Anisotropy and dispersion in rotating fluids

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Abstract

We study dispersion effects in the rotating fluid system in the whole space. We prove global strong convergence results for Leray's solutions, as well as global regularity and uniqueness of solutions as soon as the Rossby number is small enough; the method involves anisotropic Strichartz–type estimates.

1 Introduction

In this paper, our aim is to study dispersion phenomena occuring in singular perturbation models of fluid dynamics in $\mathbf{R}^3$. We focus on the small Rossby number limit of solutions to the incompressible Navier–Stokes equations, and are going to study dispersion phenomena in the following system:

$$(RF^\varepsilon)\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu\Delta u^\varepsilon + \dfrac{u^\varepsilon \times e_3}{\varepsilon} = -\nabla p^\varepsilon \quad \text{in} \quad \mathbf{R}^3 \\ \operatorname{div} u^\varepsilon = 0 \quad \text{in} \quad \mathbf{R}^3 \\ u^\varepsilon_{|t=0} = u_0, \end{cases}$$

where $e_3 = (0,0,1)$, $\varepsilon > 0$ denotes the Rossby number and $\nu > 0$ is the viscosity. We refer to [12] and [24] for a physical justification of the model.

It is well known that dispersive effects are of great importance in the study of non linear partial differential equations. Historically, the use of dispersive effects in partial differential equations appeared in the study of the wave equation with the proof of the Strichartz estimates. The idea is that despite the fact that the wave equation is time reversible, it induces a time–decay in L^p norms, of course for p greater than 2. But those decaying properties also provide smoothing effects; it has been the beginning of a long series of works (see for instance [4], [14], [17]–[22]) where those smoothing effects are used to improve, in non linear wave equations, the classical

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results of wellposedness (by classical results, we mean results proved with energy estimates). The same has been used in the framework of non linear Schrödinger equations.

Let us have a quick look at the way Strichartz estimates are proved in the case of the wave equation. The wave equation can be reduced to the following system:

$$\partial_t u_\pm \pm i|D|u_\pm = 0 \quad \text{in} \quad \mathbf{R}\times\mathbf{R}^d, \quad \text{with} \quad |D|a = \mathcal{F}^{-1}(|\xi|\widehat{a}(\xi)). \tag{1.1}$$

So the solution is of the form

$$u(t) = \mathcal{F}^{-1}\left(e^{it|\xi|}\gamma^+(\xi) + e^{-it|\xi|}\gamma^-(\xi)\right), \tag{1.2}$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbf{R}^d$. Let us suppose that the support of the Fourier transform of the initial data γ^+ and γ^- is included in a fixed ring $\mathcal{C}$ of $\mathbf{R}^d$. The Strichartz estimate is based on the so–called "dispersive estimate"

$$\|u(t)\|_{L^\infty(\mathbf{R}^d)} \leq \frac{C}{t^{\frac{d-1}{2}}}(\|\gamma^+\|_{L^1(\mathbf{R}^d)} + \|\gamma^-\|_{L^1(\mathbf{R}^d)}). \tag{1.3}$$

Then functional analysis arguments (see for instance [11]) imply that

$$\|u\|_{L^p(\mathbf{R},L^q(\mathbf{R}^d))} \leq C(\|\gamma^+\|_{L^2(\mathbf{R}^d)} + \|\gamma^-\|_{L^2(\mathbf{R}^d)})$$

for suitable p and q, which is clearly a decay property. Using scaling arguments and Sobolev embeddings, it is easily proved that, for s strictly greater than $(d+1)/2$,

$$\|\nabla u\|_{L^2([0,T];L^\infty(\mathbf{R}^d))} \leq C_T(\|\gamma^+\|_{H^s(\mathbf{R}^d)} + \|\gamma^-\|_{H^s(\mathbf{R}^d)}).$$

Now let us return to our problem. A number of studies exist concerning systems of the type (RF^ε) in the viscous case as well as in the inviscid case ($\nu = 0$). We refer, among others, to [1]-[3], [7], [10], [13], [15], [25]. Let us introduce some notation: let P be the Leray projector onto divergence free vector fields; the fact that the operator

$$L_{RF}u^\varepsilon \stackrel{\text{def}}{=} P(u^\varepsilon \times e_3)$$

is skew–symmetric implies that if the initial data u_0 is an element of the space $L^2(\mathbf{R}^3)$, then we obtain a sequence u^ε of Leray's weak solutions, uniformly bounded in the space $L^\infty(\mathbf{R}^+; L^2(\mathbf{R}^3)) \cap L^2(\mathbf{R}^+; \dot{H}^1(\mathbf{R}^3))$. We have noted $\dot{H}^s$ the homogeneous Sobolev space of order s, defined as the closure of $\mathcal{D}(\mathbf{R}^3)$ for $\|\cdot\|_{\dot{H}^s}$, where we have defined $\|u\|_{\dot{H}^s} \stackrel{\text{def}}{=} \|\,|\xi|^s\widehat{u}(\xi)\,\|_{L^2(\mathbf{R}^3)}$, and $\widehat{u}$ is the Fourier transform of u.

The aim of this study is to analyze that system in the limit when ε goes to zero. Formally, one expects the solutions to converge to elements of the kernel of the penalization operator. That leads us to the well–known bidimensional Navier–Stokes system:

$$(NS2D)\begin{cases} \partial_t\bar{u} + \bar{u}'\cdot\nabla_h\bar{u} - \nu\Delta_h\bar{u} &= -(\nabla_h p, 0) \quad \text{in} \quad \mathbf{R}^2 \\ \operatorname{div}\bar{u} &= 0 \quad \text{in} \quad \mathbf{R}^2 \\ \bar{u}_{|t=0} &= \bar{u}_0, \end{cases}$$

where $\Delta_h = \partial_1^2 + \partial_2^2$, $\nabla_h = (\partial_1, \partial_2)$, and $u' = (u_1, u_2)$ denotes the first two components of u. In the system above, $\bar{u}_0$ is a bidimensional, divergence free vector field.

Finally, it will be useful in the following to define the solutions of the free linear equations associated with (RF^ε):

$$(FRF^\varepsilon) \begin{cases} \partial_t w^\varepsilon_{_F} - \nu \Delta w^\varepsilon_{_F} + \dfrac{1}{\varepsilon} L_{_{RF}} w^\varepsilon_{_F} &= f \quad \text{in} \quad \mathbf{R}^3 \\ w^\varepsilon_{_{F|t=0}} &= w_0, \end{cases}$$

where w_0 is a divergence free, tridimensional vector field.

In the spirit of the work of S. Ukai (see [26]) and of two of the authors (see [8]), we shall use dispersive effects in the system (FRF_ε) to prove strong convergence results.

Theorem 1 *The family u^ε converges to 0 in $L^2_{loc}(\mathbf{R}^+, L^q(\mathbf{R}^3))$ for any $q \in]2, 6[$.*

Remark. A similar theorem can be stated for the compressible Navier-Stokes equations for isentropic gases in the low Mach number limit (see [8]).

Since the limit of the system (RF^ε) is the bidimensional Navier–Stokes equation $(NS2D)$, it is natural to consider initial data of the type $u_0 = \bar{u}_0 + w_0$ where $\bar{u}_0$ depends only on two variables.

Theorem 2 *Let u_0 be a tridimensional divergence free vector field, such that*

$$u_0 = \bar{u}_0 + w_0, \quad \textit{with} \quad \bar{u}_0 \in \left(L^2(\mathbf{R}^2)\right)^3 \quad \textit{and} \qquad w_0 \in \left(\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)\right)^3,$$

with $\bar{u}_0$ and w_0 divergence free. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, there is a unique global solution u^ε to the system (RF^ε), which satisfies

$$u^\varepsilon - \bar{u} \in C^0_b(\mathbf{R}^+, \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)) \cap L^2(\mathbf{R}^+, \dot{H}^{\frac{3}{2}}(\mathbf{R}^3)),$$

$$u^\varepsilon - \bar{u} - w^\varepsilon_{_F} \to 0 \quad \textit{in} \quad L^\infty(\mathbf{R}^+, \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)) \cap L^2(\mathbf{R}^+, \dot{H}^{\frac{3}{2}}(\mathbf{R}^3)),$$

where $\bar{u}$ and $w^\varepsilon_{_F}$ are the respectively the solutions of $(NS2D)$ and (FRF^ε) associated respectively with $\bar{u}_0$ and w_0.

The statement of Theorem 2 shows clearly that anisotropic phenomena are involved in the study of (RF^ε). The solutions of the linearized equation

$$\partial_t v + L_{_{RF}} v - \nu\varepsilon \Delta v = 0 \quad \text{in} \quad \mathbf{R}^3 \tag{1.4}$$

are given in a similar way to (1.2): we have

$$v(t) = \mathcal{F}^{-1}\left(e^{\pm it\frac{\xi_3}{|\xi|} - \nu t\varepsilon|\xi|^2}\gamma_\pm\right),$$

and it is proved in this paper that operators of the following type, where ξ_3 is seen as a parameter,

$$e^{\pm it\frac{\xi_3}{(\xi_3^2-\Delta_h)^{1/2}}-\nu t\varepsilon(\xi_3^2-\Delta_h)}\gamma_\pm(\cdot,\xi_3)$$

satisfy (anisotropic) Strichartz–type estimates.

The structure of the paper is as follows. In the next section, we prove some basic Strichartz estimates (see (2.6)) for systems of the type (1.4); those estimates are then used to prove Theorem 1. Although those Strichartz estimates are not explicitly anisotropic, their proof requires an anisotropic decomposition of the frequency space, with respect to the vertical frequency variable ξ_3. That is due to the fact that the phase $\dfrac{\xi_3}{|\xi|}$ is almost constant when ξ_3 is almost equal to 0, as well as when $|\xi_3|$ is much larger than $|\xi_h|$, so in particular is almost stationary; so there is no hope of getting a dispersive effect in that case.

In the third section, we prove anisotropic Strichartz–type estimates, which moreover take into account the viscosity effect. The estimates obtained are of the type

$$\|u(t)\|_{L^\infty(\mathbf{R}^2_{x_h},L^2(\mathbf{R}_{x_3}))}\leq\frac{C}{t^{\frac14}}2^{3j}\|u_0\|_{L^1(\mathbf{R}^2_{x_h},L^2(\mathbf{R}_{x_3}))},$$

for any function whose Fourier transform is supported in $2^j\mathcal{C}\cap\Big(\mathbf{R}^2\times 2^k[a,b]\Big)$, with $a<\frac12$ and $b>2$, and $(j,k)\in\mathbf{Z}^2$.

In the fourth section, we prove Theorem 2. The method used is classical in the study of the incompressible Navier–Stokes equations, once the dispersive estimates have been obtained.

2 Convergence of Leray's solutions

The aim of this part is to prove Theorem 1. We first derive Strichartz type estimates for Coriolis force and then apply it to (RF^ε).

2.1 Strichartz type estimates for Coriolis force

In order to prove Theorem 1, we need to derive dispersion estimates for wave equations related to the rotating Navier-Stokes model (RF^ε). The proof follows closely the lines of [11] and [19].

The linearized version of (RF^ε) reads

$$\partial_t v+\frac{1}{\varepsilon}v\times e_3-\nu\Delta v+\nabla p=0,\ \ \mathrm{div}\, v=0,\ \ \text{in}\ \ \mathbf{R}_t\times\mathbf{R}^3_x,\ \ v_{|t=0}=v_0\ \ \text{in}\ \ \mathbf{R}^3_x, \tag{2.1}$$

which yields in Fourier variables $\xi\in\mathbf{R}^3$

$$\partial_t\widehat{v}+\nu|\xi|^2\widehat{v}-\frac{\xi_3\xi\times\widehat{v}}{\varepsilon|\xi|^2}=0\ \text{in}\ \mathbf{R}_t\times\mathbf{R}^3_\xi,\quad\text{and}\quad \widehat{v}_{|t=0}=\widehat{v}_0. \tag{2.2}$$

Hence we are led to studying

$$\mathcal{G}_\nu^\varepsilon(t)\colon\ f \mapsto \int_{\mathbf{R}^3_\xi} \widehat{f}(\xi) e^{\pm it\frac{\xi_3}{|\xi|}-\nu t\varepsilon|\xi|^2+ix\cdot\xi}\,d\xi = \int_{\mathbf{R}^3_\xi\times\mathbf{R}^3_y} f(y)e^{\pm it\frac{\xi_3}{|\xi|}-\nu t\varepsilon|\xi|^2+i(x-y)\cdot\xi}\,d\xi dy, \tag{2.3}$$

first considering the case when $\widehat{f}$ is supported in $\mathcal{C}_{r,R}$ for some $r<R$, defined by

$$\mathcal{C}_{r,R} = \{\xi\in\mathbf{R}^3\ /\ |\xi_3|\geq r \ \text{ and }\ |\xi|\leq R\}. \tag{2.4}$$

Let us introduce

$$K(t,\tau,z) \stackrel{\text{def}}{=} \int_{\mathbf{R}^3_\xi} \psi(\xi)e^{ita(\xi,z)+iz\cdot\xi-\tau|\xi|^2}d\xi, \quad \text{where}\ \ a(\xi,z)\stackrel{\text{def}}{=}\frac{\xi_3}{|\xi|},$$

and ψ is a function of $\mathcal{D}(\mathbf{R}^3\setminus\{0\})$, which of course depends on r and R, which is radial with respect to the horizontal variable $\xi_h=(\xi_1,\xi_2)$ and whose value is 1 near $\mathcal{C}_{r,R}$.

Lemma 1 *For any (r,R) such that $0<r<R$, a constant $C_{r,R}$ exists such that*

$$\|K(t,\tau,.)\|_{L^\infty(\mathbf{R}^3)} \leq C_{r,R}\min\{1,t^{-\frac{1}{2}}\}e^{-r^2\tau}.$$

Proof. The proof follows the lines of a stationary phase method, in a very simple way. First, using the rotation invariance in (ξ_1,ξ_2), we restrict to the case when $z_2=0$. Next, denoting $\alpha(\xi)=-\partial_{\xi_2}a(\xi)=\xi_2\xi_3/|\xi|^3$, we introduce the following differential operator:

$$\mathcal{L}\stackrel{\text{def}}{=}\frac{1}{1+t\alpha^2(\xi)}\left(1+i\alpha(\xi)\partial_{\xi_2}\right)$$

which acts on the ξ_2 variable, and satisfies $\mathcal{L}(e^{ita})=e^{ita}$. Integrating by parts, we obtain

$$K(t,\tau,z)=\int_{\mathbf{R}_{\xi_2}} e^{ita(\xi,z)+iz\cdot\xi}\left({}^t\mathcal{L}(\psi(\xi)e^{-\nu\tau|\xi|^2})\right)\,d\xi_2.$$

Easy computations then yield

$${}^t\mathcal{L}\left(\psi(\xi)e^{-\tau|\xi|^2}\right)=\left(\frac{1}{1+t\alpha^2}-i(\partial_{\xi_2}\alpha)\frac{1-t\alpha^2}{(1+t\alpha^2)^2}\right)\psi(\xi)e^{-\tau|\xi|^2}-\frac{i\alpha}{1+t\alpha^2}\partial_{\xi_2}\left(e^{-\tau|\xi|^2}\psi(\xi)\right), \tag{2.5}$$

so using the fact that ξ is in a fixed ring of $\mathbf{R}^3$, and $\psi\in\mathcal{D}(\mathbf{R}^3)$, we get

$$\|K(t,\tau,\cdot)\|_{L^\infty}\leq C_{r,R}e^{-\tau|\xi|^2}\int_{\mathbf{R}}\frac{d\xi_2}{1+t\xi_2^2},$$

which proves Lemma 1. □

Remark. Notice that the above lemma can be extended to more general functions a. Indeed, in the case when the phase a is non stationary on $\operatorname{supp}\psi\times\{z\,/\,|z|\leq\rho\}$, i.e. $\nabla_\xi a\neq 0$, multiple integrations by parts yield arbitrary polynomial time decay, whereas if ψ is supported in a neighborhood of points $\xi_0(z)$ such that $\nabla a(\xi_0(z),z)=0$ and $D^2_\xi a(\xi_0(z),z)$ has j non–zero eigenvalues, then such a stationary phase argument yields a time decay of order $t^{-j/2}$. For instance, in the case of acoustic waves occuring in the low Mach number limit of compressible flows, we obtain a decay in $t^{-\frac{d-1}{2}}$, which is closely related to Strichartz' estimate for the wave equation (see [11]).

Lemma 1 yields the following corollary in a standard way (see [11]).

Corollary 2 *For any constants r and R, let $\mathcal{C}_{r,R}$ be the domain defined in* (2.4). *Then a constant $C_{r,R}$ exists such that if w_0 and $f(t,\cdot)$ are two vector fields such that*

$$\text{Supp } \widehat{w}_0 \ \cup \ \text{Supp } \widehat{f} \ \subset \ \mathcal{C}_{r,R},$$

and if $w^\varepsilon_{_F}$ is the solution of the linear equation (FRF^ε) with forcing term f and initial data w_0, then

$$\|w^\varepsilon_{_F}\|_{L^4(\mathbf{R}^+,L^\infty)} \leq C_{r,R}\,\varepsilon^{\frac14}\,\Big(\|w_0\|_{L^2} + \|f\|_{L^1(\mathbf{R}^+,L^2)}\Big)\,. \tag{2.6}$$

2.2 Proof of Theorem 1

We are now able to prove Theorem 1. Let us define the function $u^\varepsilon_R \stackrel{\text{def}}{=} \chi\Big(\dfrac{|D|}{R}\Big)u^\varepsilon$, where we have written $|D| = (-\Delta)^{\frac12}$, and where χ is a function in $\mathcal{D}(\mathbf{R})$, whose value is 1 near the origin. By Sobolev embeddings and stating $\alpha \stackrel{\text{def}}{=} \dfrac{3}{q} - \dfrac{1}{2}$, we can write

$$\begin{aligned}
\|u^\varepsilon - u^\varepsilon_R\|_{L^2(\mathbf{R}^+,L^q(\mathbf{R}^3))} &\leq C\|u^\varepsilon - u^\varepsilon_R\|_{L^2(\mathbf{R}^+,H^{1-\alpha}(\mathbf{R}^3))}\\
&\leq CR^{-\alpha}\|u^\varepsilon\|_{L^2(\mathbf{R}^+,H^1(\mathbf{R}^3))}\\
&\leq \frac{C}{\sqrt{\nu}}R^{-\alpha}\|u_0\|_{L^2(\mathbf{R}^3)}.
\end{aligned} \tag{2.7}$$

Now let us estimate u^ε_R. We have

$$\partial_t u^\varepsilon_R - \nu\Delta u^\varepsilon_R + \frac{1}{\varepsilon}P(u^\varepsilon_R \times e_3) = f^\varepsilon_R,$$

with

$$f^\varepsilon_R(t) \stackrel{\text{def}}{=} -\chi\Big(\frac{|D|}{R}\Big)P \text{ div }(u^\varepsilon \otimes u^\varepsilon)(t),$$

which implies, by Duhamel's formula, that

$$u^\varepsilon_R = e^{t(\nu\Delta+\sigma(D))}u_{0,R} + \int_0^t e^{(t-t')(\nu\Delta+\sigma(D))}f^\varepsilon_R(t')\,dt',$$

where $\sigma(\xi)v \stackrel{\text{def}}{=} \dfrac{\xi_3}{\varepsilon|\xi|^2}\xi \times v$. Now, using the Berstein inequalities, we have, for any $r > 0$,

$$\|e^{t(\nu\Delta+\sigma(D))}u_{0,R}\|_{L^\infty(\mathbf{R}^3)} \leq C(R^2r)^{\frac12}\|e^{t(\nu\Delta+\sigma(D))}u_{0,R}\|_{L^2(\mathbf{R}^3)},$$

where $D_3 = -i\partial_3$; so with Corollary 2, we have

$$\begin{aligned}
\|e^{t(\nu\Delta+\sigma(D))}u_{0,R}\|_{L^4_T(L^\infty(\mathbf{R}^3))} &\leq \|e^{t(\nu\Delta+\sigma(D))}\chi\Big(\frac{D_3}{r}\Big)u_{0,R}\|_{L^4_T(L^\infty(\mathbf{R}^3))}\\
&\quad + C\left\|e^{t(\nu\Delta+\sigma(D))}\Big(\mathrm{Id}-\chi\Big(\frac{D_3}{r}\Big)\Big)u_{0,R}\right\|_{L^4_T(L^\infty(\mathbf{R}^3))}\\
&\leq CT^{\frac14}(R^2r)^{\frac12}\|u_0\|_{L^2}+\varepsilon^{\frac14}C_{r,R}\|u_0\|_{L^2}, \qquad (2.8)
\end{aligned}$$

where to simplify, we have noted

$$L^p_T(L^q(\mathbf{R}^d)) = L^p([0,T];L^q(\mathbf{R}^d)).$$

So that implies that $e^{t(\nu\Delta+\sigma(D))}u_{0,R}$ goes to zero in $L^4_T(L^\infty(\mathbf{R}^3))$.

Moreover, we have

$$\begin{aligned}
\|\chi\Big(\frac{D_3}{r}\Big)f^\varepsilon_R\|_{L^\infty_T(L^2(\mathbf{R}^3))} &\leq R\,\|\chi\Big(\frac{|D|}{R}\Big)\chi\Big(\frac{D_3}{r}\Big)u^\varepsilon\otimes u^\varepsilon\|_{L^\infty_T(L^2(\mathbf{R}^3))}\\
&\leq R(rR^2)^{\frac12}\|u_0\|^2_{L^2(\mathbf{R}^3)}.
\end{aligned}$$

So Corollary 2 yields, as for (2.8),

$$\|\int_0^t e^{(t-t')(\nu\Delta+\sigma(D))}f^\varepsilon_R(t')\,dt'\|_{L^4_T(L^\infty(\mathbf{R}^3))} \leq CR(rR^2)^{\frac12}T^{\frac54}\|u_0\|^2_{L^2(\mathbf{R}^3)}+\varepsilon^{\frac14}C_{r,R}T\|u_0\|^2_{L^2}.$$

Using (2.8), we get

$$\begin{aligned}
\|u^\varepsilon_R\|_{L^4_T(L^\infty(\mathbf{R}^3))} \leq CT^{\frac14}(R^2r)^{\frac12}\|u_0\|_{L^2}+\varepsilon^{\frac14}C_{r,R}\|u_0\|_{L^2}&\\
+CR(rR^2)^{\frac12}T^{\frac54}\|u_0\|^2_{L^2(\mathbf{R}^3)}+\varepsilon^{\frac14}C_{r,R}T\|u_0\|^2_{L^2}&.
\end{aligned}$$

From this, we deduce that for a fixed R, we have

$$\lim_{\varepsilon\to0}u^\varepsilon_R = 0 \quad \text{in} \quad L^2_{loc}(\mathbf{R}^+;L^\infty(\mathbf{R}^3)),$$

which with (2.7) finally implies that

$$u^\varepsilon \to 0 \quad \text{in} \quad L^2_{loc}(\mathbf{R}^+;L^q(\mathbf{R}^3)) \quad \forall q\in]2,6[.$$

That proves Theorem 1. □

3 Anisotropic Strichartz estimates

In this section, we are going to prove anisotropic, and viscous, Strichartz estimates, which will enable us in the following section to prove Theorem 2.

Let us first define some notation: for all $x \in \mathbf{R}^3$, we will note $x = (x_h, x_3)$, with $x_h = (x_1, x_2)$, and we will note $L^{p,q}_{x_h,x_3}$, for $1 \leq p, q \leq \infty$, the space defined by the norm

$$\|f\|_{L^{p,q}_{x_h,x_3}} \stackrel{\text{def}}{=} \Big\| \|f\|_{L^q_{x_3}} \Big\|_{L^p_{x_h}} .$$

Moreover, let us recall some definitions of Littlewood–Paley theory (we refer for instance to [6] for details). In the following we shall note Δ_j (resp. S_j) the Littlewood–Paley spectral localization operators, defined in the following way:

$$\forall j \in \mathbf{Z}, \quad \Delta_j \stackrel{\text{def}}{=} \varphi(2^{-j}|D|) \quad \text{and} \quad S_j = \sum_{j' \leq j-1} \Delta_{j'}, \tag{3.1}$$

where $\varphi \in \mathcal{D}(\mathbf{R})$ is such that

$$\varphi(s) = 1 \quad \text{if} \quad \frac{1}{2} \leq |s| \leq 1, \quad \text{and} \quad \varphi(s) = 0 \quad \text{near} \quad 0.$$

Finally the function φ satisfies

$$\forall t \in \mathbf{R} \setminus \{0\}, \ \sum_{j \in \mathbf{Z}} \varphi(2^{-j} t) = 1.$$

In the rest of the paper, $N_0 \in \mathbf{N}$ will be a fixed constant, depending only on φ, such that if $|j - j'| \geq N_0$, then $\operatorname{supp} \varphi(2^{-j}\cdot) \cap \operatorname{supp} \varphi(2^{-j'}\cdot) = \emptyset$. Finally as we are interested in proving anisotropic estimates, we are also led to introducing an anisotropic Littlewood–Paley decomposition, in the spirit of [16]. So let us define the operators Δ^v_k and S^v_k by

$$\forall k \in \mathbf{Z}, \quad \Delta^v_k \stackrel{\text{def}}{=} \varphi(2^{-k} D_3) \quad \text{and} \quad S^v_k = \sum_{k' \leq k-1} \Delta^v_{k'}.$$

Theorem 3 *Let $\mathcal{G}^\varepsilon_\nu$ be the operator defined in Equation (2.3); for any $p \in [1, \infty]$ and for any $\alpha > 0$, a constant C exists such that, for any vector field f, we have*

$$\left\| \Delta_j \mathcal{G}^\varepsilon_\nu \Big(\frac{\cdot}{\varepsilon}\Big) f \right\|_{L^p_T(L^\infty(\mathbf{R}^3))} \leq C 2^{j(\frac{3}{2} - \frac{2}{p})} (\varepsilon 2^{2j})^{\frac{1}{4p(1+\alpha)}} \|\Delta_j f\|_{L^2(\mathbf{R}^3)}, \tag{3.2}$$

$$\left\| \Delta_j \Delta^v_k \mathcal{G}^\varepsilon_\nu \Big(\frac{\cdot}{\varepsilon}\Big) f \right\|_{L^p_T(L^{\infty,2}_{x_h,x_3})} \leq C 2^{j(1 - \frac{2}{p})} \min \Big\{ 1, (\varepsilon 2^{2j})^{\frac{1}{4p}} 2^{\frac{1}{2p}(j-k)} \Big\} \|\Delta_j \Delta^v_k f\|_{L^2(\mathbf{R}^3)}. \tag{3.3}$$

Proof of Theorem 3. Like all Strichartz–type inequalities, (3.2) and (3.3) are the consequences of dispersive estimates. Writing

$$\mathcal{G}(\tau, \theta) f \stackrel{\text{def}}{=} \mathcal{F}^{-1} \left(e^{i\tau \frac{\xi_3}{|\xi|} - \theta |\xi|^2} \widehat{f}(\xi) \right),$$

we have the following dispersive lemma.

Lemma 3 *For any function f, we have*

$$\|\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f\|_{L^\infty(\mathbf{R}^3)} \leq C \min\left\{2^{k-j}, \tau^{-\frac{1}{2}}\right\} 2^{3j} e^{-C\theta 2^{2j}} \|f\|_{L^1(\mathbf{R}^3)}, \tag{3.4}$$

and $$\|\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f\|_{L^{\infty,2}_{x_h,x_3}} \leq C \min\left\{1, \tau^{-\frac{1}{2}} 2^{(j-k)}\right\} 2^{2j} e^{-C\theta 2^{2j}} \|f\|_{L^{1,2}_{x_h,x_3}}. \tag{3.5}$$

First, let us remark that inequality (3.4) is nothing but Lemma 1 with a precise control of the constant.

Proof. We start by the proof of (3.4), and (3.5) will be obtained by very similar methods. We can write

$$\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f(x) = K_{j,k}(\tau,\theta) * f(x),$$

with

$$K_{j,k}(\tau,\theta,x) \stackrel{\text{def}}{=} \int_{\mathbf{R}^3} e^{ix\cdot\xi + i\tau\frac{\xi_3}{|\xi|} - \theta|\xi|^2} \varphi(2^{-j}|\xi|)\varphi(2^{-k}\xi_3)\, d\xi, \tag{3.6}$$

hence all we have to prove is that

$$\|K_{j,k}(\tau,\theta,\cdot)\|_{L^\infty(\mathbf{R}^3)} \leq \frac{C}{\tau^{\frac{1}{2}}} e^{-C\theta 2^{2j}} 2^{3j}. \tag{3.7}$$

Indeed, (3.7) implies that

$$\|\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f\|_{L^\infty(\mathbf{R}^3)} \leq \frac{C}{\tau^{\frac{1}{2}}} e^{-C\theta 2^{2j}} 2^{3j} \|f\|_{L^1(\mathbf{R}^3)}, \tag{3.8}$$

and we also have

$$\|\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f\|_{L^2(\mathbf{R}^3)} \leq C e^{-C\theta 2^{2j}} \|\Delta_j \Delta_k^v f\|_{L^2(\mathbf{R}^3)}. \tag{3.9}$$

Moreover, Bernstein's inequality yields

$$\|\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f\|_{L^\infty(\mathbf{R}^3)} \leq C 2^{\frac{3j}{2}} 2^{\frac{k-j}{2}} \|\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f\|_{L^2(\mathbf{R}^3)}, \tag{3.10}$$

so with (3.9), we have

$$\begin{aligned} \|\Delta_j \Delta_k^v \mathcal{G}(\tau,\theta) f\|_{L^\infty(\mathbf{R}^3)} &\leq C 2^{\frac{3j}{2}} 2^{\frac{k-j}{2}} e^{-C\theta 2^{2j}} \|\Delta_j \Delta_k^v f\|_{L^2(\mathbf{R}^3)} \\ &\leq C e^{-C\theta 2^{2j}} 2^{3j} 2^{k-j} \|f\|_{L^1(\mathbf{R}^3)}. \end{aligned} \tag{3.11}$$

Inequality (3.11) associated with (3.8) implies that (3.4) is proved, except for the fact that we still have to prove (3.7). Using the rotation invariance of $K_{j,k}$, we have

$$K_{j,k}(\tau,\theta,x) = \int_{\mathcal{B}(0,C2^j)} e^{ix_1\xi_1} \left(\int_{\mathbf{R}^2} e^{i\tau\frac{\xi_3}{|\xi|} - \theta|\xi|^2} \varphi(2^{-j}|\xi|)\varphi(2^{-k}\xi_3)\, d\xi_2 d\xi_3 \right) d\xi_1.$$

Then we write $K_{j,k}(\tau,\theta,x) = 2^{3j} \widetilde{K}_{j,k}(\tau, 2^{2j}\theta, 2^j x)$, with

$$\widetilde{K}_{j,k}(\tau,\eta,y) \stackrel{\text{def}}{=} \int_{\mathcal{B}(0,C)} e^{iy_1\xi_1} \left(\int_{\mathbf{R}^2} e^{i\tau\frac{\xi_3}{|\xi|} - \eta|\xi|^2} \varphi(|\xi|)\varphi(2^{j-k}\xi_3)\, d\xi_2 d\xi_3 \right) d\xi_1.$$

As in the proof of Lemma 1 in Section 2, we shall integrate by parts in the following integral:

$$\Phi(\xi_1,\xi_3) \stackrel{\text{def}}{=} \int_{\mathbf{R}} e^{i\tau\frac{\xi_3}{|\xi|}-\eta|\xi|^2}\varphi(|\xi|)\,d\xi_2.$$

So let us introduce the operator $\mathcal{L}$ defined as

$$\forall a, \quad \mathcal{L}a \stackrel{\text{def}}{=} \frac{1}{1+\tau\alpha^2(\xi)}\left(a+i\alpha(\xi)\partial_{\xi_2}a\right) \quad \text{with} \quad \alpha(\xi) = -\partial_{\xi_2}a(\xi) = \xi_2\xi_3/|\xi|^3.$$

An easy computation yields

$$\mathcal{L}(e^{i\tau\frac{\xi_3}{|\xi|}}) = e^{i\tau\frac{\xi_3}{|\xi|}},$$

which implies immediately that

$$\Phi(\xi_1,\xi_3) = \int_{\mathbf{R}} e^{i\tau\frac{\xi_3}{|\xi|}}\ {}^t\mathcal{L}\left(e^{-\eta|\xi|^2}\varphi(|\xi|)\right)\,d\xi_2.$$

But we can compute, for any function Ψ, as in (2.5),

$${}^t\mathcal{L}\left(\Psi(\xi)\right) = \left(\frac{1}{1+\alpha^2\tau} - i(\partial_{\xi_2}\alpha)\frac{1-\tau\alpha^2}{(1+\tau\alpha^2)^2}\right)\Psi(\xi) - \frac{i\alpha}{1+\tau\alpha^2}\partial_{\xi_2}\Psi(\xi),$$

so using the fact that ξ is in a fixed ring of $\mathbf{R}^3$, that $\varphi\in\mathcal{D}(\mathbf{R})$, we get, after elementary computations,

$$\left|{}^t\mathcal{L}\left(e^{-\eta|\xi|^2}\varphi(|\xi|)\right)\right| \leq \frac{C}{1+\tau\xi_2^2\xi_3^2}e^{-\eta|\xi|^2}.$$

Finally we have

$$\begin{aligned} \|\widetilde{K}_{j,k}(\tau,\eta,\cdot)\|_{L^\infty(\mathbf{R}^3)} &\leq Ce^{-C\eta}\int_{c\leq|\xi_2|\leq C}\int_{c2^{k-j}\leq|\xi_3|\leq C2^{k-j}}\frac{d\xi_2 d\xi_3}{1+\tau\xi_2^2\xi_3^2} \\ &\leq Ce^{-C\eta}\tau^{-\frac{1}{2}}\int_{\mathbf{R}}\frac{d\zeta_2}{1+\zeta_2^2}. \end{aligned} \tag{3.12}$$

So the lemma is proved in the case (3.4).

The proof of (3.5) is exactly along the same lines, only we keep the variable ξ_3 as a parameter in the computations above. In other words, let us define

$$I_{j,k}(\tau,\theta,x_h,\xi_3) \stackrel{\text{def}}{=} \int_{\mathbf{R}^2} e^{ix_h\cdot\xi_h+i\tau\frac{\xi_3}{|\xi|}-\theta|\xi|^2}\varphi(2^{-j}|\xi|)\varphi(2^{-k}\xi_3)\,d\xi_h.$$

Then we have

$$\|\Delta_j\Delta_k^v\mathcal{G}(\tau,\theta)f\|_{L^{\infty,2}_{x_h,x_3}} \leq C\|I_{j,k}(\tau,\theta,\cdot,\cdot)\|_{L^{\infty,\infty}_{x_h,\xi_3}}\|f\|_{L^{1,2}_{x_h,x_3}},$$

where we have used a Young inequality in the horizontal variables, and Hölder's inequality in the vertical variable, associated with Plancherel's formula.

Now all we need is an estimate of $I_{j,k}(\tau,\theta,\cdot,\cdot)$ in $L^{\infty,\infty}_{x_h,\xi_3}$, which is obtained exactly as in the isotropic case above: we have

$$I_{j,k}(\tau,\theta,x_h,\xi_3)=2^{2j}\widetilde{I}_{j,k}(\tau,2^{2j}\theta,2^jx_h,2^{-j}\xi_3)$$

with

$$\widetilde{I}_{j,k}(\tau,\eta,y_h,\zeta_3)\stackrel{\rm def}{=}\varphi(2^{j-k}\zeta_3)\int_{\mathbf{R}^2}e^{iy_h\cdot\zeta_h+i\tau\frac{\zeta_3}{|\zeta|}-\eta|\zeta|^2}\varphi(|\zeta|)\,d\zeta_h.$$

An identical computation as for (3.12) yields

$$\begin{aligned}\|\widetilde{I}_{j,k}(\tau,\eta,y_h,\zeta_3)\|_{L^{\infty,\infty}_{y_h,\zeta_3}}&\leq Ce^{-C\eta}\left\|\varphi(2^{j-k}\zeta_3)\int_{c\leq|\xi_2|\leq C}\frac{d\zeta_2}{1+\tau\zeta_2^2\zeta_3^2}\right\|_{L^\infty_{\zeta_3}}\\&\leq Ce^{-C\eta}\tau^{-\frac12}\|\varphi(2^{j-k}\zeta_3)\zeta_3^{-1}\|_{L^\infty_{\zeta_3}}\\&\leq Ce^{-C\eta}2^{(j-k)}\tau^{-\frac12}.\end{aligned}$$

So finally we have obtained that

$$\|\Delta_j\Delta_k^v\mathcal{G}(\tau,\theta)f\|_{L^{\infty,2}_{x_h,x_3}}\leq Ce^{-C\theta2^{2j}}2^{2j}2^{(j-k)}\tau^{-\frac12}\|f\|_{L^{1,2}_{x_h,x_3}}.\tag{3.13}$$

Now to conclude to (3.5), we write similar estimates as (3.10),(3.11): we have, by Sobolev embeddings,

$$\begin{aligned}\|\Delta_j\Delta_k^v\mathcal{G}(\tau,\theta)f\|_{L^{\infty,2}_{x_h,x_3}}&\leq C2^j\|\Delta_j\Delta_k^v\mathcal{G}(\tau,\theta)f\|_{L^2(\mathbf{R}^3)}\\&\leq C2^je^{-C\theta2^{2j}}\|\Delta_j\Delta_k^vf\|_{L^2(\mathbf{R}^3)}\\&\leq C2^{2j}e^{-C\theta2^{2j}}\|\Delta_j\Delta_k^vf\|_{L^{1,2}_{x_h,x_3}},\end{aligned}$$

which with (3.13) implies that

$$\|\Delta_j\Delta_k^v\mathcal{G}(\tau,\theta)f\|_{L^{\infty,2}_{x_h,x_3}}\leq Ce^{-C\theta2^{2j}}2^{2j}\min\left\{\frac{1}{\tau^{\frac12}}2^{(j-k)},1\right\}\|f\|_{L^{1,2}_{x_h,x_3}}.$$

The lemma is proved. □

End of the proof of Theorem 3. To prove the result, we are going to use duality arguments, based on the isotropic estimate (3.2). In the following, we shall call

$$\mathcal{B}\stackrel{\rm def}{=}\left\{\Psi\in\mathcal{D}(\mathbf{R}^3),\|\Psi\|_{L^\infty_T(L^1(\mathbf{R}^3))}\leq1\right\}.\tag{3.14}$$

The following set of equalities is standard:

$$\begin{aligned}\|\Delta_j\mathcal{G}_\nu^\varepsilon f\|_{L^1_T(L^\infty(\mathbf{R}^3))}&=\sup_{\Psi\in\mathcal{B}}\int_{\mathbf{R}^+}<\Delta_j\mathcal{G}_\nu^\varepsilon(t)f,\Psi(t)>\,dt\\&=\sup_{\Psi\in\mathcal{B}}(2\pi)^{-3}\int_{\mathbf{R}^+\times\mathbf{R}^3}\varphi(2^{-j}|\xi|)e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|}-\nu t|\xi|^2}\widehat{f}(\xi)\widehat{\Psi}(t,\xi)\,dtd\xi\\&=\sup_{\Psi\in\mathcal{B}}(2\pi)^{-3}\sum_{k\leq j}\int_{\mathbf{R}^+\times\mathbf{R}^3}\widehat{f}_{j,k}(\xi)\widehat{\Psi}_{j,k}(t,\xi)e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|}-\nu t|\xi|^2}\,dtd\xi,\end{aligned}$$

where we have defined $f_{j,k} = \Delta_j \Delta_k^v f$ and $\Psi_{j,k} = \Delta_j \Delta_k^v \Psi$. So it follows that

$$\|\Delta_j \mathcal{G}_\nu^\varepsilon f\|_{L^1_T(L^\infty(\mathbf{R}^3))} \leq \sup_{\Psi \in \mathcal{B}} (2\pi)^{-3} \sum_{k \leq j} \|f_{j,k}\|_{L^2(\mathbf{R}^3)} \left\| \int_{\mathbf{R}^+} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi|^2} \widehat{\Psi}_{j,k}(t,\xi)\, dt \right\|_{L^2(\mathbf{R}^3)}.$$

But we have

$$\begin{aligned} &\left\| \int_{\mathbf{R}^+} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi|^2} \widehat{\Psi}_{j,k}(t,\xi)\, dt \right\|^2_{L^2(\mathbf{R}^3)} \\ &= \int_{(\mathbf{R}^+)^2 \times \mathbf{R}^3} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi|^2} \widehat{\Psi}_{j,k}(t,\xi) e^{i\frac{s}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu s|\xi|^2} \overline{\widehat{\Psi}}_{j,k}(s,\xi)\, dt ds d\xi \\ &\leq \int_{(\mathbf{R}^+)^2} \|\Psi_{j,k}(s,\cdot)\|_{L^1(\mathbf{R}^3)} \|\Psi_{j,k}(t,\cdot)\|_{L^1(\mathbf{R}^3)} \min\left\{2^{k-j}, \frac{\varepsilon^{\frac12}}{|t-s|^{\frac12}}\right\} 2^{3j} e^{-\nu(t+s)2^{2j}}\, dt ds, \end{aligned} \tag{3.15}$$

where we have used (3.4). So

$$\|\Delta_j \mathcal{G}_\nu^\varepsilon f\|^2_{L^1_T(L^\infty(\mathbf{R}^3))} \leq C \sum_{k \leq j} 2^{3j} \|f_{j,k}\|^2_{L^2(\mathbf{R}^3)} \int_{(\mathbf{R}^+)^2} \min\left\{2^{k-j}, \frac{\varepsilon^{\frac12}}{|t-s|^{\frac12}}\right\} e^{-\nu(t+s)2^{2j}}\, dt ds. \tag{3.16}$$

On the one hand, the upper bound (3.16) can be estimated directly in the following way:

$$2^{3j} \int_{\mathbf{R}^+ \times \mathbf{R}^+} \min\left\{2^{k-j}, \frac{\varepsilon^{\frac12}}{|t-s|^{\frac12}}\right\} e^{-\nu(t+s)2^{2j}}\, dt ds \leq C 2^{k-j} 2^{-j}. \tag{3.17}$$

On the other hand, we can write

$$2^{3j} \int_{\mathbf{R}^+ \times \mathbf{R}^+} \frac{1}{|t-s|^{\frac12}} e^{-\nu(t+s)2^{2j}}\, dt ds = \int_{\mathbf{R} \times \mathbf{R}} \frac{1}{|t'-s'|^{\frac12}} e^{-\nu(t'+s')}\, dt' ds', \tag{3.18}$$

which finally implies that

$$\left\| \int_{\mathbf{R}^+} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi|^2} \widehat{\Psi}_{j,k}(t,\xi)\, dt \right\|^2_{L^2(\mathbf{R}^3)} \leq C 2^{k-j} 2^{-j} \min\left\{1, (\varepsilon 2^{2j})^{\frac12} 2^{(j-k)}\right\}.$$

Using the fact that

$$\sum_k \|\Delta_j \Delta_k^v f\|^2_{L^2(\mathbf{R}^3)} = \|\Delta_j f\|^2_{L^2(\mathbf{R}^3)},$$

we have finally

$$\|\Delta_j \mathcal{G}_\nu^\varepsilon f\|_{L^1_T(L^\infty(\mathbf{R}^3))} \leq C 2^{-\frac{j}{2}} \|\Delta_j f\|_{L^2(\mathbf{R}^3)} \sum_{k \leq j} 2^{\frac{k-j}{2}} \min\left\{1, (\varepsilon 2^{2j})^{\frac14} 2^{\frac{j-k}{2}}\right\}.$$

Finally we find, for any $r > 1$,

$$\|\Delta_j \mathcal{G}_\nu^\varepsilon f\|_{L^1_T(L^\infty(\mathbf{R}^3))} \leq C 2^{-\frac{j}{2}} \|\Delta_j f\|_{L^2(\mathbf{R}^3)} \sum_{k \leq j} 2^{\frac{k-j}{2}(1-\frac1r)} (\varepsilon 2^{2j})^{\frac{1}{4r}},$$

which means in other words that for any $\alpha > 0$, choosing $r = 1 + \alpha$, we have

$$\|\Delta_j \mathcal{G}_\nu^\varepsilon f\|_{L^1_T(L^\infty(\mathbf{R}^3))} \leq C_\alpha 2^{-\frac{j}{2}} \|\Delta_j f\|_{L^2(\mathbf{R}^3)} (\varepsilon 2^{2j})^{\frac{1}{4(1+\alpha)}}. \tag{3.19}$$

Now to get the L^p_T estimate, we just have to interpolate (3.19) with

$$\|\Delta_j \mathcal{G}_\nu^\varepsilon f\|_{L^\infty_T(L^2(\mathbf{R}^3))} \leq C \|\Delta_j f\|_{L^2(\mathbf{R}^3)}, \tag{3.20}$$

which yields

$$\forall p \in [1, \infty],\ \forall \alpha > 0, \quad \|\Delta_j \mathcal{G}_\nu^\varepsilon f\|_{L^p_T(L^q(\mathbf{R}^3))} \leq C_\alpha 2^{-\frac{j}{2p}} (\varepsilon 2^{2j})^{\frac{1}{4p(1+\alpha)}} \|\Delta_j f\|_{L^2(\mathbf{R}^3)},$$

with $1/q = 1/2 - 1/2p$. Then Bernstein's inequality implies the result in the isotropic case.

We are finally left with the anisotropic case. To prove estimate (3.3), the computation is identical to the one leading to estimate (3.2): we just have to change the set $\mathcal{B}$ defined in (3.14) into

$$\mathcal{B}' \stackrel{\text{def}}{=} \left\{ \Psi \in \mathcal{D}(\mathbf{R}^3), \|\Psi\|_{L^\infty_T(L^{1,2}_{x_h,x_3})} \leq 1 \right\}.$$

Then estimate (3.15) becomes, with (3.5),

$$\left\| \int_{\mathbf{R}^+} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi|^2} \widehat{\Psi}_{j,k}(t,\xi)\, dt \right\|^2_{L^2(\mathbf{R}^3)}$$

$$\leq \int_{(\mathbf{R}^+)^2} \|\Psi_{j,k}(s,\cdot)\|_{L^{1,2}_{x_h,x_3}} \|\Psi_{j,k}(t,\cdot)\|_{L^{1,2}_{x_h,x_3}} \min\left\{1, \frac{\varepsilon^{\frac{1}{2}} 2^{(j-k)}}{|t-s|^{\frac{1}{2}}}\right\} 2^{2j} e^{-\nu(t+s)2^{2j}}\, dtds.$$

So finally, using (3.17) and (3.18) we have

$$\|\Delta_j \Delta_k^v \mathcal{G}_\nu^\varepsilon f\|_{L^1_T(L^{\infty,2}_{x_h,x_3})} \leq C \min\left\{1, (\varepsilon 2^{2j})^{\frac{1}{4}} 2^{\frac{j-k}{2}}\right\} 2^{-j} \|\Delta_j \Delta_k^v f\|_{L^2(\mathbf{R}^3)}.$$

The end of the proof is then exactly that of the isotropic case, and we obtain estimate (3.3).

So Theorem 3 is proved. $\square$

4 Global existence and convergence of smooth solutions

Let us recall that we have noted w^ε_F for the solution of the linear equation

$$\left\{ \begin{array}{rcl} \partial_t w^\varepsilon_F - \nu \Delta w^\varepsilon_F + \dfrac{1}{\varepsilon} L_{RF} w^\varepsilon_F &=& 0 \\ w^\varepsilon_{F|t=0} &=& w_0\,. \end{array} \right.$$

Classical results on the heat equation imply that

$$\forall t \in \mathbf{R}^+, \quad \|w^\varepsilon_F(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)} + 2\nu \int_0^t \|w^\varepsilon_F(s)\|^2_{\dot{H}^{\frac{3}{2}}(\mathbf{R}^3)}\, ds \leq \|w_0\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)}.$$

Moreover, it is obvious that

$$\lim_{\substack{N\to+\infty\\ N_3\to-\infty}} (\mathrm{Id}-S^v_{N_3})S_N w_0 = w_0 \quad \text{in} \quad \dot H^{\frac12}(\mathbf{R}^3).$$

Consequently, for any $\eta>0$, there exist two integers N and N_3 such that if

$$w^\varepsilon_{_{Fm}} \stackrel{\text{def}}{=} (\mathrm{Id}-S^v_{N_3})S_N w^\varepsilon_{_F},$$

then

$$\forall t\in\mathbf{R}^+, \quad \|w^\varepsilon_{_F}(t)-w^\varepsilon_{_{Fm}}(t)\|^2_{\dot H^{\frac12}(\mathbf{R}^3)} + 2\nu\int_0^t \|w^\varepsilon_{_F}(s)-w^\varepsilon_{_{Fm}}(s)\|^2_{\dot H^{\frac32}(\mathbf{R}^3)}\,ds \le \eta. \tag{4.1}$$

Now let us define $w^\varepsilon \stackrel{\text{def}}{=} u^\varepsilon - \bar u - w^\varepsilon_{_{Fm}}$, where let us recall that $\bar u$ is the solution of the bidimensional Navier–Stokes equation.

Then the function w^ε satisfies the following system:

$$\left\{\begin{array}{rcl} \partial_t w^\varepsilon - \nu\Delta w^\varepsilon + \dfrac1\varepsilon L_{_{RF}} w^\varepsilon & = & Q(w^\varepsilon, w^\varepsilon+2\bar u+2w^\varepsilon_{_{Fm}}) + Q(w^\varepsilon_{_{Fm}}, w^\varepsilon_{_{Fm}}) + 2\,Q(\bar u, w^\varepsilon_{_{Fm}})\\ w^\varepsilon_{|t=0} & = & (\mathrm{Id}-S_N(\mathrm{Id}-S^v_{N_3}))w_0\,, \end{array}\right. \tag{4.2}$$

where we have written $Q(a,b) = A_{\ell,m}(D)a^m b^\ell$, where $A_{\ell,m}(D)$ is a Fourier multiplier of order 1.

Classical results on the incompressible bidimensional Navier–Stokes equations (see [23] forinstance) imply that

$$\forall t\in\mathbf{R}^+, \quad \|\bar u(t)\|^2_{L^2(\mathbf{R}^2)} + 2\nu\int_0^t \|\nabla\bar u(s)\|^2_{L^2(\mathbf{R}^2)}\,ds = \|\bar u_0\|^2_{L^2(\mathbf{R}^2)}. \tag{4.3}$$

It follows from (4.1) that if we prove that (4.2) is well posed, for ε small enough, in the space $C^0_b(\mathbf{R}^+,\dot H^{\frac12}(\mathbf{R}^3))\cap L^2(\mathbf{R}^+,\dot H^{\frac32}(\mathbf{R}^3))$, and that

$$w^\varepsilon \to 0 \quad \text{in} \quad L^\infty(\mathbf{R}^+,\dot H^{\frac12}(\mathbf{R}^3))\cap L^2(\mathbf{R}^+,\dot H^{\frac32}(\mathbf{R}^3)), \tag{4.4}$$

then Theorem 2 is proved.

Let us now apply the operators Δ_j defined earlier (see (3.1)), to equation (4.2), and write an energy estimate in $L^2(\mathbf{R}^3)$.

In order to estimate the terms in (4.2) in which the bidimensional function $\bar u$ appears, which is less smooth than w^ε, we use Lemma 5.1 of [9]: it states that the product of any tridimensional function by $\bar u$ is half a derivative smoother than if $\bar u$ were tridimensional; the setting in [9] is that of the torus, but the results extend unchanged to the whole space.

Let us therefore recall the rules of product in Sobolev spaces, which we shall be using extensively throughout the proofs:

$$\forall s,t<\frac d2,\ s+t>0, \quad \|ab\|_{\dot H^{s+t-\frac d2}(\mathbf{R}^d)} \le C_{s,t}\|a\|_{\dot H^s(\mathbf{R}^d)}\|b\|_{\dot H^t(\mathbf{R}^d)},$$

$$\forall s,t<1,\ s+t>0,\quad \|ab\|_{\dot H^{s+t-1}(\mathbf{R}^3)}\leq C_{s,t}\|a\|_{\dot H^s(\mathbf{R}^2)}\|b\|_{\dot H^t(\mathbf{R}^3)}.$$

Moreover, we shall also often be using the classical fact (see [6]) that

$$\forall d\geq 1,\quad \|v\|_{\dot H^s(\mathbf{R}^d)}\sim \|2^{js}\|\Delta_j v\|_{L^2(\mathbf{R}^d)}\|_{\ell^2(\mathbf{Z})},$$

$$\text{and}\quad \|\nabla\Delta_j v\|_{L^2(\mathbf{R}^d)}\geq c2^j\|\Delta_j v\|_{L^2(\mathbf{R}^d)}.$$

Now let us apply operator Δ_j to equation (4.2) satisfied by w^ε. We then take the scalar product in $L^2(\mathbf{R}^3)$ with $\Delta_j w^\varepsilon$, and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_j w^\varepsilon(t)\|^2_{L^2(\mathbf{R}^3)}+c\nu 2^{2j}\|\Delta_j w^\varepsilon(t)\|^2_{L^2(\mathbf{R}^3)}\leq C\, c_j^\varepsilon(t)2^{\frac{j}{2}}\|F^\varepsilon(t)\|_{\dot H^{-\frac12}(\mathbf{R}^3)}\|\Delta_j w^\varepsilon(t)\|_{L^2(\mathbf{R}^3)}\\ +C\, c_j^\varepsilon(t)2^j\|\bar F^\varepsilon(t)\|_{\dot H^{-1}(\mathbf{R}^3)}\|\Delta_j w^\varepsilon(t)\|_{L^2(\mathbf{R}^3)}\\ +C\, c_j^\varepsilon(t)2^{\frac{j}{2}}\|w^\varepsilon(t)\|_{\dot H^1(\mathbf{R}^3)}\left(\|w^\varepsilon(t)\|_{\dot H^1(\mathbf{R}^3)}+\|\bar u(t)\|_{\dot H^{\frac12}(\mathbf{R}^2)}\right)\|\Delta_j w^\varepsilon(t)\|_{L^2(\mathbf{R}^3)},\end{aligned}\tag{4.5}$$

where we have written

$$F^\varepsilon\stackrel{\text{def}}{=}2\,Q(w^\varepsilon,w^\varepsilon_{Fm})+Q(w^\varepsilon_{Fm},w^\varepsilon_{Fm}),\quad\text{and}\quad \bar F^\varepsilon\stackrel{\text{def}}{=}2\,Q(\bar u,w^\varepsilon_{Fm}).$$

Here like throughout the paper, $c_j^\varepsilon(t)$ is a sequence in $\ell^2(\mathbf{Z})$, of norm 1, and C denotes any "universal" constant.

To simplify the notation of the space–time norms, we shall write

$$L^p_T(\dot H^s(\mathbf{R}^d))=L^p([0,T];\dot H^s(\mathbf{R}^d))\quad\text{a nd}\quad L^p(\dot H^s(\mathbf{R}^d))=L^p(\mathbf{R}^+;\dot H^s(\mathbf{R}^d)).$$

Now we notice that estimate (4.5) has exactly the same form as estimate (B.8) of [10] (in the proof of Lemma B.1), with the following correspondance of notation:

$$\sigma=\frac12,\quad q=j,\quad A=w^\varepsilon,\quad |B(t)|_{\sigma+\frac12}=\|\bar u(t)\|_{\dot H^{\frac12}(\mathbf{R}^2)},\quad\text{and}\quad W=F^\varepsilon.$$

The only difference with estimate (B.8) is the presence of an extra forcing term, $\bar F^\varepsilon$, but it is an easy adaptation of the computations of [10] to see that as in Lemma B.1 of [10], we have

$$X_4^\varepsilon(t)e^{-C\bar U_4^4(t)}\leq C\|F^\varepsilon\|_{L^2_t(\dot H^{-\frac12}(\mathbf{R}^3))}+C\|\bar F^\varepsilon\|_{L^4_t(\dot H^{-1}(\mathbf{R}^3))}+C(X_4^\varepsilon)^2(t),\tag{4.6}$$

and

$$X_2^\varepsilon(t)+X_\infty^\varepsilon(t)\leq C\|F^\varepsilon\|_{L^2_t(\dot H^{-\frac12}(\mathbf{R}^3))}+C\|\bar F^\varepsilon\|_{L^4_t(\dot H^{-1}(\mathbf{R}^3))}+CX_4^\varepsilon(t)(X_4^\varepsilon(t)+\bar U_4(t)),\tag{4.7}$$

where we have noted

$$X_4^\varepsilon(t)=\|w^\varepsilon\|_{L^4_t(\dot H^1(\mathbf{R}^3))},\quad X_2^\varepsilon(t)=\|w^\varepsilon\|_{L^2_t(\dot H^{\frac32}(\mathbf{R}^3))},$$

$$X_\infty^\varepsilon(t)=\|w^\varepsilon\|_{L^\infty_t(\dot H^{\frac12}(\mathbf{R}^3))},\quad\text{and}\quad \bar U_4(t)=\|\bar u\|_{L^4_t(\dot H^{\frac12}(\mathbf{R}^2))}.$$

So to conclude, all we need is an estimate on F^ε in $L^2(\dot H^{-\frac12}(\mathbf{R}^3))$ and on $\bar F^\varepsilon$ in $L^4(\dot H^{-1}(\mathbf{R}^3))$. Those estimates are given in the following proposition, the proof of which is postponed for a while.

Proposition 4 *Under the assumptions of Theorem 2, we have*

$$\|F^\varepsilon\|_{L^2_T(\dot H^{-\frac12}(\mathbf{R}^3))} \leq \eta_1(\varepsilon) X_4^\varepsilon(t) + \eta_2(\varepsilon), \quad \textit{and} \quad \|\bar F^\varepsilon\|_{L^4_T(\dot H^{-1}(\mathbf{R}^3))} \leq \eta_3(\varepsilon),$$

with $\lim\limits_{\varepsilon\to 0} \eta_i = 0$, *for* $i \in \{1,2,3\}$.

Let us now conclude the proof of the theorem. Estimate (4.6), along with Proposition 4, implies that

$$X_4^\varepsilon(t)\Big(e^{-C\bar U_4^4(t)} - \eta_1(\varepsilon)\Big) \leq C(X_4^\varepsilon)^2(t) + \eta(\varepsilon),$$

with $\eta(\varepsilon) \stackrel{\text{def}}{=} \eta_2(\varepsilon) + \eta_3(\varepsilon)$. But the energy equality in $(NS2D)$ recalled in (4.3) implies that

$$\forall t \geq 0, \quad \bar U_4^4(t) \leq C\nu^{-2}\|\bar u_0\|^4_{L^2(\mathbf{R}^2)},$$

so let us choose $\varepsilon_0 > 0$ such that

$$\forall \varepsilon \leq \varepsilon_0, \quad \eta_1(\varepsilon) \leq \frac14 \exp\left(-C\nu^{-2}\|\bar u_0\|^4_{L^2(\mathbf{R}^2)}\right).$$

We get, for any $\varepsilon \leq \varepsilon_0$

$$X_4^\varepsilon(t) \leq Ce^{C\nu^{-2}\|\bar u_0\|^4_{L^2(\mathbf{R}^2)}}\Big((X_4^\varepsilon)^2(t) + \eta(\varepsilon)\Big).$$

So we have, for any $\varepsilon \leq \varepsilon_0$,

$$X_4^\varepsilon(t) \leq Ce^{C\nu^{-2}\|\bar u_0\|^4_{L^2(\mathbf{R}^2)}}\eta(\varepsilon).$$

Finally, from inequalities (4.6) and (4.7), we obtain

$$\lim_{\varepsilon\to 0} w^\varepsilon = 0 \quad \text{in the space} \quad L^\infty(\dot H^{\frac12}(\mathbf{R}^3)) \cap L^2(\dot H^{\frac32}(\mathbf{R}^3)).$$

The global well–posedness of (4.2) for ε small enough follows by classical arguments; this concludes the proof of the theorem. □

Proof of Proposition 4. Let us recall that

$$F^\varepsilon = 2\,Q(w^\varepsilon, w^\varepsilon_{Fm}) + Q(w^\varepsilon_{Fm}, w^\varepsilon_{Fm}) \quad \text{and} \quad \bar F^\varepsilon = 2\,Q(\bar u, w^\varepsilon_{Fm}).$$

It is here that we are going to use the Strichartz estimates proved in Section 3, namely estimates (3.2) and (3.3); that is the reason why we got rid of the very low and very high frequencies of the linear solution above: the function w^ε_{Fm} has frequencies such that

$$2^{N_3} \leq |\xi| \leq 2^N \quad \text{and} \quad 2^{N_3} \leq |\xi_3|,$$

so estimates (3.2) and (3.3) simply become respectively

$$\|\Delta_j w^\varepsilon_{Fm}\|_{L^p_T(L^\infty(\mathbf{R}^3))} \leq C(\varepsilon 2^{2N})^{1/(4p\alpha)}\|\Delta_j w_{0,m}\|_{L^2(\mathbf{R}^3)}, \quad \forall \alpha > 1, \tag{4.8}$$

$$\text{and} \quad \|w^\varepsilon_{Fm}\|_{L^p_T(L^{\infty,2}_{x_h,x_3})} \leq C(\varepsilon 2^{2N})^{1/4p}\|w_{0,m}\|_{L^2(\mathbf{R}^3)}, \tag{4.9}$$

where we have omitted the dependence of the constants on N and N_3, which are fixed once and for all.

Let us start by estimating $Q(w^\varepsilon_{Fm}, w^\varepsilon_{Fm} + w^\varepsilon)$ in $L^2_T(H^{-\frac{1}{2}}(\mathbf{R}^3))$. We are going to use the paraproduct algorithm introduced by J.-M. Bony in [5]: we have, with the notation introduced in (3.1), and with the usual summation convention,

$$Q(c,b) = A_{\ell,m}(D)(T'_{b^\ell}b^m + T_{b^m}c^\ell), \tag{4.10}$$

with

$$T'_c b = \sum_j S_{j+2}c\,\Delta_j b \quad \text{and} \quad T_b c = \sum_j S_{j-1}b\,\Delta_j c.$$

We can write, for any $\alpha > 1$ and according to (4.8),

$$\begin{aligned}
2^{-\frac{j}{2}}\|\Delta_j A_{\ell,m}(D)T'_{w^{\varepsilon,\ell}_{Fm}} b^m\|_{L^2_T(L^2(\mathbf{R}^3))} &\leq C\sum_{j'\geq j-N_0}\|S_{j'}w^\varepsilon_{Fm}\|_{L^4_T(L^\infty(\mathbf{R}^3))}\|\Delta_{j'}b\|_{L^4_T(L^2(\mathbf{R}^3))}2^{\frac{j}{2}}\\
&\leq C_\varepsilon \sum_{\substack{j'\geq j-N_0\\ N_3\leq j''\leq \min\{j'-1,N\}}}\|\Delta_{j''}w_0\|_{L^2(\mathbf{R}^3)}2^{\frac{j}{2}}\|\Delta_{j'}b\|_{L^4_T(L^2(\mathbf{R}^3))}\\
&\leq C_\varepsilon\|w_{0,m}\|_{L^2(\mathbf{R}^3)}\sum_{\substack{j'\geq j-N_0\\ N_3\leq j''\leq \min\{j'-1,N\}}}2^{\frac{j}{2}}2^{\frac{j''-N}{2}}\|\Delta_{j'}b\|_{L^4_T(L^2(\mathbf{R}^3))}\\
&\leq C_\varepsilon\|w_{0,m}\|_{L^2(\mathbf{R}^3)}\sum_{j'\geq j-N_0}2^{-N/2}2^{j'}\|\Delta_{j'}b\|_{L^4_T(L^2(\mathbf{R}^3))}2^{\frac{j-j'}{2}},
\end{aligned}$$

where C_ε denotes a constant multiplied by any positive power of $\varepsilon 2^{2N}$. So we have obtained, using Young's inequality,

$$\begin{aligned}
!\|A_{\ell,m}(D)T'_{w^{\varepsilon,\ell}_{Fm}}(w^\varepsilon + w^\varepsilon_{Fm})^m\|^2_{L^2_T(\dot H^{-\frac{1}{2}}(\mathbf{R}^3))} &\leq C^2_\varepsilon\|w_{0,m}\|^2_{L^2(\mathbf{R}^3)}\\
&\quad\times\left(\|w_{0,m}\|^2_{L^2(\mathbf{R}^3)} + \|w^\varepsilon\|^2_{L^4_T(\dot H^1(\mathbf{R}^3))}\right).
\end{aligned} \tag{4.11}$$

As to the term $T_{b^m}w^{\varepsilon,\ell}_{Fm}$, let us define

$$\mathcal{C}_j \stackrel{\text{def}}{=} 2^{-\frac{j}{2}}\|\Delta_j A_{\ell,m}(D)T_{b^m}w^{\varepsilon,\ell}_{Fm}\|_{L^2_T(L^2(\mathbf{R}^3))}.$$

We have

$$\begin{aligned}
\mathcal{C}_j &\leq C\sum_{\substack{|j-j'|\leq N_0\\ N_3\leq j'\leq N}}2^{\frac{j}{2}}\|S_{j'}b\|_{L^4_T(L^2(\mathbf{R}^3))}\|\Delta_{j'}w^\varepsilon_{Fm}\|_{L^4_T(L^\infty(\mathbf{R}^3))}\\
&\leq C_\varepsilon\sum_{\substack{|j-j'|\leq N_0\\ N_3\leq j'\leq N}}2^{\frac{j}{2}}\|\Delta_{j''}w_0\|_{L^2(\mathbf{R}^3)}\|S_{j'}b\|_{L^4_T(L^2(\mathbf{R}^3))}\\
&\leq C_\varepsilon\|w_{0,m}\|_{L^2(\mathbf{R}^3)}\|b\|_{L^4_T(\dot H^1(\mathbf{R}^3))}.
\end{aligned}$$

So with (4.11), we obtain finally

$$
\begin{aligned}
\|Q(w^\varepsilon_{Fm}, w^\varepsilon + w^\varepsilon_{Fm})\|^2_{L^2_T(\dot H^{-\frac12}(\mathbf{R}^3))} &\leq C^2_\varepsilon \|w_{0,m}\|^2_{L^2(\mathbf{R}^3)} \\
&\quad \times \Big(\|w_{0,m}\|^2_{L^2(\mathbf{R}^3)} + \|w^\varepsilon\|^2_{L^4_T(\dot H^1(\mathbf{R}^3))} \Big). \quad (4.12)
\end{aligned}
$$

Now to end the proof of the proposition, we need to estimate the term $Q(w^\varepsilon_{Fm}, \bar u)$; it is with that estimate in mind that we presented an anisotropic Strichartz estimate in Section 3. We have simply

$$
\begin{aligned}
\|A_{\ell,m}(D) w^{\varepsilon,\ell}_{Fm}\, \bar u^m\|_{L^4_T(\dot H^{-1}(\mathbf{R}^3))} &\leq C \|w^{\varepsilon,\ell}_{Fm}\, \bar u^m\|_{L^4_T(L^2(\mathbf{R}^3))} \\
&\leq C \|\bar u\|_{L^\infty_T(L^2(\mathbf{R}^2))} \|w^\varepsilon_{Fm}\|_{L^4_T(L^{\infty,2}_{x_h,x_3})} \\
&\leq C_\varepsilon \|\bar u_0\|_{L^2(\mathbf{R}^2)} \|w_{0,m}\|_{L^2(\mathbf{R}^3)},
\end{aligned}
$$

according to (4.9). That estimate, along with (4.12), proves Proposition 4. □

References

[1] A. Babin, A. Mahalov, and B. Nicolaenko, Global Splitting, Integrability and Regularity of 3D Euler and Navier–Stokes Equations for Uniformly Rotating Fluids, *European Journal of Mechanics*, **15**, 1996, pages 291-300.

[2] A. Babin, A. Mahalov, and B. Nicolaenko, Resonances and Regularity for Boussinesq Equations, *Russian Journal of Mathematical Physics*, **4**, 1996, pages 417-428.

[3] A. Babin, A. Mahalov, B. Nicolaenko, and Y. Zhou, On the Asymptotic Regimes and the Strongly Stratified Limit of Rotating Boussinesq Equations, *Journal of Theoretical and Computational Fluid Dynamics,* **9**, 1997, pages 223-251.

[4] H. Bahouri and J.-Y. Chemin, Équations d'ondes quasilinéaires et estimation de Strichartz, *Prépublication du Laboratoire d'Analyse Numérique, to appear in American Journal of Mathematics*, **121**, 1999.

[5] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales de l'École Normale Supérieure*, **14**, 1981, pages 209–246.

[6] J.-Y. Chemin, *Fluides parfaits incompressibles*, Astérisque, **230**, 1995.

[7] J.-Y. Chemin, À propos d'un problème de pénalisation de type antisymétrique, *Journal de Mathématiques Pures et Appliquées*, **76**, 1997, pages 739–755.

[8] B. Desjardins and E. Grenier, Low Mach number limit of compressible flows in the whole space, Proceedings of the Royal Society of London, **455**, 1999, pages 2271–2279.

[9] I. Gallagher, The Tridimensional Navier–Stokes Equations with Almost Bidimensional Data: Stability, Uniqueness and Life Span, *International Mathematics Research Notices*, **18**, 1997, pages 919-935.

[10] I. Gallagher, Applications of Schochet's Methods to Parabolic Equations, *Journal de Mathématiques Pures et Appliquées*, **77**, 1998, pages 989-1054.

[11] J. Ginibre and G. Velo, Generalized Strichartz inequalities for the wave equation, *Journal of Functional Analysis*, **133**, 1995, pages 50-68.

[12] H.P. Greenspan, The theory of rotating fluids, *Cambridge monographs on mechanics and applied mathematics*, 1969.

[13] E. Grenier, Oscillatory perturbations of the Navier Stokes equations. *Journal de Mathématiques Pures et Appliquées*, **76**, 1997, pages 477-498.

[14] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Mathématiques et Applications, **26**, Springer, 1996.

[15] D. Iftimie, La résolution des équations de Navier–Stokes dans des domaines minces et la limite quasigéostrophique, Thèse de l'Université Paris 6, 1997.

[16] D. Iftimie, The resolution of the Navier–Stokes equations in anisotropic spaces, *Revista Matematica Ibero–Americana*, **15**, 1999, pages 1-36.

[17] F. John, Existence for large times of strict solutions of non linear wave equations in three space dimensions, *Communications in Pure and Applied Mathematics*, **40**, 1987, pages 79–109.

[18] L. Kapitanski, Some generalization of the Strichartz-Brenner inequality, *Leningrad Mathematical Journal*, **1**, 1990, pages 693–721.

[19] M. Keel and T. Tao, Endpoint Strichartz Estimates, *American Journal of Mathematics*, **120**, 1998, pages 955–980.

[20] S. Klainerman, Global existence for nonlinear wave equations, *Communications in Pure and Applied Mathematics*, **33**, 1980, pages 43–101.

[21] S. Klainerman, Uniform decay estimates and the Lorentz invariance of the classical wave equation *Communications in Pure and Applied Mathematics*, **38**, 1985, pages 321–332.

[22] S. Klainerman and M. Machedon, Remark on Strichartz type inequalites (with an appendix of J. Bourgain and D. Tataru), *International Mathematical Research Notices*, **5**, 1996, pages 201–220.

[23] P.-L. Lions, Mathematical Topics in Fluid Dynamics, Vol. 1 Incompressible Models, *Oxford University Press*, 1998.

[24] J. Pedlosky, Geophysical fluid dynamics, *Springer*, 1979.

[25] S. Schochet, Fast singular limits of hyperbolic PDEs. *J. Diff. Equ.* **114**, 1994, pages 476 – 512.

[26] S. Ukai, The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.* **26**, no. 2, 1986, pages 323 – 331.