarrow [M_h^3(\text{singular})]$.

The system of holes h appearing in M_h^3 or $[M_h^3]$ is the $\pi_1 M^3$-quotient of the $\pi_1 M^3$-invariant system of holes h appearing in $\widetilde{M}_h^3$ or $[\widetilde{M}_h^3]$, when the $\pi_1 M^3$-invariant $S_u \widetilde{M}^3$ is constructed.

C) If one takes the universal covering spaces of (38) and if one also applies the functoriality properties of our structures, one finds that

$$\|S_u \widetilde{M}^3\| \overset{\text{DIFF}}{=\!=} \|S_u M^3\|^{\sim} \overset{\text{DIFF}}{=\!=} |S_b M^3|^{\sim} \overset{\text{DIFF}}{=\!=} |S_b \widetilde{M}^3|. \tag{39}$$

D) The M^3 and the object in (38) were highly non simply connected,but $\widetilde{M}^3$ is simply-connected, with a geometrically simply-connected $|S_b \widetilde{M}^3|$, i.e. we can invoke now lemma 6 and get what we want.

7. Some final comments. Philosophically speaking the proof of $\pi_1^\infty \widetilde{M}^3 = 0$ had to make use of 3-dimensionality certainly, but then also of discrete symmetry with compact fundamental domains. Those particular ingredients only appeared in the last section, as far as this paper is concerned. As a more personal comment, the main stumbling block in trying to get theorem **A** has always been, for me, how to make use of the symmetry with compact fundamental domain of $\widetilde{M}^3$. Finally the solution was to project everything down, via equivariance in a compact surrounding where automatically $\pi_1^\infty = 0$, use this to solve the corresponding problem there, and then pull it back as the real solution to the real problem, in the real world.

At least at a superficial level, the present paper looks quite closely related to my program for the Poincaré Conjecture: Geometric simple connectivity, zipping, thickening, a.s.o. certainly play a main role in both cases and, during the years, the two projects have fed each other, interracting with feed-back. But looking from a closer range, only the most elementary parts of the Poincaré Program are eventually used here, and conversely: anyway, the Poincaré Program involves, almost exclusively, smooth dimension four, with its own very specific problem, in particular with a DIFF versus TOP distinction and a correponding failure of the Hauptvermutung, which are really very serious and the main problem which one has to face there is COHERENCE (or lack of it). By contrast, for the π_1^∞ story, we can afford to work in as high dimensions as we wish to, the difficulties coming now from the ***bad locus*** $\Sigma(\infty)$. So here is a, quite philosophical digression concerning, let's say, its ontological status: is the bad locus really there?

To begin with, we will start with a highly conjectural, but still mathematical thought: something $\Sigma(\infty)$-like should be an invariant which any group G carries and (conjecturally) the present work could be rephrased as as proving that if $G = \pi_1 M^3$, then this invariant vanishes (in the sense that some S_u without $\Sigma(\infty)$ can be found). But, possibly, for the Gromov hyperbolic Coxeter-type G's appearing in the work of Mike Davis [**Da**], [**DaS**], this invariant can be detected and proven to be non-vanishing.

So, this purely hypothetical picture would say that for any individual $\widetilde{M}^3$, the $\Sigma(\infty)$ is actually not there. How this is supposed to be reconciled with the idea that, nervertheless, $\Sigma(\infty)$ is unavoidable (see, for instance, the section on the Whitehead Nightmare in [**Po5**]), is the object of the next, indeed much more philosophical-speculative thoughts. There is the concept of $\widetilde{M}^3$ (comprehension) and there is also the totality of $\widetilde{M}^3$'s for all the individual M^3's (extension). I might also add, in the same breath, that there is the concept of homotopy

3-sphere Σ^3 and then there is also the totaly of individual Heegard splittings which define simply-connected 3-manifolds or, for that matter, any other kind of concrete terrestrial embodiments of the notion of Σ^3, but that is another story.

My guess at this point, is that from a realist viewpoint (and I use this word here in its thousand years old medieval scholastic meaning of John Scot Erigena), something like $\Sigma(\infty)$ is somewhere hidden deep inside the $\widetilde{M}^3$-concept although it might well be absent at the nominalist [1] level of the extensionality of $\widetilde{M}^3$ [2].

[1] What I am hinting at here is the old debate, nominalism versus realism, triggered by the impact on the Schoolmen of a few pages from a (bad) translation of a comment of a translation from Aristoteles, a fascinating story indeed, but this is not the proper place to retell it.

[2] These thoughts might seem too lofty but they have to be taken with a grain of salt and, even at best, I do not claim much universality, as far as they are concerned. But there is also a completely pratical issue involved here. When working on a problem, like let us say $\pi_1 \widetilde{M}^3$, does one keep in one's drawer a little zoo of $\widetilde{M}^3$'s, with their beautifully balaced symmetries, but from which no pathology, and hence no Wisdon, can be expected, or a gallery of monsters, which no decent zoo might want to house, but which can serve for building and testing your theory?

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