HOLOMORPHIC EQUIVARIANT ANALYTIC TORSIONS

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ABSTRACT. The purpose of this paper is to construct and compare two natural definitions of the equivariant holomorphic torsion. The comparison formula is shown to be compatible with the embedding formulas obtained by the first author for analytic torsion forms and equivariant analytic torsion.

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INTRODUCTION

The purpose of this paper is to establish a formula relating two natural versions of equivariant Quillen metrics. These Quillen metrics are global spectral invariants, and the formula relating them is a local formula.

The equivariant Quillen metrics refine two versions of the Lefschetz formulas, the Lefschetz fixed point formulas of [ABo1], and the Kirillov like formulas of [BeV]. Namely, let (X, ω^X) be a compact Kähler manifold, let (E, h^E) be a holomorphic Hermitian vector bundle on X. Let G be a compact Lie group acting holomorphically on X, E, and assume the action preserves ω^X, h^E. Then G acts of $H^{\cdot}(X,E)$. Let $L(g) = \mathrm{Tr_s}^{H^{\cdot}(X,E)}[g]$ be the virtual character of the action of G on $H^{\cdot}(X,E)$. Then the Lefschetz formulas of Atiyah-Bott [ABo1] assert that if $g \in G$, if X_g is the fixed point set of g, then

$$L(g) = \int_{X_g} \mathrm{Td}_g(TX)\mathrm{ch}_g(E). \tag{0.1}$$

In (0.1), $\mathrm{Td}_g(TX)$ and $\mathrm{ch}_g(E)$ are equivariant versions of the Todd genus of TX and of the Chern character of E. These are cohomology classes on X_g, which depend explicitly on the angles of the action of g on the normal bundle $N_{X_g/X}$ and on $E|_{X_g}$.

There is another cohomological expression for the Lefschetz trace. In fact let $\mathfrak{g}$ be the Lie algebra of G. Then if $K \in \mathfrak{g}$, for $|K|$ small enough, we have a Kirillov like formula

$$L(e^K) = \int_X \mathrm{Td}_K(TX)\mathrm{ch}_K(E). \tag{0.2}$$

In (0.2), $\mathrm{Td}_K(TX)$ and $\mathrm{ch}_K(E)$ are the Todd genus of TX, and the Chern character of E in equivariant cohomology. More generally, if $g \in G$, if $Z(g) \in G$ is the centralizer of g, and if $\mathfrak{z}(g)$ is its Lie algebra, a similar formula expresses $L(ge^K)$ as the integral over X_g of a characteristic class in equivariant cohomology. As explained by Atiyah-Bott [ABo2] and Berline-Vergne in [BeV], the equality of the right-hand sides of (0.1) and (0.2) can be proved directly by using a localization formula, whose proof in differential form version was given by Duistermaat-Heckman [DuH] and Berline-Vergne [BeV]. In [B3], formula (0.2) was proved by a heat equation method.

Let $\lambda_G(E)$ be the inverse of the equivariant determinant of $H^{\cdot}(X,E)$, introduced in [B13]. If G is trivial, then $\lambda_G(E)$ is just the inverse of the determinant of the cohomology. More generally, $\lambda_G(E)$ is the direct sum of the inverses of the determinants of the irreducible representations of G which appear in $H^{\cdot}(X,E)$. In [B13], a Quillen metric was constructed on $\lambda_G(E)$. In fact let $|\,|_{\lambda_G(E)}$ be the L_2 metric on $\lambda_G(E)$ one obtains by identifying $H^{\cdot}(X,E)$ to the corresponding harmonic elements in the Dolbeault complex $(\Omega^{\cdot}(X,E), \overline{\partial}^X)$. Let $\theta(\omega^X, h^E)(g)(s) = -\mathrm{Tr_s}[Ng(D^{X,2})^{-s}], s \in \mathbf{C}, \mathrm{Re}(s) \gg 0$ be the equivariant Ray-Singer torsion. Then in [B13], the Quillen metric $\|\,\|_{\lambda_G(E)}$ is defined by the formula

$$\log(\|\,\|^2_{\lambda_G(E)})(g) = \log(|\,|^2_{\lambda_G(E)})(g) - \frac{\partial}{\partial s}\theta(\omega^X, h^E)(g)(0). \tag{0.3}$$

For a precise interpretation of (0.3), we refer to Section 1.4. When G is trivial, $\theta(\omega^X, h^E)(s)$ is just the standard Ray-Singer analytic torsion introduced in [RS], and $\|\,\|_{\lambda_G(E)}$ is the Quillen metric [Q2, BGS1] on the inverse of the determinant of the cohomology.

Standard Quillen metrics have remarkable properties, which were established by Bismut-Gillet-Soulé [BGS1] and Bismut-Lebeau [BL]. Namely their variation in terms of (ω^X, h^E) can be evaluated in terms of Bott-Chern classes [BoCh, BGS1]. There is a curvature theorem for Quillen metrics [BGS1], which refines the theorem of Riemann-Roch-Grothendieck at the level of differential forms. Also their behaviour under complex immersions has been studied in [BL]. Namely, let $i : Y \to X$ be an embedding of complex manifolds. Let F be a holomorphic vector bundle on Y, and let (E, v) be a holomorphic complex of vector bundles on X which is a resolution of i_*F. Let $\lambda(E), \lambda(F)$ be the determinants of the cohomology of E, F. Then by [KnM], there is a canonical isomorphism $\lambda(E) \simeq \lambda(F)$. In [BL], a local formula has been given for the ratio of corresponding Quillen metrics in terms of the Bott-Chern currents of [BGS2, BGS3] and of the Gillet-Soulé additive genus [GS3] associated to the formal power series $R(x)$ given in terms of the Riemann zeta function $\zeta(s)$ by the formula

$$R(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \left(\sum_{j=1}^{n} \frac{1}{j}\zeta(-n) + \frac{2\partial\zeta}{\partial s}(-n) \right) \frac{x^n}{n!}. \tag{0.4}$$

Using the main result of [BL], Gillet and Soulé [GS4] proved a Riemann-Roch formula in Arakelov geometry for the first Chern class. They had conjectured such a formula in [GS3]. In this formula, the genus R appears as a correction to natural arithmetic characteristic classes. In [B14], the main result of [BL] was extended to the analytic torsion forms of [BK], and Gillet-Soulé [GS5] extended their conjectural Riemann-Roch formula to arbitrary Chern classes. In [F], Faltings suggested an alternative strategy to the proof of this Riemann-Roch formula for arbitrary Chern classes. Also Roessler [Ro] has given a K-theoretic version of this result.

In [B12, B13], the results of [BGS1, BL] were extended to equivariant Quillen metrics. In particular the immersion result of [BL] was extended to the equivariant setting. The natural extension of the R genus of [GS3] to the equivariant setting has been constructed in [B12]. In fact let $L(y,s) = \sum_{n=1}^{+\infty} e^{iny}/n^s$ be the Lerch series, let $\zeta(y,s), \eta(y,s)$ be its real and imaginary parts. Set

$$\begin{aligned} R(\theta, x) = \sum_{\substack{n\geq 0\\ n \text{ even}}} i \left\{ \sum_{j=1}^{n} \frac{1}{j}\eta(\theta, -n) + 2\frac{\partial\eta}{\partial s}(\theta, -n) \right\} \frac{x^n}{n!} \\ + \sum_{\substack{n\geq 0\\ n \text{ odd}}} \left\{ \sum_{j=1}^{n} \frac{1}{j}\zeta(\theta, -n) + 2\frac{\partial\zeta}{\partial s}(\theta, -n) \right\} \frac{x^n}{n!}. \end{aligned} \tag{0.5}$$

Observe that $R(x) = R(0, x)$. A Riemann-Roch formula in equivariant Arakelov geometry has been conjectured in [B12, Section 7 e)], in which the genus $R(\theta, x)$ would appear as a correction to not yet constructed arithmetic characteristic classes. In [B13], the behaviour of equivariant Quillen metrics under immersions was studied. As anticipated in [B12], the genus $R(\theta, x)$ appears in the comparison formula.

In [KRo1], Köhler and Roessler have established the Riemann-Roch formula conjectured in [B12], under the natural assumption that X is an arithmetic variety, equipped with a projective action of Spec $(\mathbf{Z}[t]/(1 - t^n))$. Various applications of this formula have been given by Köhler and Roessler [KRo2] and Kaiser-Köhler [KaK].

A byproduct of the above considerations is that the equivariant Ray-Singer analytic torsion and the associated Quillen metrics provide a refinement on the Lefschetz fixed point formulas of [ABo1] at the level of differential forms.

Let now $p : P \overset{G_{\mathbf{C}}}{\to} S$ be a $G_{\mathbf{C}}$ principal bundle. By [GuSt], $G_{\mathbf{C}}$ acts holomorphically on X and the action lifts to E. Put

$$V = P \times_{G_{\mathbf{C}}} X. \tag{0.6}$$

Let $q : Q \overset{G}{\to} S$ be a G-reduction of P to a G-principal bundle. This reduction induces a connection on P, and therefore a connection on the fibration $\pi : V \overset{X}{\to} S$. Assume that G acts on (X, ω^X) with a moment map μ. One shows easily that the the Kirillov like formulas give a form of the theorem of Riemann-Roch-Grothendieck for the map π. It is also easy to see that $\pi : V \overset{X}{\to} S$ is a Kähler fibration in the sense of [BGS1]. Namely the form ω^X is the restriction to the fibres X of a closed $(1,1)$ form on V.

To the above data, one can then associate the analytic torsion forms $T(\omega^X, h^E)$ of Bismut-Köhler [BK]. These are real forms on S, which are sums of forms of type (p,p) one can attach naturally to any Kähler fibration. Over S, they solve a $\overline{\partial}\partial$ equation, which is a transgression in any degree of the Riemann-Roch-Grothendieck formula at the level of differential forms. The component of degree 0 in $T(\omega^X, h^E)$ is just the Ray-Singer analytic torsion of the fibres X. It turns out that in the above situation, the forms $T(\omega^X, h^E)$ are closed. More precisely, let $\mathfrak{g}$ be the Lie algebra of G, let Θ be the curvature of the given connection on $q : Q \overset{G}{\to} S$. Then we show in Section 2.6 that there is an ad-invariant formal power series $\widetilde{\theta}(\omega^X, h^E)(K)(s), K \in \mathfrak{g}, \mathrm{Re}(s) \gg 0$ such that

$$T(\omega^X, h^E) = \frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)\left(-\frac{\Theta}{2i\pi}\right)(0). \tag{0.7}$$

In Section 2, we prove that $\frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)\,(K)\,(0)$ makes sense as an analytic function defined on a neighborhood of 0 in $\mathfrak{g}$. The proof involves non trivial estimates, established in Section 7. More generally, given $g \in G$ we construct $\frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)\,(g, K)\,(0)$, with K lying in a neighborhood of 0 in $\mathfrak{z}(g)$. Using this function, in Section 2, we define an equivariant Quillen metric in infinitesimal form, and we prove corresponding anomaly formulas and comparison formulas under embeddings.

The infinitesimal equivariant Ray-Singer torsion and the corresponding Quillen metrics provide a refinement on the Kirillov like formulas at the level of differential forms.

As explained before, the purpose of this paper is to compare the equivariant Quillen metric and the equivariant Quillen metric in infinitesimal form.

In [B9], Bismut constructed equivariant Bott-Chern currents which refine the formulas of Duistermaat-Heckman [DuH] and Berline-Vergne [BeV]. Given $K \in \mathfrak{g}$ with $|K|$ small enough, if X_K is the zero set of the associated vector field K^X, a current $S_K(X, \omega^X)$ on X is constructed such that

$$\frac{\overline{\partial}_K \partial_K}{2i\pi} S_K(X, \omega^X) = 1 - \frac{\delta_{X_K}}{c_{\max,\, K}(N_{X_K/X}, h^{N_{X_K/X}})}. \tag{0.8}$$

In (0.8), $\overline{\partial}_K, \partial_K$ are equivariant refinements of the $\overline{\partial}, \partial$ operators, and $c_{\max,K}(N_{X_K/X}, h^{N_{X_K/X}})$ is an equivariant Euler form of the normal bundle $N_{X_K/X}$. Also anomaly formulas were established in [B9] in terms of equivariant Bott-Chern classes, and the behaviour of $S_K(X, \omega^X)$ under complex immersions was studied. More precisely, if $\theta \in \mathbf{R}^*, x \in \mathbf{C}$, put

$$R^\theta(x) = \frac{1}{x + i\theta}\left(2\Gamma'(1) - \log(\theta^2) - \log\left(1 + \frac{x}{i\theta}\right)\right). \tag{0.9}$$

In the main formula in [B9], the additive genus associated to $R^\theta(x)$ appears as a defect in an embedding formula.

To state our main result, we still need to introduce one piece of data, namely the function $I(\theta, \theta', x)$ given by

$$I(\theta, \theta', x) = \sum_{\substack{k \in \mathbf{Z} \\ 2k\pi + \theta \neq 0}} \frac{\log\left(1 + \frac{\theta'}{2k\pi + \theta}\right)}{i(2k\pi + \theta + \theta') + x}. \tag{0.10}$$

We identify $I(\theta, \theta', .)$ to the corresponding additive genus, as explained below.

Take now $g \in G, K_0 \in \mathfrak{z}(g)$. For $z \in \mathbf{R}^*$, put $K = zK_0$. Set $X_{g,K} = X_g \cap X_K$. Let ∇^{TX} be the holomorphic Hermitian connection on TX. Then g and $\nabla^{TX}K^X$ act on $N_{X_K/X}|_{X_{g,K}}$ as locally constant commuting operators, and $\nabla^{TX}_{\cdot}K^{\dot{X}}$ is invertible. Let $e^{i\theta}, 0 \leq \theta \leq 2\pi, i\theta', \theta' \in \mathbf{R}^*$ be the corresponding distinct eigenvalues of $g, \nabla^{TX}_{\cdot}K^X$. Then $N_{X_K/X}|_{X_{g,K}}$ splits as a direct sum of eigenbundles indexed by θ, θ'. Let $I_{g,K}(N_{X_K/X})$ be the characteristic class on $X_{g,K}$ which is the sum, indexed by (θ, θ'), of the corresponding additive genera.

Recall that $K = zK_0$. The main result of this paper is as follows.

Theorem 0.1. *For $z \in \mathbf{R}^*$, if $|z|$ is small enough, the following identity holds*

$$\log\left(\frac{\|\ \|^{\sim}_{\lambda_G(E)}(g, K)}{\|\ \|_{\lambda_G(E)}(ge^K)}\right)^2 = \int_{X_g} \mathrm{Td}_{g,K}(TX, h^{TX})\mathrm{ch}_{g,K}(E, h^E)S_K(X_g, \omega^{X_g}) - \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)I_{g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E). \tag{0.11}$$

We will now put the main result in perspective from the point of view of Quillen metrics and Arakelov geometry. We will also give its relation to results of Goette [Go] on equivariant η-invariants, and finally, we will explain the main techniques used in the paper.

1. Compatibility of Theorem 0.1 to known results on Quillen metrics.

In [B8], it was shown that, formally, the Ray-Singer torsion can be thought of as the integral over the loop space LX of a current $S_K(LX, \omega^{LX})$. This is an elaboration on ideas of Witten and Atiyah [A], who showed that by viewing a manifold X as the submanifold of the loop space LX where the canonical vector field defining the obvious action of S^1 on LX vanishes, at least formally, the Atiyah-Singer index theorem could be obtained from the McKean-Singer formula [MKeS] by a localization formula in equivariant cohomology.

In [B10], it was shown that at least formally, the embedding formula for Quillen metrics [BL] is an infinite dimensional version of the embedding formula for the currents $S_K(X, \omega^X)$established in [B9], this assertion being also valid for the intermediate steps of the proof. The analogy remains valid for the immersion result of

[B13]. In particular, in [B12], extending results in [B10], the following formula was given for $R(\theta, x)$,

$$R(\theta, x) = \sum_{\substack{k\in\mathbf{Z}\\ 2\pi k+\theta\neq 0}} R^{2\pi k+\theta}(x). \tag{0.12}$$

The fact that the genus $R(\theta, x)$ is a special case of the genus which appears in [B9] is an indication which makes the similarity between the results of [B9] and [BL, B13] plausible.

In connection with the above facts, we show in Section 4 that if $\theta \notin 2\pi\mathbf{Z}$,

$$I(\theta, \theta', x) = R(\theta, x + i\theta') - R(\theta + \theta', x), \tag{0.13}$$

and also that

$$I(0, \theta', x) = R(x + i\theta') - R(\theta', x) + R^{\theta'}(x). \tag{0.14}$$

Equations (0.13), (0.14) allow us to show in Section 5 that Theorem 0.1 is compatible with the embedding formulas of [BL, B13, B9].

Also since the embedding formulas of [BL, B13] are one of the blocks leading to the proof of a Riemann-Roch-Grothendieck formula in Arakelov geometry in [GS4, GS5] and in [KRo1], in Section 5, we are led to speculate on an evaluation of the current $S_K(X, \omega^X)$ in terms of arithmetic characteristic classes.

2.Equivariant eta invariants and equivariant analytic torsion.

In [Go], Goette arrived at a similar class of problems on equivariant eta invariants from a different point of view. In [D], extending earlier work by Atiyah-Patodi-Singer [APS] on the index theorem for manifolds with boundary, Donnelly proved a Lefschetz formula for manifolds with boundary. The contribution of the boundary is expressed as the equivariant η-invariant of the boundary. On the other hand, in [BC], Bismut and Cheeger proved a families index theorem for even dimensional manifolds with boundary. The contribution of the boundary is an even form $\widetilde{\eta}$, whose component in degree 0 is just the eta invariant of the boundary.

Assume now that the family of manifolds with boundary comes from a G-fibration. Along the lines given above, one then obtains two formulas for $L(e^K)$, the first formula being the one given in [D], and involving the equivariant eta invariant of the boundary, the second one coming from [BC], and involving an infinitesimal version of the eta invariant. These two formulas being equal tautologically, one then obtains a formula only involving the boundary, relating the equivariant eta invariant to its infinitesimal form in terms of a Chern-Simons current on the boundary.

In [Go], this relation was extended to manifolds which are not equivariantly cobordant, but only as an identity of formal power series in K, when the corresponding vector field is nowhere vanishing.

Our work can be considered as an extension of [Go] in the sense that the group G always has fixed points, since it acts on X with a moment map, and also because we deal with a more complicated sort of invariant, the analytic torsion. In principle our techniques should lead to a general proof of the result established in [Go]. No exotic class like $I(\theta, \theta', x)$ should appear in the final formula.

Our results do in fact bear some direct relation to corresponding results on eta invariants via the holonomy theorem of Bismut-Freed [BF].

3.The analytic techniques used in the proof.

Now we explain briefly some of the analytic difficulties in our proof of Theorem 0.1. The general outline of the proof is similar to the proof of corresponding results in [BL, B13, B14]. Namely, in Section 6, we produce a closed 1-form $\gamma_{t,v}$ on $\mathbf{R}_+^* \times \mathbf{R}_+^*$, whose integral on a lower triangular contour gives 0. By pushing the boundary of the contour to $+\infty$ in the v-direction and to 0 in the t-direction, we obtain our formula.

Some analytic techniques are inspired from [BL, B13]. Namely we combine the Getzler rescaling in local index theory [Ge, BeGeV] with Lax-Milgram and commutator techniques to obtain the required estimates. As in [BL, B13] finite propagation speed of solutions of hyperbolic equations [ChP, T] plays an important role, in order to show that the required estimates can be adequately localized, and to obtain Gaussian decay of certain rescaled kernels in directions normal to submanifolds like X_g or $X_{g,K}$.

Still there is a new difficulty with respect to the above references. In fact, observe that the Kirillov like formula given in (0.2) only makes sense for $|K|$ small enough. Also our main formula (0.11) in Theorem 0.1 is true only for $|z|$ small enough. More ominously, (0.11) only makes sense for $|z|$ small enough. This already indicates that the proofs themselves, and more precisely the required estimates, will only be valid for $|z|$ small enough.

Now, we will explain where the difficulty appears in the proof. Let $D^X = \overline{\partial}^X + \overline{\partial}^{X*}$ be the Hodge-Dolbeault operator acting on the Dolbeault complex $\Omega^{\cdot}(X,E) = C_\infty\left(\Lambda(T^{*(0,1)}X) \otimes E\right)$. Recall that $\Lambda(T^{*(0,1)}X) \otimes E$ is a $T_{\mathbf{R}}X$-Clifford module. Let $c(K^X)$ be the Clifford multiplication operator. Then D^X is a self-adjoint operator, and $c(K^X)$ is a skew-adjoint operator. For $T \in \mathbf{R}$, put

$$D_T^X = D^X + T\frac{c(K^X)}{2\sqrt{2}}. \tag{0.15}$$

Then

$$D_T^{X,2} = \left(D^{X,2} - T^2\frac{|K^X|^2}{8}\right) + T\left[D^X, \frac{c(K^X)}{2\sqrt{2}}\right]. \tag{0.16}$$

In (0.16), $D^{X,2} - T^2\frac{|K^X|^2}{8}$ is a second order self-adjoint elliptic operator with a lower bound which tends to $-\infty$ as $|T| \to +\infty$, and $T\left[D^X, \frac{c(K^X)}{2\sqrt{2}}\right]$ is a first order skew-adjoint operator. These two facts work against us in the course of the proof.

To make the difficulty more concrete, consider the following toy model, which appears in a limit situation, after adequate rescaling of the coordinates. Let A be a skew-adjoint matrix acting on $\mathbf{C} \simeq \mathbf{R}^2$, with eigenvalue $i\theta$, with $\theta \in \mathbf{R}^*$. Let Z be the generic element of $\mathbf{R}^2$. Let e_1, e_2 be an orthonormal basis of $\mathbf{R}^2$. For $x \in \mathbf{R}, y \in \mathbf{R}_+^*$, consider the differential operator

$$\mathcal{L} = -\frac{1}{2}\left(\nabla_{e_i} + x\frac{\langle AZ, e_i\rangle}{2}\right)^2 + \frac{y}{2}|AZ|^2. \tag{0.17}$$

Then

$$\begin{aligned} \operatorname{Re}(\mathcal{L}) &= -\frac{1}{2}\Delta + \left(\frac{4y - x^2}{8}\right)|AZ|^2, \\ \operatorname{Im}(\mathcal{L}) &= -\frac{x}{2}\nabla_{AZ}. \end{aligned} \tag{0.18}$$

By (0.18), we see that if $x^2 - 4y > 0$, the operator $\mathrm{Re}(\mathcal{L})$, while having a self-adjoint extension, is not lower bounded. In this case, it is not possible to define a priori a honest heat kernel for $e^{-\mathcal{L}}$. A natural way to overcome the difficulty is to replace A by zA, with $z \in \mathbf{C}$. When $z = i$, the operator $\mathcal{L}$ becomes a honest self-adjoint operator, which has a lower bound. Its heat kernel $p_1(Z, Z')$ with respect to $dZ'/(2\pi)$ depends analytically on $z \in \mathbf{C}$ on a neighborhood of 0. For $z \in \mathbf{C}$ and $|z|$ small enough, it is then possible to define $p_1(Z, Z')$ as an analytic function of A, for $|A|$ small enough. In particular,

$$p_1(0,0) = \frac{\sqrt{x^2 - 4y}\,\theta/2}{\sin\left(\sqrt{x^2 - 4y}\,\theta/2\right)}, \tag{0.19}$$

which is well defined for θ small enough.

In our geometric context, the analogue of the condition $x^2 - 4y > 0$ is always verified. However we are not allowed to change A into iA as we did before, this transformation destroying the geometry of the situation. In our paper, we use another method, which we briefly describe on the toy model we just considered. The idea is to express the function $\exp(-a^2)$ as an infinite sum of functions whose Fourier transform have compact support. Using finite propagation speed for solutions of hyperbolic equations [ChP, T], we write the heat kernel $p_1(Z, Z')$ as an infinite sum of kernels which only see the operator $\mathcal{L}$ on a bounded domain, on which $\mathcal{L}$ has a lower bound. The question is then to prove that the above infinite sum converges. This last fact can be proved by abstract functional analytic methods, for $|\theta|$ small enough. The point is that the above method can still be used in our geometric context. This is why we often express our kernels as infinite sums of kernels evaluated on truncated operators.

In [Go], where equivariant η-invariants were considered, the above difficulties were overcome using an involved power series argument partly inspired from [BeGeV].

Our paper is organized as follows. In Section 1, we recall the construction in [B13] of the equivariant holomorphic torsion and of the corresponding Quillen metrics, and we state its main properties. In Section 2, we construct the equivariant infinitesimal analytic torsion forms, and we describe the properties of the corresponding Quillen metrics. The proof of the estimates which are needed in the construction is deferred to Section 7. In Section 3, we recall the construction in [B9] of the equivariant Bott-Chern currents $S_K(X, \omega^X)$.

In Section 4, we introduce the genus $I(\theta, \theta', x)$, which we will obtain through evaluations of characteristic classes involving harmonic oscillators. In the proof of our main formula, the genus $I(\theta, \theta', x)$ will appear precisely via these characteristic classes. This Section is closely related to earlier work in [B7, B12], where the genera $R(x)$ and $R(\theta, x)$ were also obtained as characteristic classes involving harmonic oscillators.

In Section 5, we verify the compatibility of Theorem 0.1 to the immersion formulas for Quillen metrics, and also to the immersion formula of [B9] for Bott-Chern currents. We also speculate on an expression of $S_K(X, \omega^X)$ in terms of arithmetic classes, in connection of the Riemann-Roch formula of Köhler and Roessler [KRo1].

In Section 6, we prove Theorem 0.1. The proof relies on two intermediate results, the proof of which is deferred to Sections 8-9.

In Section 7, we prove the estimates which are needed to show the existence of the infinitesimal equivariant torsion. The proof is given in much detail, since most arguments will be used in a more complicate context in Sections 8 and 9.

Sections 8 and 9 are devoted to the proof of the intermediate results mentioned above. We have tried to give as many details as necessary, still referring when necessary to [BL, B13] to avoid tedious repetitions.

We assume the reader to be somewhat familiar with the superconnection formalism of Quillen [Q1]. In particular, if $\mathcal{A}$ is a $\mathbf{Z}_2$-graded algebra, if $A, B \in \mathcal{A}$,then $[A, B]$ denotes the supercommutator of A and B. Also $\mathrm{Tr_s}$ is our notation for the supertrace.

The results contained in this paper have been announced in [BGo].

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1. The equivariant holomorphic analytic torsion

In this Section, we recall the main results in [B12, B13] on the equivariant Quillen metrics. In particular, we introduce the $R(x)$ genus of Gillet-Soulé [GS3, GS4] and its equivariant extension $R(\theta, x)$ introduced in [B12].

This Section is organized as follows. In Section 1.1, we briefly recall various properties of Clifford algebras. In Section 1.2, we introduce the equivariant Quillen metrics. In Section 1.3, we give the asymptotics of certain supertraces. In Section 1.4, we state the anomaly formulas for equivariant Quillen metrics. Finally in Section 1.6, we state the main result of [B13], which describes the behaviour of the equivariant Quillen metrics under complex immersions.

1.1. **Complex vector spaces and Clifford algebras.** Let V be a finite dimensional complex Hermitian vector space of dimension ℓ, let $V_{\mathbf{R}}$ be the underlying real vector space. We denote by $\langle\ \rangle$ the Hermitian product on V or the associated scalar product on $V_{\mathbf{R}}$. Let $J \in \operatorname{End}(V_{\mathbf{R}})$ be the corresponding complex structure. Then J is antisymmetric, and $J^2 = -1$. Let $c(V_{\mathbf{R}})$ be the Clifford algebra of $V_{\mathbf{R}}$, i.e. the algebra spanned over $\mathbf{R}$ by $1, X \in V_{\mathbf{R}}$ and the relations for $X, Y \in V_{\mathbf{R}}$,

$$XY + YX = -2\langle X, Y\rangle. \tag{1.1}$$

Then $\Lambda(V^{*,(0,1)})$ is a $c(V_{\mathbf{R}})$-Clifford module. Namely if $X \in V$, let $\overline{X} \in \overline{V}$ be the conjugate of X, and let $X^* \in \overline{V}^*$ correspond to X by the Hermitian product $\langle\ \rangle$. If $X \in V$, set

$$c(X) = \sqrt{2}X^*\wedge,\ c(\overline{X}) = -\sqrt{2}i_{\overline{X}}. \tag{1.2}$$

Then the operators in (1.2) act on $\Lambda(V^{*(0,1)})$. We extend (1.2) to a linear map from $V_{\mathbf{R}} \otimes \mathbf{C}$ into $\operatorname{End}\big(\Lambda(V^{*(0,1)})\big)$. This map extends to a map from $c(V_{\mathbf{R}}) \otimes \mathbf{C}$ into $\operatorname{End}\big(\Lambda(V^{*(0,1)})\big)$, i.e. $\Lambda(V^{*(0,1)})$ is now a $c(V_{\mathbf{R}})$ module.

If $A \in \operatorname{End}(V)$, then A acts naturally as a derivation on $\Lambda\big(V^{*,(0,1)}\big)$. More precisely, let $e_1, \dots, e_{2\ell}$ be an orthonormal basis of $V_{\mathbf{R}}$. Then

$$A|_{\Lambda(V^{*,(0,1)})} = \frac{1}{4}\langle Ae_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\operatorname{Tr}[A]. \tag{1.3}$$

1.2. **Equivariant Quillen metrics.** Here we follow [B13]. Let X be a complex manifold of dimension ℓ. Let E be a holomorphic vector bundle on X.

Let G be a compact Lie group, and let $\mathfrak{g}$ be its Lie algebra. We assume that G acts on the left on X by holomorphic diffeomorphisms, and that this action lifts holomorphically to E. Then G acts on $H^{\cdot}(X, E)$.

Let $(\Omega^{\cdot}(X, E), \overline{\partial}^X)$ be the Dolbeault complex of smooth sections of $\Lambda(T^{*(0,1)}X)\otimes E$ on X. Then

$$H^{\cdot}(\Omega^{\cdot}(X, E), \overline{\partial}^X) \simeq H^{\cdot}(X, E). \tag{1.4}$$

Clearly G acts on $(\Omega^{\cdot}(X, E), \overline{\partial}^X)$ by the formula

$$(gs)(x) = g_* s(g^{-1}x), \tag{1.5}$$

and (1.4) is an identity of G-spaces.

Let h^{TX}, h^E be smooth G-invariant Hermitian metrics on TX, E. Let dv_X be the corresponding volume form on X. Let $\langle , \rangle_{\Lambda(T^{*(0,1)}X)\otimes E}$ be the corresponding

Hermitian product on $\Lambda(T^{*(0,1)}X) \otimes E$. If $s, s' \in \Omega^{\cdot}(X, E)$, put

$$\langle s, s' \rangle = \int_X \langle s, s' \rangle_{\Lambda(T^{*(0,1)}X)\otimes E} \frac{dv_X}{(2\pi)^{\dim X}}. \tag{1.6}$$

Then (1.6) is a G-invariant Hermitian product on $\Omega^{\cdot}(X, E)$.

Let $\overline{\partial}^{X*}$ be the formal adjoint of $\overline{\partial}^X$ with respect to (1.6). Put

$$D^X = \overline{\partial}^X + \overline{\partial}^{X*}. \tag{1.7}$$

Then D^X is a first order elliptic operator. By Hodge theory,

$$\ker D^X \simeq H^{\cdot}(X, E). \tag{1.8}$$

Also D^X commutes with G, so that G acts on $\ker D^X$. Then (1.8) is an identification of G-spaces. Also $\ker D^X$ inherits a G invariant Hermitian product from the Hermitian product (1.6) on $\Omega^{\cdot}(X, E)$. Let $h^{H^{\cdot}(X,E)}$ be the corresponding Hermitian metric on $H^{\cdot}(X, E)$.

Let $\widehat{G}$ be the set of equivalence classes of irreducible representations of G. If $W \in \widehat{G}$, let χ_W be the character of G associated to W. Then we have the isotypical decomposition

$$H^{\cdot}(X, E) = \oplus_{W \in \widehat{G}} \mathrm{Hom}_G(W, H^{\cdot}(X, E)) \otimes W, \tag{1.9}$$

which is orthogonal with respect to $h^{H^{\cdot}(X,E)}$. If $W \in \widehat{G}$, put

$$\lambda_W(E) = (\det(\mathrm{Hom}_G(W, H^{\cdot}(X, E)) \otimes W))^{-1}. \tag{1.10}$$

Then $\lambda_W(E)$ is a complex line. Set

$$\lambda_G(E) = \oplus_{W \in \widehat{G}} \lambda_W(E). \tag{1.11}$$

Let $| \; |_{\lambda_W(E)}$ be the metric induced by $h^{H^{\cdot}(X,E)}$ on $\lambda_W(E)$.

Definition 1.1. Set

$$\log\left(| \; |^2_{\lambda_G(E)}\right) = \sum_{W \in \widehat{G}} \log(| \; |^2_{\lambda_W(E)}) \frac{\chi_W}{\mathrm{rk}(W)}. \tag{1.12}$$

The symbol $| \; |^2_{\lambda_G(E)}$ will be called an equivariant L_2 metric on $\lambda_G(E)$.

Let $\ker(D^X)^{\perp}$ be the orthogonal vector space to $\ker(D^X)$ in $\Omega^{\cdot}(X, E)$. Then $D^{X,2}$ acts as an invertible operator on $\ker(D^X)^{\perp}$. Let $(D^{X,2})^{-1}$ denote the inverse of $D^{X,2}$ acting on $\ker(D^X)^{\perp}$.

Take $g \in G$. Let X_g be the fixed point set of g in X. Then X_g is a complex totally geodesic submanifold of X.

Let N be the number operator of $\Omega^{\cdot}(X, E)$, i.e. N acts by multiplication by k on $\Omega^k(X, E)$. By standard heat equation methods [Gi], [BeGeV], we know that as $t \to 0$, for any $k \in \mathbf{N}$,

$$\mathrm{Tr_s}[Ng \exp(-tD^{X,2}] = \frac{a_\ell}{t^\ell} + \ldots a_0 + a_1 t + \ldots a_k t^k + o(t^k). \tag{1.13}$$

Definition 1.2. For $g \in G, s \in \mathbf{C}, \mathrm{Re}(s) > \ell$, put

$$\theta(\omega^X, h^E)(g)(s) = -\mathrm{Tr_s}[Ng(D^{X,2})^{-s}]. \tag{1.14}$$

By (1.13),$\theta(\omega^X, h^E)(g)(s)$ extends to a meromorphic function of $s \in \mathbf{C}$, which is holomorphic at $s = 0$. In particular, $g \in G \mapsto \frac{\partial}{\partial s}\theta(\omega^X, h^E)(g)(0) \in \mathbf{C}$ is a central function. When $g = 1$, it was introduced by Ray and Singer [RS]. This function is called the Ray-Singer equivariant analytic torsion.

Definition 1.3. For $g \in G$, put

$$\log(\| \ \|^2_{\lambda_G(E)})(g) = \log(| \ |^2_{\lambda_G(E)})(g) - \frac{\partial}{\partial s}\theta(\omega^X, h^E)(g)(0). \tag{1.15}$$

The symbol $\| \ \|_{\lambda_G(E)}$ will be called an equivariant Quillen metric on $\lambda_G(E)$.

1.3. **The asymptotics of certain supertraces.**

Let J^{TX} be the complex structure of $T_{\mathbf{R}}X$. Let ω^X be the Kähler form associated to the metric h^{TX}, so that if $U, V \in T_{\mathbf{R}}X$,

$$\omega^X(U, V) = \langle U, J^{TX}V\rangle. \tag{1.16}$$

In this subsection, we assume that the metric h^{TX} is Kähler, i.e. the form ω^X is closed. In the sequel TX will denoted the complex tangent bundle to X, and $T_{\mathbf{R}}X$ the corresponding real tangent bundle. A similar notation will be used for other complex vector bundles.

Let ∇^{TX}, ∇^E be the holomorphic Hermitian connections on (TX, h^{TX}), (E, h^E), and let R^{TX}, R^E be their curvatures. Let $\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$ be the corresponding connection on $\Lambda(T^{*(0,1)}X) \otimes E$. By [H], since h^{TX} is a Kähler metric, $\sqrt{2}D^X$ is a Dirac operator. In fact recall that by Section 1.1, $\Lambda(T^{*(0,1)}X) \otimes E$ is a $c(T_{\mathbf{R}}X)$-Clifford module. Let $e_1, \dots, e_{2\ell}$ be an orthonormal basis of $T_{\mathbf{R}}X$. Then D^X is given by the formula

$$D^X = \sum_1^{2\ell} \frac{c(e_i)}{\sqrt{2}} \nabla_{e_i}^{\Lambda(T^{*(0,1)}X)\otimes E}. \tag{1.17}$$

Take $g \in G$. Then

$$TX_g = \{U \in TX|_{X_g}, \ gU = U\}. \tag{1.18}$$

Let $N_{X_g/X}$ be the normal bundle to X_g in X. Then g acts on $N_{X_g/X}$. Let $e^{i\theta_1}, \dots, e^{i\theta_q}$, $(0 < \theta_j < 2\pi)$ be the locally constant distinct eigenvalues of g acting on $N_{X_g/X}$, and let $N^{\theta_1}_{X_g/X}, \dots, N^{\theta_q}_{X_g/X}$ be the corresponding eigenbundles. Then $N_{X_g/X}$ splits holomorphically as

$$N_{X_g/X} = \oplus_{1\le j\le q} N^{\theta_j}_{X_g/X}. \tag{1.19}$$

Also we have the holomorphic splitting

$$TX = TX_g \oplus N_{X_g/X}. \tag{1.20}$$

Moreover the splittings (1.19) and (1.20) are orthogonal. Let $h^{TX_g}, h^{N^{\theta_1}_{X_g/X}} \dots$ be the Hermitian metrics induced by h^{TX} on $TX_g, N^{\theta_1}_{X_g/X} \dots$. Then $\nabla^{TX}|_{X_g}$ induces the holomorphic Hermitian connections $\nabla^{TX_g}, \nabla^{N^{\theta_1}_{X_g/X}} \dots$ on (TX_g, h^{TX_g}), $(N^{\theta_1}_{X_g/X}, h^{N^{\theta_1}_{X_g/X}}) \dots$. Let $R^{TX_g}, R^{N^{\theta_1}_{X_g/X}} \dots$ be their curvatures.

Definition 1.4. Let P^X be the vector space of smooth forms on X, which are sums of forms of type (p, p). Let $P^{X,0}$ be the subspace of the $\omega \in P^X$ such that there exist smooth forms α, β on X with $\omega = \partial\alpha + \overline{\partial}\beta$.

If A is a (q,q) matrix, put

$$\mathrm{Td}(A)=\det\left(\frac{A}{1-e^{-A}}\right),\ \mathrm{ch}(A)=\mathrm{Tr}[\exp(A)],\ c_{\max}(A)=\det(A). \tag{1.21}$$

The genera associated to Td, ch and $c_{\max}$ are called the Todd genus, the Chern character and the maximal Chern class.

Definition 1.5. Set

$$\begin{aligned}
\mathrm{Td}_g(TX,h^{TX}) &= \mathrm{Td}(\frac{-R^{TX_g}}{2i\pi})\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{c_{\max}}\right)\left(\frac{-R^{N^{\theta_j}_{X_g/X}}}{2i\pi}+i\theta_j\right),\\
\mathrm{Td}'_g(TX,h^{TX}) &= \frac{\partial}{\partial b}\left[\mathrm{Td}\left(\frac{-R^{TX_g}}{2i\pi}+b\right)\right.\\
&\times \left.\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{c_{\max}}\right)\left(\frac{-R^{TX_g}}{2i\pi}+i\theta_j+b\right)|_{b=0}\right],\\
(\mathrm{Td}_g^{-1})'(TX,h^{TX}) &= \frac{\partial}{\partial b}\left[\mathrm{Td}^{-1}\left(\frac{-R^{TX_g}}{2i\pi}+b\right)\right.\\
&\times \left.\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{c_{\max}}\right)^{-1}\left(\frac{-R^{N^{\theta_j}_{X_g/X}}}{2i\pi}+i\theta_j+b\right)|_{b=0}\right],\\
\mathrm{ch}_g(E,h^E) &= \mathrm{Tr}\left[g\exp\left(\frac{-R^E|_{X_g}}{2i\pi}\right)\right].
\end{aligned} \tag{1.22}$$

The forms in (1.22) are closed forms on X_g and lie in P^{X_g}, and their cohomology class does not depend on the G-invariant metrics h^{TX}, h^E. We denote by $\mathrm{Td}_g(TX)$, $\mathrm{Td}'_g(TX),\dots,\mathrm{ch}_g(E)$ these cohomology classes, two of which appear in the Lefschetz formulas of Atiyah-Bott[ABo1]. In fact, if $g\in G$, put

$$L(g)=\mathrm{Tr_s}^{H^{\cdot}(X,E)}[g]. \tag{1.23}$$

Then the Lefschetz formulas of [ABo1] assert that

$$L(g)=\int_{X_g}\mathrm{Td}_g(TX)\mathrm{ch}_g(E). \tag{1.24}$$

Now we state a result from [B13, Theorem 8.3].

Theorem 1.6. *There exist* $D_{-1}(g), D_0(g)\dots$ *such that as* $t\to 0$,

$$\mathrm{Tr_s}[N\exp(-tD^{X,2})]=\frac{D_{-1}(g)}{t}+D_0(g)+\ldots+D_k(g)t^k+\mathcal{O}(t^k). \tag{1.25}$$

In particular,

$$\begin{aligned}
D_{-1}(g) &= \int_{X_g}\frac{\omega^X}{2\pi}\mathrm{Td}_g(TX)\mathrm{ch}_g(E),\\
D_0(g) &= \int_{X_g}\mathrm{Td}_g(TX)\left(\dim(X)-\left(\frac{\mathrm{Td}'}{\mathrm{Td}}\right)_g(TX)\right)\mathrm{ch}_g(E).
\end{aligned} \tag{1.26}$$

Moreover there is $c>0$ *such that as* $t\to+\infty$,

$$\mathrm{Tr_s}[Ng\exp(-tD^{X,2})]=\mathrm{Tr_s}^{H^{\cdot}(X,E)}[Ng]+\mathcal{O}(e^{-ct}). \tag{1.27}$$

Proposition 1.7. *The following identity holds,*

(1.28)
$$\begin{aligned}\frac{\partial}{\partial s}\theta(\omega^X,h^E)(g)(0) = -\int_0^1 \left(\mathrm{Tr_s}\left[Ng\exp(-tD^{X,2})\right] - \frac{D_{-1}(g)}{t} - D_0(g)\right)\frac{dt}{t}\\ -\int_1^{+\infty}\left(\mathrm{Tr_s}\left[Ng\exp(-tD^{X,2})\right] - \mathrm{Tr_s}{}^{H^{\cdot}(X,E)}[Ng]\right)\frac{dt}{t} + \\ D_{-1}(g) + \Gamma'(1)(D_0(g) - \mathrm{Tr_s}{}^{H^{\cdot}(X,E)}[Ng]).\end{aligned}$$

Proof. This follows easily from (1.25), (1.27). ☐

1.4. **Anomaly formulas for equivariant Quillen metrics.** Let $(h^{'TX},h^{'E})$ be another couple of G-invariant metrics on TX,E. We denote with a $'$ the objects which we just considered which are attached to $h^{'TX},h^{'E}$.

By [BGS1, Part I], there are uniquely defined classes $\widetilde{\mathrm{Td}}_g(TX,h^{TX},h^{'TX})$ and $\widetilde{\mathrm{ch}}_g(E,h^E,h^{'E})$ in $P^{X_g}/P^{X_g,0}$ such that

$$\begin{aligned}\frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}_g(TX,h^{TX},h^{'TX}) &= \mathrm{Td}_g(TX,h^{'TX}) - \mathrm{Td}_g(TX,h^{TX}),\\ \frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{ch}}_g(E,h^E,h^{'E}) &= \mathrm{ch}_g(E,h^{'E}) - \mathrm{ch}_g(E,h^E).\end{aligned} \tag{1.29}$$

Now we recall a result in [B13, Theorem 2.5].

Theorem 1.8. *Assume that the metrics h^{TX} and $h^{'TX}$ are Kähler. Then*

$$\begin{aligned}\log\left(\frac{\|\ \|'_{\lambda_G(E)}}{\|\ \|_{\lambda_G(E)}}\right)^2 = \int_{X_g}\widetilde{\mathrm{Td}}_g(TX,h^{TX},h^{'TX})\mathrm{ch}_g(E,h^E)\\ + \int_{X_g}\mathrm{Td}(TX,h^{'TX})\widetilde{\mathrm{ch}}_g(E,h^E,h^{'E}).\end{aligned} \tag{1.30}$$

1.5. The genera R.

Definition 1.9. For $\theta\in\mathbf{R}^*, x\in\mathbf{C}$, put

$$R^\theta(x) = \frac{1}{x+i\theta}\left(2\Gamma'(1) - \log(\theta^2) - \log\left(1+\frac{x}{i\theta}\right)\right). \tag{1.31}$$

Let $\zeta(y,s),\eta(y,s)$ be the real and imaginary parts of the Lerch series,

$$\begin{aligned}\zeta(y,s) &= \sum_{n=1}^{+\infty}\frac{\cos(ny)}{n^s},\\ \eta(y,s) &= \sum_{n=1}^{+\infty}\frac{\sin(ny)}{n^s}.\end{aligned} \tag{1.32}$$

Then if $y\notin 2\pi\mathbf{Z}$, $s\mapsto\zeta(y,s)(y,s)$ extends to a holomorphic function of $s\in\mathbf{C}$, if $y\in 2\pi\mathbf{Z}$, $s\mapsto\zeta(y,s)$ extends to a meromorphic function on $\mathbf{C}$ with a simple pole at $s=1$. Also $s\mapsto\eta(y,s)$ extends to a holomorphic on $\mathbf{C}$.

Definition 1.10. For $\theta \in \mathbf{R}, x \in \mathbf{C}, |x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$ if $\theta \notin 2\pi\mathbf{Z}$, set

$$R(\theta, x) = \sum_{\substack{n\geq 0\\ n \text{ even}}} i \left\{ \sum_{j=1}^{n} \frac{1}{j}\eta(\theta, -n) + 2\frac{\partial \eta}{\partial s}(\theta, -n) \right\} \frac{x^n}{n!} + \sum_{\substack{n\geq 0\\ n \text{ odd}}} \left\{ \sum_{j=1}^{n} \frac{1}{j}\zeta(\theta, -n) + 2\frac{\partial \zeta}{\partial s}(\theta, -n) \right\} \frac{x^n}{n!}. \tag{1.33}$$

By [B12, Theorems 7.2 and 7.8], the series (1.33) is uniformly convergent on its domain of definition. Now we recall the definition of the series $R(x)$ by Gillet and Soulé [GS3]. Let $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ be the Riemann zeta function.

Definition 1.11. For $x \in \mathbf{C}, |x| < 2\pi$, set

$$R(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \left(\sum_{j=1}^{n} \frac{1}{j}\zeta(-n) + \frac{2\partial\zeta}{\partial s}(-n) \right) \frac{x^n}{n!}. \tag{1.34}$$

Clearly

$$R(0, x) = R(x). \tag{1.35}$$

Definition 1.12. For $x \in \mathbf{C}$, $|x| < 2\pi$, set

$$\rho(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} 2\zeta'(-n)\frac{x^n}{n!}. \tag{1.36}$$

Now we recall results from [B12, Theorems 7.9, 7.10 and 7.13].

Theorem 1.13. *For $\theta \in \mathbf{R}, x \in \mathbf{C}, |x| < 2\pi$ if $\theta \in 2\pi\mathbf{Z}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$if $\theta \notin 2\pi\mathbf{Z}$, then $R(\theta, x)$ is given by*

$$R(\theta, x) = \sum_{\substack{k\in\mathbf{Z}\\ 2k\pi+\theta\neq 0}} R^{2k\pi+\theta}(x). \tag{1.37}$$

If $\theta \in]-2\pi, 2\pi[\setminus\{0\}$, if $x \in \mathbf{C}$, $|x| < \inf_{k\in\mathbf{Z}} |\theta + 2k\pi|$, then

$$R(\theta, x) = R(x + i\theta) - \sum_{k\in\mathbf{Z}^*} \frac{\log(1 + \theta/2k\pi)}{2ik\pi + i\theta + x} + R^\theta(x). \tag{1.38}$$

Also for $\theta \in \mathbf{R}$,

$$R(\theta, 0) = \frac{2i\partial\eta}{\partial s}(\theta, 0)\,, \qquad R(\theta, 0) = \rho(i\theta) + \frac{2\Gamma'(1) - \log(\theta^2)}{i\theta}. \tag{1.39}$$

In the sequel, we identify $R(\theta, x), R(x)$ to the corresponding additive genera.

1.6. **Equivariant immersions and Quillen metrics.** Let $i : Y \mapsto X$ be an embedding of complex manifolds, let $N_{Y/X}$ be the normal bundle to Y in X. We assume that G also acts holomorphically on Y. Let F be a holomorphic vector bundle on Y, let

$$(E,v) : 0 \to E_m \underset{v}{\to} E_{m-1} \dots \to \dots \underset{v}{\to} E_0 \to 0. \tag{1.40}$$

be a G-complex of holomorphic vector bundles on X, which, together with a holomorphic restriction map $r : E_0|_Y \mapsto F$ provides a G-equivariant resolution of $i_*(\mathcal{O}_Y(F))$, i.e. we have the exact sequence of sheaves

$$0 \to \mathcal{O}_X(E_m) \underset{v}{\to} \mathcal{O}_X(E_{m-1}) \dots \to \dots \underset{v}{\to} \mathcal{O}_X(E_0) \to i_*\mathcal{O}_Y(F) \to 0. \tag{1.41}$$

In particular, the complex (E,v) is acyclic on $X \setminus Y$.

Let $N_{Y/X}$ be the normal bundle to Y. If $y \in Y$, let $H_y(E,v)$ be the homology of the complex $(E,v)_y$. If $y \in Y, u \in N_{Y/X}$, let $\partial_u v$ be the derivative of v in any holomorphic trivialization of E near y. Then using the local uniqueness of resolutions [Se], the following results were proved in [B6].

- The $H_y(E,v)$ are the fibres of a holomorphic $\mathbf{Z}$ -graded vector bundle $H((E,v))$ on Y. The map $\partial_u v$ acts on $H_y(E,v)$ as a chain map, and this action does not depend on the trivialization of (E,v) near y, and only depends on the image z of u in $N_{Y/X}$. From now on, we will write $\partial_z v$ instead of $\partial_u v$.
- Let π be the projection $N_{Y/X} \mapsto Y$. Then over $N_{Y/X}$, we have a canonical identification of $\mathbf{Z}$-graded chain complexes

$$(\pi^* H(E,v), \partial_z v) \simeq (\pi^* \left(\Lambda(N^*_{Y/X}) \otimes E\right), \sqrt{-1} i_z). \tag{1.42}$$

Let $H^\cdot(X,E)$ be the hypercohomology of E. Then by [B13, Section 3], we have the canonical isomorphism of G-spaces

$$H^\cdot(X,E) \simeq H^\cdot(Y,F). \tag{1.43}$$

If, given $W \in \widehat{G}$, λ_W, μ_W are complex lines, if $\lambda = \oplus_W \lambda_W, \mu = \oplus_{W\in\widehat{G}} \mu_W$, put

$$\lambda^{-1} = \oplus_{W\in\widehat{G}} \lambda^{-1}, \ \lambda \otimes \mu = \oplus_{W\in\widehat{G}} \left(\lambda_W \otimes \mu_W\right). \tag{1.44}$$

Now, we construct the $\lambda_G(E_i), 1 \leq i \leq m$, $\lambda_G(F)$ as in Section 1.2. Put

$$\lambda_G(E) = \otimes_{1\leq i\leq m} \left(\lambda_G(E_i)\right)^{(-1)^i}. \tag{1.45}$$

Then in [B13, eq. (3.15)], using in particular(1.43), one shows easily that we have a canonical isomorphism

$$\lambda_G(E) \simeq \lambda_G(F). \tag{1.46}$$

Let $h^E = \oplus_{1\leq i\leq m} h^{E_i}, h^F$ be G-invariant Hermitian metrics on $E = \oplus_{\leq i\leq m} E_i, F$. Let $h^{N_{Y/X}}$ be a G-invariant Hermitian metric on $N_{Y/X}$. By finite dimensional Hodge theory , the $\mathbf{Z}$ -graded vector bundle $H(E,v)$ inherits a G-invariant metric $h^{H(E,v)}$ from the metric h^E. Also the metrics $h^F, h^{N_{Y/X}}$ induce a Hermitian metric on $\Lambda(N^*_{Y/X}) \otimes F$.

We will say that the metric h^E verify assumption (A) with respect to the metrics $h^F, h^{N_{Y/X}}$ if the identification (1.42) is an isometry. By [B6, Proposition 1.6] and [B13, Proposition 3.5], given G-invariant Hermitian metrics $h^F, h^{N_{Y/X}}$ on $F, N_{Y/X}$, there exist a G-invariant Hermitian metric $h^E = \oplus_{1\leq i\leq m} h^{E_i}$ which verifies assumption (A) with respect to $h^F, h^{N_{Y/X}}$.

Let now h^{TX}, h^{TY} be G-invariant Kähler metrics on TX, TY. We identify the normal bundle $N_{Y/X}$ to the orthogonal bundle to TY in $TX|_Y$. Then the metric h^{TX} induces a G-invariant Hermitian metric $h^{N_{Y/X}}$ on $N_{Y/X}$. Let $h^E = \oplus_{1\leq i\leq m} h^{E_i}, h^F$ be G-invariant Hermitian metrics on E, F . We assume that h^E, h^F verify assumption (A) with respect to $h^F, h^{N_{Y/X}}$.

Take $g \in G$. Then $Y_g \subset X_g$. Let δ_{Y_g} be the current of integration on Y_g. In [BGS2, Section 6], and following earlier work by Bismut-Gillet-Soulé [BGS2] in the case $g = 1$, a current $T_g(X_g, h^E)$ on X_g is constructed, which is a sum of currents of type (p,p), whose wave front set is included in $N^*_{Y_g/X_g,\mathbf{R}}$, which verifies the equation of currents

$$\frac{\overline{\partial}\partial}{2i\pi} T_g(X_g, h^E) = \mathrm{Td}_g^{-1}(N_{Y/X}, h^{N_{Y/X}}) \mathrm{ch}_g(F, h^F) \delta_{Y_g} - \mathrm{ch}_g(E, h^E). \tag{1.47}$$

Clearly g acts on $TX|_{X_g}$. If $e^{i\theta_j}, 0 \leq \theta_j < 2\pi, 1 \leq j \leq q$ are the locally constant distinct eigenvalues of g, and if $TX|^{\theta_j}_{X_g}, 1 \leq j \leq q$ are the corresponding eigenbundles, then

$$TX|_{X_g} = \oplus TX|^{\theta_j}_{X_g}. \tag{1.48}$$

Definition 1.14. Put

$$R_g(TX) = \sum R(\theta_j, TX|^{\theta_j}_{X_g}). \tag{1.49}$$

Then $R_g(TX)$ is a cohomology class on X_g. Similarly we can define the cohomology classes $R_g(TY), R_g(N_{Y/X})$ on Y_g.

Consider the exact sequence of holomorphic Hermitian vector bundles on Y_g,

$$0 \to TY|_{Y_g} \to TX|_{Y_g} \to N_{Y/X}|_{Y_g} \to 0. \tag{1.50}$$

Let $\widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}}) \in P^{Y_g}/P^{Y_g,0}$ be the Bott-Chern class of [BGS1] such that

$$\frac{\overline{\partial}\partial}{2i\pi} \widetilde{\mathrm{Td}}_g(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}}) = \mathrm{Td}_g(TX|_{Y_g}) - \mathrm{Td}_g(TY|_{Y_g}) \mathrm{Td}_g(N_{Y/X}|_{Y_g}). \tag{1.51}$$

Now we state the main result in [B13].

Theorem 1.15. *For $g \in G$, the following identity holds,*

$$\begin{aligned}
\log\left(\frac{\|\ \|_{\widetilde{\lambda}_G(E)}}{\|\ \|_{\lambda_G(F)}}\right)^2(g) &= -\int_{X_g}\mathrm{Td}_g(TX,h^{TX})T_g(X_g,h^E)\\
&+\int_{Y_g}\mathrm{Td}_g^{-1}(N_{Y/X},h^{N_{Y/X}})\widetilde{\mathrm{Td}}_g(TY|_{Y_g},TX|_{Y_g},h^{TX|_{Y_g}})\mathrm{ch}_g(F,h^F)\\
&-\int_{Y_g}\mathrm{Td}_g(TY)R_g(N_{Y/X})\mathrm{ch}_g(F),\\
\log\left(\frac{\|\ \|_{\widetilde{\lambda}_G(E)}}{\|\ \|_{\lambda_G(F)}}\right)^2(g) &= -\int_{X_g}\mathrm{Td}_g(TX,h^{TX})T_g(X_g,h^E)\\
&+\int_{Y_g}\mathrm{Td}_g^{-1}(N_{Y/X},h^{N_{Y/X}})\widetilde{\mathrm{Td}}_g(TY|_{Y_g},TX|_{Y_g},h^{TX|_{Y_g}})\mathrm{ch}_g(F,h^F)\\
&-\int_{X_g}\mathrm{Td}_g(TX)R_g(TX)\mathrm{ch}_g(E)+\int_{Y_g}\mathrm{Td}_g(TY)R_g(TY)\mathrm{ch}_g(F)\ .
\end{aligned}\tag{1.52}$$

2. The equivariant infinitesimal analytic torsion forms

In this Section, we construct the equivariant analytic torsion in infinitesimal form, and we establish its main properties. As explained in the introduction, it is here crucial that the given Lie group G acts on the manifold X with a moment map.

This Section is organized as follows. In Section 2.1, we introduce the moment map $\langle \mu, K\rangle$ which is associated to the action of G. In Section 2.2, as a motivation for our construction, we introduce a new metric on the trivial bundle, which depends on $T \in \mathbf{R}$, and we construct the corresponding Dirac operator. In Section 2.3, we recall the Lefschetz fixed point formulas of Atiyah-Bott [ABo1] and the corresponding delocalized Kirillov formulas, relating them through the localization formulas of Duistermaat-Heckman [DuH], and the equivariant cohomology formalism of Berline-Vergne [BeV]. In Section 2.4, we recall the heat equation proof of the Kirillov formula given in [B3]. In Section 2.5, we briefly recall the construction by Bismut-Köhler [BK] in the case where the structure group of the considered fibration reduces to a compact Lie group G. In Section 2.6, we construct the infinitesimal equivariant analytic torsion forms, and the associated infinitesimal equivariant Quillen metric. In Section 2.7, we give the corresponding anomaly formulas. Finally in Section 2.8, we give a formula which describes the behaviour of these Quillen metrics by immersion.

2.1. **Complex manifolds and moment maps.**

We make the same assumptions as in Section 1.2. Let ω^{TX} be a closed G-invariant real 2-form on X which is of complex type (1,1), which is the Kähler form of a Hermitian metric h^{TX} on TX. If J^{TX} is the complex structure of $T_{\mathbf{R}}X$, if $U, V \in T_{\mathbf{R}X}$,

$$\omega^X(U,V) = \langle U, J^{TX}V\rangle_{h^{TX}}. \tag{2.1}$$

Then ω^X is a symplectic form on X, and G acts on X by symplectic diffeomorphisms.

If $K \in \mathfrak{g}$, let K^X be the corresponding vector field on X. If $K, K' \in \mathfrak{g}$, then

$$[K^X, K'^X] = -[K, K']^X. \tag{2.2}$$

Let $\mu : X \mapsto \mathfrak{g}^*$ be a smooth moment for the action of G on (X, ω^X). Namely μ is such that

- If $g \in G, x \in X$,

$$\mu(gx) = g.\mu(x) \tag{2.3}$$

- If $K \in \mathfrak{g}$, then

$$d\langle \mu, K\rangle + i_{K^X}\omega^X = 0. \tag{2.4}$$

In particular, if $K, K' \in \mathfrak{g}$,

$$\langle \mu, [K, K']\rangle = -\omega^X(K^X, K'^X). \tag{2.5}$$

Recall that ∇^{TX} is the holomorphic hermitian connection on (TX, h^{TX}). Since (X, ω^X) is Kähler, ∇^{TX} induces on $T_{\mathbf{R}}X$ the corresponding Levi-Civita connection.

Definition 2.1. If $K \in \mathfrak{g}$, set

$$m^{TX}(K) = \nabla^{TX}_{\cdot} K^X. \tag{2.6}$$

The vector field K^X is Killing and holomorphic, and the connection ∇^{TX} induces the Levi-Civita connection on $T_{\mathbf{R}}X$. Therefore $m^{TX}(K)$ is a skew-adjoint section of $\mathrm{End}(TX)$, which is also the vertical part with respect to ∇^{TX} of the lift K^{TX} of K^X to TX. The Lie derivative operator L_{K^X} is given by

$$L_K = \nabla^{TX}_{K^X} - m^{TX}(K). \tag{2.7}$$

Let K^E be the vector field induced by the action of K on E. Let $m^E(K) \in \mathrm{End}(E)$ be the vertical part of K^E with respect to the G-invariant connection ∇^E in E. Then $m^E(K)$ is a skew-adjoint section of $\mathrm{End}(E)$. Moreover since the connections ∇^{TX} and ∇^E are G-invariant, by [BeGeV, Section 7],

$$\begin{aligned} \nabla^{TX}_{\cdot} m^{TX}(K) + i_{K^X} R^{TX} &= 0, \\ \nabla^{E}_{\cdot} m^{E}(K) + i_{K^X} R^{E} &= 0. \end{aligned} \tag{2.8}$$

Let ∇^{LTXE} be the connection on $\Lambda(T^{*(0,1)}X) \otimes E$ induced by ∇^{TX}, ∇^E. Then the operator L_K defined in (2.7) extends to an operator L_K acting on $\Omega^{\cdot}(X,E)$, given by

$$L_K = \nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{K^X} - m^{TX}(K) - m^E(K). \tag{2.9}$$

2.2. **A new metric on the trivial line bundle.** Let $(\mathbf{C}, |\ |)$ be the trivial hermitian line bundle.

Definition 2.2. If $K \in \mathfrak{g}, T \geq 0$, let $|\ |_T$ be the metric on the trivial bundle

$$|\ |_T = e^{-T\langle \mu, K\rangle/2} |\ |. \tag{2.10}$$

Now, we equip the trivial line bundle with the metric $|\ |_T$. Equivalently, the hermitian product on $\Omega^{\cdot}(X,E)$ in (1.6) is now given by

$$\langle s, s'\rangle_T = \int_X \langle s, s'\rangle_{\Lambda(T^{*(0,1)}X)\otimes E} e^{-T\langle \mu, K\rangle} \frac{dv_X}{(2\pi)^{\dim X}}. \tag{2.11}$$

Let $\overline{\partial}^{X*}_T$ be the formal adjoint of $\overline{\partial}^X$ with respect to (2.11). Then

$$\overline{\partial}^{X*}_T = e^{T\langle \mu, K\rangle} \overline{\partial}^{X*} e^{-T\langle \mu, K\rangle}. \tag{2.12}$$

Proposition 2.3. *The following identity holds*

$$\overline{\partial}^{X*}_T = \overline{\partial}^{X*} - T\sqrt{-1} i_{K^{X(0,1)}}. \tag{2.13}$$

Proof. By (2.12) ,

$$\overline{\partial}^{X*}_T = \overline{\partial}^{X*} + T i_{(\nabla_{\cdot}\langle \mu, K\rangle)^{(0,1)}}. \tag{2.14}$$

By (2.1), (2.4), we get

$$(\nabla_{\cdot}\langle \mu, K\rangle)^{(0,1)} = -\sqrt{-1} K^{X(0,1)}. \tag{2.15}$$

Then (2.13) follows from (2.14) and (2.15). □

Put

$$\begin{aligned} \delta^X_T &= e^{-\frac{T}{2}\langle \mu, K\rangle} \overline{\partial}^X e^{\frac{T}{2}\langle \mu, K\rangle}, \\ \delta^{X*}_T &= e^{\frac{T}{2}\langle \mu, K\rangle} \overline{\partial}^{X*} e^{-\frac{T}{2}\langle \mu, K\rangle}, \\ A_T &= \delta^X_T + \delta^{X*}_T. \end{aligned} \tag{2.16}$$

Then δ_T^{X*} is the adjoint of δ_T^X with respect to the standard hermitian product $\langle\ \rangle$ on $\Omega^{\cdot}(X,E)$.

Proposition 2.4. *The map*

$$s\in\Omega^{\cdot}(X,E)\mapsto e^{-\frac{T}{2}\langle\mu,K\rangle}s\in\Omega^{\cdot}(X,E) \tag{2.17}$$

is an isomorphism between the hermitian complexes $(\Omega^{\cdot}(X,E),\overline{\partial}^X,\langle\ \rangle_T)$ *and* $(\Omega^{\cdot}(X,E),\delta_T^X,\langle\ \rangle)$.

Proof. This is a trivial result, whose proof is left to the reader. □

Proposition 2.5. *The following identities hold,*

$$\begin{aligned}\delta_T^X&=\overline{\partial}^X+\frac{T}{2}\sqrt{-1}K^{X(1,0)}\wedge,\\ \delta_T^{X*}&=\overline{\partial}^{X*}-\frac{T}{2}\sqrt{-1}i_{K^{X(0,1)}},\\ A_T&=D^X+T\sqrt{-1}\frac{c(K^X)}{2\sqrt{2}}.\end{aligned} \tag{2.18}$$

Proof. The second identity was already proved in Proposition 2.5. The proof of the remaining identities is similar. □

Clearly, the objects which have been defined in (2.18) also make sense for $T\in\mathbf{C}$. In particular

$$A_{-\sqrt{-1}T}=D^X+T\frac{c(K^X)}{2\sqrt{2}}. \tag{2.19}$$

However, for $T\neq 0$, the operator $A_{-\sqrt{-1}T}$ is not self-adjoint.

2.3. Lefschetz and Kirillov formulas.

Recall that by (1.23),

$$L(g)=\operatorname{Tr_s}^{H^{\cdot}(X,E)}[g]. \tag{2.20}$$

Then $L(g)$ is the character of the representation of G on the $\mathbf{Z}$-graded vector space $\ker(D^X)$. The McKean-Singer heat equation formula [MKeS] asserts that for any $t>0$,

$$L(g)=\operatorname{Tr_s}[g\exp(-tD^{X,2})]. \tag{2.21}$$

By making $t\to 0$ in (2.21), and using local index theory [ABoP], [Gi], [B1], [BeGeV], one obtains the Atiyah-Bott-Lefschetz formula [ABo1],

$$L(g)=\int_{X_g}\operatorname{Td}_g(TX,h^{TX})\operatorname{ch}_g(E,h^E), \tag{2.22}$$

which can also be written as in (1.24) in the form

$$L(g)=\int_{X_g}\operatorname{Td}_g(TX)\operatorname{ch}_g(E). \tag{2.23}$$

Let $(DR(X),d)$ be the de Rham complex of smooth complex differential forms on X, equipped with the de Rham operator d. Let $(DR^G(X),d)$ be the subcomplex of the G-invariant forms. The de Rham operator d splits as

$$d=\partial+\overline{\partial}. \tag{2.24}$$

Now we follow Berline-Vergne [BeV], and Bismut [B8, B9].

Definition 2.6. Set

$$\begin{aligned} d_K &= d - 2i\pi i_{K^X}, \\ \partial_K &= \partial - 2i\pi i_{K^{X(0,1)}}, \\ \overline{\partial}_K &= \overline{\partial} - 2i\pi i_{K^{X(1,0)}}. \end{aligned} \tag{2.25}$$

Clearly,

$$\begin{aligned} d_K &= \partial_K + \overline{\partial}_K, \\ d_K^2 &= -2i\pi L_{K^X}. \end{aligned} \tag{2.26}$$

In particular when acting on G-invariant forms,

$$d_K^2 = 0. \tag{2.27}$$

The cohomology classes of $(DR^G(X), d_K)$ are called equivariant cohomology classes.

Since K^X is a holomorphic vector field,

$$\partial_K^2 = 0,\ \overline{\partial}_K^2 = 0. \tag{2.28}$$

By (2.26), (2.28),

$$\begin{aligned} [\partial_K, \overline{\partial}_K] &= -2i\pi L_{K^X}, \\ [L_{K^X}, \partial_K] = 0,\ &[L_{K^X}, \overline{\partial}_K] = 0. \end{aligned} \tag{2.29}$$

In particular when acting on G-invariant forms, the operators ∂_K and $\overline{\partial}_K$ anticommute.

Put

$$R_K^{TX} = R^{TX} - 2i\pi m^{TX}(K),\ R_K^E = R^E - 2i\pi m^E(K). \tag{2.30}$$

Then R_K^{TX} and R^E are called the equivariant curvatures of TX and E.

Take $g \in G$. Let $Z(g) \subset G$ be the centralizer of g, and let $\mathfrak{z}(g)$ be its Lie algebra. Then

$$\mathfrak{z}(g) = \{K \in \mathfrak{g},\ g.K = K\}. \tag{2.31}$$

In the sequel, we always take $g \in G, K \in \mathfrak{z}(g)$. Put

$$X_K = \{x \in X,\ K^X(x) = 0\}. \tag{2.32}$$

Then X_K, which is the fixed point set of the group generated by K, is a complex totally geodesic submanifold of X. Set

$$X_{g,K} = X_g \cap X_K. \tag{2.33}$$

Then $X_{g,K}$ is a complex totally geodesic submanifold of X. More precisely, if $K_0 \in \mathfrak{z}(g)$ and, for $z \in \mathbf{R}$, $K = zK_o$, for z small enough,

$$X_{g,K} = X_{ge^K}. \tag{2.34}$$

Observe that $m^{TX}(K)|_{X_g}$ acts on TX_g and on $N_{X_g/X}$. Also it preserves the splitting (1.19) of $N_{X_g/X}$. Let $m^{TX_g}(K)$ and $m^{N^{\theta_j}_{X_g/X}}(K)$, $1 \leq j \leq q$ be the restriction of $m^{TX}(K)|_{X_g}$ to $TX_g, N^{\theta_j}_{X_g/X}$. Then $m^{TX_g}(K)$ and the$m^{N^{\theta_j}_{X_g/X}}(K)$ are just analogues of $m^E(K)$. We define the corresponding equivariant curvatures $R_K^{TX_g}, R_K^{N^{\theta_j}_{X_g/X}}$.

Definition 2.7. For $K \in \mathfrak{z}(g)$ small enough, set

$$\mathrm{Td}_{g,K}(TX,h^{TX}) = \mathrm{Td}(\frac{-R_K^{TX_g}}{2i\pi})\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{c_{\max}}\right)\left(\frac{-R_K^{N_{X_g/X}^{\theta_j}}}{2i\pi}+i\theta_j\right),$$

$$\mathrm{Td}'_{g,K}(TX,h^{TX}) = \frac{\partial}{\partial b}\Bigg[\mathrm{Td}(\frac{-R_K^{TX_g}}{2i\pi}+b) \times\prod_{j=1}^{q}\left(\frac{\mathrm{Td}}{c_{\max}}\right)\left(\frac{-R_K^{N_{X_g/X}^{\theta_j}}}{2i\pi}+i\theta_j+b\right)\Bigg]|_{b=0}, \tag{2.35}$$

$$\mathrm{ch}_{g,K}(E,h^{E}) = \mathrm{Tr}\left[\exp(-\frac{R_K^E}{2i\pi})\right].$$

The restriction on K is needed because the denominator of $\mathrm{Td}(x)$ vanishes for $x \in 2i\pi\mathbf{Z}$.

The forms $\mathrm{Td}_{g,K}(TX,h^{TX})$ and $\mathrm{ch}_{g,K}(E,h^E)$ are forms on X_g, which lie in P^{X_g}. They are G-invariant. Moreover by [BeV], [BeGeV, Theorem 7.7],

$$d_K\mathrm{Td}_{g,K}(TX,h^{TX}) = 0,\ d_K\mathrm{ch}_{g,K}(E,h^E) = 0, \tag{2.36}$$

and their equivariant cohomology classes do not depend on the G- invariant metrics h^{TX} and h^E.

The localization formulas in equivariant cohomology of Duistermaat-Heckman [DuH], Berline-Vergne [BeV] imply that

$$\int_{X_g}\mathrm{Td}_{g,K}(TX,h^{TX})\mathrm{ch}_{g,K}(E,h^E) = \int_{X_{g,K}}\mathrm{Td}_{ge^K}(TX,h^{TX})\mathrm{ch}_{ge^K}(E,h^E). \tag{2.37}$$

From the Atiyah-Bott-Lefschetz formulas in (2.23) and from (2.37), we recover the Kirillov formulas in the manner of Berline-Vergne [BeV]. Namely, if $K \in \mathfrak{z}(g)$ is small enough,

$$L(ge^K) = \int_{X_g}\mathrm{Td}_{g,K}(TX,h^{TX})\mathrm{ch}_{g,K}(E,h^E). \tag{2.38}$$

2.4. The heat equation proof of the Kirillov formulas.

Now, we will briefly explain the direct heat equation proof of (2.38) given in [B3], which is an analytic analogue of the heat equation proof of the Atiyah-Bott-Lefschetz formulas outlined in (2.21),(2.22).

Definition 2.8. For $t > 0$, put

$$C_{K,t} = \sqrt{t}D^X + \frac{c(K^X)}{2\sqrt{2t}}. \tag{2.39}$$

Equivalently

$$C_{K,t} = \sqrt{t}D^X_{1/t}. \tag{2.40}$$

A first trivial step in the heat equation proof of (2.38) is an obvious extension of the McKean-Singer formula

$$L(ge^K) = \mathrm{Tr_s}[g\exp(-L_K - C_{K,t}^2)]. \tag{2.41}$$

For $g = 1$, one then shows in [B3] that 'fantastic cancellations' still occur in supertrace of the kernel of the operator $g\exp(-L_K - C^2_{K,t})$, and that for K small enough, the limit as $t \to 0$ exists, and is given by the right-hand side of (2.38). We thus get a direct proof of (2.38) in the case $g = 1$. The general case will be dealt with in Section 7.

2.5. **Analytic torsion forms and compact Lie groups.** In this Section, we assume that G is connected. Then by Guillemin-Sternberg [GuSt, Proposition 4.1], G has a unique complexification $G_{\mathbf{C}}$. Namely $G_{\mathbf{C}}$ is a connected Lie group, whose Lie algebra $\mathfrak{g}_{\mathbf{C}}$ is the complexification of $\mathfrak{g}$, and G is a maximal compact subgroup of $G_{\mathbf{C}}$. By [GuSt, Theorem 4.4], the holomorphic action of G on X extends to a holomorphic action of $G_{\mathbf{C}}$ on X. Similarly, by proceeding as in [GuSt, Theorem 5.1], this action of $G_{\mathbf{C}}$ lifts to a holomorphic action on E. Let $Z(g)_{\mathbf{C}}$ be the centralizer of g in $G_{\mathbf{C}}$.

We will assume that $Z(g)$ is connected. By [Bou, Corollaire 5.3.1], this is always the case if G is simply connected. Needless to say, if $g = 1$, $Z(g) = G$ is connected.

Let $G_{\mathbf{C}} = PG$ be the Cartan decomposition of $G_{\mathbf{C}}$. If $h = pq$ is the Cartan decomposition of $h \in Z(g)_{\mathbf{C}}$, then p, q commute with g. Therefore $Z(g)_{\mathbf{C}}$ is connected, and is the complexification of $Z(g)$.

Let S be a complex manifold. Let $p : P \overset{Z(g)_{\mathbf{C}}}{\to} S$ be a $Z(g)_{\mathbf{C}}$ holomorphic principal bundle over S. By [GuSt, p. 527], $Z(g)_{\mathbf{C}}/Z(g)$ is contractible.Therefore the $Z(g)_{\mathcal{C}}$-bundle P can be reduced to a $Z(g)$-bundle $q : Q \overset{Z(g)_{\mathbf{C}}}{\to} S$. Then the $Z(g)$-bundle Q is canonically equipped with a connection which lifts to a holomorphic connection on the $Z(g)_{\mathbf{C}}$-bundle P, for which g is parallel. Let θ be the connection form on Q, and let Θ be its curvature. Then Θ is a $(1,1)$ form with values in the vector bundle $Q \times_{Z(g)} \mathfrak{g}$.

Put

$$V = P \times_{Z(g)_{\mathbf{C}}} X. \tag{2.42}$$

Then $\pi : V \overset{X}{\to} S$ is a holomorphic fibration with compact fibre X. Also g acts fibrewise on V. Moreover

$$V = Q \times_{Z(g)} X \tag{2.43}$$

Therefore V is also equipped with a connection, i.e. we have a splitting

$$TV = T^H V \oplus TX, \tag{2.44}$$

and g is parallel with respect to this connection.

Let $P^{TX} : T_{\mathbf{R}}V \to T_{\mathbf{R}}X$ be the obvious projection with respect to the splitting (2.44). If $U \in T_{\mathbf{R}}S$, let $U^H \in T^H V$ be the horizontal lift of U. If $U, V \in T_{\mathbf{R}}S$, put

$$T(U,V) = -P^{TX}[U^H, V^H]. \tag{2.45}$$

One verifies easily that T is a tensor.

Proposition 2.9. *The tensor T is a (1,1) form on S with values in vector fields along the fibres X. More precisely*

$$T = \Theta^X. \tag{2.46}$$

Proof. Equation (2.46) follows from (2.45). □

Observe that by (2.3), $\langle \mu, \Theta^X \rangle$ is a well-defined differential form on S.

Definition 2.10. Let ω^V be the 2-form on V

$$\omega^V = \omega^X + \pi^* \langle \mu, \Theta^X \rangle. \tag{2.47}$$

Proposition 2.11. *The 2-form ω^V is a real closed (1,1) form on V.*

Proof. This follows easily from equation (2.4). □

Observe that:

- The restriction of the form ω^V to the fibres X is just the Kähler form along the fibres.
- The vector bundle $T^H V$ is the orthogonal bundle with respect to ω^V.

By [BGS1, part II, Definition 1.4], we find that $(\pi, h^{TX}, T^H V)$ is a Kähler fibration, and that ω^V is an associated (1,1) form. If ω^H is the restriction of ω^V to $T^H V$, then

$$\omega^H = \langle \mu, \Theta^X \rangle. \tag{2.48}$$

It is a general fact for Kähler fibrations that if $U, V \in T_{\mathbf{R}} S$, then $\omega^H(U^H, V^H)$ is a Hamiltonian for the vector field $T(U,V)$, which is of course the case here.

The vector bundle $P \times_{Z(g)_{\mathbf{C}}} E$ is a holomorphic vector bundle on V. We still denote it by E. Clearly it is also given by $Q \times_{Z(g)} E$. Therefore E is now a holomorphic hermitian vector bundle on V. Similarly $Z(g)_{\mathbf{C}}$ acts on $\Omega^{\cdot}(X,E)$. Therefore $\Omega^{\cdot}(X,E)$ descends to a ‘ holomorphic ’ vector bundle on S, given by $P \times_{Z(g)_{\mathbf{C}}} \Omega^{\cdot}(X,E)$, which is also $Q \times_{Z(g)} \Omega^{\cdot}(X,E)$, i.e. $\Omega^{\cdot}(X,E)$ is now a holomorphic hermitian vector bundle on S.

The given connection on Q induces the holomorphic hermitian connection ∇^E on (E, h^E), and the holomorphic hermitian connection $\nabla^{\Omega^{\cdot}(X,E)}$ on $(\Omega^{\cdot}(X,E), h^{\Omega^{\cdot}(X,E)})$.

The operator $\overline{\partial}^X$ being $Z(g)_{\mathbf{C}}$-invariant descends to an operator acting on the vector bundle $\Omega^{\cdot}(X,E)$. Therefore $(\Omega^{\cdot}(X,E), \overline{\partial}^X)$ is a a holomorphic complex of infinite dimensional vector bundles on S. The adjoint $\overline{\partial}^{X*}$ being $Z(g)$-invariant descends to the corresponding adjoint.Therefore $D^X = \overline{\partial}^X + \overline{\partial}^{X*}$ acts along the fibres $\Omega^{\cdot}(X,E)$.

Now, we recall the definition of the Levi-Civita superconnection [B4, Section 3].

Definition 2.12. For $t > 0$, let B_t be the Levi-Civita superconnection

$$B_t = \sqrt{t} D^X + \nabla^{\Omega^{\cdot}(X,E)} - \frac{c(T)}{2\sqrt{2t}}. \tag{2.49}$$

Clearly, if

$$\begin{aligned} B_t'' &= \sqrt{t}\,\overline{\partial}^X + \nabla^{\Omega^{\cdot}(X,E)''} - \frac{c(T^{(1,0)})}{2\sqrt{2t}}, \\ B_t' &= \sqrt{t}\,\overline{\partial}^{X*} + \nabla^{\Omega^{\cdot}(X,E)'} - \frac{c(T^{(0,1)})}{2\sqrt{2t}}, \end{aligned} \tag{2.50}$$

then

$$B_t = B_t'' + B_t'. \tag{2.51}$$

By [BGS1, part II, Theorem 2.6],

$$\begin{aligned} B_t^{''2} = 0,\ B_t^{'2} &= 0, \\ B_t^2 &= [B_t'', B_t']. \end{aligned} \tag{2.52}$$

By [BGS1, part II, Theorem 2.6] or [B14, eq. (2.35)]

$$\begin{aligned} B_t'' &= \exp(-\frac{i\langle\mu,\Theta^X\rangle}{2t})t^{N/2}(\nabla^{\Omega^{\cdot}(X,E)''}+\overline{\partial}^X)t^{-N/2}\exp(i\frac{\langle\mu,\Theta^X\rangle}{2t}), \\ B_t' &= \exp(i\frac{\langle\mu,\Theta^X\rangle}{2t})t^{-N/2}(\nabla^{\Omega^{\cdot}(X,E)'}+\overline{\partial}^{X*})t^{N/2}\exp(-i\frac{\langle\mu,\Theta^X\rangle}{2t}). \end{aligned} \tag{2.53}$$

Recall that if $K \in \mathfrak{g}$, the operator L_K was defined in (2.9). One verifies easily that

$$\nabla^{\Omega^{\cdot}(X,E),2} = -L_\Theta. \tag{2.54}$$

Therefore, by proceeding as in [B4, Remark III.2], we get

$$B_t^2 = -L_\Theta + C_{\Theta,t}^2. \tag{2.55}$$

Definition 2.13. For $t > 0$, put

$$N_t = N + i\frac{\langle\mu,\Theta\rangle}{t}. \tag{2.56}$$

Let $\varphi \in \mathrm{End}(\Lambda(T^*_{\mathbf{R}}(S)) \otimes_{\mathbf{R}} \mathbf{C})$ be given by

$$\alpha \mapsto (2i\pi)^{-\deg(\alpha)/2}\alpha. \tag{2.57}$$

Definition 2.14. Let α_t, γ_t be the forms on S,

$$\alpha_t = \varphi\mathrm{Tr_s}[g\exp(-B_t^2)] \ \gamma_t = \varphi\mathrm{Tr_s}[N_t g\exp(-B_t^2)]. \tag{2.58}$$

Theorem 2.15. *The forms α_t and γ_t are real and closed, and lie in P^S. More precisely,*

$$\begin{aligned} \alpha_t &= L\left(g\exp(-\frac{\Theta}{2i\pi})\right), \\ \gamma_t &= \mathrm{Tr_s}\left[\left(N+\frac{\langle\mu,\Theta\rangle}{2\pi t}\right)g\exp\left(\frac{L_\Theta}{2i\pi}-(\sqrt{t}D^X-\frac{c(\Theta^X)}{2\sqrt{2t}})^2\right)\right]. \end{aligned} \tag{2.59}$$

Proof. By (2.55),

$$\alpha_t = \varphi\mathrm{Tr_s}\left[\exp\left(L_\Theta-(\sqrt{t}D^X-\frac{c(\Theta^X)}{2\sqrt{2t}})^2\right)\right]. \tag{2.60}$$

Using (2.41), (2.60), we get the first equation in (2.59). The proof of the second equation in (2.59) is similar. Finally Chern-Weil theory shows that the forms α_t and γ_t are closed. The proof of our Theorem is completed. □

Remark 2.16. By (2.38) and (2.59), it follows that

$$\alpha_t = \int_{X_g}\mathrm{Td}_{g\exp(-\Theta/2i\pi)}(TX,h^{TX})\mathrm{ch}_{g\exp(-\Theta/2i\pi)}(E), \tag{2.61}$$

This is a form of the theorem of Riemann-Roch-Grothendieck , but also a special case of the local families index theorem[B4], where the forms α_t remain constant.

Remark 2.17. In a more general context, it is shown in [BGS1, part II, Theorem 2.9] that

$$\frac{\partial}{\partial t}\alpha_t = -\frac{1}{t}\frac{\overline{\partial}\partial}{2i\pi}\gamma_t. \tag{2.62}$$

This equation is here tautologically verified.

Clearly $G_{\mathbf{C}}$ acts on $H^{\cdot}(X,E)$, and G acts isometrically on $(H^{\cdot}(X,E),h^{H^{\cdot}(X,E)})$. Therefore $(H^{\cdot}(X,E),h^{H^{\cdot}(X,E)})$ descends to a holomorphic hermitian vector bundle on S.

By [BGS1, part II, Theorems 2.11 and 2.16] in the case where $g=1$ and by [B13, Theorem 8.3] in the general case, we know that as $t\to 0$,

$$\gamma_t=\frac{C_{-1}}{t}+C_0+\sum_{j=1}^{k}C_kt^k, \tag{2.63}$$

and, as $t\to\infty$,

$$\gamma_t=\mathrm{Tr_s}^{H^{\cdot}(X,E)}[Ng\exp(-\Theta/2i\pi)]+\mathcal{O}(1/\sqrt{t}). \tag{2.64}$$

Definition 2.18. For $s\in\mathbf{C}, 0<\mathrm{Re}(s)<1/2$, put

$$R(s)=-\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}(\gamma_t-\gamma_{+\infty})dt. \tag{2.65}$$

By (2.63), (2.64), $R(s)$ extends to a meromorphic function, of $s\in\mathbf{C},\mathrm{Re}(s)<1/2$, which is holomorphic at $s=0$.

Definition 2.19. Set

$$T(\omega^X,h^E)(g)=\frac{\partial}{\partial s}R(s)|_{s=0}. \tag{2.66}$$

Then $T(\omega^X,h^E)(g)$ is an even form on S which lies in P^S.

Theorem 2.20. *The form $T(\omega^X,h^E)(g)$ is closed. Its cohomology class does not depend on the reduction of P to Q.*

Proof. By Theorem 2.15, the forms γ_t are closed. Therefore the form $T(\omega^X,h^E)(g)$ is also closed. The same argument shows that the cohomology class of $T(\omega^X,h^E)(g)$ does not depend on Q. The proof of our Theorem is completed. □

2.6. **The equivariant infinitesimal analytic torsion.** We make the same assumptions as in Sections 2.1-2.4, the results of Section 2.5 being only intended as a motivation for what follows.

Definition 2.21. For $t>0, K\in\mathfrak{g}$, put

$$\begin{aligned}\alpha_t(g,K)&=\mathrm{Tr_s}\left[g\exp(-L_K-C^2_{K,t})\right],\\ \gamma_t(g,K)&=\mathrm{Tr_s}\left[(N-i\frac{\langle\mu,K\rangle}{t})g\exp(-L_K-C^2_{K,t})\right].\end{aligned} \tag{2.67}$$

Clearly the above definitions extend to the case where $K\in\mathfrak{g}_{\mathbf{C}}$.

Theorem 2.22. *For $t>0, K\in\mathfrak{z}(g)_{\mathbf{C}}$,*

$$\alpha_t(g,K)=L(ge^K). \tag{2.68}$$

If $|K|$ is small enough, there exist $\gamma\in]0,1]$ and $C_1(g,K),\dots,C_0(g,K)\in\mathbf{C}$ such that for $t\in]0,1]$,

$$\gamma_t(g,K)=\frac{C_{-1}(g,K)}{t}+C_0(g,K)+\mathcal{O}(t^\gamma). \tag{2.69}$$

Moreover

$$\begin{aligned} C_{-1}(g,K) &= \int_{X_g} \left(\frac{\omega^X}{2\pi} - i\langle \mu, K\rangle\right) \mathrm{Td}_{g,K}(TX,h^{TX})\mathrm{ch}_{g,K}(E,h^E), \\ C_0(g,K) &= \int_{X_g} \mathrm{Td}_{g,K}(TX)\left(\dim(X) - \left(\frac{\mathrm{Td}'}{\mathrm{Td}}\right)_{g,K}(TX)\right)\mathrm{ch}_{g,K}(E). \end{aligned} \tag{2.70}$$

Finally, if $K \in \mathfrak{z}(g)_{\mathbf{C}}$, *as* $t \to +\infty$,

$$\gamma_t(g,K) = \mathrm{Tr_s}^{H^{\cdot}(X,E)}[Nge^K] + \mathcal{O}(1/\sqrt{t}). \tag{2.71}$$

Proof. The fact that α_t does not depend on t was already stated in (2.41). Equations (2.70) and (2.71) will be proved in Section 7. □

Remark 2.23. Needless to say, one can also show that for any $k \in \mathbf{N}$, as $t \to 0$, $\gamma_t(g,K)$ has a Taylor expansion to arbitrary order $k \in \mathbf{N}$.

Definition 2.24. For $K \in \mathfrak{z}(g)_{\mathbf{C}}, |K|$ small enough, for $s \in \mathbf{C}, 0 < \mathrm{Re}(s) < 1/2$, set

$$\widetilde{\theta}(\omega^X,h^E)(g,K)(s) = -\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\big(\gamma_t(g,K) - \gamma_{+\infty}(g,K)\big)dt. \tag{2.72}$$

By (2.69), (2.71), $\widetilde{\theta}(\omega^X,h^E)(g,K)(s)$ extends to a holomorphic function of s near $s=0$.

By comparing (1.14) and (2.72), we find that

$$\theta(\omega^X,h^E)(g)(s) = \widetilde{\theta}(\omega^X,h^E)(g,0)(s). \tag{2.73}$$

Then if $K \in \mathfrak{z}(g)_{\mathbf{C}}$, $\widetilde{\theta}(\omega^X,h^E)(g,zK)$ depends analytically on $z \in \mathbf{C}$, for $|z|$ small enough. One then verifies easily that with the notation in (2.65),

$$T(\omega^X,h^E)(g) = \frac{\partial}{\partial s}\widetilde{\theta}(\omega^X,h^E)(g,-\frac{\Theta}{2i\pi})(0), \tag{2.74}$$

where (2.74) is now an equality of power series which contain only a finite number of terms. Also (2.74) makes clear that the cohomology class of $T(\omega^X,h^E)$ does not depend on the choice of the reduction Q of P.

Proposition 2.25. *The following identity holds,*

$$\begin{aligned} \frac{\partial}{\partial s}\widetilde{\theta}(\omega^X,h^E)(g,K)(0) = -\int_0^1 \Big(\gamma_t(g,K) - \frac{C_{-1}(g,K)}{t} - C_0(g,K)\Big)\frac{dt}{t} \\ -\int_1^{+\infty}\big(\gamma_t(g,K)-\gamma_{+\infty}(g,K)\big)\frac{dt}{t}+C_{-1}(g,K)+\Gamma'(1)(C_0(g,K)-\mathrm{Tr_s}^{H^{\cdot}(X,E)}[Nge^K]). \end{aligned} \tag{2.75}$$

Proof. This follows easily from (2.69), (2.70). □

In the sequel, $\frac{\partial}{\partial s}\widetilde{\theta}(\omega^X,h^E)(g,K)(0)$ will be called the equivariant infinitesimal Ray-Singer torsion.

Now we imitate Definition 1.3.

Definition 2.26. Put

$$\log(\| \ \|^{\sim}_{\lambda_G(E)})^2(g,K) = \log(| \ |^2_{\lambda_W(E)})(ge^K) - \frac{\partial}{\partial s}\widetilde{\theta}(\omega^X,h^E)(g,K)(0). \tag{2.76}$$

The symbol $\| \|^{\sim}_{\lambda_G(E)}$ will be called an equivariant infinitesimal Quillen metrics. By (2.73)

$$\log(\| \|^{2}_{\lambda_G(E)})(g) = \log(\| \|^{\sim}_{\lambda_G(E)})^{2}(g,0). \tag{2.77}$$

The main purpose of this paper is to compare $\log(\| \|^{2}_{\lambda_G(E)})(ge^{K})$ and $\log(\| \|^{\sim,2}_{\lambda_G(E)})$ (g,K).

2.7. **Anomaly formulas for equivariant infinitesimal Quillen metrics.** Let ω'^{X} be another G-invariant Kähler form on X, and let h'^{E} be a second G-invariant hermitian metric on E. We denote $\| \|'^{\sim}_{\lambda_G(E)}$ the corresponding families Quillen metric. By [BGS1, part I], for $|K|$ small enough, we can define Bott-Chern classes $\widetilde{\mathrm{Td}}_{g,K}(TX,h^{TX}) \in P^{X_g}/P^{X_g,0}$ which verify the obvious analogue of (1.29). Similarly the class $\widetilde{\mathrm{ch}}_{g,K}(E,h^{E},h'^{E})$ was already considered in (1.29).

Theorem 2.27. *For $K \in \mathfrak{z}(g)_{\mathbf{C}}, |K|$ small enough, the following identity holds*

$$\begin{aligned}\log\left(\frac{\| \|'^{\sim}_{\lambda_G(E)}}{\| \|^{\sim}_{\lambda_G(E)}}\right)^{2}(g,K) = &\int_{X_g} \widetilde{\mathrm{Td}}_{g,K}(TX,h^{TX},h'^{TX})\mathrm{ch}_{g,K}(E,h^{E}) \\ &+ \int_{X_g} \mathrm{Td}_{g,K}(TX,h'^{TX})\widetilde{\mathrm{ch}}_{g,K}(E,h^{E},h'^{E}).\end{aligned} \tag{2.78}$$

Proof. When $K=0$, our Theorem is just Theorem 1.8. The general case can be proved by using the methods of [BGS1, part III], [B13]. □

2.8. **Equivariant immersions and equivariant infinitesimal Quillen metrics.** Now, we suppose that besides the above assumptions, the assumptions of Section 1.6 also hold. Let $K \in \mathfrak{z}(g)_{\mathbf{C}}$. Then one can easily modify the construction of the current $T_g(E,h^{E})$ on X_g given in [B13, Section 6], and, for $|K|$ small enough, construct a current $T_{g,K}(X_g,h^{E})$ on X_g which verifies the equation of currents

(2.79)
$$\frac{\overline{\partial}_K \partial_K}{2i\pi} T_{g,K}(X_g,h^{E}) = \mathrm{Td}^{-1}_{g,K}(N_{Y/X},h^{N_{Y/X}})\mathrm{ch}_{g,K}(F,h^{F})\delta_{Y_g} - \mathrm{ch}_{g,K}(E,h^{E}).$$

In fact in [B13], one needs just to replace the considered curvatures by the corresponding equivariant curvatures. Similarly $R_{g,K}(N_{Y/X},h^{N_{Y/X}})$ is the analogue of $R_g(N_{Y/X},h^{N_{Y/X}})$, where the curvature $R^{N_{Y/X}}$ is replaced by the equivariant curvature $R^{N_{Y/X}}_{K}$. Let $R_{g,K}(N_{Y/X})$ be the corresponding equivariant cohomology class on X_g.

Theorem 2.28. *For $K \in \mathfrak{z}(g)_{\mathbf{C}}, |K|$ small enough, the following equivalent identities hold,*

$$\begin{aligned}
\log\left(\frac{\| \|^{\sim}_{\lambda_G(E)}}{\| \|^{\sim}_{\lambda_G(F)}}\right)^2 (g,K) = &-\int_{X_g} \mathrm{Td}_{g,K}(TX,h^{TX}) T_{g,K}(X_g,h^E) \\
&+\int_{Y_g} \mathrm{Td}^{-1}_{g,K}(N_{Y/X},h^{N_{Y/X}}) \widetilde{\mathrm{Td}}_{g,K}(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}}) \mathrm{ch}_{g,K}(F,h^F) \\
&-\int_{Y_g} \mathrm{Td}_{g,K}(TX) R_{g,K}(N_{Y/X}) \mathrm{ch}_{g,K}(F), \\
\log\left(\frac{\| \|^{\sim}_{\lambda_G(E)}}{\| \|^{\sim}_{\lambda_G(F)}}\right)^2 (g,K) = &-\int_{X_g} \mathrm{Td}_{g,K}(TX,h^{TX}) T_{g,K}(E,h^E) \\
&+\int_{Y_g} \mathrm{Td}^{-1}_{g,K}(N_{Y/X},h^{N_{Y/X}}) \widetilde{\mathrm{Td}}_{g,K}(TY|_{Y_g}, TX|_{Y_g}, h^{TX|_{Y_g}}) \mathrm{ch}_{g,K}(F,h^F) \\
&-\int_{X_g} \mathrm{Td}_{g,K}(TX) R_{g,K}(TX) \mathrm{ch}_{g,K}(E) + \int_{Y_g} \mathrm{Td}_{g,K}(TY) R_{g,K}(TY) \mathrm{ch}_{g,K}(F).
\end{aligned} \tag{2.80}$$

Proof. When $K = 0$, the above result is just the result of [B13], which was stated in Theorem 1.15. In infinitesimal form, i.e. in the context of Section 2.5, the above result was proved in [B14] when $g = 1$. One minor observation with respect to [B14] is that the identity proved there (which is an identity in $P^S/P^{S,0}$) becomes here an identity of complex numbers. Establishing (2.80) in full generality requires some non trivial analysis, whose necessity is best revealed by the fact that it only makes sense for $|K|$ small. Most of the extra analysis which is needed with respect to [BL], [B13], [B14] is done in Section 7. Finally, an alternative method is to notice that both sides are analytic in K near $K = 0$. Using [B14] is then enough to establish (2.80). The proof of our Theorem is completed. □

3. Equivariant Bott-Chern currents

In the Section, we recall the construction of the equivariant Bott-Chern currents $S_K(X,\omega^X)$ given in [B8, B9] and we state the main results which have been obtained on these currents. These Bott-Chern currents are an essential ingredient in the formula which is the main result of this paper.

This Section is organized as follows. In Section 3.1, we construct the currents $S_K(X,\omega^X)$. In Section 3.2, we state the main result of [B9], which gives a formula describing the behaviour of such currents by a complex immersion. Finally, in Section 3.3, we specialize the construction to the case where the group G acts on X with a moment map.

3.1. Construction of the Bott-Chern currents.

Let X be a compact complex manifold. Let ω^X be the Kähler form of a Kähler metric on TX. Let G be a compact connected Lie group acting holomorphically on X and preserving ω^X. Let $\mathfrak{g}$ be the Lie algebra of G. Otherwise, we keep the notation of Sections 1 and 2, and in particular the notation of Section 2.3. The major difference is that, for the moment, we do not assume that a moment map is attached to the action of G on X.

Definition 3.1. Let P_K^X be the vector space of the smooth K^X-invariant forms on X, which are sums of forms of type (p,p). Let $P_K^{X,0}$ be the vector space of the $\alpha \in P_K^X$ which can be written in the form $\alpha = \partial\beta + \overline{\partial}\gamma$, where β,γ are smooth K^X-invariant forms on X.

If $K=0$, we will write P^X instead of P_K^X. By (2.29), when acting on P_K^X, the operators ∂_K and $\overline{\partial}_K$ anticommute.

We will now prove an identity already established in [B8, eq. (14)] and in [B9, eq. (2.12)]. Let $K^{'X} \in T^*_{\mathbf{R}}X$ be the 1-form which is dual to K^X by the metric h^{TX}. Since ∇^{TX} induces the Levi-Civita connection on $T_{\mathbf{R}}X$ and since K^X is a Killing vector field, if $U,V \in T_{\mathbf{R}}X$,

$$dK^{'X}(U,V) = 2\langle \nabla_U^{TX} K^X, V\rangle. \tag{3.1}$$

Proposition 3.2. *The following identity holds*

$$\frac{\overline{\partial}_K\partial_K}{2i\pi}\frac{\omega^X}{2\pi} = d_K\frac{K^{'X}}{4i\pi}. \tag{3.2}$$

Proof. By (2.1),

$$K^{'X} = i(i_{K^{(1,0)}} - i_{K^{(0,1)}})\omega^X. \tag{3.3}$$

Since ω^X is closed, we can rewrite (3.3) in the form

$$K^{'X} = \frac{1}{2\pi}(\partial_K - \overline{\partial}_K)\omega^X. \tag{3.4}$$

Using (2.29), (3.4), we get (3.2). The proof of our Proposition is completed. □

Definition 3.3. For $t>0$, let b_t, d_t be the even forms on X,

$$\begin{aligned} b_t &= \exp\left(\frac{\overline{\partial}_K\partial_K}{2i\pi t}\frac{\omega^X}{2\pi}\right), \\ d_t &= -\frac{\omega^X}{2\pi t}\exp\left(\frac{\overline{\partial}_K\partial_K}{2i\pi t}\frac{\omega^X}{2\pi}\right). \end{aligned} \tag{3.5}$$

We define $\varphi \in \operatorname{End}\big(\Lambda(T^*{}_{\mathbf{R}}X) \otimes_{\mathbf{R}} \mathbf{C}\big)$ as in (2.57). With the notation in [B9, Definition 2.2],

$$b_t = \varphi^{-K}\alpha_t,\ d_t = \frac{1}{2i\pi}\varphi^{-K}\gamma_t. \tag{3.6}$$

Now we state a result in [B8, Proposition 5], [B9, Theorem 2.5].

Theorem 3.4. *For $t > 0$, the forms b_t and d_t lie in P^X_K. For $t > 0$, the forms b_t are d_K closed, and their d_K cohomology class does not depend on t. More precisely*

$$\frac{\partial}{\partial t}b_t = \frac{\overline{\partial}_K\partial_K}{2i\pi t}d_t. \tag{3.7}$$

Definition 3.5. Let P^X_{K,X_K} be the set of K-invariant currents on X which are sums of currents of type (p,p), whose wave-front set is included in $N^*_{X_K/X,\mathbf{R}}$. Let $P^{X,0}_{K,X_K}$ be the set of currents $\alpha \in P^X_{K,X_K}$ such that there are K^X-invariant currents β,γ on X, whose wave-front set is included in $N^*_{X_K/X,\mathbf{R}}$, with $\alpha = \partial_K\beta + \overline{\partial}_K\gamma$.

Let $h^{N_{X_K/X}}$ be the hermitian metric induced by ω^X on $N_{X_K/X}$, which is identified to the orthogonal bundle to TX_K in $TX|_{X_K}$.

By [B5, Theorem 1.3], [B9, Theorem 2.5 and Remark 2.6], there are currents $\rho_1,\dots,\rho_k\dots$ in P^X_{K,X_K}such that if η is a smooth form on X,

$$\int_X \eta b_t = \int_{X_K} \frac{\eta}{c_{\max,K}(N_{X_K/X},h^{N_{X_K/X}})} + \sum_{j=1}^{k}\int_X \eta\rho_j t^j + o(t^k). \tag{3.8}$$

By (3.8), we see that as $t \to 0$,

$$\int_X \eta d_t = -\int_{X_K} \frac{\eta\omega^X/2\pi}{c_{\max,\,K}(N_{X_K/X},h^{N_{X_K/X}})}\frac{1}{t} - \sum_{j=0}^{k}\int_X \eta\frac{\omega^X}{2\pi}\rho_{j+1}t^j + o(t^k). \tag{3.9}$$

If A is an invertible (q,q) matrix, put

$$\begin{aligned} c'_{\max}(A) &= \frac{\partial}{\partial b}\det(A+b)|_{b=0}, \\ c^{-1,'}_{\max}(A) &= \frac{\partial}{\partial b}[\det(A+b)]^{-1}|_{b=0}. \end{aligned} \tag{3.10}$$

Let δ_{X_K} be the current of integration on X_K. The following result was proved in [B8, eq. (40)-(49)] and in [B9, Theorem 2.7].

Theorem 3.6. *The following identity holds*

$$-\frac{\omega^X}{2\pi}\rho_1 = (c_{\max})^{-1,'}_K(N_{X_K/X},h^{N_{X_K/X}})\delta_{X_K} \text{ in } P^X_K/P^{X,0}_K. \tag{3.11}$$

Let η be a smooth form on X. By (3.9), the function of $s \in \mathbf{C}, \operatorname{Re}(s) > 1$,

$$F^1_\eta(s) = \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\{\int_X \eta d_t\}dt, \tag{3.12}$$

extends to a meromorphic function, which is holomorphic at $s = 0$. Also, for $s \in \mathbf{C}$, $\operatorname{Re}(s) < 1$,

$$F^2_\eta(s) = \frac{1}{\Gamma(s)}\int_1^{+\infty} t^{s-1}\{\int_X \eta d_t\}dt \tag{3.13}$$

is holomorphic

Definition 3.7. Let $S_K(X,\omega^X)$ be the current on X, such that

$$\int_X \eta S_K(X,\omega^X) = \frac{\partial}{\partial s}(F_\eta^1 + F_\eta^2)(0). \tag{3.14}$$

Using (3.6), with the notation in [B9, Definition 2.9],

$$S_K(X,\omega^X) = \frac{1}{2i\pi}\varphi^{-K} S_{\omega^X}. \tag{3.15}$$

Now we state a result in [B9, Proposition 2.11].

Proposition 3.8. *The following identity holds*

$$\begin{aligned}(3.16)\int_X \eta S_K(X,\omega^X) = \int_0^1 \{\int_X \eta \left(d_t + \frac{(\omega^X/2\pi)\delta_{X_K}}{c_{\max,\,K}(N_{X_K/X}, h^{N_{X_K/X}})}\frac{1}{t} + \frac{\omega^X}{2\pi}\rho_1\right)\}\frac{dt}{t} \\ + \int_1^{+\infty}\{\int_X \eta d_t\}\frac{dt}{t} + \int_{X_K} \eta \frac{\omega^X/2\pi}{c_{\max,\,K}(N_{X_K/X}, h^{N_{X_K/X}})} + \Gamma'(1)\int_X \eta\frac{\omega^X}{2\pi}\rho_1.\end{aligned}$$

Proof. This follows from (3.9), (3.12), (3.13). ☐

Now we state the result of [B8, Theorem 6], [B9, Theorem 2.12].

Theorem 3.9. *The current $S_K(X,\omega^X)$ lies in P_K^X. Moreover*

$$\frac{\overline{\partial}_K\partial_K}{2i\pi}S_K(X,\omega^X) = 1 - \frac{\delta_{X_K}}{c_{\max,\,K}(N_{X_K/X}, h^{N_{X_K/X}})}. \tag{3.17}$$

Let ω'^X be another G-invariant Kähler form on X. Let $h'^{N_{X_K/X}}$ be the corresponding metric on $N_{X_K/X}$. Let $\widetilde{c}_{\max,K}^{-1}(N_{X_K/X}, h^{N_{X_K/X}}, h'^{N_{X_K/X}}) \in P^{X_K}/P^{X_K,o}$ be the Bott-Chern class such that

$$\begin{aligned}\frac{\overline{\partial}\partial}{2i\pi}\widetilde{c}_{\max,K}^{-1}(N_{X_K/X}, h^{N_{X_K/X}}, h'^{N_{X_K/X}}) = c_{\max,K}^{-1}(N_{X_K/X}, h'^{N_{X_K/X}}) - \\ c_{\max,K}^{-1}(N_{X_K/X}, h^{N_{X_K/X}}).\end{aligned} \tag{3.18}$$

The following result was established in [B8, Theorem 7].

Theorem 3.10. *The following identity holds,*

$$\begin{aligned}S_K(X,\omega'^X) - S_K(X,\omega^X) = -\widetilde{c}_{\max,K}^{-1}(N_{X_K/X}, h^{N_{X_K/X}}, h'^{N_{X_K/X}})\delta_{X_K} \\ \text{in } P^{X_K}/P^{X_K,0}.\end{aligned} \tag{3.19}$$

3.2. **Bott-Chern currents, excess normal bundles and immersion formulas.** Let (H, h^H) be a holomorphic Hermitian vector bundle on X. Suppose that G acts on (H, h^H) . Let σ be a G-invariant holomorphic section of H. Put

$$Y = \sigma^{-1}(0). \tag{3.20}$$

Assume that $\partial\sigma|_Y$ is surjective, so that Y is a complex G-invariant submanifold of X, and that $\partial\sigma : N_{Y/X} \mapsto H|_Y$ identifies the vector bundles $N_{Y/X}$ and $H|_Y$. Let $h^{N_{Y/X}}$ be the corresponding Hermitian metric on $N_{Y/X}$. Let $i : Y \mapsto X$ be the obvious embedding. Put $\omega^Y = i^*\omega^X$.

We will use for Y the same notation as for X. Since Y is G-invariant, the vector field K^X restricts on Y to the vector field K^Y. We still denote by i the embedding $Y_K \mapsto X_K$.

Consider the short exact sequence of vector bundles on Y_K,

$$0 \to N_{Y_K/Y} \to N_{X_K/X} \to \widetilde{N} \to 0. \tag{3.21}$$

Equation (3.21) defines the vector bundle $\widetilde{N}$. Observe that for reasons of dimension, $\widetilde{N}$ is in general nonzero. This vector field appears as an excess normal bundle. It measures the defect of transversality between X_K and Y.

Observe that we have the holomorphic splitting of vector bundles on Y_K,

$$N_{Y/X}|_{Y_K} = N_{Y_K/X_K} \oplus \widetilde{N}. \tag{3.22}$$

In fact, observe that K acts on $N_{Y/X}|_{Y_K}$. Then N_{Y_K/X_K} is the eigenbundle associated to the eigenvalue 0, and $\widetilde{N}$ is the direct sum of eigenbundles associated to nonzero eigenvalues. Also the splitting (3.22) is orthogonal with respect to the metric $\utilde{h^{N_{Y/X}}}$.

Let $c^{-1}_{\max,K}(N_{Y_K/Y}, N_{X_K/X}, h^{N_{X_K/X}})$ be the Bott-Chern class on Y_K such that

$$\begin{aligned}\frac{\overline{\partial}\partial}{2i\pi}\widetilde{c^{-1}}_{\max,K}(N_{Y_K/Y}, N_{X_K/X}, h^{N_{X_K/X}}) = c^{-1}_{\max,K}(N_{X_K/X}, h^{N_{X_K/X}}) -\\ c^{-1}_{\max,K}(N_{Y_K/Y}, h^{N_{Y_K/Y}})c^{-1}_{\max,K}(\widetilde{N}, h^{\widetilde{N}}).\end{aligned} \tag{3.23}$$

Clearly, the vector bundle $\widetilde{N}$ splits holomorphically as

$$\widetilde{N} = \oplus_{\theta'_{j'}} \widetilde{N}^{\theta'_{j'}}, \tag{3.24}$$

so that $\nabla^{TX}K^X = m^{TX}(K)$ acts on $\widetilde{N}^{\theta'_{j'}}$ by multiplication by $i\theta'_{j'}$. Of course the splitting is orthogonal. Let $h^{\widetilde{N}^{\theta'_{j'}}}$ be the corresponding metric on $\widetilde{N}^{\theta'_{j'}}$.

Definition 3.11. Put

$$R^K(\widetilde{N}, h^{\widetilde{N}}) = \sum R^{\theta'_{j'}}(\widetilde{N}^{\theta'_{j'}}, h^{\widetilde{N}^{\theta'_{j'}}}). \tag{3.25}$$

Let $R^K(\widetilde{N})$ be the associated cohomology class.

We define $P^X_{K,Y}$ as in Definition 3.5, by replacing X_K by Y. By [B9, Section 2 e)], there is an Euler-Green current $\widetilde{e}_K(X, h^H)$ on X, which lies in $P^X_{K,Y}$, such that

$$\frac{\overline{\partial}_K\partial_K}{2i\pi}\widetilde{e}^K(X, h^H) = \delta_Y - c_{\max,K}(H, h^H). \tag{3.26}$$

The construction of $\widetilde{e}_K(X, h^H)$ extends the construction of standard Euler-Green currents (with $K = 0$) given in [BGS3, Section 3 f)].

We define $P^{X_K}_{Y_K}$ as before. By [B9, Section 2 e)], there is a current $e_K(X_K, h^H)$ on X_K, which lies in $P^{X_K}_{Y_K}$, such that

$$\frac{\overline{\partial}_K\partial_K}{2i\pi}\widetilde{e}^K(X_K, h^H) = c_{\max,K}(\widetilde{N}, h^{\widetilde{N}})\delta_{Y_K} - i^*c_{\max,K}(H, h^H). \tag{3.27}$$

Note that since K^X vanishes on X_K, in (3.27), we may replace $\overline{\partial}_K, \partial_K$ by $\overline{\partial}, \partial$.

Let $P^X_{K,X_K\cup Y}$ be the set of K-invariant currents on X which are sums of currents of type (p,p), whose wave front set is included in $N^*_{X_K/X,\mathbf{R}} + N^*_{Y/X,\mathbf{R}}$. Let $P^{X,0}_{K,X_K\cup Y}$ be the set of currents γ in $P^X_{K,X_K\cup Y}$ such that there exists K-invariant currents α, β, whose wave front set is included in $N^*_{X_K/X,\mathbf{R}} + N^*_{Y/X,\mathbf{R}}$, such that $\gamma = \partial_K\alpha + \overline{\partial}_K\beta$.

Now we recall the main result given in [B9, Theorem 3.2].

Theorem 3.12. *The following identity of currents on X holds,*

$$\begin{aligned}(3.28)\quad & -\widetilde{e}_K(X,h^H) - S_K(X,\omega^X)c_{\max,K}(H,h^H) + S_K(Y,\omega^Y)\delta_Y \\ & + \frac{\widetilde{e}_K(X_K,h^H)}{c_{\max,K}(N_{X_K/X},h^{N_{X_K/X}})}\delta_{X_K} - c_{\max,K}(\widetilde{N},h^{\widetilde{N}})\widetilde{c^{-1}_{\max,K}}(N_{Y_K/Y},N_{X_K/X},h^{N_{X_K/X}})\delta_{Y_K} \\ & + c^{-1}_{\max,K}(N_{Y_K/Y})R_K(\widetilde{N})\delta_{Y_K} = 0 \text{ in } P^X_{K,X_K\cup Y}/P^{X,0}_{K,X_K\cup Y}.\end{aligned}$$

Equivalently

$$\begin{aligned}(3.29)\quad & -\widetilde{e}_K(X,h^H) - S_K(X,\omega^X)c_{\max,K}(H,h^H) + S_K(Y,\omega^Y)\delta_Y \\ & + \frac{\widetilde{e}_K(X_K,h^H)}{c_{\max,K}(N_{X_K/X},h^{N_{X_K/X}})}\delta_{X_K} - c_{\max,K}(\widetilde{N},h^{\widetilde{N}})\widetilde{c^{-1}_{\max,K}}(N_{Y_K/Y},N_{X_K/X},h^{N_{X_K/X}})\delta_{Y_K} \\ & + c^{-1}_{\max,K}(N_{X_K/X})R_K(N_{X_K/X})c_{\max,K}(H)\delta_{X_K} \\ & - c^{-1}_{\max,K}(N_{Y_K/Y})R_K(N_{Y_K/Y})\delta_{Y_K} = 0 \text{ in } P^X_{K,X_K\cup Y}/P^{X,0}_{K,X_K\cup Y}.\end{aligned}$$

Remark 3.13. In Section 5, we will show that Theorem 3.12 is compatible with a conjectural Riemann-Roch like formula, in which the current $S_K(X,\omega^X)$ can be expressed as as the difference of two 'exotic' classes.

3.3. **Bott-Chern currents and moment maps.**

Now we make the same assumptions as in Sections 1 and 2. In particular we assume that G acts on X with a moment map μ.

First we establish a modification of Proposition 3.2.

Proposition 3.14. *The following identity holds*

$$\frac{\overline{\partial}_K\partial_K}{2\pi}\langle\mu,K\rangle = d_K\frac{K^{'X}}{4i\pi}. \tag{3.30}$$

Proof. By (2.4),

$$d_K(\omega^X - 2i\pi\langle\mu,K\rangle) = 0 \tag{3.31}$$

From (3.2),(3.31), we get (3.30). □

Definition 3.15. For $t>0$, let $\tilde{d}_t$ be the even form on X,

$$\tilde{d}_t = -i\frac{\langle\mu,K\rangle}{t}\exp\left(\frac{\overline{\partial}_K\partial_K}{2\pi t}\langle\mu,K\rangle\right). \tag{3.32}$$

Now we give an analogue of Theorem 3.4.

Proposition 3.16. *The following identity holds*

$$\frac{\partial}{\partial t}b_t = \frac{\overline{\partial}_K\partial_K}{2i\pi t}\tilde{d}_T. \tag{3.33}$$

Proof. Equation (3.33) follows easily from (3.30). □

Clearly, when replacing d_t by $\tilde{d}_t$, the analogue of (3.9) holds, when replacing $\omega^X/2\pi$ by $i\langle\mu,K\rangle$.

Theorem 3.17. *The following identity holds,*

$$-i\langle\mu,K\rangle\rho_1 = (c_{\max})^{-1,'}_K(N_{X_K/X},h^{N_{X_K/X}})\delta_{X_K} \text{ in } P^X_K/P^{X,0}_K. \tag{3.34}$$

Proof. We give two proofs of (3.34). First one can reproduce word for word the proof of [B8, eq. (40)-(49)] and of [Theorem 2.7][B9], and still get (3.34). Another indirect proof is to observe that the d_K cohomology class of $t(d_t - \tilde{d}_t)$ is constant. Therefore one deduces easily (3.34) from (3.11). □

By replacing d_t in (3.12)-(3.13) by $\tilde{d}_t$, one constructs a current $\widetilde{S}_K(X,\omega^X)$ as in Definition 3.7. Equation (3.16) still holds for $\widetilde{S}_K(X,\omega^X)$, when replacing d_t by $\tilde{d}_t$ and ω^X by $2i\pi\langle\mu,K\rangle$. Also Theorem 3.9 still holds for $\widetilde{S}_K(X,\omega^X)$.

An explanation for the above results is as follows.

Theorem 3.18. *The following identity holds*

$$S_K(X,\omega^X) = \widetilde{S}_K(X,\omega^X) \,\mathrm{in}\, P_K^X/P_K^{X,0}. \tag{3.35}$$

Proof. Clearly

$$d_t - \tilde{d}_t = (-\omega^X/2\pi + i\langle\mu,K\rangle)b_t/t. \tag{3.36}$$

Now, we can reproduce the construction of the current $S_K(X,\omega^X)$ in (3.12)-(3.14), by replacing d_t by b_t/t. Let $T_K(X,\omega^X)$ be the corresponding current. We claim that

$$T_K(X,\omega^X) = 0 \text{ in } P_K^X/P_K^{X,0}. \tag{3.37}$$

In fact, this follows from the identities in s, which are valid in the obvious domains of definition

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-2}dt = \frac{1}{\Gamma(s)(s-1)}\,,\ \frac{1}{\Gamma(s)}\int_1^{+\infty} t^{s-2}dt = -\frac{1}{\Gamma(s)(s-1)}. \tag{3.38}$$

From (3.31), (3.36) and (3.38), we get (3.35). The proof of our Theorem is completed. □

Observe that by the analogue of (3.16), if η is a smooth form on X,

$$\begin{aligned}\int_X \eta\widetilde{S}_K(X,\omega^X) = \int_0^1 \left\{\int_X \eta(-i\langle\mu,K\rangle)\exp\big(t\frac{\overline{\partial}_K\partial_K}{2\pi}\langle\mu,K\rangle\big)\right\}dt \\ +\int_1^{+\infty}\left\{\int_X \eta(-i\langle\mu,K\rangle)\left(\exp\big(t\frac{\overline{\partial}_K\partial_K}{2\pi}\langle\mu,K\rangle\big) - \frac{\delta_{X_K}}{c_{\max,\,K}(N_{X_K/X},h^{N_{X_K/X}})} - \frac{\rho_1}{t}\right)\right\}dt \\ +\int_{X_K}\eta\frac{i\langle\mu,K\rangle}{c_{\max,\,K}(N_{X_K/X},h^{N_{X_K/X}})} + \Gamma'(1)\int_X \eta i\langle\mu,K\rangle\rho_1.\end{aligned} \tag{3.39}$$

Equation (3.39) is interesting. In fact let $\mathbf{C},|\ |$ be the trivial hermitian line bundle. Then

$$c_{1,K}(\mathbf{C},e^{T\langle\mu,K\rangle/2}|\ |) = i\frac{\overline{\partial}_K\partial_K}{2\pi}T\langle\mu,K\rangle. \tag{3.40}$$

Clearly, on X, we can define equivariant Bott-Chern currents by the method of [BGS1, part I], by simply replacing the operators $\partial,\overline{\partial}$ by the operators $\partial_K,\overline{\partial}_K$. They will be also noted with a $\widetilde{\ }$. Also by [BGS1, part I, Theorem 1.27], we have the identity

$$\widetilde{\mathrm{ch}}_K(\mathbf{C},|\ |,e^{T\langle\mu,K\rangle/2}|\ |) = -\int_0^T\langle\mu,K\rangle\exp\big(ti\frac{\overline{\partial}_K\partial_K}{2\pi}\langle\mu,K\rangle\big)dt \,\mathrm{in}\, P_K^X/P_K^{X,0}. \tag{3.41}$$

Of course the left-hand side of (3.40) still makes sense when replacing $\langle \mu, K\rangle$ by $-i\langle \mu, K\rangle$. By (3.39),(3.40), we see that the current $\widetilde{S}_K(X, \omega^X)$ is a normalized limit as $T \to +\infty$ of Bott-Chern currents $-\widetilde{\mathrm{ch}}(\mathbf{C}, |\ |, e^{-iT\langle \mu, K\rangle/2}|\ |)$.

4. Harmonic oscillator and the genus I

In this Section, we construct the genus I which will appear in our main formula. The genus I appears as the integral of an auxiliary genus Φ, which itself appears in a computation involving the heat kernels of semigroups which are associated to an harmonic oscillator. As explained in the Introduction, this harmonic oscillator is not lower bounded, because the corresponding quadratic potential is in general negative. The associated heat kernel is then defined via analytic continuation. This is a strong signal of difficulties to come, the estimates in the proof of our main result being made more difficult because the quadratic potential has the 'wrong' sign.

This Section is organized as follows. In Section 4.1, we introduce the function $I(\theta, \theta', x)$, and in Section 4.2, the function $\Phi_s(\theta, \theta', x)$. In Section 4.3, we define the associated additive genera $\Phi_{s,g,B}$ and $I_{g,B}$. In Section 4.4, using Mehler's formula, we compute the kernel of the semigroup associated to a harmonic oscillator. In Section 4.5, we obtain the genus $\Phi_{s,g,B}$ as the supertrace of the heat kernel associated to an harmonic oscillator. Finally in Section 4.6, under the assumptions of Sections 1 and 2, we construct the characteristic classes $I_{g,K}(N_{X_K/X})$ and $\Phi_{s,g,K}(N_{X_K/X})$.

4.1. **The function I.**

Definition 4.1. For $\theta \in \mathbf{R}$, $\theta' \in \mathbf{R}$, $|\theta'|$ small enough, and $x \in \mathbf{C}$, $|x|$ small enough, set

$$I(\theta, \theta', x) = \sum_{\substack{k \in \mathbf{Z} \\ 2k\pi + \theta \neq 0}} \frac{\log\left(1 + \frac{\theta'}{2k\pi + \theta}\right)}{i(2k\pi + \theta + \theta') + x}. \tag{4.1}$$

Recall that the function $R^\theta(x)$ was introduced in Definition 1.9.

Theorem 4.2. *The function $I(\theta, \theta', x)$ is periodic in θ of period 2π. If $\theta \notin 2\pi\mathbf{Z}$,*

$$I(\theta, \theta', x) = R(\theta, x + i\theta') - R(\theta + \theta', x). \tag{4.2}$$

Moreover $I(\theta, 0, x) = 0$. For $\theta' \in]-2\pi, 2\pi[\backslash 0$, $x \in \mathbf{C}$, $|x| < \inf |\theta' + 2k\pi|$, then

$$I(0, \theta', x) = R(x + i\theta') - R(\theta', x) + R^{\theta'}(x). \tag{4.3}$$

Proof. First we consider the case $\theta \notin 2\pi\mathbf{Z}$. We may and we will assume that $\theta \in]0, 2\pi[$. Clearly

$$I(\theta, \theta', x) = \sum_{k \in \mathbf{Z}^*} \frac{\log\left(1 + (\theta + \theta')/2k\pi\right) - \log\left(1 + \theta/2k\pi\right)}{i(2k\pi + \theta + \theta') + x} + \frac{\log(1 + \theta'/\theta)}{i(\theta + \theta') + x}. \tag{4.4}$$

Using (1.38), (4.4), we get (4.2). Clearly $I(\theta, 0, x) = 0$. We will now assume that θ', x are taken as indicated before (4.3). Then

$$I(0, \theta', x) = \sum_{k \in \mathbf{Z}^*} \frac{\log\left(1 + \theta'/2k\pi\right)}{i(2k\pi + \theta') + x}. \tag{4.5}$$

From (1.38), (4.5), we get (4.3). The proof of our Proposition is completed. □

4.2. The function Φ_s.

Definition 4.3. For $\theta \notin 2\pi\mathbf{Z}$, and $|\theta'|, |x|$ small, or for $\theta \in 2\pi\mathbf{Z}$, $|\theta'|, |x|$ small, with $x + i\theta' \neq 0$, and $s \in]0,1]$, set

$$\Phi_s(\theta,\theta',x) = \frac{1}{2}\left[\coth(\frac{i\theta + si\theta'}{2}) - \coth(\frac{i\theta + i\theta' + x}{2})\right]\frac{si\theta'}{(1-s)i\theta' + x}. \tag{4.6}$$

Observe that even if $x + (1-s)\theta'$ may vanish, the expression (4.6) is still well-defined by analytic continuation. In particular $\Phi_s(\theta,\theta',x)$ depends smoothly on $s \in]0,1]$. Also $\Phi_s(\theta,\theta',x)$ is periodic in θ of period 2π. Finally if $\theta \notin 2\pi\mathbf{Z}$,

$$\Phi_s(\theta,0,x) = 0. \tag{4.7}$$

Proposition 4.4. *As $s \in]0,1] \to 0$,*

$$\Phi_s(\theta,\theta',x) = 1_{\theta\in 2\pi\mathbf{Z}}\frac{1}{i\theta' + x} + \mathcal{O}(s). \tag{4.8}$$

Proof. Clearly, if $\theta \notin 2\pi\mathbf{Z}$, as $s \to 0$,

$$\Phi_s(\theta,\theta',x) = \mathcal{O}(s). \tag{4.9}$$

Also

$$\Phi_s(0,\theta',x) = \frac{1}{2}\left[\coth(si\theta'/2) - \coth((i\theta' + x)/2)\right]\frac{si\theta'}{(1-s)i\theta' + x}. \tag{4.10}$$

From (4.9), (4.10), we get (4.8). The proof of our Theorem is completed. □

Proposition 4.5. *The following identity holds,*

$$\Phi_s(\theta,\theta',x) = \sum_{k\in\mathbf{Z}}\frac{s\theta'}{(2\pi k + s\theta' + \theta)(2i\pi k + i\theta + i\theta' + x)}. \tag{4.11}$$

Proof. We use (4.6) and the identity

$$\frac{1}{2}\coth(x/2) = \sum_{k\in\mathbf{Z}}\frac{1}{2i\pi k + x}. \tag{4.12}$$

□

Theorem 4.6. *The following identity holds,*

$$\int_0^1\Big(\Phi_s(\theta,\theta',x) - 1_{\theta\in 2\pi\mathbf{Z}}\frac{1}{i\theta' + x}\Big)\frac{ds}{s} = I(\theta,\theta',x). \tag{4.13}$$

Proof. By (4.11),

$$\Phi_s(\theta,\theta',x) - 1_{\theta\in 2\pi\mathbf{Z}}\frac{1}{i\theta' + x} = \sum_{\substack{k\in\mathbf{Z}\\ 2k\pi+\theta\neq 0}}\frac{s\theta'}{(2k\pi + s\theta' + \theta)(2ik\pi + i\theta + i\theta' + x)}. \tag{4.14}$$

By (4.1),(4.14), we get (4.13). □

4.3. **The genera $\Phi_{s,g,B}, I_{g,B}$.** Let X be a complex manifold. Let N be a holomorphic vector bundle on X. Let h^N be a hermitian metric on N.

Let $g \in \mathrm{Aut}(N)$ be a holomorphic isometry of (N, h^N). Let $e^{i\theta_1}, \ldots, e^{i\theta_q}, 0 \leq \theta_j < 2\pi$ be the locally constant distinct eigenvalues of the action of g on N. Let B be a holomorphic skew-adjoint section of $\mathrm{End}(N)$, which commutes with g. Let $i\theta'_1, \ldots, i\theta'_{q'}, \theta'_{j'} \in \mathbf{R}$ be the locally constant eigenvalues of the action of B on N. Then N splits orthogonally as

$$N = \oplus_{\substack{1 \leq j \leq q \\ 1 \leq j' \leq q'}} N^{\theta_j, \theta'_{j'}}. \tag{4.15}$$

We make the basic assumption that in (4.15), θ and θ' do not vanish together.

Set

$$\begin{aligned} N^+ &= \oplus_{\substack{1 \leq j \leq q \\ 1 \leq j' \leq q', \theta'_{j'} \neq 0}} N^{\theta_j, \theta'_{j'}}, \\ N^0 &= \oplus_{1 \leq j' \leq q'} N^{0, \theta'_{j'}}. \end{aligned} \tag{4.16}$$

Then N^+, N^0 are holomorphic subbundles of N and $N^0 \subset N^+$. Let h^{N^+}, h^{N^0} be the metrics on N^+, N^0 induced by h^N on N^+, N^0.

Let $\nabla^{N^{\theta_j, \theta'_{j'}}}$ be the holomorphic Hermitian connection on $N^{\theta_j, \theta'j'}$, and let $R^{N^{\theta_j, \theta'j'}}$ be its curvature.

If A is a (q, q) invertible matrix, using the notation in (3.10), we get

$$\frac{c'_{\max}}{c_{\max}}(A) = \mathrm{Tr}[A^{-1}]. \tag{4.17}$$

Definition 4.7. Put

$$\begin{aligned} \Phi_{s,g,B}(N, h^N) &= \sum_{\substack{1 \leq j \leq q \\ 1 \leq j' \leq q'}} \mathrm{Tr}\left[\Phi_s\left(\theta_j, \theta'_{j'}, -\frac{R^{N^{\theta_j, \theta_{j'}}}}{2i\pi}\right)\right], \\ I_{g,B}(N, h^N) &= \sum_{\substack{1 \leq j \leq q \\ 1 \leq j' \leq q'}} \mathrm{Tr}\left[I\left(\theta_j, \theta'_{j'}, -\frac{R^{N^{\theta_j, \theta_{j'}}}}{2i\pi}\right)\right], \\ \left(\frac{c'_{\max}}{c_{\max}}\right)_B (N^0, h^{N^0}) &= \sum_{1 \leq j' \leq q'} \mathrm{Tr}\left[\left(i\theta'_{j'} - \frac{R^{N^{0, \theta'_{j'}}}}{2i\pi}\right)^{-1}\right]. \end{aligned} \tag{4.18}$$

Observe that by Theorem 4.2, and by (4.7), if $\theta \notin 2\pi\mathbf{Z}$, then $I(\theta, 0, x) = 0, \Phi_s(\theta, 0, x) = 0$, so that

$$\begin{aligned} I_{g,B}(N, h^N) &= I_{g,B}(N^+, h^{N^+}), \\ \Phi_{s,g,B}(N, h^N) &= \Phi_{s,g,B}(N^+, h^{N^+}). \end{aligned} \tag{4.19}$$

Proposition 4.8. *As $s \in]0, 1] \to 0$,*

$$\Phi_{s,g,B}(N, h^N) = \left(\frac{c'_{\max}}{c_{\max}}\right)_B (N^0, h^{N^0}) + \mathcal{O}(s). \tag{4.20}$$

Proof. This follows from Proposition 4.4 . ☐

Theorem 4.9. *The following identity holds,*

$$\int_0^1 \Big(\Phi_{s,g,B}(N,h^N) - \Big(\frac{c'_{\max}}{c_{\max}}\Big)_B (N^0,h^{N^0})\Big)\frac{ds}{s} = I_{g,B}(N^+,h^{N^+}). \tag{4.21}$$

Proof. This follows from Theorem 4.6 and from (4.20). □

4.4. **The heat kernel of the harmonic oscillator and its analytic extension.**

For $u \in \mathbf{C}, \eta \in \mathbf{C}, x \in \mathbf{C}$, put

$$\sigma(u,\eta,x) = 4\sinh\left(\frac{x-2\eta+\sqrt{x^2+4u}}{4}\right)\sinh\left(\frac{-x+2\eta+\sqrt{x^2+4u}}{4}\right). \tag{4.22}$$

Note that $\sqrt{x^2+4u}$ is well defined up to sign, but that $\sigma(u,\eta,x)$ does not depend on the choice of the square root. In particular $\sigma(u,\eta,x)$ is a holomorphic function of its arguments, which is periodic in η of period $2i\pi$. By [B12, Proposition 4.2],

$$\sigma(u,i\theta,x) = (\theta^2+i\theta x+u)\prod_{k\in\mathbf{Z}^*}\left(\frac{(\theta+2k\pi)^2+i(\theta+2k\pi)x+u}{4k^2\pi^2}\right), \tag{4.23}$$

and if $\theta \notin 2\pi\mathbf{Z}$,

$$\sigma(u,i\theta,x) = 4\sin^2(\theta/2)\prod_{k\in\mathbf{Z}}\big(1+\frac{ix}{\theta+2k\pi}+\frac{u}{(\theta+2k\pi)^2}\big). \tag{4.24}$$

Let V be a finite dimensional complex hermitian vector space. Let F, H be commuting elements of $\mathrm{End}(V)$, with F self-adjoint and H skew-adjoint. Let $e_1,\dots,e_{2\ell}$ be an orthonormal base of $V_{\mathbf{R}}$. Consider the differential operator

$$\mathcal{L} = -\frac{1}{2}(\nabla_{e_i}+\frac{1}{2}\langle HZ,e_i\rangle)^2+\frac{1}{2}\langle FZ,Z\rangle. \tag{4.25}$$

Put

$$P = H^2+4F. \tag{4.26}$$

Then P is a self-adjoint element of $\mathrm{End}(V)$. Clearly

$$\mathcal{L} = -\frac{1}{2}\Delta-\frac{1}{2}\nabla_{HZ}+\frac{1}{8}\langle PZ,Z\rangle. \tag{4.27}$$

Set

$$\mathcal{M} = \mathcal{L}+\frac{1}{2}\nabla_{HZ}. \tag{4.28}$$

Then

$$[\mathcal{M},\nabla_{HZ}] = 0. \tag{4.29}$$

Also the operator $\mathcal{M}$, with domain $\mathcal{C}_0(V_{\mathbf{R}},\mathbf{R})$, is a formally self-adjoint operator. If $P \geq 0$, the operator $\mathcal{M}$, when viewed as an unbounded operator acting on $L_2(V_{\mathbf{R}})$, has a self-adjoint closure. When $P > 0$, the operator $\mathcal{M}$ is a harmonic oscillator. When $P \geq 0$, let Q be the obvious nonnegative self-adjoint square root of P.

When $P \geq 0$, let $p(Z,Z'), Z, Z' \in V_{\mathbf{R}}$, be the smooth kernel of the operator $\exp(-\mathcal{L})$ with respect to $dv_V/(2\pi)^{\dim V}$. By using Mehler's formula [GlJ, Theorem

1.5.10] as in [B7, eq. (6.37), (6.38)], [B12, eq. (4.48), (4.49)] we get

$$p(Z,Z') = \det\left(\frac{Q/2}{\sinh(Q/2)}\right)\exp\left\{-\frac{1}{2}\left\langle\frac{Q/2}{\tanh(Q/2)}Z,Z\right\rangle -\frac{1}{2}\left\langle\frac{Q/2}{\tanh(Q/2)}Z',Z'\right\rangle + \left\langle\frac{Q/2}{\sinh(Q/2)}e^{H/2}Z,Z'\right\rangle\right\}. \tag{4.30}$$

Observe that in (4.30), $p(Z,Z')$ depends analytically on P, since all the functions of Q which appear are in fact functions of $Q^2 = P$. Therefore the kernel $p(Z,Z')$ extends to a holomorphic function of (F,H), which is well-defined as long as for any $k \in \mathbf{N}^*$, $-4k^2\pi^2$ is not an eigenvalue of P, although the operator $\mathcal{M}$ is not lower bounded. Still, in this case, we will say that $p(Z,Z')$ is the smooth kernel associated to the operator $\exp(-\mathcal{L})$.

A functional integral interpretation of the above is as follows. Let $Z,Z' \in V_{\mathbf{R}}$, and let $\mathfrak{P}_{(Z,Z')}$ be the probability law of the Brownian bridge $Z_s, s \in [0,1]$, with $Z_0 = Z, Z_1 = Z'$. Then by using Feynman-Kac formula and the Ito calculus as in [B7, eq. (4.11)], we find that if $P \geq 0$, , so that the operator $\exp(-\mathcal{L})$ is well defined, then

$$p(Z,Z') = \exp(-|Z-Z'|^2/2)E^{\mathfrak{P}_{(Z,Z')}}\left[\exp\left(\int_0^1\langle HZ_s,dZ_s\rangle/2 - \int_0^1\langle FZ_s,Z_s\rangle/2\right)\right]. \tag{4.31}$$

In (4.31), $E^{\mathfrak{P}_{(Z,Z')}}$ is the expectation operator with respect to $\mathfrak{P}_{(Z,Z')}$, and the integral

$$\int_0^1\langle HZ_s,dZ_s\rangle$$

is an Ito or Stratonovitch stochastic integral. More generally, the condition $P > -4\pi^2$ guarantees that the expression in the right-hand side of (4.31) is integrable with respect to $\mathfrak{P}_{(Z,Z')}$.

Proposition 4.10. *If F is positive definite, there exists $C > 0, C' > 0$ such that for any skew-adjoint $H \in \operatorname{End}(V)$ commuting with F such that $H^2 \geq -4\pi^2$, then*

$$|p(Z,Z)| \leq C\exp\big(-C'|Z|^2\big). \tag{4.32}$$

Proof. By (4.30),

$$p(Z,Z) = \det\left(\frac{Q/2}{\sinh(Q/2)}\right) \exp\left(-\frac{1}{2}\left\langle\frac{Q/2}{\sinh(Q/2)}4\sinh\big((H+Q)/4\big)\sinh\big((H-Q)/4\big)Z,Z\right\rangle\right). \tag{4.33}$$

Clearly $\frac{Q/2}{\sinh(Q/2)}$ is positive definite and remains uniformly bounded. The operator $4\sinh\big((H+Q)/2\big)\sinh\big((H-Q)/2\big)$ is self-adjoint and nonnegative. We claim that it has a strictly positive lower bound, which is uniformly bounded away from 0. In fact, by (4.23),

$$4\sinh\big((H+Q)/4\big)\sinh\big((H-Q)/4\big) = F\prod_{k\in\mathbf{Z}^*}\big(1+\frac{iH}{2k\pi}+\frac{F}{4k^2\pi^2}\big). \tag{4.34}$$

By (4.34), we get the required uniform lower bound. Using (4.33), we thus get (4.32). The proof of our Proposition is completed. □

4.5. **Harmonic oscillator and the genus $\Phi_{s,g,B}$.** Let X be a complex manifold. Let

$$\mathcal{E} : 0 \to L \to M \to N \to 0 \tag{4.35}$$

be a short exact sequence of holomorphic vector bundles on X. We assume that the sequence (4.35) splits holomorphically, i.e.

$$M = L \oplus N, \tag{4.36}$$

Let h^M be a hermitian metric on M which is such that the splitting (4.36) is orthogonal. Let h^L, h^N be the corresponding metrics on L, N. Let $g \in \mathrm{Aut}(M)$ be a holomorphic isometry of M, which preserves L and N. Let $e^{i\theta_1}, \dots, e^{i\theta_q}$, $0 \leq \theta_j < 2\pi$ be the distinct eigenvalues of the action of g on $\mathcal{E}$. Then the exact sequence $\mathcal{E}$ splits into a direct sum

$$\mathcal{E} = \oplus_{1 \leq j \leq q} \mathcal{E}^{\theta_j}, \tag{4.37}$$

so that g acts on $\mathcal{E}^{\theta_j}$ by multiplication by $e^{i\theta_j}$, and the splitting (4.37) is orthogonal. Let B be a holomorphic skew-adjoint section of $\mathrm{End}(M)$ which also preserves L and N and which commutes with g. Let $i\theta'_1, \dots, i\theta'_{q'}$, $\theta'_j \in \mathbf{R}$ be the distinct eigenvalues of B acting on $\mathcal{E}$. Then the splitting (4.37) can be refined into

$$\mathcal{E} = \oplus_{\substack{1 \leq j \leq q \\ 1 \leq j' \leq q'}} \mathcal{E}^{\theta_j, \theta'_{j'}}, \tag{4.38}$$

so that on $\mathcal{E}^{\theta_j, \theta'_{j'}}$, g acts by multiplication by $e^{i\theta_j}$, and B by multiplication by $i\theta'_{j'}$. Of course, the splitting (4.38) is still orthogonal. Let $h^{N^{\theta_j, \theta'_{j'}}}$ be the corresponding metric on $N^{\theta_j, \theta'_{j'}}$. Let $\nabla^N, \nabla^{N^{\theta_j \theta'_{j'}}}$ be the holomorphic hermitian connection on $(N, h^N), (N^{\theta_j, \theta'_{j'}}, h^{N^{\theta_j, \theta'_{j'}}})$, and let $R^N, R^{N^{\theta_j, \theta'_{j'}}}$ be the corresponding curvatures .

We make the basic assumption that

$$\mathcal{E}^{0,0} = L. \tag{4.39}$$

Equivalently, in M, N is the direct sum of the eigenbundles of g and B associated to $\theta > 0$ or $\theta' \neq 0$.

Let J^M be the complex structure of $M_{\mathbf{R}}$. Let $da, \overline{da}$ be odd Grassmann variables. Since $\mathbf{C} \simeq \mathbf{R}^2$, then $\Lambda(\mathbf{R}^2) \otimes_{\mathbf{R}} \mathbf{C}$ is the algebra generated by $1, da$ and $\overline{da}$. Moreover $\Lambda^1(\mathbf{R}^2) \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C}(da, \overline{da})$. In the sequel, we will consider the graded tensor product $\big(\Lambda(\mathbf{R}^2) \otimes_{\mathbf{R}} \mathbf{C}\big) \widehat{\otimes} c(N^*_{\mathbf{R}})$, so that $da, \overline{da}$ will anticommute with the odd elements of $c(N^*_{\mathbf{R}})$.

Let $\ell'' = \dim L, \ell = \dim M$. Let $e_1, \dots, e_{2\ell''}$ be an orthonormal basis of $L_{\mathbf{R}}$, let $e_{2\ell''+1}, \dots, e_{2\ell}$ be an orthonormal basis of $N_{\mathbf{R}}$.

Recall that $\Lambda(N^{*(0,1)})$ is a $c(N_{\mathbf{R}})$ Clifford module. To make our notation simpler, we will assume that if $U \in L_{\mathbf{R}}$, then $c(U)$ acts like 0 on $\Lambda(N^{*(0,1)})$.

Definition 4.11. For $s \in]0,1]$, let L_0^s be the differential operator acting on $\mathcal{C}_\infty(M_{\mathbf{R}}, \mathbf{C}(da,\overline{da})\widehat{\otimes}\Lambda(N^{*(0,1)}))$,

$$\begin{aligned} L_0^s = &-\frac{1}{2}\left(\nabla_{e_i} + \frac{1}{2}\langle(R^M-(1+s)B)Z, e_i\rangle\right)^2 + \frac{s}{2}|BZ|^2 \\ &-\frac{s}{4}\langle Be_i, e_j\rangle c(e_i)c(e_j) + \frac{s}{2}\langle R^M BZ, Z\rangle - \frac{s}{2}idac(J^M BZ) - \frac{is}{2}da\overline{da}\langle J^M BZ, Z\rangle \\ &+\frac{1}{2}\mathrm{Tr}[R^M] - \frac{1}{2}\mathrm{Tr}[B]. \end{aligned} \tag{4.40}$$

Using the notation in (4.25), set

$$\begin{aligned} F &= -sB^2 + s(R^M - iJ^M da\overline{da})B, \\ H &= R^M - (1+s)B. \end{aligned} \tag{4.41}$$

Put

$$P = H^2 + 4F \tag{4.42}$$

A straightforward computation shows that

$$P = (R^M - (1-s)B)^2 - 4siJ^M Bda\overline{da}. \tag{4.43}$$

Let $P^{(0)}$ be the component of degree 0 of P in the considered Grassmann variables. Then

$$P^{(0)} = (1-s)^2 B^2. \tag{4.44}$$

Therefore, if $|B|$ is small enough, for $s \in]0,1]$, then $P^{(0)} > -4\pi^2$. For $|B|$ small enough, we can then use the results of Section 4.4.

Definition 4.12. For $|B|$ small enough, let $q^s(Z,Z'), Z, Z' \in M_{\mathbf{R}}$ be the smooth kernel of the operator $\exp(-L_0^s)$ with respect to the volume $dv_M(Z')/(2\pi)^{\dim M}$.

Then $q^s(Z,Z') \in \big(\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}\big)\widehat{\otimes}\Lambda(T^*_{\mathbf{R}}X)\widehat{\otimes}\mathrm{End}(\Lambda(N^{*(0,1)}))$. In the sequel, we will take the supertrace of $gq^s(g^{-1}Z,Z)$. In the case where $N=0$, we just consider $q^s(Z,Z) \in \Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}$.

Theorem 4.13. *For $s \in]0,1]$, the form $\mathrm{Tr_s}[gq^s(g^{-1}Z,Z)]$ is invariant by translations by elements of $L_{\mathbf{R}}$. If $|B|$ is small enough, for any $s \in]0,1]$, there is $C>0, C'>0$ such that if $Z \in N_{\mathbf{R}}$, then*

$$|q^s(g^{-1}Z,Z)| \leq C\exp(-C'|Z|^2). \tag{4.45}$$

Moreover, if $|B|$ is small enough, the following identity holds,

$$\begin{aligned} \varphi\int_N \mathrm{Tr_s}[gq^s(g^{-1}Z,Z)]\frac{dv_N(Z)}{(2\pi)^{\dim N}} = \mathrm{Td}_{ge^B}(M,h^M) \\ \big(1+da\overline{da}\Phi_{s,g,B}(N^+,h^{N^+})\big). \end{aligned} \tag{4.46}$$

Proof. To prove (4.45) and (4.46), we may as well assume that g acts on M by multiplication by $e^{i\theta}, 0 \leq \theta < 2\pi$ and B by multiplication by $i\theta'$.

Now we use the notation in (4.22)-(4.47). By the above, it follows that $\sigma(F,i\theta,H) \in \mathrm{End}(M)$ is well-defined. Put

$$\Sigma(F,i\theta,H) = \det[\sigma(F,i\theta,H)]. \tag{4.47}$$

If $\theta' = 0$, i.e. if $B = 0$, then

$$P = R^{M,2}, \tag{4.48}$$

of which a trivial square root Q is R^M. If $\theta' \neq 0$, then B is invertible. If $s \in]0,1[$, a square root Q of P is given by

$$Q = R^M - (1-s)B - da\overline{da}(R^M - (1-s)B)^{-1} 2isJ^M B. \tag{4.49}$$

More generally, in all cases (including the case where $s = 1$), we temporarily assume that R^M is invertible (so that $R^M(R^M)^{-1} = 1$), and write a square root Q of P as in (4.49). Using (4.49), we get the identity ,

$$\begin{aligned} \frac{H+Q}{2} &= R^M - B - da\overline{da}\frac{isJ^M B}{R^M - (1-s)B}, \\ \frac{-H+Q}{2} &= sB - da\overline{da}\frac{isJ^M B}{R^M - (1-s)B}. \end{aligned} \tag{4.50}$$

Let $\mathcal{L}^s$ be the differential operator defined as in (4.25), with $V = M$, and F, H as given in (4.41). Let $p^s(Z, Z')$ be the smooth kernel associated with the operator $\exp(-\mathcal{L}^s)$, with respect to the volume $dv_M(Z')/(2\pi)^{\dim M}$. By (4.30), we get

$$p^s(g^{-1}Z, Z) = \det\left[\frac{Q/2}{\sinh(Q/2)}\right] \exp\left(-\frac{1}{2}\left\langle \frac{Q/2}{\sinh(Q/2)}\sigma(F, i\theta, H)Z, Z\right\rangle\right). \tag{4.51}$$

First we consider the case where $\theta = 0, \theta' = 0$, i.e. $M = L$. Then

$$\sigma(F, 0, H) = 0. \tag{4.52}$$

Recall that $\widehat{A}$ is the multiplicative genus associated to $\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}$. By (4.51),(4.52), we obtain

$$p^s(Z, Z) = \widehat{A}(-R^L). \tag{4.53}$$

Using (4.40), (4.53), we get

$$\varphi q^s(Z, Z) = \mathrm{Td}(-R^L/2i\pi). \tag{4.54}$$

So we have established our Theorem when $\theta = 0, \theta' = 0$.

Now we assume that θ and θ' do not vanish together so that $M = N$. We claim that if $|B|$ is small enough, if $s \in]0,1]$, then

$$\left(\frac{Q/2}{\sinh(Q/2)}\right)^{(0)}$$

and

$$\sigma(F, i\theta, H)^{(0)}$$

are positive definite operators. By (4.49), this is clear for the first operator. Also observe that by (4.50),

$$\sigma(F, 0, H)^{(0)} = 4\sinh(-B/2)\sinh(sB/2), \tag{4.55}$$

which is indeed positive for B invertible if$|B|$ is small enough. A similar argument can also be used in the case where $\theta \notin 2\pi\mathbf{Z}$.

From (4.51), we get (4.45) and we obtain

$$\int_{N_{\mathbf{R}}} p^s(g^{-1}Z,Z)\frac{dv_N(Z)}{(2\pi)^{\dim N}} = \frac{1}{\Sigma(F,i\theta,H)}. \tag{4.56}$$

By (4.22), (4.47)-(4.50), we get

$$\begin{aligned}\Sigma(F,i\theta,H) = \det\Bigg[4\sinh\left(\frac{1}{2}\left(R^N - B - i\theta + da\overline{da}\left(sB/\left(R^N-(1-s)B\right)\right)\right)\right)\\ \sinh\left(\frac{1}{2}\left(sB+i\theta+da\overline{da}sB/\left(R^N-(1-s)B\right)\right)\right)\Bigg].\end{aligned} \tag{4.57}$$

Equivalently,

$$\begin{aligned}\Sigma(F,i\theta,H) = \det\big[4\sinh\big((-R^N+B+i\theta)/2\big)\sinh\left(-(sB+i\theta)/2\right)\big]\\ \left(1-\frac{da\overline{da}}{2}\mathrm{Tr}\left[\frac{sB}{R^N-(1-s)B}\left(\coth\big((-R^N+B+i\theta)/2\big)-\right.\right.\right.\\ \left.\left.\left.\coth\left((sB+i\theta)/2\right)\right)\right]\right).\end{aligned} \tag{4.58}$$

From (4.58), we get

$$\begin{aligned}\frac{1}{\Sigma(F,i\theta,H)} = \left[\det\left(4\sinh\big((-R^N+B+i\theta)/2\big)\sinh\left(-(sB+i\theta)/2\right)\right)\right]^{-1}\\ \left(1+\frac{da\overline{da}}{2}\mathrm{Tr}\left[\frac{sB}{R^N-(1-s)B}\left(\coth\big((-R^N+B+i\theta)/2\big)-\coth\left((sB+i\theta)/2\right)\right)\right]\right).\end{aligned} \tag{4.59}$$

Moreover by [B7, eq. (5.6)],

$$\begin{aligned}\mathrm{Tr_s}^{\Lambda(N^{*(0,1)})}\left[g\exp\big(s/4\sum\langle Be_i,e_j\rangle c(e_i)c(e_j)\rangle\big)\right] =\\ e^{i\theta\dim N/2}\det[2\sinh\left(-(sB+i\theta)/2\right)].\end{aligned} \tag{4.60}$$

From (4.40), (4.56)-(4.60), we find that when $\theta > 0$,

$$\begin{aligned}\varphi\int_{N_{\mathbf{R}}}\mathrm{Tr_s}[gq^s(g^{-1}Z,Z)]\frac{dv_N(Z)}{(2\pi)^{\dim N}} = \mathrm{Td}_{ge^B}(N,h^N)\\ \left(1+\frac{da\overline{da}}{2}\mathrm{Tr}\left[\left(-\coth\big((-R^N/2i\pi+B+i\theta)/2\big)+\coth\left((sB+i\theta)/2\right)\right)\frac{sB}{-R^N/2i\pi+(1-s)B}\right]\right).\end{aligned} \tag{4.61}$$

From (4.6), (4.19), (4.54), '(4.61), we get (4.46). The proof of our Theorem is completed. □

4.6. **The characteristic classes $I_{g,K}(N_{X_K/X})$ and $\Phi_{s,g,K}(N_{X_K/X})$.** We make the same assumptions and we use the same notation as in Sections 1 and 2. Let $g \in G, K_0 \in \mathfrak{z}(g)$. Take $z \in \mathbf{R}^*, K = zK_0$. On $X_{g,K}$, g acts on $N_{X_K/X}$, with locally constant distinct eigenvalues $\exp^{i\theta_1},\dots,\exp^{i\theta_q}$. Similarly $B = \nabla_{\cdot}^{TX}K^X$ acts as a skew-adjoint morphism of $N_{X_K/X}$ which commutes with g, whose eigenvalues are also locally constant.

For $z \in \mathbf{R}^*$ close enough to 0, we can then define the characteristic classes $I_{g,B}(N_{X_K/X}), \Phi_{s,g,B}(N_{X_K/X})$ on X_g. In the sequel, we will note these classes $I_{g,K}(N_{X_K/X}), \Phi_{s,g,K}(N_{X_K/X})$.

5. A COMPARISON FORMULA FOR THE EQUIVARIANT TORSIONS

In this Section we state our main result, which is a local formula for the ratio of the infinitesimal equivariant Quillen metric to the equivariant Quillen metric. In this local formula, the current $S_K(X,\omega^X)$ constructed in Section 3 and the genus I introduced in Section 4 appear. Also we verify the compatibility of our formula to the immersion formulas for the Quillen metrics and for the current $S_K(X,\omega^X)$. Finally, we speculate on the arithmetic consequences of our formulas, in particular in connection with recent work by Köhler-Roessler [KRo1, KRo2] and Kaiser-Köhler [KaK]. We show that at least formally, the known results on Quillen metrics and analytic torsion forms are consequences of a formula expressing the current $S_K(X,\omega^X)$ in terms of arithmetic characteristic classes.

This Section is organized as follows. In Section 5.1, we state the formula comparing the two sorts of equivariant Quillen metrics. In Section 5.2, we show that this result combined with the immersion formulas for these Quillen metrics implies a non trivial identity on currents. In Section 5.3, we prove a nontrivial identity on Bott-Chern currents with excess normal bundle, which is essentially a restatement of the results given in Section 3.2 on excess formulas for the current $S_K(X,\omega^X)$. In Section 5.4, we give a direct proof of the compatibility of our main formula with the immersion formulas using these excess formulas. Finally in Section 5.5, we speculate on the arithmetic consequences of our identity.

5.1. **The main result.** For convenience, we state again the main result of this paper, already given in Theorem 0.1. Here we take $g \in G, K_0 \in \mathfrak{z}(g)$. If $z \in \mathbf{R}$, we take $K = zK_0$.

Theorem 5.1. *For $z \in \mathbf{R}^*$, if $|z|$ is small enough, the following identity holds*

$$\tag{5.1} \log\left(\frac{\| \ \|^{\sim}_{\lambda_G(E)}(g,K)}{\| \ \|_{\lambda_G(E)}(ge^K)}\right)^2 = \int_{X_g} \mathrm{Td}_{g,K}(TX,h^{TX})\mathrm{ch}_{g,K}(E,h^E)S_K(X_g,\omega^{X_g}) \\ - \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)I_{g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E).$$

5.2. **Compatibility with the Immersion formulas.** We temporarily assume that besides the above assumptions, the assumptions of Sections 1.6 and 2.8 also hold. Using Theorem 2.28, we find that for $|K|$ small enough,

$$\tag{5.2} \log\left(\frac{\| \ \|^{\sim}_{\lambda_G(E)}}{\| \ \|^{\sim}_{\lambda_G(F)}}\right)^2(g,K) - \log\left(\frac{\| \ \|_{\lambda_G(E)}}{\| \ \|_{\lambda_G(F)}}\right)^2(ge^K) = \\ -\int_{X_g} \mathrm{Td}_{g,K}(TX,h^{TX})T_{g,K}(X_g,h^E) + \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX,h^{TX})T_{ge^K}(X_{g,K},h^E) \\ + \int_{Y_g} \mathrm{Td}^{-1}_{g,K}(N_{Y/X},h^{N_{Y/X}})\widetilde{\mathrm{Td}}_{g,K}(TY|_{Y_g},TX|_{Y_g,},h^{TX|_{Y_g}})\mathrm{ch}_{g,K}(F,h^F) \\ - \int_{Y_{g,K}} \mathrm{Td}^{-1}_{ge^K}(N_{Y/X},h^{N_{Y/X}})\widetilde{\mathrm{Td}}_{ge^K}(TY|_{Y_{g,K}},TX|_{Y_{g,K}},h^{TX|_{Y_{g,K}}})\mathrm{ch}_{ge^K}(F,h^F) \\ -\int_{Y_g} \mathrm{Td}_{g,K}(TX)R_{g,K}(N_{Y/X})\mathrm{ch}_{g,K}(F) + \int_{Y_{g,K}} \mathrm{Td}_{ge^K}(TX)R_{ge^K}(N_{Y/X})\mathrm{ch}_{ge^K}(F).$$

On the other hand, by using Theorem 5.1, we find that

$$\begin{aligned}(5.3)\quad \log\left(\frac{\| \ \|^{\sim}_{\lambda_G(E)}}{\| \ \|^{\sim}_{\lambda_G(F)}}\right)^2(g,K)-\log\left(\frac{\| \ \|_{\lambda_G(E)}}{\| \ \|_{\lambda_G(F)}}\right)^2(ge^K)= \\ \int_{X_g}\mathrm{Td}_{g,K}(TX,h^{TX})\mathrm{ch}_{g,K}(E,h^E)S_K(X_g,\omega^{X_g}) \\ -\int_{Y_g}\mathrm{Td}_{g,K}(TY,h^{TY})\mathrm{ch}_{g,K}(F,h^F)S_K(Y_g,\omega^{Y_g}) \\ -\int_{X_{g,K}}\mathrm{Td}_{ge^K}(TX)I_{g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E)+\int_{Y_{g,K}}\mathrm{Td}_{ge^K}(TY)I_{g,K}(N_{Y_K/Y})\mathrm{ch}_{ge^K}(F).\end{aligned}$$

Now, we will give a direct explanation for the compatibility of formulas (5.2) and (5.3).

5.3. **An identity of Bott-Chern currents with an excess normal bundle.**

Consider the short exact sequence of vector bundles on $Y_{g,K}$,

$$(5.4)\qquad 0\to N_{Y_{g,K}/Y_g}\to N_{X_{g,K}/X_g}\to\widetilde{N}\to 0.$$

Equation (5.4) defines the vector bundle $\widetilde{N}$. This vector bundle is an excess normal bundle.

Observe that we have the holomorphic splitting

$$(5.5)\qquad N_{Y_g/X_g}=N_{Y_{g,K}/X_{g,K}}\oplus\widetilde{N}.$$

In fact, observe that K acts on N_{Y_g/X_g}. Then $N_{Y_{g,K}/X_{g,K}}$ is the eigenbundle associated to the eigenvalue 0, and $\widetilde{N}$ is the direct sum of eigenbundles associated to nonzero eigenvalues. Also the splitting (5.5) is orthogonal with respect to the obvious metric $h^{N_{Y_g/X_g}}$.

Let $\widetilde{c}^{-1}_{\max,K}(N_{Y_{g,K}/Y_g},N_{X_{g,K}/X_g},h^{N_{X_{g,K}/X_g}})$ be the Bott-Chern class on $X_{g,K}$, such that

$$\begin{aligned}(5.6)\quad \frac{\overline{\partial}\partial}{2i\pi}\widetilde{c}^{-1}_{\max,K}(N_{Y_{g,K}/Y_g},N_{X_{g,K}/X_g},h^{N_{X_{g,K}/X_g}})=c^{-1}_{\max,K}(N_{X_{g,K}/X_g},h^{N_{X_{g,K}/X_g}})- \\ c_{\max,K}(N_{Y_{g,K}/Y_g},h^{N_{Y_{g,K}/Y_g}})c^{-1}_{\max,K}(\widetilde{N},h^{\widetilde{N}}).\end{aligned}$$

Clearly, the vector bundle $\widetilde{N}$ splits holomorphically as

$$(5.7)\qquad \widetilde{N}=\oplus_{\theta'_{j'}}\widetilde{N}^{\theta'_{j'}},$$

so that $\nabla^{TX}K^X=m^{TX}(K)$ acts on $\widetilde{N}^{\theta'_{j'}}$ by multiplication by $i\theta'_{j'}$. Of course the splitting is orthogonal. Let $h^{\widetilde{N}^{\theta'_{j'}}}$ be the corresponding metric on $\widetilde{N}^{\theta'_{j'}}$.

Definition 5.2. Put

$$(5.8)\qquad R^K(\widetilde{N},h^{\widetilde{N}})=\sum R^{\theta'_{j'}}(\widetilde{N}^{\theta'_{j'}},h^{\widetilde{N}^{\theta'_{j'}}}).$$

Let $R^K(\widetilde{N})$ be the associated cohomology class.

Theorem 5.3. *For $z \in \mathbf{R}, |z|$ small enough, the following identity of currents holds,*

$$\begin{aligned}(5.9)\qquad & - T_{g,K}(X_g, h^E) - S_K(X_g, \omega^{X_g})\mathrm{ch}_{g,K}(E, h^E) + \\ & \frac{S_K(Y_g, \omega^{Y_g})\mathrm{ch}_{g,K}(F, h^F)}{\mathrm{Td}_{g,K}(N_{Y/X}, h^{N_{Y/X}})}\delta_{Y_g} + \frac{T_{ge^K}(X_{g,K}, h^E)}{c_{\max,K}(N_{X_{g,K}/X_g}, h^{N_{X_{g,K}/X_g}})}\delta_{X_{g,K}} \\ & - c_{\max,K}(\widetilde{N}, h^{\widetilde{N}})\widetilde{c^{-1}_{\max,K}}(N_{Y_{g,K}/Y_g}, N_{X_{g,K}/X_g}, h^{N_{X_{g,K}/X_g}}) \\ & \mathrm{Td}^{-1}_{g,K}(N_{Y/X}, h^{N_{Y/X}})\mathrm{ch}_{g,K}(F, h^F)\delta_{Y_{g,K}} \\ & + c^{-1}_{\max,K}(N_{Y_{g,K}/Y_g})\mathrm{Td}^{-1}_{g,K}(N_{Y/X})R_K(\widetilde{N})\mathrm{ch}_{g,K}(F)\delta_{Y_{g,K}} = 0 \text{ in } P^{X_g}_{K,X_K \cup Y_g}/P^{X_g,0}_{K,X_K \cup Y_g}.\end{aligned}$$

Proof. This result has been essentially proved in [B9, Theorem 3.2], this result being also restated in Theorem 3.12. Here we will use the notation in Section 3.2. In fact assume temporarily that $g = 1$, and that the complex (E, v) is a Koszul complex $(\Lambda(H^*), \sqrt{-1}i_\sigma)$. Here H is a holomorphic vector bundle on X on which G acts, σ is a G-invariant section of H. Then $d\sigma|_Y$ gives a G-equivariant identification

$$(5.10)\qquad H|_Y \simeq N_{Y/X}.$$

We assume that the metric h^E comes from a G-invariant metric h^H on H, which is such that the identification (5.10) is an isometry. Then by [BGS3, Theorem 3.17],

$$(5.11)\qquad T_K(X, h^E) = \mathrm{Td}^{-1}_K(H, h^H)\widetilde{e}^K(X, h^H) \text{ in } P^X_{K,Y}/P^{X,0}_{K,Y}.$$

In [BGS3], (5.11) has only been proved in the case where $K = 0$, but the proof immediately extends to the general case. The same argument shows that

$$(5.12)\qquad T_{e^K}(X_K, h^E) = \mathrm{Td}^{-1}_K(H, h^H)\widetilde{e}_K(X_K, h^H) \text{ in } P^{X_K}_{K,Y_K}/P^{X_K,0}_{K,Y_K}.$$

Then using Theorem 3.12, and (5.11), (5.12), we get (5.9) in this special case. In the case where $g = 1$, but the complex (E, v) is not necessarily Koszul, one can reproduce word for word the proof of [B9, Theorem 3.2] and obtain (5.9) in full generality.

When g is not necessarily equal to the identity, a combination of the arguments of [B9], [BGS3, Section 3 f)] and [B13, Section 6] leads easily to (5.9). Details are left to the reader. □

5.4. **A direct proof of the compatibility.** Now we will show how to verify directly the equality of the right-hand sides of (5.2) and (5.3) using (5.9).

In fact we multiply (5.9) by the form $\mathrm{Td}_{g,K}(TX, h^{TX})$, and integrate on X_g. We claim that this identity, when combined with Theorem 4.2, leads to the equality of the right-hand sides of (5.2) and (5.3). In fact one uses the obvious identity over $X_{g,K}$,

$$(5.13)\qquad \frac{\mathrm{Td}_{g,K}(TX, h^{TX})}{c_{\max,K}(N_{X_{g,K}/X_g}, h^{N_{X_{g,K}/X_g}})} = \mathrm{Td}_{ge^K}(TX, h^{TX}).$$

Also, by Theorem 3.9, we get

$$
\begin{aligned}
(5.14)\quad \int_{Y_g}\mathrm{Td}_{g,K}(TX,h^{TX})\frac{S_K(Y_g,\omega^{Y_g})}{\mathrm{Td}_{g,K}(N_{Y/X},h^{N_{Y/X}})}\mathrm{ch}_{g,K}(F,h^F) = {}&\\
\int_{Y_g}\mathrm{Td}_{g,K}(TY,h^{TY})S_K(Y_g,\omega^{Y_g})\mathrm{ch}_{g,K}(F,h^F)&\\
+\int_{Y_g}\mathrm{Td}^{-1}_{g,K}(N_{Y/X},h^{N_{Y/X}})\widetilde{\mathrm{Td}}_{g,K}(TY|_{Y_g},TX|_{Y_g},h^{TX|_{Y_g}})\mathrm{ch}_{g,K}(F,h^F)&\\
-\int_{Y_{g,K}}\Big(\mathrm{Td}_{g,K}(N_{Y/X},h^{N_{Y/X}})c_{\max,K}(N_{Y_{g,K}/Y_g},h^{N_{Y_{g,K}/Y_g}})\Big)^{-1}&\\
\widetilde{\mathrm{Td}}_{g,K}(TY|_{Y_g},TX|_{Y_g},h^{TX|_{Y_g}})\mathrm{ch}_{g,K}(F,h^F).&
\end{aligned}
$$

We claim that on $Y_{g,K}$,

$$
\begin{aligned}
(5.15)\qquad&\\
c_{\max,K}(\widetilde{N},h^{\widetilde{N}})\widetilde{c^{-1}_{\max,K}}(N_{Y_{g,K}/Y_g},N_{X_{g,K}/X_g},h^{N_{X_{g,K}/X_g}})\mathrm{Td}^{-1}(N_{Y/X},h^{N_{Y/X}})&\\
\mathrm{Td}_{g,K}(TX,h^{TX})+\Big(\mathrm{Td}_{g,K}(N_{Y/X},h^{N_{Y/X}})c_{\max,K}(N_{Y_{g,K}/Y_g},h^{N_{Y_{g,K}/Y_g}})\Big)^{-1}&\\
\widetilde{\mathrm{Td}}_{g,K}(TY|_{Y_g},TX|_{Y_g},h^{TX|_{Y_g}}) = {}&\\
\mathrm{Td}^{-1}_{ge^K}(N_{Y/X},h^{N_{Y/X}})\widetilde{\mathrm{Td}}_{ge^K}(TY|_{Y_{g,K}},TX|_{Y_{g,K}},h^{TX_{Y_{g,K}}})\text{in } P^{Y_{g,K}}/P^{Y_{g,K},0}.&
\end{aligned}
$$

In fact, let $N^{\perp}_{Y/X}$ be the direct sum of the eigenbundles of g associated to eigenvalues of g not equal to 1. Then we have the holomorphic splitting of vector bundles over $Y_{g,K}$,

$$
(5.16)\qquad N_{Y/X} = N_{Y_{g,K}/X_{g,K}} \oplus \widetilde{N} \oplus N^{\perp}_{Y/X}.
$$

Also the splitting in (5.16) is orthogonal. Using (5.16), we get identities of forms on $Y_{g,K}$ which extend (5.13),

$$
\begin{aligned}
(5.17)\qquad \mathrm{Td}_{ge^K}(TY,h^{TY}) &= \frac{\mathrm{Td}_{g,K}(TY,h^{TY})}{c_{\max,K}(N_{Y_{g,K}/Y_g},h^{N_{Y_{g,K}/Y_g}})},\\
\mathrm{Td}_{ge^K}(N_{Y/X},h^{N_{Y/X}}) &= \frac{\mathrm{Td}_{g,K}(N_{Y/X},h^{N_{Y/X}})}{c_{\max,K}(\widetilde{N},h^{\widetilde{N}})}.
\end{aligned}
$$

Using (5.13) and (5.17), when applying the operator $\frac{\overline{\partial}\partial}{2i\pi}$ to both sides of (5.15), we get a known identity. The identity (5.15) itself then follows from the uniqueness of Bott-Chern classes.

Also observe that by (1.47) and by Theorem 3.9,

$$
\begin{aligned}
\textstyle\int_{Y_g}\mathrm{Td}_{g,K}(TX)R_{g,K}(N_{Y/X})\mathrm{ch}_{g,K}(F)-\int_{Y_{g,K}}\mathrm{Td}_{ge^K}(TX)R_{ge^K}(N_{Y/X})\mathrm{ch}_{ge^K}(F)&\\
(5.18)\quad = \textstyle\int_{Y_{g,K}}\mathrm{Td}_{ge^K}(TX)\big(R_{g,K}(N_{Y/X})-R_{ge^K}(N_{Y/X})\big)\,\mathrm{ch}_{ge^K}(F),&\\
\textstyle\int_{X_{g,K}}\mathrm{Td}_{ge^K}(TX)I_{g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E)-\int_{Y_{g,K}}\mathrm{Td}_{ge^K}(TY)I_{g,K}(N_{Y_K/Y})\mathrm{ch}_{ge^K}(F)&\\
= \textstyle\int_{Y_{g,K}}\mathrm{Td}_{ge^K}(TY)I_{g,K}(N_{X_K/X}\backslash N_{Y_K/Y})\mathrm{ch}_{ge^K}(F).&
\end{aligned}
$$

Now using Theorem 4.2, we get

$$
(5.19)\qquad I_{g,K}(N_{X_K/X}\backslash N_{Y_K/Y}) = R_{g,K}(N_{Y/X})-R_{ge^K}(N_{Y/X})+R^K(\widetilde{N}).
$$

From (3.28), (5.14), (5.15), (5.18), (5.19), we have thus obtained a direct proof of the equality of the right-hand sides of (5.2) and (5.3).

5.5. **Riemann-Roch for the equivariant Quillen metrics.** Suppose that X is an arithmetic variety in the sense of Gillet-Soulé [GS1, GS2, GS3], i.e. X is an integral regular flat projective scheme over $\mathrm{Spec}(\mathbf{Z})$. Let ω^X be a Kähler form on $X_{\mathbf{C}}$ which is invariant under complex conjugation. Let $\left(E, h^E\right)$ be an algebraic Hermitian vector bundle on X. Here h^E is a Hermitian metric on $X_{\mathbf{C}}$ which is also invariant under complex conjugation. Let ω^X be a Kähler form on $X_{\mathbf{C}}$ which is invariant under complex conjugation. Put

$$\lambda = \det\left(H^{\cdot}(X,E)\right). \tag{5.20}$$

We equip $\lambda_{\mathbf{C}}$ with the Quillen metric $\| \ \|$ associated to ω^X, h^E.

The Hermitian vector bundles TX and E map to the Grothendieck group $\widehat{K}_0(X)$ of Hermitian vector bundles introduced by Gillet and Soulé in [GS2]. In [GS1, GS2], Gillet-Soulé also defined arithmetic Chow groups $\widehat{CH}(X)$ and a Chern character map $\widehat{\mathrm{ch}} : \widehat{K}_0(X) \mapsto \widehat{CH}(X)$. In particular, there is a map a from the vector space of sums of smooth real (p,p) forms modulo forms of the type $\partial\alpha + \overline{\partial}\beta$ into $\widehat{CH}(X)$. Other characteristic classes like the Todd class $\widehat{\mathrm{Td}}$ were constructed as well. The map $f : X \mapsto \mathrm{Spec}(\mathbf{Z})$ maps $\widehat{CH}(X)$ into $\widehat{CH}(\mathrm{Spec}(\mathbf{Z}))$. Also $\widehat{CH}^1(\mathrm{Spec}(\mathbf{Z})) \simeq \mathbf{R}$.

Let $\widehat{\deg}(\lambda) \in \mathbf{R}$ be the Arakelov degree of λ Then the Riemann-Roch theorem of Gillet-Soulé [GS4, Theorem 7] asserts in particular that

$$\widehat{\deg}(\lambda) = f_*\left(\widehat{\mathrm{Td}}(TX, h^{TX})\widehat{\mathrm{ch}}(E,h^E) - a\big(\mathrm{Td}(TX)R(TX)\mathrm{ch}(E)\big)\right)^{(1)}. \tag{5.21}$$

More generally if X and Y are non singular quasiprojective arithmetic varieties and if $f : X \mapsto Y$ is a projective morphism, let $\widehat{f_!} : \widehat{K}_0(X) \mapsto \widehat{K}_0(Y)$ be the direct image in $\widehat{K}_0$ theory. Note that the definition of $\widehat{f_!}$ incorporates the higher analytic torsion forms constructed in Bismut-Köhler [BK]. As an example, if for $i > 0$, $R^i\pi_*E = 0$, let $h^{R^0\pi_*E}$ be the obvious L_2 metric on $R^0\pi_*E$. By definition,

$$f_!(E) = \left(R^0\pi_*E, h^{R^0\pi_*E}\right) - a\big(T(\omega^X, h^E)\big). \tag{5.22}$$

As conjectured by Gillet-Soulé in [GS3], and proved by them in [GS5] using results of [B14] on the behaviour of the higher analytic torsion forms under embeddings (the results of [B14] extend the results of Bismut-Lebeau [BL], where the case where the base is a point was considered), (5.21) extends to the equality

$$\widehat{\mathrm{ch}}(f_!(E)) = f_*\left(\widehat{\mathrm{Td}}(TX, h^{TX})\widehat{\mathrm{ch}}(E,h^E) - a\big(\mathrm{Td}(TX)R(TX)\mathrm{ch}(E)\big)\right) \text{ in } \widehat{CH}(Y)_{\mathbf{Q}}. \tag{5.23}$$

A K-theoretic version of (5.23) has been given by Roessler [Ro].

Assume now that X is an arithmetic variety taken as before. Put $T = \mathrm{Spec}(t, t^{-1})$, and assume that X is equipped with a projective action of T. This induces in particular an action of $\mathbf{C}^*$ on $X_{\mathbf{C}}$. Of course $S^1 \subset \mathbf{C}^*$. We assume that E is T-equivariant, and we also suppose that the considered metrics are S^1-invariant.

Take $n \in \mathbf{N}$. Let U be the total space of the universal line bundle on $\mathbf{P}^n$. Then $\mathrm{Spec}(t,t^{-1})$ acts on U^* and X. Form the quotient[1] $V = U^* \times_{\mathrm{Spec}(t,t^{-1})} X$. Then $f : V \mapsto \mathbf{P}^n$ has fibre X. Let Θ be the curvature of the connection of the universal

---

[1] As pointed out by D. Roessler, quotients have to be handled carefully.

line bundle on $\mathbf{P}^n$ equipped with the Fubini-Study metric. Then by (2.74), the analytic torsion forms $T(\omega^X, h^E)$ associated to this projection (these are forms on $\mathbf{P}^n_{\mathbf{C}}$) are given by

$$T(\omega^X, h^E) = \frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)(1, -\frac{\Theta}{2i\pi})(0). \tag{5.24}$$

On the other hand, in [B12, Section 7 e)], Bismut conjectured an arithmetic Lefschetz formula with group actions, in which the genus $R(x)$ of Gillet and Soulé would be replaced by the genus $R(\theta, x)$. Using the results of [B13] described in Section 1.6, Köhler and Roessler [KRo1] have established such a formula, which takes the form

$$\widehat{\mathrm{ch}}_g(f_!(E)) = f_* \left(\widehat{\mathrm{Td}}_g(TX)\widehat{\mathrm{ch}}_g(E) - a\big(\mathrm{Td}_g(TX)R_g(TX)\mathrm{ch}_g(E)\big)\right). \tag{5.25}$$

In (5.25), if f is the projection $X \mapsto \mathrm{Spec}(\mathbf{Z})$, $\widehat{\mathrm{Td}}_g(TX), \widehat{\mathrm{ch}}_g(E)$ are arithmetic versions of the Atiyah-Bott characteristic forms evaluated on the fixed point set of g on X. The precise assumptions are given in [KRo1]. It is very likely that a similar statement holds when $\mathrm{Spec}(\mathbf{Z})$ is replaced by an arbitrary base Y. In (5.25), if for $i > 0$, $R^i f_* E = 0$, then

$$f_!(E) = \left(R^0 f_* E, h^{R^0 f_* E}\right) - a\left(T(\omega^X, h^E)\right)(.). \tag{5.26}$$

In (5.26), $\left(R^0 f_* E, h^{R^0 f_* E}\right)$ is considered as a G-equivariant Hermitian vector bundle, and $T(\omega^X, h^E)(.)$ is viewed as a form on Y depending on $g \in G$.

Now in the special case considered after (5.23), $R^{\cdot} f_* E$ is given by

$$R^{\cdot} f_*(E) = U^* \times_{\mathrm{Spec}(t,t^{-1})} H^{\cdot}(X, E). \tag{5.27}$$

From (5.27), we find that $R^{\cdot} f_*(E)$ can be obtained via the character of the representation of T on $H^{\cdot}(X, E)$, which is given by the Lefschetz fixed point formula.

Let us now apply the Köhler-Roessler formula (5.25) in degree 1, and also the Riemann-Roch formula of Gillet-Soulé given in (5.23) to the projection $f : V \mapsto \mathbf{P}^n$. Using Theorems 4.2 and 5.1, we should get the following conjectural equality in $\mathbf{R}$,

(5.28)

$$\int_X \mathrm{Td}_K(TX, h^{TX})\mathrm{ch}_K(E, h^E)S_K(X, \omega^X) = \int_X \widehat{\mathrm{Td}}_K(TX, h^{TX})\widehat{\mathrm{ch}}_K(E, h^E)$$
$$- \int_{X_K} \widehat{\mathrm{Td}}_{e^K}(TX, h^{TX})\widehat{\mathrm{ch}}_{e^K}(E, h^E) + \int_{X_K} \mathrm{Td}_{e^K}(TX)R^K(N_{X_K/X})\mathrm{ch}_{e^K}(E).$$

In (5.28), K is a rational element of $\mathbf{R}$, the Lie algebra of S^1. Also $\widehat{\mathrm{Td}}_K(TX, h^{TX})$ and $\widehat{\mathrm{ch}}_K(E, h^E)$ are still not defined equivariant arithmetic characteristic classes. More generally, one should expect that if α is an arbitrary equivariant characteristic class,

(5.29)

$$\int_X \alpha(K)S_K(TX, h^{TX}) = \int_X \widehat{\alpha}(K) - \int_{X_K} \widehat{\alpha}(K)\widehat{c^{-1}}_{\max,K}(N_{X_K/X}, h^{N_{X_K/X}})$$
$$+ \int_{X_K} \alpha(K)c^{-1}_{\max,K}(N_{X_K/X})R^K(N_{X_K/X}).$$

In fact a direct motivation for suspecting that (5.29) should be true is the immersion result for the currents $S_K(X, \omega^X)$ [B9, Theorem 3.2], which was stated in Theorem 3.12.

It is very likely that a formula like (5.29) can be given a direct proof, independent of Riemann-Roch, by using a technique of deformation to the normal cone.

However, it should be pointed out that part of Bismut's work on Quillen metrics was itself motivated by the fact that at least formally, the theorem of Riemann-Roch in Arakelov geometry is itself a consequence of a formula like (5.29) in infinite dimensions.This has been explained in some detail in [B9, B10].

Namely let (X, ω^X) be a compact Kähler manifold, and let LX be its loop space, i.e. the set of smooth functions from S^1 into X. Then S^1 acts on LX by $k_s x_. = x_{.+s}$. The generating vector field for this action of S^1 is just $K(x_.) = \frac{dx}{dt}_.$. The Kähler ω^X lifts to a S^1 -invariant Kähler form ω^{LX} on LX. Moreover K is a holomorphic vector field on LX (for a natural holomorphic structure on LX).

In [A], Atiyah explained an argument of Witten showing that at least formally, starting from the classical McKean Singer index formula [MKeS] for the Euler characteristic $\chi\big(\det TX)^{-1/2}\big)$(this is just equation (2.21), with $g = 1$) could be written formally in the form

$$\chi\big(\det TX)^{-1/2}\big) = \int_{LX} \exp\big(-(d + i_K)K'/2t\big). \tag{5.30}$$

Also in [A], it was observed that if $\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}$ is identified with the corresponding multiplicative genus, then one has the formal equality of cohomology classes on X,

$$c^{-1}_{\max,K}(X/LX) = \widehat{A}(TX). \tag{5.31}$$

Recall that b_t was defined in (3.5). In [B9, B8], identity (5.31) was extended to the formal equality

$$\chi(E) = \int_{LX} b_t \beta\left(L\big(E \otimes \big(\det TX\big)^{1/2}\big)\right). \tag{5.32}$$

In (5.32), $\beta\left(L\big(E \otimes \big(\det TX\big)^{1/2}\big)\right)$ is a natural lift of the Chern character form of $E \otimes \big(\det TX\big)^{1/2}$ to LX into a $d - 2i\pi i_K$-closed form, which was constructed in [B2].

In [B8], it was shown that at least formally

$$\frac{\partial}{\partial s}\theta(\omega^X, h^E) = \int_{LX} S_K(LX, \omega^{LX})\beta\left(L\big(E \otimes \big(\det TX\big)^{1/2}\big)\right). \tag{5.33}$$

In [B10], the main result of Bismut-Lebeau [BL], which is Theorem 1.15 for $g = 1$, was obtained as a formal consequence of the main result of [B9] stated in Theorem 3.12. In particular, an interpretation given in [B10] of the genus R of Gillet and Soulé [GS3] is that

$$R^K(N_{X/LX}) = R(TX) \tag{5.34}$$

One can then show that at least formally, equation (5.29) 'implies' the Riemann-Roch theorem of Gillet and Soulé [GS4] in the form stated in (5.23).

The theory has almost come full circle. An infinite dimensional version of infinite dimensional equivariant cohomology was a motivation behind some of the proofs needed in establishing analytic aspects of the theory of Quillen metrics and higher analytic torsion forms.

6. A PROOF OF THE COMPARISON FORMULA

The purpose of this Section is to establish our main result Theorem 5.1. The general organization of the Section is similar to [BL, Section 6]. In particular, we prove our main result by deforming a contour integral.

This Section is organized as follows. In Section 6.1, we construct a closed 1-form $\gamma_{t,v}$ on $\mathbf{R}_+^* \times \mathbf{R}_+^*$. In Section 6.2, we introduce a contour depending on two parameters,(a, A), with $0 < a < 1 < A < +\infty$, on which the integral of $\gamma_{t,v}$ vanishes identically. We thus get an identity $\sum_{k=1}^3 I_k^0 = 0$. In Section 6.3, we state two key intermediate results, whose proof is delayed to Sections 8 and 9. In Section 6.4, we study the asymptotics of the I_k^0's, as $A \to +\infty, a \to 0$. Finally, in Section 6.5, we obtain a final identity, which is shown to be equivalent to Theorem 5.1.

In this Section, we make the same assumptions and we use the same notation as in Section 5.

6.1. **A closed one-form.**

Definition 6.1. Set

$$D_T^X = D^X + T\frac{c(K^X)}{2\sqrt{2}}. \tag{6.1}$$

With the notation in (2.19),

$$D_T^X = A_{-\sqrt{-1}T}. \tag{6.2}$$

Proposition 6.2. *Let $\alpha_{t,T}$ be the 1-form on $\mathbf{R}_+^* \times \mathbf{R}$,*

$$\alpha_{t,T} = \frac{dt}{t}\operatorname{Tr_s}[Ng\exp(-L_K - tD_T^{X,2})] + dT\operatorname{Tr_s}[i\langle\mu, K\rangle g\exp(-L_K - tD_T^{X,2})]. \tag{6.3}$$

Then $\alpha_{t,T}$ is a closed form.

Proof. Replace temporarily T by $\sqrt{-1}T$, i.e. D_T^X by A_T, which was introduced in (2.16), (2.18). Then the proof of our Proposition is the same as the proof of [B6, Theorem 2.2], [BL, Theorem 3.3] or [1, Theorem 5.6]. The comparison with [1] is especially relevant, since, in view of (2.10), $\langle\mu, K\rangle$ plays here the role of $f/2$ in [1]. Changing back T into $-\sqrt{-1}T$, we obtain the desired result. Needless to say, a direct simple proof can be given. □

Definition 6.3. Let $\gamma_{t,v}$ be the 1-form on $\mathbf{R}_+^* \times \mathbf{R}_+^*$,

$$\begin{aligned}\gamma_{t,v} = \frac{dt}{t}\operatorname{Tr_s}\big[\big(N - i\frac{\langle\mu, K\rangle}{t}\big)g\exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\big]\\ + \frac{dv}{v^2}\operatorname{Tr_s}\big[i\langle\mu, K\rangle g\exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\big].\end{aligned} \tag{6.4}$$

Proposition 6.4. *The 1-form $\gamma_{t,v}$ is closed.*

Proof. In (6.3), we make the change of variables

$$T = \frac{1}{t} - \frac{1}{v}, \tag{6.5}$$

and we use Proposition 6.2. □

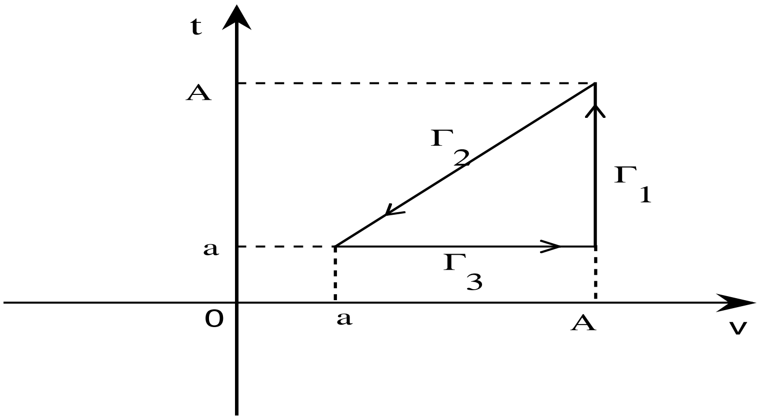

FIGURE 6.1

6.2. **A contour integral.** Take $0 < a < 1 < A < +\infty$. Let $\Gamma = \Gamma_{a,A}$ be the oriented contour in $\mathbf{R}^2$ indicated in Figure 6.1. Set

$$I_k^0 = \int_{\Gamma_k} \gamma,\ 1 \le k \le 3. \tag{6.6}$$

Theorem 6.5. *The following identity holds,*

$$\sum_{k=1}^{3} I_k^0 = 0. \tag{6.7}$$

Proof. This follows from Proposition 6.4. □

In the sequel, we will make $A \to +\infty, a \to 0$ in this order in identity (6.7), and we will ultimately obtain Theorem 5.1.

6.3. **Two intermediate results.** We use the notation of Section 3. Set

$$\begin{aligned}\widetilde{e}_v &= \int_{X_g} \mathrm{Td}_{g,K}(TX, h^{TX})\mathrm{ch}_{g,K}(E, h^E)\widetilde{d}_v,\\ \widetilde{D}_{-1}(g,K) &= \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)\mathrm{ch}_{ge^K}(E) i\langle\mu, K\rangle,\\ E_0(g,K) &= \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)\left(\frac{c'_{\max}}{c_{\max}}\right)_K (N_{X_{g,K}/X_g})\mathrm{ch}_{ge^K}(E).\end{aligned} \tag{6.8}$$

Then by the analogue of (3.9) for $\widetilde{d}_v$ and by (3.34), as $v \to 0$,

$$\widetilde{e}_v = -\frac{\widetilde{D}_{-1}(ge^K)}{v} - E_0(g,K) + \mathcal{O}(v). \tag{6.9}$$

Also observe that by (4.20), as $s \to 0$,

$$\int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)\Phi_{s,g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E) = E_0(g,K) + \mathcal{O}(s). \tag{6.10}$$

Recall that $g \in G, K_0 \in \mathfrak{z}(g)$, and that $K = zK_0$, with $z \in \mathbf{R}^*$. We now state two intermediate results which will be needed in our proof of Theorem 5.1.

Theorem 6.6. *For $z \in \mathbf{R}^*$, if $|z|$ is small enough, given $v \in \mathbf{R}^*_+$, as $t \to 0$,*

$$\mathrm{Tr_s}\Big[i\frac{\langle \mu, K\rangle}{v} g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\Big] \to -\widetilde{e}_v. \tag{6.11}$$

Moreover, given $z \in \mathbf{R}^*$, and $|z|$ small enough, there exist $C > 0, \gamma \in]0,1]$ such that for $t \in]0,1], v \in [t,1]$,

$$\Big|\mathrm{Tr_s}\Big[i\frac{\langle \mu, K\rangle}{v} g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\Big] + \widetilde{e}_v\Big| \leq C\big(t/v\big)^{\gamma}. \tag{6.12}$$

Given $z \in \mathbf{R}^*$, if $|z|$ is small enough, there exists $C > 0$ such that for $t \in]0,1], v \geq 1$,

$$\Big|\mathrm{Tr_s}\Big[i\frac{\langle \mu, K\rangle}{v} g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\Big]\Big| \leq \frac{C}{v}. \tag{6.13}$$

Theorem 6.7. *If $z \in \mathbf{R}^*$, if $|z|$ is small enough, given $v \in [1, +\infty[$, as $t \to 0$,*

$$\begin{aligned}\mathrm{Tr_s}\Big[i\frac{\langle \mu, K\rangle}{tv} g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})}\big)\Big] &= \frac{\widetilde{D}_{-1}(ge^K)}{tv} \\ &+ \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)\Phi_{1/v,g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E) + o(1).\end{aligned} \tag{6.14}$$

6.4. **The asymptotics of the $I_k^{0'}$s.** We now study the I_k^0's. It will be understood in the sequel that in all our statement, $z \in \mathbf{R}^*$ will be such that $|z|$ is small enough.

1) The term I_1^0.

Clearly,

$$I_1^0 = \int_a^A \mathrm{Tr_s}\Big[\big(N - \frac{i\langle \mu, K\rangle}{t}\big) g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{A}}\big)\Big]\frac{dt}{t}. \tag{6.15}$$

$\alpha) A \to +\infty$

As $A \to +\infty$,

$$\begin{aligned}\mathrm{Tr_s}\Big[\big(N - \frac{i\langle \mu, K\rangle}{t}\big) g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{A}}\big)\Big] &\to \\ \mathrm{Tr_s}\Big[\big(N - \frac{i\langle \mu, K\rangle}{t}\big) g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}}\big)\Big] &= \gamma_t(g,K).\end{aligned} \tag{6.16}$$

Clearly,

$$\sqrt{t}D^X_{\frac{1}{t}-\frac{1}{A}} = \sqrt{t}D^X + \big(1 - \frac{t}{A}\big)\frac{c(K^X)}{2\sqrt{2t}}. \tag{6.17}$$

By (6.17) and by Theorem 7.7, for $|z| \leq 1$, there is $C > 0$ such that for $t \in [1, +\infty[, A \in [t, +\infty[$,

$$\Big|\mathrm{Tr_s}\Big[\big(N - \frac{i\langle \mu, K\rangle}{t}\big) g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{A}}\big)\Big] - \mathrm{Tr_s}^{H^{\cdot}(X,E)}[Nge^K]\Big| \leq \frac{C}{\sqrt{t}}. \tag{6.18}$$

By (6.15)-(6.18), we find that as $A \to +\infty$,

$$\begin{aligned} I_1^0 - \mathrm{Tr_s}^{H^{\cdot}(X,E)}[Nge^K]\log(A) &\to \\ I_1^1 = \int_a^1 \gamma_t(g,K)\frac{dt}{t} &+ \int_1^{+\infty}\big(\gamma_t(g,K) - \gamma_{+\infty}(g,K)\big)\frac{dt}{t}.\end{aligned} \tag{6.19}$$

$\beta)a \to 0$

By Theorem 2.22, as $a \to 0$,

$$(6.20)\quad I_1^1 - \frac{C_{-1}(g,K)}{a} + C_0(g,K)\log(a) \to$$
$$I_1^2 = \int_0^1 \Big(\gamma_t(g,K) - \frac{C_{-1}(g,K)}{t} - C_0(g,K)\Big)\frac{dt}{t} + \int_1^{+\infty}\Big(\gamma_t(g,K) - \gamma_{+\infty}(g,K)\Big)\frac{dt}{t} - C_{-1}(g,K).$$

$\gamma)$Evaluation of I_1^2

Proposition 6.8. *The following identity holds,*

$$(6.21)\qquad I_1^2 = -\frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)(g,K)(0) + \Gamma'(1)\big(C_0(g,K) - \mathrm{Tr_s}^{H^\cdot(X,E)}[Nge^K]\big).$$

Proof. This follows from Proposition 2.25 and from (6.20). □

2) The term I_2^0

We have the identity

$$(6.22)\qquad I_2^0 = -\int_a^A \mathrm{Tr_s}\Big[Ng\exp\big(-L_K - tD^{X,2}\big)\Big]\frac{dt}{t}.$$

$\alpha)A \to +\infty$

As $A \to +\infty$,

$$(6.23)$$
$$I_2^0 + \mathrm{Tr_s}^{H^\cdot(X,E)}[Nge^K]\log(A) \to I_2^1 = -\int_a^1 \mathrm{Tr_s}\Big[Ng\exp\big(-L_K - tD^{X,2}\big)\Big]\frac{dt}{t}$$
$$-\int_1^{+\infty}\Big(\mathrm{Tr_s}\Big[Ng\exp\big(-L_K - tD^{X,2}\big)\Big] - \mathrm{Tr_s}^{H^\cdot(X,E)}[Nge^K]\Big)\frac{dt}{t}.$$

$\beta)a \to 0$

By Theorem 1.6, as $a \to 0$,

$$(6.24)\quad I_2^1 + \frac{D_{-1}(ge^K)}{a} - D_0(ge^K)\log(a) \to$$
$$I_2^2 = -\int_0^1 \Big(\mathrm{Tr_s}\Big[Ng\exp\big(-L_K - tD^{X,2}\big)\Big] - \frac{D_{-1}(ge^K)}{t} - D_0(ge^K)\Big)\frac{dt}{t}$$
$$-\int_1^{+\infty}\Big(\mathrm{Tr_s}\Big[Ng\exp\big(-L_K - tD^{X,2}\big)\Big] - \mathrm{Tr_s}^{H^\cdot(X,E)}[Nge^K]\Big)\frac{dt}{t} + D_{-1}(ge^K).$$

$\gamma)$Evaluation of I_2^3

Proposition 6.9. *The following identity holds,*

$$(6.25)\qquad I_2^3 = \frac{\partial}{\partial s}\theta(\omega^X, h^E)(ge^K)(0) + \Gamma'(1)(-D_0(ge^K) + \mathrm{Tr_s}^{H^\cdot(X,E)}[Nge^K]).$$

Proof. This follows from Proposition 1.7 and from (6.24). □

3) The term I_3^0

We have the obvious

$$I_3^0 = \int_a^A \mathrm{Tr_s}\left[i\langle \mu, K\rangle g \exp\left(-L_K - aD^{X,2}_{\frac{1}{a}-\frac{1}{v}}\right)\right]\frac{dv}{v^2}. \tag{6.26}$$

$\alpha) A \to +\infty$

As $A \to +\infty$,

$$I_3^0 \to I_3^1 = \int_a^{+\infty} \mathrm{Tr_s}\left[i\langle \mu, K\rangle g \exp\left(-L_K - aD^{X,2}_{\frac{1}{a}-\frac{1}{v}}\right)\right]\frac{dv}{v^2}. \tag{6.27}$$

$\beta) a \to 0$

Set

$$\begin{aligned} J_1 &= -\int_a^1 \widetilde{e}_v \frac{dv}{v}, \\ J_2 &= \int_1^{+\infty} \mathrm{Tr_s}\Big[\frac{i\langle \mu, K\rangle}{v} g \exp\big(-L_K - aD^{X,2}_{\frac{1}{a}-\frac{1}{v}}\big)\Big]\frac{dv}{v}, \\ J_3 &= \int_1^{1/a}\Big(\mathrm{Tr_s}\Big[\frac{i\langle \mu, K\rangle}{av} g \exp\big(-L_K - aD^{X,2}_{\frac{1}{a}(1-\frac{1}{v})}\big)\Big] + \widetilde{e}_{av}\Big)\frac{dv}{v}. \end{aligned} \tag{6.28}$$

Clearly

$$I_3^1 = J_1 + J_2 + J_3. \tag{6.29}$$

By (6.9), as $a \to 0$,

$$\begin{aligned} & J_1 - \frac{\widetilde{D}_{-1}(ge^K)}{a} + E_0(g,K)\log(a) \to \\ & \qquad J_1^1 = -\int_0^1 \Big(\widetilde{e}_v + \frac{\widetilde{D}_{-1}(ge^K)}{v} + E_0(g,K)\Big)\frac{dv}{v} - \widetilde{D}_{-1}(ge^K). \end{aligned} \tag{6.30}$$

By (6.11), as $a \to 0$,

$$\mathrm{Tr_s}\Big[\frac{i\langle \mu, K\rangle}{v} g \exp\big(-L_K - aD^{X,2}_{\frac{1}{a}-\frac{1}{v}}\big)\Big] \to -\widetilde{e}_v. \tag{6.31}$$

Also by (6.13), there exist $C > 0$ such that for $v \geq 1, a \in]0,1]$,

$$\Big|\mathrm{Tr_s}\Big[\frac{i\langle \mu, K\rangle}{v} g \exp\Big(-L_K - aD^{X,2}_{\frac{1}{a}-\frac{1}{v}}\Big)\Big]| \leq \frac{C}{v}. \tag{6.32}$$

From (6.28), (6.31), (6.32), we find that as $a \to 0$,

$$J_2 \to J_2^1 = -\int_1^{+\infty} \widetilde{e}_v \frac{dv}{v}. \tag{6.33}$$

By (6.12), there exists $C > 0, \gamma \in]0,1]$ such that for $1 \leq v \leq 1/a$,

$$\Big|\mathrm{Tr_s}\Big[\frac{i\langle \mu, K\rangle}{av} g \exp\Big(-L_K - aD^{X,2}_{\frac{1}{a}(1-\frac{1}{v})}\Big)\Big] + \widetilde{e}_{av}\Big| \leq \frac{C}{v^\gamma}. \tag{6.34}$$

Using (6.9), (6.14), (6.28) and (6.34), we find that as $a \to 0$,

$$J_3 \to J_3^1 = \int_1^{+\infty}\Big(\int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)\Phi_{1/v,g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E) - E_0(g,K)\Big)\frac{dv}{v}. \tag{6.35}$$

From (6.29), (6.30), (6.33), (6.35), we see that as $a \to 0$,

$$I_2^1 - \frac{\widetilde{D}_{-1}(ge^K)}{a} + E_0(g,K)\log(a) \to I_3^2 = J_1^1 + J_2^1 + J_3^1. \tag{6.36}$$

γ)Evaluation of I_3^2

Proposition 6.10. *The following identity holds,*

$$\begin{aligned} I_3^2 = -\int_{X_g} \mathrm{Td}_{ge^K}(TX, h^{TX})\mathrm{ch}_{ge^K}(E, h^E)\widetilde{S}_K(X_g, \omega^{X_g}) \\ + \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX) I_{g,K}(N_{X_K/X}) + \Gamma'(1)E_0(g,K). \end{aligned} \tag{6.37}$$

Proof. By (4.21) (6.30), (6.33), (6.35), (6.36), we get

$$\begin{aligned} I_3^2 = -\int_0^1 \Big(\widetilde{e}_v + \frac{\widetilde{D}_{-1}(ge^K)}{v} + E_0(g,K)\Big)\frac{dv}{v} - \int_1^{+\infty} \widetilde{e}_v \frac{dv}{v} \\ - \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX) I_{g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E) - \widetilde{D}_{-1}(ge^K). \end{aligned} \tag{6.38}$$

Using the analogue of (3.16) for $\widetilde{S}_K(X_g, \omega^{X_g})$, (3.34) and (6.38), we get (6.37). ☐

6.5. The final identity.

Theorem 6.11. *The following identity holds,*

$$\sum_{k=1}^{3} I_k^2 = 0. \tag{6.39}$$

Proof. We will check that, as should be the case by equation (6.7) in Theorem 6.5, the divergences of the I_k^0's add up to 0.
α)$A \to +\infty$
By (6.19), (6.23), we get the diverging terms

$$-\mathrm{Tr_s}^{H^\cdot(X,E)}[Nge^K]\log(A) + \mathrm{Tr_s}^{H^\cdot(X,E)}[Nge^K]\log(A) = 0 \tag{6.40}$$

Therefore

$$\sum_{k=1}^{3} I_k^1 = 0. \tag{6.41}$$

β)$a \to 0$
By (6.20), (6.24), (6.36), we get the diverging terms

$$\begin{aligned} -\frac{C_{-1}(g,K)}{a} + C_0(g,K)\log(a) + \frac{D_{-1}(ge^K)}{a} - D_0(ge^K)\log(a) \\ - \frac{\widetilde{D}_{-1}(ge^K)}{a} + E_0(g,K)\log(a). \end{aligned} \tag{6.42}$$

Now by (3.17), (3.31),

$$\int_{X_g}\Big(\frac{\omega^X}{2\pi}-i\langle\mu,K\rangle\Big)\mathrm{Td}_{g,K}(TX,h^{TX}))\mathrm{ch}_{g,K}(E,h^E)= \\ \int_{X_{g,K}}\Big(\frac{\omega^X}{2\pi}-i\langle\mu,K\rangle\Big)\mathrm{Td}_{ge^K}(TX,h^{TX})\mathrm{ch}_{ge^K}(E,h^E). \tag{6.43}$$

Equation (6.43) says that, as should be the case, the coefficient of $\frac{1}{a}$ in (6.42) vanishes identically. A similar argument shows that the coefficient of $\log(a)$ also vanishes. We thus get (6.39). The proof of our Theorem is completed. □

Now we use Propositions 6.8-6.10 and Theorem 6.11. Since the coefficient of $\log(a)$ in (6.42) vanishes, we find that the coefficient of $\Gamma'(1)$ in the left-hand side of (6.39) vanishes. Identity (6.39) is then equivalent to (5.1), i.e. we have completed the proof of Theorem 5.1.

7. The construction of the families equivariant analytic torsion forms

The purpose of this Section is in particular to establish Theorem 2.22. This result implies that the infinitesimal equivariant torsion $\frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)(g,K)(0)$ introduced in Section 2.6 is well defined. Also we introduce many of the tools which will be needed to complete the proof of the main result of this paper. Many of our tools were already used in [BL].

One of the main points is to show that $\frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)(g,K)(0)$ is well defined as a function of $K \in \mathfrak{g}$, for $|K|$ small enough. By using the techniques of [BGS1], [BK], [BeGeV], it is relatively easy to define $\frac{\partial}{\partial s}\widetilde{\theta}(\omega^X, h^E)(g,K)(0)$ as a formal power series. The fact it exists as a function requires nontrivial estimates.

This Section is organized as follows. In Section 7.1, we study the behaviour of $\gamma_t(g,K)$ as $t \to +\infty$. The proofs use estimates on trace class operators. The fact that $C_{K,t}$ is the sum of a self-adjoint part $\sqrt{t}D^X$ and of a skew-adjoint part $c(K^X)/2\sqrt{2t}$ plays an important role. In Sections 7.2-7.14, we study of $\gamma_t(g,K)$ as $t \to 0$. In Section 7.3, by using finite propagation speed techniques for solutions of hyperbolic equations [ChP], [T], we show that our problem can be localized near the fixed point set X_g. Let $L^{3,t}_{x,K}$ be a rescaled version of the operator $L_K + C^2_{K,t}$ near $x \in X_g$. The problem is now reduced to understanding the behaviour as $t \to 0$ of a kernel $\widetilde{F}_t(L^{3,t}_{x,K})(Z,Z')$ whose support is close to the diagonal.

Recall that equation (2.69) involves an asymptotic expansion of $\gamma_t(g,K)$ as $t \to 0$, and the first two terms have to be evaluated explicitly. Sections 7.4-7.13 are devoted to obtaining the first term in this asymptotic expansion. In Section 7.4, we state a Lichnerowicz formula. In Section 7.5, we introduce a local coordinate system near X_g. In Section 7.6, given $x \in X_g$, we replace the manifold X by $(T_{\mathbf{R}}X)_x$. In Section 7.7, we perform a Getzler rescaling [Ge] on certain Clifford variables.

From Section 7.8 on, we use functional analytic methods inspired from [BL, Section 11]. In Section 7.8, we introduce a natural family of norms, and we prove that the rescaled operators $L^{3,t}_{x,K}$ verify corresponding elliptic estimates. In Section 7.9, we truncate the operator $L^{3,t}_{x,K}$, and we express $\widetilde{F}_t(L^{3,t}_{x,K})(Z,Z')$ as an infinite sum, for the reasons outlined in the Introduction.

In Section 7.10, we give estimates on the resolvent of the truncated operators, and in Section 7.11, we prove corresponding uniform regularizing properties. In Section 7.12, we establish uniform estimates on the truncated kernels. Finally, in Section 7.13, we obtain the first term in the asymptotic expansion of $\gamma_t(g,K)$ as $t \to 0$.

In Section 7.14, we obtain the second term in the asymptotic expansion of $\gamma_t(g,K)$. In fact we use an algebraic identity taken from [BGS1], to reduce the problem to a new one, which is accessible to the techniques we just described.

In our proofs, the constants $C > 0$ may vary from line to line.

7.1. **The behaviour of the supertraces as $t \to \infty$.**

Recall that

$$\gamma_t(g,K) = \operatorname{Tr_s}\left[(N - i\frac{\langle \mu, K\rangle}{t})g\exp(-L_K - C^2_{K,t})\right]. \tag{7.1}$$

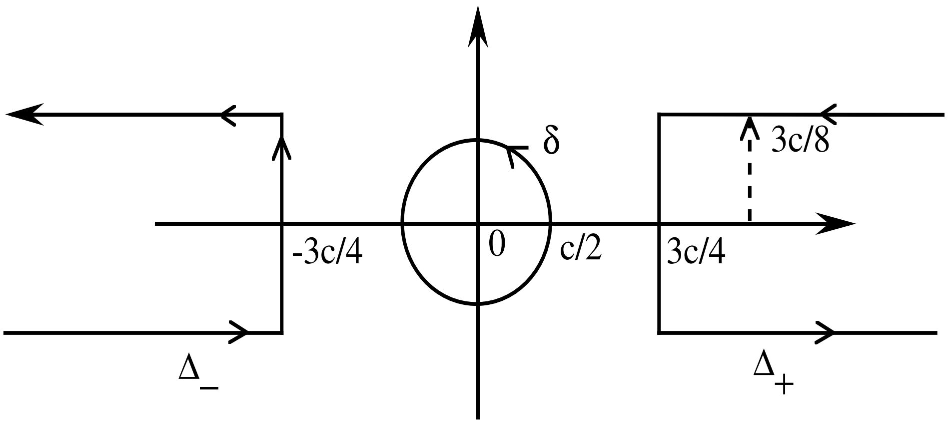

Figure 7.1

Theorem 7.1. *There exists* $\beta > 0, C > 0$ *such that if* $K \in \mathfrak{z}(g), |K| \leq \beta$, $t \geq 1$,

$$|\gamma_t(g, K) - \operatorname{Tr_s}^{H^{\cdot}(X,E)}[Nge^K]| \leq \frac{C}{\sqrt{t}}. \tag{7.2}$$

Proof. This subsection is devoted to the proof of Theorem 7.1. □

Let F^s be the Sobolev space of sections of $\Lambda(T^{*(0,1)}X) \otimes E$ on X. Then

$$F^\infty = \Omega^{\cdot}(X, E). \tag{7.3}$$

The Hilbert space F^0 is naturally equipped with the norm $|\ |_0$ associated to the Hermitian product (1.6). More generally, for $s \in \mathbf{R}\backslash\{0\}$, let $|\ |_s$ be a norm on F^s. We may and we will assume that $|\ |_s$ increases with s. Let $c \in]0, 1]$ be such that

$$\operatorname{Sp}(D^X) \cap \{\lambda \in \mathbf{R}, |\lambda| \leq 2c\} \subset 0. \tag{7.4}$$

Consider the contour in $\mathbf{C}$ indicated in Figure 7.1. Let U be the elements of $\mathbf{C}$ which lie outside the domain bounded by Δ and δ. In particular $\pm i\infty \in U$.

If $A \in \mathcal{L}(F^s, F^{s'})$, let $\|A\|^{s,s'}$ be the norm of A with respect to the given norms on F^s, $F^{s'}$.

Proposition 7.2. *There exist* $C > 0$, $C' > 0$ *such that if* $t \geq 1$, $\lambda \in U$,

$$\begin{aligned} &\|(\lambda - \sqrt{t}D^X)^{-1}\|^{0,0} \leq C, \\ &\|(\lambda - \sqrt{t}D^X)^{-1}\|^{0,1} \leq C(1 + \frac{|\lambda|}{\sqrt{t}}). \end{aligned} \tag{7.5}$$

Proof. Using (7.4), the first inequality in (7.5) follows. Also

$$\begin{aligned} (\lambda - \sqrt{t}D^X)^{-1} = &\frac{1}{\sqrt{t}}(\sqrt{-1} - D^X)^{-1} \\ &+ (\sqrt{-1} - \lambda/\sqrt{t})(\sqrt{-1} - D^X)^{-1}(\lambda - \sqrt{t}D^X)^{-1}. \end{aligned} \tag{7.6}$$

Moreover

$$\|(\sqrt{-1} - D^X)^{-1}\|^{0,1} \leq C'. \tag{7.7}$$

From the first inequality in (7.5), and from (7.6), (7.7), we get the second inequality in (7.5). The proof of our Proposition is completed. □

Definition 7.3. For $t \geq 1$, $\lambda \in U$, put

$$m_t(\lambda) = 1 - \frac{c(K^X)}{2\sqrt{2t}}(\lambda - \sqrt{t}D^X)^{-1}. \tag{7.8}$$

Proposition 7.4. *Given $M > 0$, there exist $t_0 \geq 1$, $C > 0$ such that for $K \in \mathfrak{g}$, $|K| \leq M$, $t \geq t_0$, $\lambda \in U$, then $m_t(\lambda) \in \mathcal{L}(F^0)$ is invertible and moreover*

$$\|m_t^{-1}(\lambda) - 1\|^{0,0} \leq \frac{C}{\sqrt{t}}. \tag{7.9}$$

Proof. By (7.5), there exists $C' > 0$ such that for $K \in \mathfrak{g}$, $|K| \leq M$, $t \geq 1$,

$$\|m_t(\lambda) - 1\|^{0,0} \leq \frac{C'}{\sqrt{t}}. \tag{7.10}$$

Equation (7.9) follows from (7.10). □

Definition 7.5. If $q \in \mathbf{R}$, $q \geq 1$, let $\mathcal{L}_q(F^0)$ be the space of the $A \in \mathcal{L}(F^0)$ such that

$$\|A\|_q = \left\{\operatorname{Tr}\left[(A^*A)^{q/2}\right]\right\}^{1/q} < +\infty. \tag{7.11}$$

Then $\mathcal{L}_q(F^0)$ is a vector space and $\| \ \|_q$ is a norm on $\mathcal{L}_q(F^0)$. Similarly, if $A \in \mathcal{L}(F^0)$, let $\|A\|_\infty$ be the usual norm of A.

Let p be the orthogonal projection operator from F^0 on $\ker D^X$. Put

$$p^\perp = 1 - p. \tag{7.12}$$

Theorem 7.6. *Given $M > 0$, there exist $t_0 \geq 1$, $C > 0$ such that for $K \in \mathfrak{g}$, $|K| \leq M$, $t \geq t_0$, $\lambda \in U$, then $\lambda - C_{K,t}$ is invertible, an moreover*

$$\|(\lambda - C_{K,t})^{-1}\|_\infty \leq C. \tag{7.13}$$

Let q be an integer such that $q \geq 2\dim X + 1$. Then there exists $C' > 0$ such that if $K \in \mathfrak{g}$, $|K| \leq M$, $t \geq t_0$, $\lambda \in U$, $|\lambda| \leq c\sqrt{t}$,

$$\|(\lambda - C_{K,t})^{-1} - p/\lambda\|_q \leq \frac{C'}{\sqrt{t}}(1 + |\lambda|). \tag{7.14}$$

There exist $C'' > 0$ such that if $K \in \mathfrak{g}$, $|K| \leq M$, $t \geq t_0$, $\lambda \in U$, $|\lambda| \leq c\sqrt{t}$,

$$\|(\lambda - C_{K,t})^{-q} - p/\lambda^q\|_1 \leq \frac{C''}{\sqrt{t}}(1 + |\lambda|). \tag{7.15}$$

Finally there exist $C''' > 0$ such that if $K \in \mathfrak{g}$, $|K| \leq M$, $t \geq t_0$, $\lambda \in U$,

$$\|(\lambda - C_{K,t})^{-1}\|_q \leq C(1 + |\lambda|). \tag{7.16}$$

Proof. Clearly

$$\lambda - C_{K,t} = m_t(\lambda)(\lambda - \sqrt{t}D^X). \tag{7.17}$$

Using Propositions 7.2 and 7.4, and (7.17), we find that for $t \geq t_0$, $\lambda - C_{K,t}$ is invertible. More precisely

$$(\lambda - C_{K,t})^{-1} = (\lambda - \sqrt{t}D^X)^{-1} m_t^{-1}(\lambda). \tag{7.18}$$

By (7.5), (7.9), (7.18), we get (7.13).

By (7.5), (7.6), for $q \geq 2\dim X + 1$,

$$\begin{aligned} \|(\lambda - \sqrt{t}D^X)^{-1}\|_q &\leq C(1 + |\lambda|)\|(\sqrt{-1} - D^X)^{-1}\|_q \\ &\leq C'(1 + |\lambda|). \end{aligned} \tag{7.19}$$

Using (7.9), (7.18), (7.19), we get

$$\begin{aligned}\|(\lambda - C_{K,t})^{-1} - (\lambda - \sqrt{t}D^X)^{-1}\|_q \\ \leq \|(\lambda - \sqrt{t}D^X)^{-1}\|_q \|m_t^{-1}(\lambda) - 1\|_\infty \leq \frac{C}{\sqrt{t}}(1+|\lambda|).\end{aligned} \tag{7.20}$$

If $\mu \in \mathrm{Sp}(D^X)$, $\mu \neq 0$, then $|\mu| \geq 2c$. So if $\mu' \in \mathbf{C}$, $|\mu'| \leq c$, then

$$|\mu - \mu'| \geq \frac{|\mu|}{2}. \tag{7.21}$$

Let $(D^X)^{-1}$ be the inverse of D^X on the orthogonal space $(\ker D^X)^\perp$ to $\ker D^X$ in F^0. From (7.21), we find that if $\lambda \in U$, $|\lambda| \leq c\sqrt{t}$, then

$$\|p^\perp(\lambda - \sqrt{t}D^X)^{-1}\|_q \leq \frac{2}{\sqrt{t}}\|p^\perp (D^X)^{-1}\|_q \leq \frac{C}{\sqrt{t}}. \tag{7.22}$$

Equation (7.22) is equivalent to

$$\|(\lambda - \sqrt{t}D^X)^{-1} - \frac{p}{\lambda}\|_q \leq \frac{C}{\sqrt{t}}. \tag{7.23}$$

By (7.20), (7.23), we get (7.14).

If $1 \leq r \leq q$, if $\lambda \in U$, $|\lambda| \leq c\sqrt{t}$, by (7.14),

$$\begin{aligned}\left\|(\lambda - C_{K,t})^{-(r-1)}\left((\lambda - C_{K,t})^{-1} - \frac{p}{\lambda}\right)\frac{p}{\lambda^{q-r}}\right\|_1 \\ \leq C\|(\lambda - C_{K,t})^{-1}\|_q^{r-1}\|(\lambda - C_{K,t})^{-1} - p/\lambda\|_q \leq \frac{C}{\sqrt{t}}(1+|\lambda|).\end{aligned} \tag{7.24}$$

From (7.24), we get (7.15).

Fix $\lambda_0 \in U$. If $\lambda \in U$,

$$(\lambda - C_{K,t})^{-1} = (\lambda_0 - C_{K,t})^{-1} + (\lambda_0 - \lambda)(\lambda - C_{K,t})^{-1}(\lambda_0 - C_{K,t})^{-1}. \tag{7.25}$$

By (7.13), (7.14), (7.25), we get (7.16). The proof of our Theorem is completed. □

Theorem 7.7. *Let $M > 0$. There exist $C > 0$ such that for $K \in \mathfrak{z}(g)$, $|K| \leq M$, $t \geq 1$, then*

$$\left\|\exp\left(-C_{K,t}^2\right) - p\right\|_1 \leq \frac{C}{\sqrt{t}}. \tag{7.26}$$

Proof. By Theorem 7.6, for $|K| \leq M$, $t \geq t_0$,

$$\exp(-C_t^2) = \frac{1}{2i\pi}\int_{\Delta \cup \{\delta\}} \exp(-\lambda^2)(\lambda - C_{K,t})^{-1} d\lambda. \tag{7.27}$$

Take $q \in \mathbf{N}$, $q \geq 2\dim X + 1$. Let f_q be the unique holomorphic function on $\mathbf{C}\backslash\sqrt{-1}\mathbf{R}$ which has the following two properties :

$$\begin{aligned}\lim_{\lambda \to \pm\infty} f_q(\lambda) &= 0 \\ \frac{f_q^{(q-1)}(\lambda)}{(q-1)!} &= \exp(-\lambda^2).\end{aligned} \tag{7.28}$$

Clearly, there exist $c > 0$, $C > 0$ such that if $\lambda \in \Delta$,

$$|f_q(\lambda)| \leq c\exp(-C|\lambda|^2). \tag{7.29}$$

We have the identity

$$\frac{1}{2i\pi}\int_{\Delta}\exp(-\lambda^2)(\lambda-C_{K,t})^{-1}d\lambda=\frac{1}{2i\pi}\int_{\Delta}f_q(\lambda)(\lambda-C_{K,t})^{-q}d\lambda. \tag{7.30}$$

By inequality (7.15) in Theorem 7.6 and by (7.29), for $t\geq t_0$,

$$\left\|\frac{1}{2i\pi}\int_{\Delta\cap\{\lambda;|\lambda|\leq c\sqrt{t}\}}f_q(\lambda)\left((\lambda-C_{K,t})^{-q}-\frac{p}{\lambda^q}\right)d\lambda\right\|_1\leq\frac{C}{\sqrt{t}}. \tag{7.31}$$

Also by (7.16), (7.29),

$$\begin{aligned}&\left\|\frac{1}{2i\pi}\int_{\Delta\cap\{\lambda;|\lambda|\geq c\sqrt{t}\}}f_q(\lambda)(\lambda-C_{K,t})^{-q}-\frac{p}{\lambda^q}\right\|_1\\&\leq C\int_{\Delta\cap\{\lambda;|\lambda|\geq c\sqrt{t}\}}|f_q(\lambda)|\ \left\|(\lambda-C_{K,t})^{-q}-\frac{p}{\lambda^q}\right\|_1d\lambda\leq C\exp(-C't).\end{aligned} \tag{7.32}$$

Finally by the theorem of residues,

$$\frac{1}{2i\pi}\int_{\Delta}f_q(\lambda)\frac{d\lambda}{\lambda^q}=0. \tag{7.33}$$

By (7.30)-7.33), we find that for $t\geq t_0$,

$$\left\|\frac{1}{2i\pi}\int_{\Delta}\exp(-\lambda^2)(\lambda-C_{K,t})^{-1}d\lambda\right\|_1\leq\frac{C}{\sqrt{t}}. \tag{7.34}$$

Let now $g_q(\lambda)$ be a holomorphic function on $\mathbf{C}$ such that

$$\frac{g_q^{(q-1)}(\lambda)}{(q-1)!}=\exp(-\lambda^2). \tag{7.35}$$

Then

$$\frac{1}{2i\pi}\int_{\delta}\exp(-\lambda^2)(\lambda-C_{K,t})^{-1}d\lambda=\frac{1}{2i\pi}\int_{\delta}g_q(\lambda)(\lambda-C_{K,t})^{-q}d\lambda. \tag{7.36}$$

Moreover

$$\frac{1}{2i\pi}\int_{\delta}\frac{g_q(\lambda)}{\lambda^q}d\lambda=\frac{1}{2i\pi}\int_{\delta}\exp(-\lambda^2)\frac{d\lambda}{\lambda}=1. \tag{7.37}$$

By (7.15) and (7.36), (7.37), we get

$$\left\|\frac{1}{2i\pi}\int_{\delta}\exp(-\lambda^2)(\lambda-C_{K,t})^{-1}d\lambda-p\right\|_1\leq\frac{C}{\sqrt{t}}. \tag{7.38}$$

By (7.27), (7.34), (7.38), we get (7.26).
The proof of our Theorem is completed. □

Remark 7.8. Equation (7.2) in Theorem 7.1 is a simple consequence of Theorem 7.7.

7.2. The asymptotics of the supertraces as $t \to 0$.

Theorem 7.9. *There exist $\beta > 0, C > 0, \gamma \in]0,1]$ such that if $K \in \mathfrak{g}, |K| \leq \beta$,*

$$\left|\alpha_t(g,K) - \int_{X_g} \mathrm{Td}_{g,K}(TX)\mathrm{ch}_{g,K}(E)\right| \leq Ct^{\gamma}, \tag{7.39}$$
$$\left|\gamma_t(g,K) - \frac{C_{-1}(g,K)}{t} - C_0(g,K)\right| \leq Ct^{\gamma}.$$

Proof. The remainder of the Section is devoted to the proof of Theorem 7.9. ☐

Remark 7.10. By (2.68), $\alpha_t(g,K)$ does not depend on t. The first inequality in (7.39) then leads to a proof of the Kirillov formulas in (2.38). Still because $\alpha_t(g,K)$ does not depend on t, (2.38) follows from an equality of formal power series in the variable $z \in \mathbf{C}$, when replacing K by zK. A proof of Kirillov's formula along these lines has been given by Bismut [B3] by changing z into iz, and by Berline-Getzler-Vergne [BeGeV, Section 8.3] by proving the convergence as $t \to 0$ of the corresponding power series. Still $\gamma_t(g,K)$ cannot be dealt with by such arguments.

7.3. Finite propagation speed and localization.

Put

$$\mathcal{A}_t = t(L_K + C^2_{K,t}). \tag{7.40}$$

Equivalently

$$\mathcal{A}_t = tL_K + t^2 D^{X,2}_{1/t}. \tag{7.41}$$

Theorem 7.11. *Given $\beta > 0$, there exist $C_1 > 0, C_2 > 0, C_2'(\beta) > 0, C_3(\beta) > 0, C_4 > 0, C_5(\beta) > 0$ such that if $K \in \mathfrak{g}, |K| \leq \beta, t \in]0,1]$, if $s, s' \in \Omega^{\cdot}(X,E)$,*

$$\begin{aligned}
\mathrm{Re}\langle \mathcal{A}_t s, s\rangle &\geq C_1 t^2 |s|_1^2 - (C_2 t^2 + C_2'(\beta))|s|_0^2,\\
|\mathrm{Im}\langle \mathcal{A}_t s, s\rangle| &\leq C_3(\beta) t |s|_1 |s|_0,\\
|\langle \mathcal{A}_t s, s'\rangle| &\leq C_4(t|s|_1 + C_5(\beta)|s|_0)(t|s'|_1 + C_5(\beta)|s'|_0).
\end{aligned} \tag{7.42}$$

Moreover, as $\beta \to 0$, $C_2'(\beta), C_3(\beta), C_5(\beta) \to 0$.

Proof. Clearly

$$\mathcal{A}_t = t^2 D^{X,2} + t(L_K + [D^X, \frac{c(K^X)}{2\sqrt{2}}]) - \frac{|K^X|^2}{8}. \tag{7.43}$$

Moreover $D^{X,2}$ is an elliptic second order differential operator, and $L_K, [D^X, \frac{c(K^X)}{2\sqrt{2}}]$ are first order skew-adjoint differential operators. So we find in particular that

$$\begin{aligned}
\mathrm{Re}\langle \mathcal{A}_t s, s\rangle &= \left\langle (t^2 D^{X,2} - \frac{|K^X|^2}{8})s, s\right\rangle,\\
\mathrm{Im}\langle \mathcal{A}_t s, s\rangle &= t\left\langle (L_K + [D^X, \frac{c(K^X)}{2\sqrt{2}}])s, s\right\rangle.
\end{aligned} \tag{7.44}$$

From (7.43), (7.44), we get the first two equations in (7.42). The proof of the third equation is similar. The last statement in our Theorem is trivial. ☐

In the sequel, we take $\beta > 0$, and we assume that $|K| \leq \beta$.

For $c > 0$, put

$$\begin{aligned} U_c &= \{\lambda \in \mathbf{C},\, \mathrm{Re}(\lambda) \leq \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2\}, \\ V_c &= \{\lambda \in \mathbf{C},\, \mathrm{Re}(\lambda) \geq \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2\}, \\ \Gamma_c &= \{\lambda \in \mathbf{C},\, \mathrm{Re}(\lambda) = \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2\}. \end{aligned} \tag{7.45}$$

Note that U_c, V_c, Γ_c are the images of $\{\lambda \in \mathbf{C}, |\mathrm{Im}\lambda| \geq c\}$, $\{\lambda \in \mathbf{C}, |\mathrm{Im}\lambda| \leq c\}$, $\{\lambda \in \mathbf{C}, |\mathrm{Im}\lambda| = c\}$ by the map $\lambda \to \lambda^2$.

Theorem 7.12. *There exists $C > 0$ such that given $c \in]0,1]$, for $\beta > 0$ and $t \in]0,1]$ small enough, if $\lambda \in U_c$, the resolvent $(\lambda - \mathcal{A}_t)^{-1}$ exists, extends to a continuous operator from F^{-1} into F^1, and moreover,*

$$\begin{aligned} \|(\lambda - \mathcal{A}_t)^{-1}\|^{0,0} &\leq \frac{2}{c^2}, \\ \|(\lambda - \mathcal{A}_t)^{-1}\|^{-1,1} &\leq \frac{C}{c^2 t^4}(1 + |\lambda|)^2. \end{aligned} \tag{7.46}$$

Proof. We use a method due to Lax-Milgram. By (7.42), we observe that if $\lambda \in \mathbf{R}, \lambda \leq -(C_2 t^2 + C_2'(\beta))$, then

$$\mathrm{Re}\langle(\mathcal{A}_t - \lambda)s, s\rangle \geq C_1 t^2 |s|_0^2, \tag{7.47}$$

so that

$$|s|_0 \leq \frac{1}{C_1 t^2} |(\mathcal{A}_t - \lambda)s|_0. \tag{7.48}$$

Also since $\mathcal{A}_t$ is elliptic of order 2, given $\lambda \in \mathbf{C}$, there exists $C_1(\lambda, t) > 0$ such that

$$|s|_2 \leq C_1(\lambda, t)(|(\lambda - \mathcal{A}_t)s|_0 + |s|_0). \tag{7.49}$$

From (7.48), (7.49), we find that if $\lambda \in \mathbf{R}, \lambda \leq -(C_2 t^2 + C_2'(\beta))$, the resolvent $(\lambda - \mathcal{A}_t)^{-1}$ exists.

Take now $\lambda = a + ib$, $a, b \in \mathbf{R}$. By (7.42),

$$\begin{aligned} |\langle(\mathcal{A}_t - \lambda)s, s\rangle| \geq \sup\big\{C_1 t^2 |s|_1^2 - (C_2 t^2 + C_2'(\beta) + a)|s|_0^2, \\ - C_3(\beta) t |s|_1 |s|_0 + |b||s|_0^2\big\}. \end{aligned} \tag{7.50}$$

Set

$$C(\lambda, t) = \inf_{\substack{u \in \mathbf{R} \\ u \geq 1}} \sup\big\{C_1 (tu)^2 - (C_2 t^2 + C_2'(\beta) + a), -C_3(\beta) tu + |b|\big\}. \tag{7.51}$$

Since $|s|_0 \leq |s|_1$, using (7.50), (7.51) we get

$$|\langle(\mathcal{A}_t - \lambda)s, s\rangle| \geq C(\lambda, t)|s|_0^2. \tag{7.52}$$

Now we fix $c \in]0,1]$. Suppose that $\lambda \in U_c$, i.e.

$$a \leq \frac{b^2}{4c^2} - c^2. \tag{7.53}$$

Assume that $u \in \mathbf{R}$ is such that

$$|b| - C_3(\beta) tu \leq c^2. \tag{7.54}$$

Then by (7.53), (7.54),

$$\begin{aligned}(7.55)\quad C_1(tu)^2-(C_2t^2+C_2'(\beta)+a)&\geq C_1(tu)^2-\frac{b^2}{4c^2}+c^2-C_2t^2-C_2'(\beta)\\ &\geq (C_1-\frac{C_3^2(\beta)}{4c^2})(tu)^2-\frac{C_3(\beta)}{2}tu+\frac{3}{4}c^2-C_2t^2-C_2'(\beta).\end{aligned}$$

For β small enough,

$$C_1\geq C_1-\frac{C_3(\beta)^2}{4c^2}\geq\frac{C_1}{2}. \tag{7.56}$$

The discriminant Δ of the polynomial in the variable tu in the right-hand side of (7.55) is given by

$$\Delta=\frac{C_3(\beta)^2}{4}-4(\frac{3}{4}c^2-C_2t^2-C_2'(\beta))(C_1-\frac{C_3(\beta)^2}{4c^2}). \tag{7.57}$$

Clearly, for β, t small enough,

$$\Delta\leq -2c^2C_1. \tag{7.58}$$

From (7.55)-(7.58), we get

$$C_1(tu)^2-(C_2t^2+C_2'(\beta)+a)\geq\frac{c^2}{2}. \tag{7.59}$$

Ultimately, by (7.51)-(7.59), we find that for $\beta>0, t\in]0,1]$ small enough, if $\lambda\in U_c$,

$$C(\lambda,t)\geq\frac{c^2}{2}. \tag{7.60}$$

From (7.52), (7.60), we deduce that given $c>0$, for $t>0,\beta>0$ small enough,

$$|\langle(\mathcal{A}_t-\lambda)s,s\rangle|\geq\frac{c^2}{2}|s|_0^2. \tag{7.61}$$

By (7.61), we get

$$|(\lambda-\mathcal{A}_t)s|_0\geq\frac{c^2}{2}|s|_0. \tag{7.62}$$

By (7.62), we deduce that if $\lambda\in U_c$, if the resolvent $(\lambda-\mathcal{A}_t)^{-1}$ exists, then

$$\|(\lambda-\mathcal{A}_t)^{-1}\|^{0,0}\leq\frac{2}{c^2}. \tag{7.63}$$

From (7.63), we find that if $\lambda'\in\mathbf{C}, |\lambda'-\lambda|<\frac{c^2}{2}$, the resolvent $(\lambda'-\mathcal{A}_t)^{-1}$ still exists. We saw that if $\lambda\in\mathbf{R},\lambda\leq-(C_2t^2+C_2'(\beta))$, the resolvent $(\lambda-\mathcal{A}_t)^{-1}$ exists. From the above, we deduce that for any $\lambda\in U_c$, the resolvent $(\lambda-\mathcal{A}_t)^{-1}$ exists and (7.63) holds, i.e. we have established the first inequality in (7.46). Incidentally, observe that (7.49) and (7.62) also imply the existence of the resolvent $(\lambda-\mathcal{A}_t)^{-1}$.

Clearly $\mathcal{A}_t$ is a linear map from F^1 into F^{-1}. By the first inequality in (7.42), if $\lambda_0\in\mathbf{R},\lambda_0\leq -C_2t^2-C_2'(\beta)$, then

$$|s|_1\leq\frac{1}{C_1t^2}|(\lambda-\mathcal{A}_t)s|_{-1}. \tag{7.64}$$

So by (7.49), (7.64), we see that if $\lambda_0 \in \mathbf{R}, \lambda_0 \leq -(C_2 t^2 + C_2'(\beta))$, the resolvent $(\lambda_0 - \mathcal{A}_t)^{-1}$ exists, is one to one from F^{-1} into F^1, and moreover

$$\|(\lambda_0 - \mathcal{A}_t)^{-1}\|^{-1,1} \leq \frac{1}{C_1 t^2}. \tag{7.65}$$

Take $\lambda \in U_c, \lambda_0 \in \mathbf{R}, \lambda_0 \leq -(C_2 t^2 + C_2'(\beta))$. Then

$$(\lambda - \mathcal{A}_t)^{-1} = (\lambda_0 - \mathcal{A}_t)^{-1} + (\lambda_0 - \lambda)(\lambda - \mathcal{A}_t)^{-1}(\lambda_0 - \mathcal{A}_t)^{-1}. \tag{7.66}$$

By (7.63)-(7.66), $(\lambda - \mathcal{A}_t)^{-1}$ is a linear continuous map from F^{-1} into F^0 such that

$$\|(\lambda - \mathcal{A}_t)^{-1}\|^{-1,0} \leq \frac{1}{C_1 t^2}(1 + \frac{2}{c^2}|\lambda - \lambda_0|). \tag{7.67}$$

Moreover we can interchange λ and λ_0 in (7.66). Using (7.65) and (7.67), we now get

$$\|(\lambda - \mathcal{A}_t)^{-1}\|^{-1,1} \leq \frac{1}{C_1 t^2} + \frac{|\lambda - \lambda_0|}{(C_1 t^2)^2}(1 + \frac{2}{c^2}|\lambda - \lambda_0|). \tag{7.68}$$

By (7.68), we get the second inequality in (7.46). The proof of our Theorem is completed. □

Theorem 7.13. *Given $q \geq 2\dim X + 1$, there exist $C > 0, C' > 0$ such that given $c > 0$, for $\beta > 0, t \in]0,1]$ small enough, if $\lambda \in U_c$, the resolvent $(\lambda - \mathcal{A}_t)^{-1}$ exists, and moreover*

$$\begin{aligned} \|(\lambda - \mathcal{A}_t)^{-1}\|_q &\leq \frac{C}{c^2 t^4}(1 + |\lambda|)^2, \\ \|(\lambda - \mathcal{A}_t)^{-q}\|_1 &\leq \frac{C^q}{(c^2 t^4)^q}(1 + |\lambda|)^{2q}. \end{aligned} \tag{7.69}$$

Proof. Under the conditions of Theorem 7.12, we get

$$\|(\sqrt{-1} - D^X)(\lambda - \mathcal{A}_t)^{-1}\|_\infty \leq \frac{C}{c^2 t^4}(1 + |\lambda|)^2. \tag{7.70}$$

By (7.70), for $q \geq 2\dim X + 1$,

$$\begin{aligned} \|(\lambda - \mathcal{A}_t)^{-1}\|_q &\leq \|\sqrt{-1} - D^X)^{-1}\|_q \|(\sqrt{-1} - D^X)(\lambda - \mathcal{A}_t)^{-1}\|_\infty \\ &\leq \frac{C}{c^2 t^4}(1 + |\lambda|)^2, \end{aligned} \tag{7.71}$$

which is just the first inequality in (7.69) The second inequality is now trivial. □

Let a_X be the injectivity radius of X. Let $\alpha \in]0, \frac{a_X}{8}]$. The precise value of α will be fixed later. The constants $C, C' \ldots$may depend on the choice of α.

Let $f : \mathbf{R} \to [0,1]$ be a smooth even function such that

$$\begin{aligned} f(s) &= 1 \text{ for } |s| \leq \frac{\alpha}{2}, \\ &= 0 \text{ for } |s| \geq \alpha. \end{aligned} \tag{7.72}$$

Set

$$g(s) = 1 - f(s). \tag{7.73}$$

Definition 7.14. For $t>0, a\in \mathbf{C}$, put

$$F_t(a)=\int_{-\infty}^{+\infty}\exp(is\sqrt{2}a)\exp\left(-\frac{s^2}{2}\right)f(\sqrt{t}s)\frac{ds}{\sqrt{2\pi}},\tag{7.74}$$
$$G_t(a)=\int_{-\infty}^{+\infty}\exp(is\sqrt{2}a)\exp\left(-\frac{s^2}{2}\right)g(\sqrt{t}s)\frac{ds}{\sqrt{2\pi}}.$$

Then $F_t(a), G_t(a)$ are even holomorphic functions of a such that

$$\exp(-a^2)=F_t(a)+G_t(a).\tag{7.75}$$

Moreover F_t and G_t both lie in the Schwartz space $S(\mathbf{R})$.

Put

$$I_t(a)=\int_{-\infty}^{+\infty}\exp\left(is\sqrt{2}\frac{a}{t}\right)\exp\left(-\frac{s^2}{2t}\right)g(s)\frac{ds}{\sqrt{2\pi t}}.\tag{7.76}$$

Then

$$I_t(a)=G_t(\frac{a}{\sqrt{t}}).\tag{7.77}$$

By (7.72), (7.76), we find that given $m,m'\in\mathbf{N}$, there exist $C>0, C'>0$ such that if $t\in]0,1], a\in\mathbf{C}, |\mathrm{Im}(a)|\leq\frac{\alpha}{8}$,

$$|a|^m|I_t^{(m')}(a)|\leq C\exp(-C'/t).\tag{7.78}$$

Clearly, there exist uniquely defined holomorphic functions $\widetilde{F}_t(a), \widetilde{G}_t(a), \widetilde{I}_t(a)$ such that

$$F_t(a)=\widetilde{F}_t(a^2),\ G_t(a)=\widetilde{G}_t(a^2),\ I_t(a)=\widetilde{I}_t(a^2).\tag{7.79}$$

By (7.75),(7.77),

$$\exp(-a)=\widetilde{F}_t(a)+\widetilde{G}_t(a),\tag{7.80}$$
$$\widetilde{I}_t(a)=\widetilde{G}_t(a/t).$$

By (7.78), we find that if $\lambda\in V_{\alpha/8}$, then

$$|\lambda|^m|\widetilde{I}_t^{(m')}(\lambda)|\leq C\exp(-C'/t).\tag{7.81}$$

For $q\in\mathbf{N}$, let $\widetilde{I}_{t,q}(a)$ be the holomorphic function on $\mathbf{C}$, which is characterized by the following two properties:

$$\lim_{\lambda\to+\infty}\widetilde{I}_{t,q}(\lambda)=0,\tag{7.82}$$
$$\frac{\widetilde{I}_{t,q}^{(q)}(\lambda)}{(q-1)!}=\widetilde{I}_t(\lambda).$$

By (7.81), if $\lambda\in V_{\alpha/8}$,

$$|\lambda|^m|\widetilde{I}_{t,q}^{(q)}(\lambda)|\leq C\exp(-C'/t).\tag{7.83}$$

By (7.80),

$$\exp(-L_K-C_{K,t}^2)=\widetilde{F}_t(L_K+C_{K,t})+\widetilde{I}_t(\mathcal{A}_t).\tag{7.84}$$

Theorem 7.15. *There exist $\beta>0, C>0, C'>0$ such that if $K\in\mathfrak{g}, |K|\leq\beta, t\in]0,1]$,*

$$\|\widetilde{I}_t(\mathcal{A}_t)\|_1\leq C\exp(-C'/t).\tag{7.85}$$

Proof. In Theorem 7.13, we take $c = \frac{\alpha}{8}$. Then given $\beta > 0, t \in]0,1]$ small enough, Theorem 7.13 holds. By (7.81),

$$\widetilde{I}_t(\mathcal{A}_t) = \frac{1}{2i\pi}\int_{\Gamma_{\alpha/8}} \widetilde{I}_t(\lambda)(\lambda - \mathcal{A}_t)^{-1} d\lambda. \tag{7.86}$$

From (7.83), (7.86), for $q \in \mathbf{N}$,

$$\widetilde{I}_t(\mathcal{A}_t) = \frac{1}{2i\pi}\int_{\Gamma_{\alpha/8}} \widetilde{I}_{t,q}(\lambda)(\lambda - \mathcal{A}_t)^{-q} d\lambda. \tag{7.87}$$

Now in (7.87), we take $q \geq 2\dim X + 1$, and we use the second inequality in (7.69) and (7.83). Then (7.85) follows.

The proof of our Theorem is completed. □

By (7.84), (7.85), we find that to establish (7.39) in Theorem 7.9, we may as well replace $\exp(-L_K - C^2_{K,t})$ by $\widetilde{F}_t(L_K + C_{K,t})$.

By (7.74),

$$\widetilde{F}_t(L_K + C^2_{K,t}) = 2\int_0^{+\infty} \cos\left(s\sqrt{2(L_K + C^2_{K,t})}\right) \exp(-s^2/2))f(\sqrt{t}s)\frac{ds}{\sqrt{2\pi}}. \tag{7.88}$$

The principal symbol of the operator $2(L_K + C^2_{K,t})$ is scalar and equal to $t|\xi|^2$. Also $f(\sqrt{t}s)$ vanishes for $|\sqrt{t}s| \geq \alpha$.

For $\epsilon > 0, x \in X$, let $B^X(x,\epsilon)$ be the open ball of center x and radius ϵ.

Let dv_X be the volume form on X associated to the metric h^{TX}. Other volume forms will be denoted in the same way.

Let $\widetilde{F}_t(L_K + C^2_{K,t})(x,x')(x,x' \in X)$ be the smooth kernel associated to the operator $\widetilde{F}_t(L_K + C^2_{K,t})$ with respect to $\frac{dv_X(x')}{(2\pi)^{\dim X}}$. Using finite propagation speed for solutions of hyperbolic equations [ChP, Section 7.8], [T, Section 4.4], we find that $\widetilde{F}_t(L_K + C^2_{K,t})(x,.)$ depends only on the restriction of the operator $L_K + C^2_{K,t}$ to the ball $B^X(x,\alpha)$. Therefore, we have shown that the proof of (7.39) can be made local on X.

Clearly the kernel of $g\widetilde{F}_t(L_K + C^2_{K,t})$ is given by $g\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}x,x')$. By the above, it follows that $g\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}x,x')$ vanishes if $d^X(g^{-1}x,x) \geq \alpha$.

Now we explain our choice of α. Given $\epsilon > 0$, let $\mathcal{U}_\epsilon$ be the ϵ-neighborhood of X_g in $N_{X_g/X}$. Recall that $N_{X_g/X}$ is identified with the orthogonal bundle to TX_g in $TX|_{X_g}$. There exists $\epsilon_0 \in]0, a_X/32]$ such that if $\epsilon \in]0, 16\epsilon_0]$, the map $(x,Z) \in N_{X_g/X,\mathbf{R}} \to \exp^X_x(Z)$ is a diffeomorphism of $\mathcal{U}_\epsilon$ on the tubular neighborhood $\mathcal{V}_\epsilon$ of X_g in X.

In the sequel, we identify $\mathcal{U}_\epsilon$ and $\mathcal{V}_\epsilon$. This identification if g-equivariant.

We will assume that $\alpha \in]0,\epsilon_0]$ is small enough so that if $x \in X, d^X(g^{-1}x,x) \leq \alpha$, then $x \in \mathcal{V}_{\epsilon_0}$.

By the above, it follows that for $\beta > 0$ small enough, our proof of (7.39) has been localized on the ϵ_0-neighborhood $\mathcal{V}_{\epsilon_0}$ of X_g.

Let $k(x,Z)$ be the smooth function on $\mathcal{U}_{\epsilon_0}$ such that

$$dv_X(x,Z) = k(x,Z)dv_{X_g}(x)dv_{N_{X_g/X}}(Z). \tag{7.89}$$

In particular

$$k|_{X_g} = 1. \tag{7.90}$$

If $\delta \in \Lambda(T^*_{\mathbf{R}}X_g)$, if $\delta^{(2\dim X_g)}$ is the component of top degree of δ, let $\delta^{\max} \in \mathbf{R}$ be given by the relation

$$\delta^{(2\dim X_g)} = \delta^{\max} dv_{X_g}. \tag{7.91}$$

Theorem 7.16. *There exist $\beta > 0, \gamma \in]0,1]$ such that if $K \in \mathfrak{z}(g), |K| \leq \beta, t \in]0,1], x \in X_g$,*

(7.92)

$$\Bigg| \frac{t^{\dim N_{X_g/X}}}{(2\pi)^{\dim X_g}} \int_{\substack{Z \in N_{X_g/X,\mathbf{R}} \\ |Z| \leq \epsilon_0/\sqrt{t}}} \operatorname{Tr_s}\big[g\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}(x,\sqrt{t}Z),(x,\sqrt{t}Z))\big]$$

$$k(x,\sqrt{t}Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} - \big\{\operatorname{Td}_{g,K}(TX,h^{TX})\operatorname{ch}_{g,K}(E,h^E)\big\}^{\max}\Bigg| \leq Ct^{\gamma},$$

$$\Bigg| \frac{t^{\dim N_{X_g/X}}}{(2\pi)^{\dim X_g}} \int_{\substack{Z \in N_{X_g/X,\mathbf{R}} \\ |Z| \leq \epsilon_0/\sqrt{t}}} t\operatorname{Tr_s}\big[(N - i\frac{\langle \mu, K\rangle}{t})g\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}(x,\sqrt{t}Z),(x,\sqrt{t}Z))\big]$$

$$k(x,\sqrt{t}Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} - \big\{(\frac{\omega^X}{2\pi} - i\langle \mu, K\rangle)\operatorname{Td}_{g,K}(TX,h^{TX})\operatorname{ch}_{g,K}(E,h^E)\big\}^{\max}\Bigg| \leq Ct^{\gamma}.$$

Proof. Sections 7.4-7.13 are devoted to the proof of our Theorem. ☐

Remark 7.17. Clearly,

$$\int_X \operatorname{Tr_s}\big[g\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}x,x)\big]\frac{dv_X(x)}{(2\pi)^{\dim X}} = \tag{7.93}$$

$$\int_{\mathcal{U}_{\epsilon_0}} \operatorname{Tr_s}\big[g\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}x,x)\big]\frac{dv_X(x)}{(2\pi)^{\dim X}} =$$

$$\int_{(x,Z)\in\mathcal{U}_{\epsilon_0\sqrt{t}}} t^{\dim N_{X_g/X}} \operatorname{Tr_s}\Big[g\widetilde{F}_t(L_K + C^2_{K,t})\big(g^{-1}(x,\sqrt{t}Z),(x,\sqrt{t}Z)\big)\Big]$$

$$k(x,\sqrt{t}Z)\frac{dv_{X_g}(x)dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim X}}.$$

Using Theorems 7.15 and 7.16, we find that for $|K| \leq \beta$,

$$\Big|\alpha_t(g,K) - \int_{X_g}\operatorname{Td}_{g,K}(TX,h^{TX})\operatorname{ch}_{g,K}(E,h^E)\Big| \leq Ct^{\gamma}. \tag{7.94}$$

A similar argument shows that

$$\big|t\gamma_t(g,K) - C_{-1}(g,K)\big| \leq Ct^{\gamma}. \tag{7.95}$$

So we have established the first inequality in (7.39) and part of the second inequality.

Now we concentrate on the proof of Theorem 7.16. The proof of Theorem 7.9 will be completed in Section 7.14.

7.4. **A Lichnerowicz formula.** Now we recall the Lichnerowicz formula established in [B3, Theorem 1.6]. Let $e_1, \dots, e_{2\ell}$ be a locally defined smooth orthonormal basis of $T_{\mathbf{R}}X$. Let (F, ∇^F) be a vector bundle with connection on X. We use the notation

$$\big(\nabla^F_{e_i}\big)^2 = \sum_{i=1}^{2\ell}\big(\nabla^F_{e_i}\big)^2 - \nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{\sum_{i=1}^{2\ell}\nabla^{TX}_{e_i}e_i}. \tag{7.96}$$

We will use formula (7.96) apllied to $\Lambda(T^{*(0,1)}X)\otimes E$ equipped with the connection $\nabla_{\cdot}^{\Lambda(T^{*(0,1)}X)\otimes E} - \frac{\langle K^X,.\rangle}{2t}$. Let H be the scalar curvature of X.

Proposition 7.18. *The following identity holds,*

$$L_K + C_{K,t}^2 = -\frac{t}{2}\left(\nabla_{e_i}^{\Lambda(T^{*(0,1)}X)\otimes E} - \frac{\langle K^X, e_i\rangle}{2t}\right)^2 + \frac{tH}{8} + \frac{t}{4}c(e_i)c(e_j)\big(R^E + \frac{1}{2}\mathrm{Tr}[R^{TX}]\big)(e_i,e_j) - \big(m^E(K) + \frac{1}{2}\mathrm{Tr}[m^{TX}(K)]\big). \tag{7.97}$$

7.5. **A local coordinate system near the fixed point set of g.** Take $x \in X_g$. Then the map $Z \in (T_{\mathbf{R}}X)_x, |Z| \le a_X/2 \to \exp_x(Z) \in X$ identifies $B^{TX_x}(0, a_X/2)$ and $B^X(x, a_X/2)$. With this identification, there exists a smooth function $k'_x(Z), Z \in B^{TX}(0, a_X/2)$ such that

$$dv_X(Z) = k'_x(Z)dv_{TX}(Z). \tag{7.98}$$

Also

$$k'_x(0) = 1. \tag{7.99}$$

Recall that h^{TX} denotes the Hermitian metric on TX. We may and we will assume that ϵ_0 is small enough so that if $Z \in (T_{\mathbf{R}}X)_x \le 4\epsilon_0$,

$$\frac{1}{2}h_x^{TX} \le h_Z^{TX} \le \frac{3}{2}h_x^{TX}. \tag{7.100}$$

Recall that K'^X is the one form dual to K^X.

Definition 7.19. Let ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$ be the connection on $\Lambda(T^{*(0,1)}X)\otimes E$,

$${}^1\nabla_{\cdot}^{\Lambda(T^{*(0,1)}X)\otimes E} = \nabla_{\cdot}^{\Lambda(T^{*(0,1)}X)\otimes E} - \frac{K'^X}{2}. \tag{7.101}$$

Using (1.3) and (7.101), we find easily that

$${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E,2} = \frac{1}{4}\langle R^{TX}e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\mathrm{Tr}[R^{TX}] + R^E - \frac{1}{2}dK'^X. \tag{7.102}$$

Definition 7.20. Let ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E,t}$ be the connection on $\Lambda(T^{*(0,1)}X)\otimes E$,

$${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E,t} = \nabla^{\Lambda(T^{*(0,1)}X)\otimes E} - \frac{K'^X}{2t}. \tag{7.103}$$

In the sequel, we will trivialize $\Lambda(T^{*(0,1)}X)\otimes E$ by parallel transport along $s \in [0,1] \to sZ$ with respect to the connection ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E,t}$. It will often be more convenient to trivialize with respect to ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$ and then to change K into K/t. Incidentally, observe that the above connections are g-invariant.

Observe that if $A \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)$,

$${}^1\nabla_{\cdot}^{\Lambda(T^{*(0,1)}X)\otimes E}A = \nabla_{\cdot}^{\Lambda(T^{*(0,1)}X)\otimes E}A. \tag{7.104}$$

In particular, if B is a smooth section of $T_{\mathbf{R}}X$,

$${}^1\nabla_{\cdot}^{\Lambda(T^{*(0,1)}X)\otimes E}c(B) = c(\nabla_{\cdot}^{TX}B). \tag{7.105}$$

We temporarily trivialize TX, E by parallel transport along $s \to sZ$ with respect to ∇^{TX}, ∇^{E}. Let Γ^{TX}, Γ^{E} be the connection forms for ∇^{TX}, ∇^{E} in this trivialization. By [ABoP, Proposition 3.7],

$$\begin{aligned} \Gamma^{TX}(Z) &= \frac{1}{2} R^{TX}(Z, .) + \mathcal{O}(|Z|^2), \\ \Gamma^{E}(Z) &= \frac{1}{2} R^{E}(Z, .) + \mathcal{O}(|Z|^2). \end{aligned} \tag{7.106}$$

Let Γ^K be the one form on $B^{TX_x}(0, \alpha)$ which vanishes at 0 and is obtained by radial integration of the 2 form dK'^X along $s \in [0,1] \to sZ$, so that

$$d\Gamma^K = dK'^X. \tag{7.107}$$

In particular, Γ^K depends linearly on K. Then, by the same result as before,

$$\Gamma^K(Z) = \frac{1}{2} dK'^X(Z, .) + \mathcal{O}(|Z|^2). \tag{7.108}$$

Let $(e_1, \dots, e_{2\ell'})$ be an orthonormal oriented basis of $(T_{\mathbf{R}} X_g)_x$, let $(e_{2\ell'+1}, \dots, e_{2\ell})$ be an orthonormal oriented basis of $(N_{X_g/X,\mathbf{R}})_x$, so that $(e_1, \dots, e_{2\ell})$ is an orthonormal oriented basis of $(T_{\mathbf{R}} X)_x$. We denote with an upper script the corresponding dual basis.

Let ${}^1\Gamma^{\Lambda(T^{*(0,1)}X) \otimes E}$ be the connection form of ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$ in the trivialization of $\Lambda(T^{*(0,1)}X) \otimes E$ associated to the connection ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$ on $\Lambda(T^{*(0,1)}X) \otimes E$. Then by (1.3),

$${}^1\Gamma^{\Lambda(T^{*(0,1)}X)\otimes E} = \frac{1}{4} \langle \Gamma^{TX} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} \operatorname{Tr}[\Gamma^{TX}] + \Gamma^E - \frac{\Gamma^K}{2}. \tag{7.109}$$

7.6. **Replacing the manifold by $(T_{\mathbf{R}} X)_x$.** Let $\gamma(s)$ be a smooth even function from $\mathbf{R}$ into $[0,1]$ such that

$$\begin{aligned} \gamma(s) &= 1 \text{ if } |s| \leq 1/2, \\ &= 0 \text{ if } |s| \geq 1. \end{aligned} \tag{7.110}$$

If $Z \in (T_{\mathbf{R}} X)_x$, put

$$\rho(Z) = \gamma\left(\frac{|Z|}{4\epsilon_0}\right). \tag{7.111}$$

Then

$$\begin{aligned} \rho(Z) &= 1 \text{ if } |Z| \leq 2\epsilon_0, \\ &= 0 \text{ if } |Z| \geq 4\epsilon_0. \end{aligned} \tag{7.112}$$

For $x \in X_g$, let $\mathbf{H}_x$ be the vector space of smooth sections of $(\Lambda(T^{*(0,1)}X) \otimes E)_x$ over $(T_{\mathbf{R}} X)_x$. Let Δ^{TX} be the standard Laplacian on the fibres of $T_{\mathbf{R}} X$.

Definition 7.21. Let $L^{1,t}_{x,K}$ be the differential operator acting on $\mathbf{H}_x$,

$$L^{1,t}_{x,K} = -(1 - \rho^2(Z)) \frac{t}{2} \Delta^{TX} + \rho^2(Z)(L_K + C^2_{K,t}). \tag{7.113}$$

Observe that if $Z \in N_{X_g/X,\mathbf{R},x}$, $|Z| \leq \epsilon_0$, if $x' \in X$ is such that $d^X(Z, x') \leq \alpha$, since $\alpha \leq \epsilon_0$, then

$$d^X(x, x') \leq 2\epsilon_0. \tag{7.114}$$

In particular x' is represented by $Z' \in (T_{\mathbf{R}}X)_x$ such that $|Z'| \leq 2\epsilon_0$, so that $\rho(Z') = 1$.

Let $\widetilde{F}_t(L^{1,t}_{x,K})(Z,Z')$ be the smooth kernel of $\widetilde{F}_t(L^{1,t}_{x,K})$ with respect to $\frac{dv_{TX}(Z')}{(2\pi)^{\dim X}}$. Using the finite propagation speed for solutions of hyperbolic equations [ChP, Section 7.8], [T, Section 4.4], we find that if $Z \in (N_{X_g/X,\mathbf{R}})_x, |Z| \leq \epsilon_0$, then

$$\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}Z, Z)k'_x(Z) = \widetilde{F}_t(L^{1,t}_{x,K})(g^{-1}Z, Z). \tag{7.115}$$

In our proof of Theorem 7.16, we can then replace $L_K + C^2_{K,t}$ by $L^{1,t}_{x,K}$.

7.7. **The Getzler rescaling.** Let Op_x be the set of scalar differential operators acting on $\mathbf{H}_x$. Then

$$L^{1,t}_{x,K} \in \mathrm{End}\Big((\Lambda(T^{*(0,1)}X) \otimes E)_x\Big) \otimes \mathrm{Op}_x. \tag{7.116}$$

For $t > 0$, let $H_t : \mathbf{H}_x \to \mathbf{H}_x$ be the linear map

$$H_t h(Z) = h(\frac{Z}{\sqrt{t}}). \tag{7.117}$$

Definition 7.22. Let $L^{2,t}_{x,K}$ be the differential operator acting on $\mathbf{H}_x$,

$$L^{2,t}_{x,K} = H_t^{-1} L^{1,t}_{x,K} H_t. \tag{7.118}$$

By (7.116), since $\mathrm{End}\big(\Lambda(T^{*(0,1)}X)\big) = c(T_{\mathbf{R}}X) \otimes_{\mathbf{R}} \mathbf{C}$,

$$L^{2,t}_{x,K} \in \big(c(T_{\mathbf{R}}X) \otimes \mathrm{End}(E)\big)_x \otimes \mathrm{Op}_x. \tag{7.119}$$

Now we introduce the Getzler rescaling [Ge] of the Clifford algebra. For $1 \leq j \leq 2\ell'$, the operators $e^j\wedge$ and i_{e_j} act as odd operators on $\Lambda\big(T^*_{\mathbf{R}}(X_g)\big)$.

Definition 7.23. For $t > 0$, put

$$c_t(e_j) = \sqrt{2/t}e^j \wedge -\sqrt{t/2}i_{e_j},\ 1 \leq j \leq 2\ell'. \tag{7.120}$$

Definition 7.24. Let $L^{3,t}_{x,K} \in \mathrm{End}\big(\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X}) \otimes E\big)_x \otimes \mathrm{Op}_x$ be the operator obtained from $L^{2,t}_{x,K}$ by replacing $c(e_j)$ by $c_t(e_j)$ for $1 \leq j \leq 2\ell'$ while leaving the $c(e_j)$'s unchanged for $2\ell' + 1 \leq j \leq 2\ell$.

Let $\widetilde{F}_t(L^{3,t}_{x,K})(Z,Z')$ be the smooth kernel associated to $\widetilde{F}_t(L^{3,t}_{x,K})$ with respect to $dv_{TX}(Z')/(2\pi)^{\dim X}$. Recall that g acts on $\big(\Lambda(N^*_{X_g/X} \otimes E\big)_x$.

We may write $\widetilde{F}_t(L^{3,t}_{x,K})(Z,Z')$ in the form

$$\widetilde{F}_t(L^{3,t}_{x,K})(Z,Z') = \sum e^{i_1} \wedge \ldots e^{i_p} i_{e_{j_1}} \ldots i_{e_{j_q}} \widetilde{F}^{j_1 \ldots j_q}_{t,i_1,\ldots i_p}, \tag{7.121}$$

$$\widetilde{F}^{j_1 \ldots j_q}_{t,i_1,\ldots i_p} \in \big(c(N_{X_g/X,\mathbf{R}}) \otimes \mathrm{End}(E)\big)_x.$$

Put

$$\big[\widetilde{F}_t(L^{3,t}_{x,K})(Z,Z')\big]^{\max} = \widetilde{F}_{t,1,2,\ldots,2\ell'}(Z,Z'). \tag{7.122}$$

More precisely, $\widetilde{F}_{t,1,2,\ldots,2\ell'}(Z,Z')$ is the coefficient of $e^1 \wedge \ldots e^{2\ell'}$ in (7.121).

Proposition 7.25. *If* $Z \in (T_{\mathbf{R}}X)_x$, $|Z| \leq \epsilon_0/\sqrt{t}$, *the following identity holds,*

$$t^{\dim N_{X_g/X}} \operatorname{Tr_s}\left[g\widetilde{F}_t(L_K + C^2_{K,t})(g^{-1}(\sqrt{t}Z), \sqrt{t}Z)\right] k'_x(\sqrt{t}Z) = \\ (-i)^{\ell'} \operatorname{Tr_s}^{\Lambda(N^*_{X_g/X})\otimes E}\left[g\widetilde{F}_t(L^{3,t}_{x,K})(g^{-1}Z, Z)\right]^{\max}. \tag{7.123}$$

Proof. Recall that all the above identifications are g-equivariant. Our identity now follows from (7.115), [Ge], [BL, Proposition 11.2]. □

Let $j : X_g \to X$ be the obvious embedding.

Definition 7.26. Let $L^{3,0}_{x,K}$ be the operator in $\left(\Lambda(T^*_{\mathbf{R}}X_g) \otimes c(N_{X_g/X})\right)_x \otimes \mathrm{Op}_x$ given by

$$L^{3,0}_{x,K} = -\frac{1}{2}\left(\nabla_{e_i} + \frac{1}{2}\langle (j^*R^{TX} - m^{TX}(K))Z, e_i\rangle\right)^2 + j^*R^E_x - m^E(K)_x \\ + \frac{1}{2}\mathrm{Tr}[j^*R^{TX} - m^{TX}(K)]. \tag{7.124}$$

In the sequel, we will write that a sequence of differential operators on $(T_{\mathbf{R}}X)_x$ converges if its coefficients converge together with their derivatives uniformly on the compact subsets in $(T_{\mathbf{R}}X)_x$.

Proposition 7.27. *As* $t \to 0$,

$$L^{3,t}_{x,K} \to L^{3,0}_{x,K}. \tag{7.125}$$

Proof. Using (3.1), (7.97), (7.106), (7.108), and proceeding as in Getzler [Ge] and in Berline-Getzler-Vergne [BeGeV, Proposition 8.16], we get (7.125). □

7.8. **A family of norms.** Now we imitate constructions in Bismut-Lebeau [BL, Section 11 k)].

Definition 7.28. If $s \in C^\infty\left(T_{\mathbf{R}}X, (\Lambda^q(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X}) \otimes E)_x\right)$ has compact support, put

$$|s|^2_{t,x,0} = \int_{T_{\mathbf{R}}X} |s|^2 \left(1 + |Z|\rho\left(\frac{\sqrt{t}Z}{2}\right)\right)^{2(2\ell'-q)} dv_{TX}(Z). \tag{7.126}$$

Let $(\mathbf{I}^0_x, |\ |_{t,x,0})$ be the Hilbert closure of the above vector space, and let $\langle\ \rangle_{t,x,0}$ be the corresponding Hermitian product.

Recall that by (7.112), if $\rho(\sqrt{t}Z) > 0$, then $|\sqrt{t}Z| \leq 4\epsilon_0$. Now we have the result in [BL, Proposition 11.24].

Proposition 7.29. *For* $t \in]0,1]$, *the following family of operators acting on* $(\mathbf{I}^0_x, |\ |_{t,x,0})$ *are uniformly bounded*

$$1_{|\sqrt{t}Z| \leq 4\epsilon_0}\sqrt{t}c_t(e_j),\ 1 \leq j \leq 2\ell', \\ 1_{|\sqrt{t}Z| \leq 4\epsilon_0}|Z|\sqrt{t}c_t(e_j),\ 1 \leq j \leq 2\ell'. \tag{7.127}$$

are uniformly bounded.

Proof. If $|\sqrt{t}Z| \leq 4\epsilon_0$, by (7.112), $\rho(\sqrt{t}Z/2) = 1$. Then our Proposition follows from the obvious inequalities under the stated conditions,

$$\frac{1}{1 + |Z|\rho(\sqrt{t}Z/2)} \leq 1,\ \frac{|Z|}{1 + |Z|} \leq 1, \\ t(1 + |Z|) \leq C,\ t|Z|(1 + |Z|) \leq C. \tag{7.128}$$

□

Definition 7.30. If $s \in C^\infty\Big(T_{\mathbf{R}}X, (\Lambda^q(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X})\otimes E)_x\Big)$ has compact support, put

$$|s|^2_{t,x,1} = |s|^2_{t,x,0} + \sum_1^{2\ell} |\nabla_{e_i} s|^2_{t,x,0}. \tag{7.129}$$

Let $(\mathbf{I}^1_x, |\,|_{t,x,1})$ be the Hilbert closure of the above vector space with respect to $|\,|_{t,x,1}$. Then $(\mathbf{I}^1_x, |\,|_{t,x,1})$ is densely embedded in $(\mathbf{I}^0_x, |\,|_{t,x,0})$ with norm smaller than 1. We identify $\mathbf{I}^0_x$ to its antidual by the Hermitian product $\langle\,\rangle_{t,x,0}$. Let $(\mathbf{I}^{-1}_x, |\,|_{t,x,-1})$ be the antidual of $(\mathbf{I}^1_x, |\,|_{t,x,1})$. Then $(\mathbf{I}^0_x, |\,|_{t,x,0})$ embeds densely in $(\mathbf{I}^{-1}_x, |\,|_{t,x,-1})$ with norm smaller than 1.

Theorem 7.31. *There exist constants $C_1 > 0, \dots, C_4 > 0$ such that if $t \in]0,1], z \in \mathbf{C}, |z| \leq 1$, if $n \in \mathbf{N}, x \in X_g$, if the support of $s, s' \in C^\infty\Big(T_{\mathbf{R}}X, (\Lambda^q(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X})\otimes E)_x\Big)$ is included in $\{Z \in (T_{\mathbf{R}}X)_x, |Z| \leq n\}$, then*

$$\begin{aligned}
&\operatorname{Re}\langle L^{3,t}_{x,zK} s, s\rangle_{t,x,0} \geq C_1 |s|^2_{t,x,1} - C_2\big(1 + |nz|^2\big)|s|^2_{t,x,0},\\
&|\operatorname{Im}\langle L^{3,t}_{x,zK} s, s\rangle_{t,x,0}| \leq C_3\big((1+|nz|)|s|_{t,x,1}|s|_{t,x,0} + |nz|^2|s|^2_{t,x,0}\big),\\
&|\langle L^{3,t}_{x,zK} s, s'\rangle_{t,x,0}| \leq C_4\big(1+|nz|^2\big)|s|_{t,x,1}|s'|_{t,x,1}.
\end{aligned} \tag{7.130}$$

Proof. By Proposition 7.18 and using (7.106), (7.108) and Proposition 7.29, we get (7.130) easily. Note that the terms containing $|nz|^2$ come from terms like

$$\left|\langle\big(\sqrt{t}\rho(\sqrt{t}Z)\frac{z\Gamma^K(\sqrt{t}Z)}{t}\big)^2 s, s\rangle_{t,x,0}\right|, \tag{7.131}$$

which can be dominated by $C(1+|nz|^2)|s|^2_{t,x,0}$. □

7.9. **The kernel** $\widetilde{F}_t(L^{3,t}_{x,K})$ as an infinite sum. Let $h : \mathbf{R} \to [0,1]$ be a smooth even function such that

$$\begin{aligned}
h(u) &= 1 \text{ for } |u| \leq 1/2,\\
&= 0 \text{ for } |u| \geq 1.
\end{aligned} \tag{7.132}$$

For $n \in \mathbf{N}$, put

$$h_n(u) = h(u + n/2) + h(u - n/2). \tag{7.133}$$

Then h_n is a smooth even function whose support is included in $[-\frac{n}{2}-1, -\frac{n}{2}+1] \cup [\frac{n}{2}-1, \frac{n}{2}+1]$.

Set

$$H(u) = \sum_{n\in\mathbf{N}} h_n(u). \tag{7.134}$$

The above sum is locally finite, and $H(u)$ is a bounded smooth even function which takes positive values and has a positive lower bound on $\mathbf{R}$.

Put

$$k_n(u) = \frac{h_n}{H}(u). \tag{7.135}$$

Then the k_n are bounded even smooth functions with bounded derivatives, and moreover

$$\sum_{n\in\mathbf{N}} k_n = 1 \tag{7.136}$$

Definition 7.32. For $t \in [0,1], n \in \mathbf{N}, a \in \mathbf{C}$, put

$$F_{t,n}(a) = \int_{-\infty}^{+\infty} \exp(is\sqrt{2}a)\exp(-\frac{s^2}{2})f(\sqrt{t}s)k_n(s)\frac{ds}{\sqrt{2\pi}}. \tag{7.137}$$

By (7.136),

$$F_t(a) = \sum_{n\in\mathbf{N}} \widetilde{F}_{t,n}(a). \tag{7.138}$$

Also, given $m, m' \in \mathbf{N}$, there exist $C > 0, C' > 0, C'' > 0$ such that for any $n \in \mathbf{N}, c > 0$,

$$\sup_{\substack{a\in\mathbf{C}\\ |\mathrm{Im}(a)|\leq c}} |a|^m|F_{t,n}^{(m')}(a)| \leq C\exp(-C'n^2 + C''c^2). \tag{7.139}$$

Let $\widetilde{F}_{t,n}(a)$ be the unique holomorphic function such that

$$F_{t,n}(a) = \widetilde{F}_{t,n}(a^2). \tag{7.140}$$

Recall that V_c was defined in (7.45). By (7.139), given $m, m' \in \mathbf{N}$, there exist $C > 0, C' > 0, C'' > 0$ such that for any $c > 0$, if $\lambda \in V_c$,

$$|\lambda|^m|\widetilde{F}_{t,n}^{(m')}(\lambda)| \leq C\exp(-C'n^2 + C''c^2). \tag{7.141}$$

By (7.138),

$$\widetilde{F}_t(a) = \sum_{n\in\mathbf{N}} \widetilde{F}_{t,n}(a). \tag{7.142}$$

Using (7.142), we get

$$\widetilde{F}_t(L_{x,K}^{3,t}) = \sum_{n\in\mathbf{N}} \widetilde{F}_{t,n}(L_{x,K}^{3,t}). \tag{7.143}$$

More precisely, by (7.141) and using standard elliptic estimates, given $t \in]0,1]$, we have the identity

$$\widetilde{F}_t(L_{x,K}^{3,t})(Z,Z') = \sum_{n\in\mathbf{N}} \widetilde{F}_{t,n}(L_{x,K}^{3,t})(Z,Z'), \tag{7.144}$$

and the series in the right-hand side of (7.144) converges uniformly together with its derivatives on the compact sets in $T_{\mathbf{R}}X$.

Definition 7.33. Put

$$L_{x,K,n}^{3,t} = -\left(1 - \gamma\left(\frac{|Z|}{2(n+2)}\right)\right)\frac{\Delta^{TX}}{2} + \gamma\left(\frac{|Z|}{2(n+2)}\right)L_{x,K}^{3,t}. \tag{7.145}$$

Observe that if $k_n(s) \neq 0$, then $|s| \leq \frac{n}{2} + 1$. Using finite propagation speed and (7.100), we find there is $C > 0$ such that if $Z \in T_{\mathbf{R}}X$, the support of $\widetilde{F}_{t,n}(L_{x,K}^{3,t})(Z,.)$ is included in $\{Z' \in T_{\mathbf{R}}X, |Z' - Z| \leq 2(\frac{n}{2}+1)\}$. Therefore, given $p \in \mathbf{N}$, if $Z \in T_{\mathbf{R}}X, |Z| \leq p$, the support of $\widetilde{F}_t(L_{x,K}^{3,t})(Z,.)$ is included in $\{Z' \in T_{\mathbf{R}}X, |Z'| \leq n + p + 2\}$.

If $|Z| \leq n+p+2$, then $\gamma\big(\frac{|Z|}{2(n+p+2)}\big) = 1$. Using finite propagation speed again, we see that if $Z \in T_{\mathbf{R}}X, |Z| \leq p$, then

$$\widetilde{F}_{t,n}(L^{3,t}_{x,K})(Z,Z') = \widetilde{F}_{t,n}(L^{3,t}_{x,K,n+p})(Z,Z'). \tag{7.146}$$

7.10. **Estimates on the resolvent of $L^{3,t}_{x,K,n}$.** Now we proceed as in [BL, Section 11 l]. From Theorem 7.31, it follows easily that

$$\begin{aligned}
&\operatorname{Re}\langle L^{3,t}_{x,zK,n}s,s\rangle_{t,x,0} \geq C_1|s|^2_{t,x,1} - C_2(1+|nz|^2)|s|^2_{t,x,0}),\\
&|\operatorname{Im}\langle L^{3,t}_{x,zK,n}s,s\rangle_{t,x,0}| \leq C_3\left((1+|nz|)|s|_{t,x,1}|s|_{t,x,0} + |nz|^2|s|^2_{t,x,0}\right),\\
&|\langle L^{3,t}_{x,zK,n}s,s'\rangle_{t,x,0}| \leq C(1+|nz|^2)|s|_{t,x,1}|s'|_{t,x,1}.
\end{aligned} \tag{7.147}$$

If $A \in \mathcal{L}(\mathbf{I}^k_x, \mathbf{I}^{k'}_x)$, $k,k' = -1,0,1$, let $\|A\|^{k,k'}_{t,x}$ be the norm of A with respect to the norms $|\;|_{t,x,k}, |\;|_{t,x,k'}$.

Theorem 7.34. *Given $\eta > 0$, there exist $c_\eta \in]0,1], C > 0, d > 0$ such that if $t \in]0,1], z \in \mathbf{C}, |z| \leq c_\eta, n \in \mathbf{N}, x \in X_g, \lambda \in U_{\eta n+d}$, the resolvent $(\lambda - L^{3,t}_{x,zK,n})^{-1}$ exists, and moreover*

$$\begin{gathered}
\|(\lambda - L^{3,t}_{x,zK,n})^{-1}\|^{0,0}_{t,x} \leq \frac{4}{(\eta n + d)^2},\\
\|(\lambda - L^{3,t}_{x,zK,n})^{-1}\|^{-1,1}_{t,x} \leq C(1+|n|^2)(1+|\lambda|^2).
\end{gathered} \tag{7.148}$$

Proof. We will use arguments which were already used in the proof of Theorem 7.12. By (7.147), if $\lambda \in \mathbf{R}, \lambda \leq -C_2(1+|nz|^2)$,

$$\operatorname{Re}\langle (L^{3,t}_{x,zK,n} - \lambda)s,s\rangle_{t,x,0} \geq C_1|s|^2_{t,x,0}, \tag{7.149}$$

so that

$$|s|_{t,x,0} \leq \frac{1}{C_1}|(L^{3,t}_{x,zK,n} - \lambda)s|_{t,x,0}. \tag{7.150}$$

Since $L^{3,t}_{x,zK,n}$ is elliptic of order 2 and coincides with $-\Delta^{TX}/2$ at infinity, there exists $C_1(\lambda,n,t)$ such that

$$|s|_2 \leq C_1(\lambda,n,t)|(L^{3,t}_{x,zK,n} - \lambda)s|_{t,x,0}. \tag{7.151}$$

From (7.150), (7.151), we find that if $\lambda \in \mathbf{R}, \lambda \leq -C_2(1+|nz|^2)$, the resolvent $(\lambda - L^{3,t}_{x,zK,n})^{-1}$ exists.

Let $\lambda = a+ib, a,b \in \mathbf{R}$. By (7.147),

$$\begin{aligned}
|\langle (L^{3,t}_{x,zK,n} - \lambda)s,s\rangle_{t,x,0}| \geq \sup\big(&C_1|s|^2_{t,x,1} - (C_2(1+|nz|^2)+a)|s|^2_{t,x,0},\\
&- C_3(1+|nz|)|s|_{t,x,1}|s|_{t,x,0} + (|b| - C_3|nz|^2)|s|^2_{t,x,0}\big).
\end{aligned} \tag{7.152}$$

Set

$$C(\lambda,n,t) = \inf_{\substack{u\in\mathbf{R}\\ u\geq 1}} \sup\left(C_1u^2 - (C_2(1+|nz|^2+a), |b| - C_3(1+|nz|)u - C_3|nz|^2\right). \tag{7.153}$$

By (7.150), (7.153), we get

$$|\langle (L^{3,t}_{x,zK,n} - \lambda)s,s\rangle_{t,x,0}| \geq C(\lambda,n,t)|s|^2_{t,x,0}. \tag{7.154}$$

Take $\eta > 0, d > 0$, and assume that $\lambda = a + ib \in U_{\eta n+d}$, i.e.

$$a \leq \frac{b^2}{4(\eta n + d)^2} - (\eta n + d)^2. \tag{7.155}$$

Suppose that u is such that

$$|b| - C_3(1 + |nz|)u \leq (\eta n + d)^2. \tag{7.156}$$

Then

$$\begin{gathered} C_1 u^2 - (C_2(1 + |nz|^2) + a) \geq C_1 u^2 - \frac{b^2}{4(\eta n + d)^2} + (\eta n + d)^2 - C_2(1 + |nz|^2) \geq \\ \left(C_1 - \frac{C_3^2(1 + |nz|)^2}{4(\eta n + d)^2}\right) u^2 - \frac{C_3}{2}(1 + |nz|)u + \frac{3}{4}(\eta n + d)^2 - C_2(1 + |nz|^2). \end{gathered} \tag{7.157}$$

If $|z| \ll \eta$, if $d > 0$ is large enough, for any $n \in \mathbf{N}$,

$$C_1 \geq C_1 - \frac{C_3^2(1 + |nz|)^2}{4(\eta n + d)^2} \geq \frac{C_1}{2}. \tag{7.158}$$

The discriminant of the polynomial in the right-hand side of (7.157) is given by

$$\Delta = \frac{C_3^2}{4}(1 + |nz|)^2 - 4\left(\frac{3}{4}(\eta n + d)^2 - C_2(1 + |nz|^2)\right)\left(C_1 - \frac{C_3^2(1 + |nz|)^2}{4(\eta n + d)^2}\right). \tag{7.159}$$

Using (7.158), we find that for $d > 0$ large enough, and $|z| \ll \eta$,

$$\Delta \leq -C_1(\eta n + d)^2. \tag{7.160}$$

Therefore a lower bound for the polynomial in the right-hand side of (7.157) is given by $\frac{(\eta n + d)^2}{4}$.

If

$$|b| - C_3(1 + |nz|)u \geq (\eta n + d)^2, \tag{7.161}$$

for $|z| \ll \eta$,

$$|b| - C_3(1 + |nz|)u - C_3|nz|^2 \geq \frac{(\eta n + d)^2}{4}. \tag{7.162}$$

It follows from the above that given $\eta > 0$, for $d > 0$ large enough and $|z| \ll \eta$, for any $n \in \mathbf{N}$, if $\lambda \in U_{\eta n+d}$,

$$C(\lambda, t, n) \geq \frac{(\eta n + d)^2}{4}. \tag{7.163}$$

From (7.154), (7.163), for $\eta > 0, t \in]0, 1], |z| \ll \eta, n \in \mathbf{N}, \lambda \in U_{\eta n+d}$,

$$|\langle (L^{3,t}_{x,zK,n} - \lambda)s, s\rangle_{t,x,0}| \geq \frac{(\eta n + d)^2}{4} |s|^2_{t,x,0}. \tag{7.164}$$

From (7.164), we deduce that if $\lambda \in U_{\eta n+d}$, if the resolvent $(\lambda - L^{3,t}_{x,K,n})^{-1}$ exists, then

$$\|(\lambda - L^{3,t}_{x,zK,n})^{-1}\|^{0,0} \leq \frac{4}{(\eta n + d)^2}. \tag{7.165}$$

From (7.165), and by proceeding as in the proof of Theorem 7.12, we find that for $t \in]0, 1], |z| \ll \eta, n \in \mathbf{N}, \lambda \in U_{\eta n+d}$, the resolvent $(\lambda - L^{3,t}_{x,zK,n})^{-1}$ exists and (7.165) holds.

Then by proceeding as in (7.64)-(7.68), we get (7.148). The proof of our Theorem is completed. □

7.11. **Regularizing properties of the resolvent of $L^{3,t}_{x,K,n}$.** Here, we proceed as in Section [BL, Section 11 m)].

Definition 7.35. Let $\mathcal{Q}_x$ be the family of operators

$$\mathcal{Q}_x = \{\nabla_{e_i}, 1 \le i \le 2\ell\}. \tag{7.166}$$

For $j \in \mathbf{N}$, let $\mathcal{Q}^j_x$ be the set of operators $Q_1 \dots Q_j$, with $Q_i \in \mathcal{Q}_x, 1 \le i \le j$.

Proposition 7.36. *Take $k \in \mathbf{N}$. There exists $C_k > 0$ such that if $t \in]0,1], z \in \mathbf{C}, |z| \le 1, x \in X_g, Q_1, \dots, Q_k \in \mathcal{Q}_x$, if $s, s' \in C^\infty\Big((T_{\mathbf{R}}X)_x, (\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X}) \otimes E)_x\Big)$, have compact support, then*

$$|\langle [Q_1, [\dots Q_k, L^{3,t}_{x,K,n}] \dots]s, s'\rangle_{t,x,0}| \ \le C_k(1+n^2)|s|_{t,x,1}|s'|_{t,x,1}. \tag{7.167}$$

Proof. We proceed as in the proof of Theorem 7.31, and we obtain (7.167) easily. □

If $s \in C^\infty\Big((T_{\mathbf{R}}X)_x, (\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X}) \otimes E)_x\Big)$ has compact support, put

$$\|s\|^2_{t,x,k} = \sum_{j=0}^{k} \sum_{Q \in \mathcal{Q}^j_x} |Qs|^2_{t,x,0}. \tag{7.168}$$

Let $(\mathbf{I}^k, \| \ \|_{t,x,k})$ be the corresponding closure with respect to the norm $\| \ \|_{t,x,k}$.

Theorem 7.37. *Given $\eta > 0, k \in \mathbf{N}$, there exist $m_k \in \mathbf{N}, C_k > 0$ such that if $t \in]0,1], z \in \mathbf{C}, |z| \le c_\eta, n \in \mathbf{N}, x \in X_g, \lambda \in U_{\eta n + d}$, the resolvent $(\lambda - L^{3,t}_{x,zK,n})^{-1}$ maps $\mathbf{I}^k_x$ into $\mathbf{I}^{k+1}_x$, and moreover*

$$\|(\lambda - L^{3,t}_{x,zK,n})^{-1}\|^{k,k+1}_{t,x} \le C_k\big((1+n^2)(1+|\lambda|^2)\big)^{m_k}. \tag{7.169}$$

Proof. In view of Theorem 7.34 and of Proposition 7.36, the proof of our Theorem is the same as the proof of [BL, Theorem 11.30]. □

7.12. **Uniform estimates on the kernel of $\widetilde{F}_{t,n}(L^{3,t}_{x,K})$.** Recall that for $c > 0$, the set $\Gamma_c \subset \mathbf{C}$ was defined in (7.45). We will now proceed as in [BL, Section 11 n)].

Theorem 7.38. *There exist $C' > 0, C'' > 0, C''' > 0$ such that for $\eta > 0$ small enough, for any $m \in \mathbf{N}$, there is $C > 0, r \in \mathbf{N}$ such that for $t \in]0,1], |z| \le c_\eta, n \in \mathbf{N}, x \in X_g, Z, Z' \in (T_{\mathbf{R}}X)_x$,*

$$\begin{aligned} \sup_{|\alpha|,|\alpha'| \le m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \widetilde{F}_{t,n}(L^{3,t}_{x,K})(Z,Z') \right| &\le C\big(1+|Z|+|Z'|\big)^r \\ &\exp\Big(-C'n^2/4 + 2C''\eta^2 \sup\big(|Z|^2, |Z^{'2}|\big) - C'''|Z-Z'|^2\Big). \end{aligned} \tag{7.170}$$

Proof. We use the notation in (7.139)-(7.142). We fix $\eta > 0$ small enough so that

$$C' - 2C''\eta^2 \ge \frac{C'}{2}. \tag{7.171}$$

By (7.141), (7.171), given $m, m' \in \mathbf{N}, d > 0$, there exists $C > 0$ such that if $\lambda \in V_{\eta(n+p)+d}$,

$$|\lambda|^m |\widetilde{F}^{(m')}_{t,n}(\lambda)| \leq C_0 \exp\left(-\frac{C'}{2}n^2 + 2C''\eta^2 p^2\right). \tag{7.172}$$

By Theorem 7.34 and by (7.172), if $\eta > 0$ is chosen as before, for c_η and d taken as in Theorem 7.34, and $z \in \mathbf{C}, |z| \leq c_\eta$,

$$\widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p}) = \frac{1}{2i\pi}\int_{\Gamma_{\eta(n+p)+d}} \widetilde{F}_{t,n}(\lambda)(\lambda - L^{3,t}_{x,zK,n+p})^{-1} d\lambda. \tag{7.173}$$

In view of (7.172), given $q \in \mathbf{N}$, there is a unique holomorphic function $\widetilde{F}_{t,n,q}(\lambda)$ defined on a neighborhood of $V_{\eta(n+p)+d}$ such that

$$\begin{aligned} &\widetilde{F}_{t,n,q}(\lambda) \to 0 \,\text{as}\, \lambda \to +\infty,\\ &\frac{\widetilde{F}^{(q-1)}_{t,n,q}(\lambda)}{(q-1)!} = \widetilde{F}_{t,n}(\lambda). \end{aligned} \tag{7.174}$$

Then $\widetilde{F}_{t,n,q}(\lambda)$ verifies bounds on $V_{\eta(n+p)+d}$ similar to (7.172).

By (7.173), (7.174), we get

$$\widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p}) = \frac{1}{2i\pi}\int_{\Gamma_{\eta(n+p)+d}} \widetilde{F}_{t,n,q}(\lambda)(\lambda - L^{3,t}_{x,zK,n+p})^{-q} d\lambda. \tag{7.175}$$

By Theorem 7.37, if $Q \in \mathcal{Q}^k_x, k \leq q$, there is m_q such that if $\lambda \in U_{\eta(n+p)+d}, z \in \mathbf{C}, |z| \leq c_\eta$,

$$\|Q(\lambda - L^{3,t}_{x,zK,n+p})^{-q}\|^{0,0}_{t,x,0} \leq C\left((1+(n+p)^2)(1+|\lambda|^2)\right)^{m_q}. \tag{7.176}$$

By introducing the adjoint operator $L^{3,t*}_{x,zK,n+p}$ and by proceeding as before, we also find that if $Q' \in \mathcal{Q}^k_x, k \leq q$,

$$\|(\lambda - L^{3,t}_{x,zK,n+p})^{-q}Q'\|^{0,0}_{t,x,0} \leq C\left((1+(n+p)^2)(1+|\lambda|^2)\right)^{m_q}. \tag{7.177}$$

Ultimately, we find that if $Q \in \mathcal{Q}^k_x, Q' \in \mathcal{Q}^{k'}_x, k + k' \leq q$,

$$\|Q(\lambda - L^{3,t}_{x,zK,n+p})^{-q}Q'\|^{0,0}_{t,x,0} \leq C\left((1+(n+p)^2)(1+|\lambda|^2)\right)^{m_q}. \tag{7.178}$$

From (7.172), (7.175), (7.178), we deduce that there exists $m_q \in \mathbf{N}$ such that if $Q \in \mathcal{Q}^k_x, Q' \in \mathcal{Q}^{k'}_x, k + k' \leq q$, if $z \in \mathbf{C}, |z| \leq c_\eta$,

$$\|Q\widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p})Q'\|^{0,0}_{t,x} \leq C(1+p)^{2m_q}\exp\bigl(-C'\frac{n^2}{4} + 2C''\eta^2p^2\bigr). \tag{7.179}$$

Let $J^0_{x,p}$ be the vector space of square integrable sections of $\Bigl(\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X})\otimes E\Bigr)_x$ over $\{Z \in T_{\mathbf{R}}X, |Z| \leq p+1\}$. If $s \in J^0_{x,p}$, put

$$|s|^2_0 = \int_{|Z|\leq p+1} |s|^2 dv_{TX}(Z). \tag{7.180}$$

By (7.126), if $s \in J^0_{x,p}$,

$$|s|_0 \leq |s|_{t,x,0} \leq C(1+p)^{2\ell'}|s|_0. \tag{7.181}$$

If $A \in \mathcal{L}(J^0_{x,p})$, let $\|A\|_{\infty,p}$ be the norm of A with respect to to $|\,|_0$. From (7.179)-(7.181), we find that if $Q \in \mathcal{Q}^k_x, Q' \in \mathcal{Q}^{k'}_x, k+k' \leq q$,

$$(7.182)\quad \|Q\widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p})Q'\|_{\infty,p} \leq C(1+p)^{2(\ell'+m_q)}\exp\bigl(-C'n^2/4+2C''\eta^2p^2\bigr).$$

Using Sobolev inequalities, we deduce from (7.182) that given $m \in \mathbf{N}$, there exist $C>0, r\in\mathbf{N}$ such that for $z \in \mathbf{C}, |z| \leq c_\eta, n \in \mathbf{N}, p \in \mathbf{N}$,

(7.183)
$$\sup_{\substack{|\alpha|,|\alpha'|\leq m\\ |Z|,|Z'|\leq p}} \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p})(Z,Z')\right| \leq C(1+p)^{(2\ell'+r)}\exp\left(-C'n^2/4+2C''\eta^2p^2\right).$$

For $h \in \mathbf{N}$, put

$$(7.184)\quad F^h_{t,n}(a) = \int_{-\infty}^{+\infty}\exp(is\sqrt{2}a)\exp\bigl((-\frac{s^2}{2}\bigr)f(\sqrt{t}s)k_n(s)\,(1-\gamma(2s/h))\,\frac{ds}{\sqrt{2\pi}}.$$

Then $F^h_{t,n}$ has the same properties as $\widetilde{F}_{t,n}$. The estimate in (7.139) is replaced by

$$(7.185)\qquad \sup_{\substack{a\in\mathbf{C}\\ |\mathrm{Im}(a)|\leq c}} |a|^m|F^{h(m')}_{t,n}(a)| \leq C\exp\bigl(-C'n^2+C''c^2-C'''h^2\bigr).$$

Let $\widetilde{F}^h_{t,n}(a)$ be the holomorphic function such that

$$(7.186)\qquad F^h_{t,n}(a) = \widetilde{F}^h_{t,n}(a^2).$$

Then $\widetilde{F}^h_{t,n}$ verifies uniform estimates similar to (7.141), with the extra factor $\exp(-C'''h^2)$.

Using finite propagation speed and (7.100), it is clear that if $Z, Z' \in (T_{\mathbf{R}}X)_x, |Z-Z'| \geq h$,

$$(7.187)\qquad \widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p})(Z,Z') = \widetilde{F}^h_{t,n}(L^{3,t}_{x,zK,n+p})(Z,Z')$$

On the other hand, by (7.185) and by proceeding as in (7.172)-(7.183), we find that for $|z| \leq c_\eta$,

$$(7.188)\quad \sup_{\substack{|\alpha|,|\alpha'|\leq m\\ |Z|,|Z'|\leq p}} \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}}\widetilde{F}^h_{t,n}(L^{3,t}_{x,zK,n+p})(Z,Z')\right| \leq$$
$$C(1+p)^{(2\ell'+r)}\exp\left(-C'n^2/4+2C''\eta^2p^2-C'''h^2\right).$$

Using (7.146), (7.187), (7.188), we get (7.170). The proof of our Theorem is completed. □

Put

$$(7.189)\qquad L^{3,0}_{x,K,n} = -\left(1-\gamma^2\left(\frac{|Z|}{2(n+2)}\right)\right)\frac{\Delta^{TX}}{2}+\gamma^2\left(\frac{|Z|}{2(n+2)}\right)L^{3,0}_{x,K}.$$

By Proposition 7.27, as $t \to 0$,

$$(7.190)\qquad L^{3,t}_{x,K,n} \to L^{3,0}_{x,zK,n}.$$

Also if s is smooth with compact support, as $s \to 0$, $|s|_{t,x,0} \to |s|_{0,x,0}, |s|_{t,x,1} \to |s|_{0,x,1}$. Let $(\mathbf{I}'^0_x, |\,|_{0,x,0})$, $(\mathbf{I}'^1_x, |\,|_{0,x,1})$be the Hilbert closures of the above s with respect to to the corresponding norms. Let $(\mathbf{I}'^{-1}_x, |\,|_{0,x,-1})$ be the antidual of $(\mathbf{I}'^1_x, |\,|_{0,x,1})$. Then we have the embeddings with norm smaller than 1,

$$\mathbf{I}'^1_x \to \mathbf{I}'^0_x \to \mathbf{I}'^{-1}_x.$$

Observe that we can take $t=0$ in (7.147). Therefore the estimates in Theorems 7.34 and 7.37 still hold when making $t=0$. Also

$$|s|_{t,x,0} \leq |s|_{0,x,0}. \tag{7.191}$$

7.13. **A proof of Theorem 7.16.**

Theorem 7.39. *There exist $C>0$ such that for $t\in]0,1], z\in \mathbf{C}, |z|\leq 1, n\in \mathbf{N}, x\in X_g$, if $s\in C^\infty\Big((T_{\mathbf{R}}X)_x, (\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X})\otimes E)_x\Big)$ has compact support, then*

$$\left|(L^{3,t}_{x,zK,n} - L^{3,0}_{x,zK,n})s\right|_{t,x,-1} \leq C\sqrt{t}(1+n^4)|s|_{0,x,1}. \tag{7.192}$$

Proof. Clearly (7.192) is equivalent to the inequality

$$\left|\langle(L^{3,t}_{x,zK,n} - L^{3,0}_{x,zK,n})s, s'\rangle_{t,x,0}\right| \leq C\left(1+\sqrt{t}n^4\right)|s|_{0,x,1}|s'|_{t,x,1}. \tag{7.193}$$

Recall that if $\gamma\big(\frac{Z}{2(n+2)}\big)\neq 0$, then $|Z|\leq 2(n+2)$. An application of Taylor's formula leads to (7.193). □

Now we prove an analogue of [BL, Theorem 11.36].

Theorem 7.40. *Given $\eta>0$, there exist $C>0, c_\eta\in]0,1], d>0, q\in\mathbf{N}$ such that if $t\in]0,1], z\in\mathbf{C}, |z|\leq c_\eta, x\in X_g, \lambda\in U_{\eta n+d}$, if s has compact support, then*

$$\left|\left((\lambda - L^{3,t}_{x,zK,n})^{-1} - (\lambda - L^{3,0}_{x,zK,n})^{-1}\right)s\right|_{t,x,0} \leq C\sqrt{t}(1+n^q)(1+|\lambda|^q)|s|_{t,x,0}. \tag{7.194}$$

Proof. We use the formula

$$(\lambda - L^{3,t}_{x,zK,n})^{-1} - (\lambda - L^{3,0}_{x,zK,n})^{-1} = (\lambda - L^{3,t}_{x,zK,n})^{-1}(L^{3,t}_{x,zK,n} - L^{3,0}_{x,zK,n})(\lambda - L^{3,0}_{x,zK,n})^{-1}. \tag{7.195}$$

Also if s is smooth with compact support, by (7.191),

$$|s|_{0,x,-1} \leq |s|_{t,x,-1} \leq |s|_{t,x,0}. \tag{7.196}$$

Using now the inequality (7.148) (also in the case $t=0$) and (7.192), (7.195), we get (7.194). □

By (7.137),

$$F_{0,n}(a) = \int_{-\infty}^{+\infty} \exp(is\sqrt{2}a)\exp(-s^2/2)k_n(s)\frac{ds}{\sqrt{2\pi}}. \tag{7.197}$$

By proceeding as in (7.139), we find that for $m, m'\in\mathbf{N}$, there exist $C>0, C'>0, C''>0, C'''>0$ such that for for $c>0$,

$$\sup_{\substack{a\in\mathbf{C}\\ |\mathrm{Im}(a)|\leq c}} |a|^m|(F_{t,n} - F_{0,n})^{(m')}(a)| \leq C\exp(-C'n^2 + C''c^2 - C'''/t). \tag{7.198}$$

Also by (7.136),

$$\sum_{n\in\mathbf{N}} F_{0,n}(a) = \exp(-a^2). \tag{7.199}$$

Moreover $\widetilde{F}_{0,n}(a)$ is the holomorphic function such that

$$F_{0,n}(a) = \widetilde{F}_{0,n}(a^2). \tag{7.200}$$

By (7.198), (7.199), we get

$$\sum_{n\in\mathbf{N}} \widetilde{F}_{0,n}(a) = \exp(-a). \tag{7.201}$$

Now we use the notation in (7.182). In particular if $A \in \mathcal{L}(J^0_{x,p})$, let $\|A\|_{\infty,p}$ be the norm of A with respect to the norm $|\ |_0$ on $J^0_{x,p}$.

Theorem 7.41. *For $\eta > 0$ small enough, there exists $C > 0, c_\eta \in]0,1], r \in \mathbf{N}$ such that if $t \in]0,1], z \in \mathbf{C}, |z| \leq c_\eta, n \in \mathbf{N}, p \in \mathbf{N}, x \in X_g$,*

$$\|\widetilde{F}_{t,n}(L^{3,t}_{x,K}) - \widetilde{F}_{0,n}(L^{3,0}_{x,K})\|_{\infty,p} \leq C\sqrt{t}(1+p)^r \exp\bigl(-C'n^2/4 + 2C''\eta^2p^2\bigr). \tag{7.202}$$

Proof. Clearly, by Theorem 7.34,

$$\begin{aligned} &\widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p}) - \widetilde{F}_{t,n}(L^{3,0}_{x,zK,n+p}) = \\ &\qquad \frac{1}{2i\pi}\int_{\Gamma_{\eta(n+p)+d}} \widetilde{F}_{t,n}(\lambda)\left((\lambda - L^{3,t}_{x,zK,n+p})^{-1} - (\lambda - L^{3,0}_{x,zK,n+p})^{-1}\right) d\lambda. \end{aligned} \tag{7.203}$$

By (7.172), (7.181), by Theorem 7.40 and by (7.203), for $\eta > 0$ small enough, and $z \in \mathbf{C}, |z| \leq c_\eta$,

$$\|\widetilde{F}_{t,n}(L^{3,t}_{x,zK,n+p}) - \widetilde{F}_{t,n}(L^{3,0}_{x,zK,n+p})\|_{\infty,p} \leq C\sqrt{t}(1+p)^q \exp\left(-C'n^2/4 + 2C''\eta^2p^2\right). \tag{7.204}$$

Also by (7.148), (7.181) and (7.198), for $\eta > 0$ small enough, $z \in \mathbf{C}, |z| \leq c_\eta$,

$$\|(\widetilde{F}_{t,n} - \widetilde{F}_{0,n})(L^{3,0}_{x,zK,n+p})\|_{\infty,p} \leq C(1+p)^{2\ell'} \exp\left(-\mathbf{C}'n^2/4 + 2C''\eta^2p^2 - \frac{C'''}{t}\right). \tag{7.205}$$

By (7.146),(7.204),(7.205), we get (7.202). The proof of our Theorem is completed. □

Theorem 7.42. *For $\eta > 0$ small enough, there is $C > 0, C' > 0, C'' > 0, C''' > 0, r \in \mathbf{N}, c_\eta \in]0,1]$ such that if $t \in]0,1], z \in \mathbf{C}, |z| \leq c_\eta, n \in \mathbf{N}, x \in X_g, Z, Z' \in (T_{\mathbf{R}}X)_x$,*

$$\begin{aligned} &\left|\left(\widetilde{F}_{t,n}(L^{3,t}_{x,K}) - \widetilde{F}_{0,n}(L^{3,0}_{x,K})\right)(Z,Z')\right| \\ &Ct^{1/4(2\ell+1)}\bigl(1+|Z|+|Z'|\bigr)^r \exp\left(-C'n^2/4 + 2C''\eta^2\sup(|Z|^2,|Z'|^2) - \frac{C'''}{2}|Z-Z'|^2\right). \end{aligned} \tag{7.206}$$

Proof. We proceed as in [BL, Section 11 p)]. Let $\phi : \mathbf{R} \to [0,1]$ be a smooth function with compact support, equal to 1 near 0, such that

$$\int_{T_{\mathbf{R}}X} \phi(|Z|)dv_{TX}(Z) = 1. \tag{7.207}$$

Take $\beta \in]0,1]$. By Theorem 7.38, there exists $C>0$ such that if $|Z|\,|Z'| \leq p, U, U' \in (\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X})\otimes E)_x$,

$$\begin{aligned}(7.208)\quad &\Big|\Big\langle \big(\widetilde{F}_{t,n}(L^{3,t}_{x,K}) - \widetilde{F}_{0,n}(L^{3,0}_{x,K})\big)(Z,Z')U,U'\Big\rangle - \int_{(T_{\mathbf{R}}X)_x\times(T_{\mathbf{R}}X)_x}\langle(\widetilde{F}_{t,n}(L^{3,t}_{x,K}) - \\ &\widetilde{F}_{0,n}(L^{3,0}_{x,K}))(Z-\widetilde{Z},Z'-\widetilde{Z}')U,U'\rangle\frac{1}{\beta^{2\ell}}\phi(\widetilde{Z}/\beta)\frac{1}{\beta^{2\ell}}\phi(\widetilde{Z}'/\beta)dv_{TX}(\widetilde{Z})dv_{TX}(\widetilde{Z}')\Big| \leq \\ &\qquad C\beta(1+p)^r\exp\left(-C'n^2/4+2C''\eta^2p^2\right)|U||U'|.\end{aligned}$$

On the other hand, by Theorem 7.41,

$$\begin{aligned}(7.209)\quad &\Big|\int_{(T_{\mathbf{R}}X)_x\times(T_{\mathbf{R}}X)_x}\langle(\widetilde{F}_{t,n}(L^{3,t}_{x,K}) - \widetilde{F}_{0,n}(L^{3,0}_{x,K}))(Z-\widetilde{Z},Z'-\widetilde{Z}')U,U'\rangle \\ &\frac{1}{\beta^{2\ell}}\phi(\widetilde{Z}/\beta)\frac{1}{\beta^{2\ell}}\phi(\widetilde{Z}'/\beta)dv_{Tx}(\widetilde{Z})dv_{TX}(\widetilde{Z}')\Big| \leq C\frac{\sqrt{t}}{\beta^{2\ell}}(1+p)^r\exp\left(-C'n^2/4+2C''\eta^2p^2\right).\end{aligned}$$

By taking $\beta = t^{1/2(2\ell+1)}$, we deduce from (7.208),(7.209) that if $|Z|,|Z'|\leq p$,

$$\begin{aligned}(7.210)\quad &\\ &\Big|(\widetilde{F}_{t,n}(L^{3,t}_{x,K}) - \widetilde{F}_{0,n}(L^{3,0}_{x,K}))(Z,Z')\Big| \leq Ct^{1/2(2\ell+1)}(1+p)^r\exp\left(-C'n^2/4+2C''\eta^2p^2\right).\end{aligned}$$

By (7.170), (7.210), we get (7.206). The proof of our Theorem is completed. ☐

Now we briefly explain how to make sense of the kernel $\exp(-L^{3,0}_{x,K})(Z,Z')$. In fact, since $L^{3,0}_{x,zK,n+p}$ coincides with $-\Delta^{TX}/2$ at ∞, the operator$\widetilde{F}_{0,n}(L^{3,0}_{x,zK,n+p})$ is well defined. Also, by proceeding as in (7.146), if $|Z|,|Z'|\leq p$, using finite propagation speed, we find that the kernel $\widetilde{F}_{0,n}(L^{3,0}_{x,zK,n+p})(Z,Z')$ does not depend on p. Finally this kernel verifies estimates similar to (7.170) for $\eta>0$ small enough and $|z|\leq c_\eta$. Therefore we may define the kernel $\exp(-L^{3,0}_{x,K})(Z,Z')$ by the formula

$$(7.211)\qquad \exp(-L^{3,0}_{x,K})(Z,Z') = \sum_{n\in\mathbf{N}}\widetilde{F}_{0,n}(L^{3,0}_{x,zK,n+p})(Z,Z'),\ |Z|,|Z'|\leq p,$$

and the series in (7.211) converges uniformly on compact subsets of $(T_{\mathbf{R}}X)_x$ together with its derivatives.

Theorem 7.43. *For $\eta>0$ small enough, there exists $c_\eta \in]0,1], r\in\mathbf{N}$ such that if $t\in]0,1], z\in\mathbf{C}, |z|\leq c_\eta, x\in X_g, Z,Z'\in T_{\mathbf{R}}X$,*

$$\begin{aligned}(7.212)\quad &\Big|\big(\widetilde{F}_t(L^{3,t}_{x,K}) - \exp(-L^{3,0}_{x,K})\big)(Z,Z')\Big| \leq \\ &\qquad Ct^{1/4(2\ell+1)}\big(1+|Z|+|Z'|\big)^r\exp\Big(2C''\eta^2\sup(|Z|^2,|Z'|^2) - \frac{C'''}{2}|Z-Z'|^2\Big).\end{aligned}$$

Proof. By (7.144), by Theorem 7.42 and by (7.211), (7.212) follows. ☐

Now there is $c>0$ such that if $Z\in N_{X_g/X,\mathbf{R}}$, then

$$(7.213)\qquad |g^{-1}Z - Z| \geq c|Z|.$$

By (7.212), (7.213), we find that there exists $C'''' > 0$ such that if $Z \in N_{X_g/X,\mathbf{R},x}$,

$$\left|\Big(\widetilde{F}_t(L^{3,t}_{x,K}) - \exp\big(-L^{3,0}_{x,K}\big)\Big)(g^{-1}Z,Z)\right| \leq \\ Ct^{1/4(2\ell+1)}\big(1+|Z|\big)^r \exp\big(2C''\eta^2|Z|^2 - C''''|Z|^2\big). \tag{7.214}$$

For $\eta > 0$ small enough,

$$2C''\eta^2 - C'''' \leq -C''''/2, \tag{7.215}$$

so that by (7.214), if $Z \in N_{X_g/X,\mathbf{R},x}$,

$$\left|\Big(\widetilde{F}_t(L^{3,t}_{x,K}) - \exp(-L^{3,0}_{x,K})\Big)\,(g^{-1}Z,Z)\right| \leq Ct^{1/4(2\ell+1)} \exp\left(-\frac{C''''}{4}|Z|^2\right). \tag{7.216}$$

Put

$$H^{TX} = j^*R^{TX} - m^{TX}(zK). \tag{7.217}$$

Clearly H^{TX} splits as

$$H^{TX} = H^{TX_g} + H^{N_{X_g/X}}. \tag{7.218}$$

By (4.25), 4.30), (7.124), for $|z|$ small enough,

$$\exp\big(-L^{3,0}_{x,K}\big)(g^{-1}Z,Z) = \det\left(\frac{H^{TX}/2}{\sinh\big(H^{TX}/2\big)}\right)\exp\big(-\frac{1}{2}\mathrm{Tr}[H^{TX}]\big) \\ \exp\left(-\left\langle\frac{H^{TX}/2}{\sinh\big(H^{TX}/2\big)}(\cosh(H^{TX}/2) - e^{H^{TX}/2}g^{-1})Z,Z\right\rangle\right)\exp\left(-j^*R^E + m^E(zK)\right). \tag{7.219}$$

We can write $g|_{N_{X_g/X}}$ in the form

$$g = e^A, \tag{7.220}$$

with $A \in \mathrm{End}(N_{X_g/X})$ skew-adjoint, parallel and commuting with H^{TX}.

By (7.219), (7.220), we get

$$\exp\big(-L^{3,0}_{x,K}\big)(g^{-1}Z,Z) = \det\left(\frac{H^{TX}/2}{\sinh\big(H^{TX}/2\big)}\right)\exp\big(-\frac{1}{2}\mathrm{Tr}[H^{TX}]\big) \\ \exp\left(-\left\langle\frac{H^{TX}/2}{\sinh(H^{TX}/2)}\big(\cosh(H^{TX}/2) - \cosh(H^{TX}/2 - A)\big)\,Z,Z\right\rangle\right) \\ \exp(-j^*R^E + m^E(zK)) = \det\left(\frac{H^{TX}/2}{\sinh\big(H^{TX}/2\big)}\right)\exp\big(-\frac{1}{2}\mathrm{Tr}[H^{TX}]\big) \\ \exp\left(-\left\langle\frac{H^{TX}/2}{\sinh(H^{TX}/2)}2\sinh(A/2)\sinh\big((H^{TX}-A)/2\big)Z,Z\right\rangle\right)\exp\big(-j^*R^E+m^E(zK)\big). \tag{7.221}$$

Now we use the same notation as in (4.44). Observe that for $z \in \mathbf{R}$, if $|z|$ is small enough, $\Big(\frac{H^{TX}/2}{\sinh(H^{TX}/2)}\Big)^{(0)}$ is positive definite. Moreover if $|z|$ is small enough,

$$\big(\sinh(A/2)\sinh\big((H^{TX}-A)/2\big)\big)^{(0)}$$

is positive definite on $N_{X_g/X}$. Using (7.221), one then finds easily that if $|z|$ is small enough, $\exp\big(-L^{3,0}_{x,K}\big)$ is a Gaussian on $N_{X_g/X,\mathbf{R},x}$, and moreover

$$\int_{N_{X_g/X}} \exp\big(-L^{3,0}_{x,K}\big)(g^{-1}Z,Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} == \det\left(\frac{H^{TX_g}}{\sinh\big(H^{TX_g}/2\big)}\right) \\ \frac{\exp\big(-\frac{1}{2}\mathrm{Tr}[H^{TX}]\big)}{\det^{N_{X_g/X}}\left[4\sinh(A/2)\sinh\big((H^{N_{X_g/X}}-A)/2\big)\right]}\exp\big(-j^*R^E+m^E(zK)\big). \tag{7.222}$$

Also

$$\mathrm{Tr_s}^{\Lambda(N^{*(0,1)}_{X_g/X})\otimes E}\big[g\exp(-j^*R^E+m^E(zK)\big]=\det(1-e^A)|_{N_{X_g/X}} \\ \mathrm{Tr}\left[g\exp\big(-j^*R^E+m^E(zK)\big)\right]. \tag{7.223}$$

Using (7.222), (7.223), we get

$$\varphi\int_{N_{X_g/X}} \mathrm{Tr_s}^{\Lambda(N^{*(0,1)}_{X_g/X})\otimes E}\left[g\exp\big(-L^{3,0}_{x,K}\big)(g^{-1}Z,Z)\right]\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} = \\ \mathrm{Td}_{g,K}(TX,h^{TX})\mathrm{ch}_{g,K}(E,h^E)\big]. \tag{7.224}$$

From (7.123), (7.216), (7.224), we find that the first inequality holds in (7.92), with K replaced by zK. It is then trivial to obtain the required uniformity. The proof of the second inequality is similar. In fact by [BGS1, II, Proposition 2.4], or by an easy direct computation,

$$N=\frac{i}{4}c(Je_k)c(e_k)+\frac{\dim X}{2}. \tag{7.225}$$

Let N^t be the operator obtained from N by the Getzler rescaling indicated above. One verifies easily that as $t\to 0$,

$$tN^t\to\omega^{X_g}. \tag{7.226}$$

By proceeding as before, we get the second inequality in (7.92).

7.14. **A proof of the second half of Theorem 7.9.** Put

$$C''_{K,t}=\sqrt{t}\bar{\partial}^{X}+\frac{c(K^{X(1,0)})}{2\sqrt{2t}}, \\ C'_{K,t}=\sqrt{t}\bar{\partial}^{X*}+\frac{c(K^{X(0,1)})}{2\sqrt{2t}}. \tag{7.227}$$

Then

$$C^2_{''K,t}=0\,,\; C^{',2}_{K,t}=0, \\ C^2_{K,t}=[C''_{K,t},C'_{K,t}]. \tag{7.228}$$

Moreover, using (2.4), we get

$$\frac{\partial}{\partial t}C''_{K,t}=-\frac{1}{2t}[C''_{K,t},N-i\frac{\langle\mu,K\rangle}{t}], \\ \frac{\partial}{\partial t}C'_{K,t}=\frac{1}{2t}[C'_{K,t},N-i\frac{\langle\mu,K\rangle}{t}]. \tag{7.229}$$

Let $da, \overline{da}$ be odd Grassmann variables as in Section 4.5. The considerations we made after (4.39) still apply. Here $da, \overline{da}$ will anticommute with the odd elements of the above endomorphism algebras, and commute with the even elements of these algebras. If $\sigma \in \Lambda\big(R^{2,*}\big) \otimes_{\mathbf{R}} \mathbf{C}$, then

$$\sigma = s_0 + da\sigma^{da} + \overline{da}\sigma^{\overline{da}} + da\overline{da}\sigma^{da\overline{da}},\ s_0, \sigma^{da}, \sigma^{\overline{da}}, \sigma^{da\overline{da}} \in \mathbf{C}. \tag{7.230}$$

Now we establish an identity proved in a related form in [BGS1, II, Theorem 2.14].

Proposition 7.44. *The following identity holds,*

$$\begin{aligned}\frac{\partial}{\partial t}t\gamma_t(g,K) = \operatorname{Tr_s}\Bigg[g\exp\Bigg(-L_K - C^2_{K,t} - \\ da\sqrt{2}t\frac{\partial}{\partial t}(C^{''}_{K,t} - C'_{K,t}) - \overline{da}\sqrt{2}t\frac{\partial}{\partial t}(C^{''}_{K,t} + C'_{K,t}) + Nda\overline{da}\Bigg)\Bigg]^{da\overline{da}}.\end{aligned} \tag{7.231}$$

Proof. Using (7.229), we get

$$\begin{aligned}\frac{\partial}{\partial t}t\operatorname{Tr_s}\Big[\Big(N - i\frac{\langle\mu,K\rangle}{t}\Big)\,g\exp\big(-L_K - C^2_{K,t}\big)\Big] = \operatorname{Tr_s}\Big[Ng\exp\big(-L_K - C^2_{K,t}\big)\Big] \\ + \frac{\partial}{\partial b}\operatorname{Tr_s}\Big[(tN - i\langle\mu,K\rangle)g\exp\big((-L_K - C^2_{K,t} - b[C_{K,t}, \frac{\partial}{\partial t}C_{K,t}\big)\Big]|_{b=0}.\end{aligned} \tag{7.232}$$

Also

$$[L_K, C_{K,t}] = 0 \tag{7.233}$$

By (7.229), (7.232), (7.233), and using the fact that $\operatorname{Tr_s}$ vanishes on supercommutators [Q1], we obtain

$$\begin{aligned}\frac{\partial}{\partial t}t\operatorname{Tr_s}\Big[\Big(N - i\frac{\langle\mu,K\rangle}{t}\Big)\,g\exp\big(-L_K - C^2_{K,t}\big)\Big] = \operatorname{Tr_s}\Big[Ng\exp\big(-L_K - C^2_{K,t}\big)\Big] \\ + \frac{\partial}{\partial b}\operatorname{Tr_s}\Big[2t^2\frac{\partial}{\partial t}(C''_{K,t} - C'_{K,t})g\exp\big(-L_K - C^2_{K,t} - b\frac{\partial}{\partial t}C_{K,t}\big)\Big],\end{aligned} \tag{7.234}$$

which is equivalent to (7.231). The proof of our Proposition is completed. ☐

The following result was proved in [BGS1, II, Theorem 2.15].

Theorem 7.45. *The following identity holds*

$$\begin{aligned}L_K + C^2_{K,t} + da\sqrt{2}t\frac{\partial}{\partial t}(C^{''}_{K,t} - C'_{K,t}) + \overline{da}\sqrt{2}t\frac{\partial}{\partial t}(C^{''}_{K,t} + C'_{K,t}) - Nda\overline{da} = \\ -\frac{t}{2}\left(\nabla^{\Lambda(T^{*(0,1)}X)\otimes E} - \frac{\langle K^X, e_i\rangle}{2t} + \frac{da}{2\sqrt{t}}ic(J^{TX}e_i) - \frac{\overline{da}}{2\sqrt{t}}c(e_i)\right)^2 + t\frac{H}{8} - \frac{\dim X}{2}da\overline{da} \\ + \frac{t}{4}c(e_i)c(e_j)\big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TX}]\big)(e_i,e_j) - (m^E(K) + \frac{1}{2}\operatorname{Tr}[m^{TX}(K)].\end{aligned} \tag{7.235}$$

Proof. When making $da = 0, \overline{da} = 0$, (7.235) is just (7.97). Using (7.225), we then get (7.235) in full generality. ☐

By proceeding as in [BGS1, II, Theorem 2.10], [BeGeV, Chapter 10] and [B14, Section 11], i.e. by using the techniques of the local families index theorem instead of the above techniques for the usual local index theorem, we see that the above methods can be applied to the operator which appears in (7.228). In particular , we find that for $z \in \mathbf{R}$, with $|z|$ small enough, $t \in]0,1]$,

$$|\frac{\partial}{\partial t} t\gamma_t(g, zK) - C_0(g, zK)| \leq Ct^{\gamma}. \tag{7.236}$$

From (7.95), (7.236), we deduce that for $|z|$ small enough, as $t \to 0$,

$$t\gamma_t(g, zK) - C_{-1}(g, zK) = C_0(g, zK)t + \mathcal{O}(t^{\gamma+1}), \tag{7.237}$$

which is just the second inequality in (7.39). This completes the proof of Theorem 7.9.

8. A PROOF OF THEOREM 6.6

The purpose of this Section is to establish Theorem 6.6. We will use some of the techniques developed in Section 7, and also the techniques of [BL, Section 11], [B13, Section 11]. It should be observed that the geometric situation is much closer to [BL] and [B13] than in Section 7, with $X_{g,K}$ playing formally the role of Y or Y_g in [BL] and [B13].

For technical reasons it is here important to split $\langle \mu, K\rangle$ into two pieces μ_1 and μ_2, where μ_1 vanishes together with its first derivatives on X_K, and μ_2 is locally constant near X_K. In fact X_K is the critical submanifold for $\langle \mu, K\rangle$. If $\langle \mu, K\rangle$ vanished identically on X_K, the proof of Theorem 6.6 would be relatively easy, as the sequel will show. Also if $\langle \mu, K\rangle$ was just any constant (forgetting for the moment the relation between $\langle \mu, K\rangle$ and K^X), the proof of Theorem 6.6 would be even easier, since the supertrace appearing in (6.11) would itself be constant. Since the above ideal assumptions are absurd anyway, we try to capture in the proof the best features of these two cases.

This Section is organized as follows. In Section 8.1, we establish a Lichnerowicz formula. In Section 8.2, we establish equation (6.11).

Sections 8.3-8.12 are devoted to the proof of the uniform estimate (6.12) in Theorem 6.6. In Section 8.3, we construct a splitting of the function $\langle \mu, K\rangle$. In Section 8.4, we show that the problem is localizable near X_g. In Section 8.5, we introduce a rescaling of the normal coordinate to $X_{g,K}$ in X_g. In Section 8.17, we construct a coordinate system near $y_0 \in X_{g,K}$ and we use an associated Getzler rescaling technique on the Clifford variables. Note here that we concentrate on the analysis near $X_{g,K}$, because what happens away of $X_{g,K}$ is easier to control. In Section 8.7, we construct a family of norms, closely related to constructions in [BL]. In Section 8.8, we establish regularizing properties for the resolvents of the considered rescaled operators. In Section 8.9, we prove uniform estimates on the truncated kernels, which are indexed by $n \in \mathbf{N}$, as in Section 7. In Section 8.10, this leads us to estimates on the full kernel. In Section 8.11, we estimate the local supertrace containing μ_1 near $X_{g,K}$, and in Section 8.12 we control the full supertrace involving μ_1. In Section 8.13, using non trivial algebraic identities still involving a Grassmann variable, we prove corresponding estimates for the supertrace associated to μ_2, which leads us to the completion of the proof of the estimate (6.12) in Theorem 6.6.

Finally in Section 8.14, we establish the estimate in (6.13). This completes the proof of Theorem 6.6.

We use the same notation as in Sections 3, 6 and 7.

8.1. A Lichnerowicz formula.

We still use the notation in (7.96).

Theorem 8.1. *The following identity holds,*

$$(8.1)\quad L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}} = -\frac{t}{2}\left(\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{e_i} - \big(\frac{1}{t}+\frac{1}{v}\big)\frac{\langle K^X, e_i\rangle}{2}\right)^2 + \frac{1}{2v}|K^X|^2 + \frac{t}{8}H + \frac{t}{4}c(e_i)c(e_j)\Big(\big(R^E + \frac{1}{2}\mathrm{Tr}[R^{TX}]\big)(e_i,e_j) - \frac{1}{v}\langle \nabla^{TX}_{e_i}K^X, e_j\rangle\Big) - \big(m^E(K) + \frac{1}{2}\mathrm{Tr}[\nabla^{TX}_{\cdot}K^X]\big).$$

Proof. By (3.1), if $U,V\in T_{\mathbf{R}}X$,

$$dK'^X(U,V)=2\langle\nabla_U^{TX}K^X,V\rangle. \tag{8.2}$$

Also observe that $\sqrt{2}D^X_{-1/v}$ is a standard Dirac operator associated to the connection $d-\frac{K'^X}{2v}$ on the trivial line bundle L. Moreover

$$\left(D^X_{-1/v}\right)_{1/t}=D^X_{\frac{1}{t}-\frac{1}{v}}. \tag{8.3}$$

Note that since G acts trivially on L, the corresponding $m^L(K)$ in the sense of Section 2.1 is obviously given by

$$m^L(K)=-\frac{|K^X|^2}{2v}. \tag{8.4}$$

Then equation (8.1) is just a special case of Proposition 7.18. □

8.2. **A proof of the convergence of the supertrace as $t\to 0$.** Recall that $\widetilde{e}_v$ was defined in (6.8).

Theorem 8.2. *For $|K|$ small enough, given $v>0$, as $t\to 0$,*

$$\operatorname{Tr_s}\left[\frac{i\langle\mu,K\rangle}{v}g\exp\left(-L_K-tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\right)\right]\to -\widetilde{e}_v. \tag{8.5}$$

Proof. As we observed in (8.3),

$$L_K+tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}=L_K+t\left(D^X_{-1/v}\right)^2_{1/t}. \tag{8.6}$$

Using the methods in the proof of Theorem 7.9, (8.4) and (8.6), we get (8.5). Details are left to the reader. □

Remark 8.3. The main difficulty in the proof of Theorem 6.6 is to establish an estimate of the rate of convergence in (8.5), which is uniform $t\in]0,1],v\in[t,1]$.

8.3. **A splitting of $\langle\mu,K\rangle$.** By (2.4),

$$d\langle\mu,K\rangle|_{X_K}=0. \tag{8.7}$$

Therefore $\langle\mu,K\rangle$ is locally constant on X_K. For $\epsilon\in]0,a^X/8]$, let $\mathcal{U}'_\epsilon$ be the ϵ-neighborhood of X_K in $N_{X_K/X}$. We choose ϵ'_0 small enough so that if $\epsilon\in]0,8\epsilon'_0]$, the map $(x,Z)\in\mathcal{U}'_\epsilon\mapsto\exp_x(Z)\in X$ is a diffeomorphism from $\mathcal{U}'_\epsilon$ into the tubular neighborhood $\mathcal{V}'_\epsilon$ of X_K in X. Recall that ϵ_0 was introduced in Section 7.3. By replacing ϵ_0 and ϵ'_0 by $\inf(\epsilon_0,\epsilon'_0)$, we may and we will assume that $\epsilon_0=\epsilon'_0$.

Since X_g is totally geodesic in X, the above identification also identifies a neighborhood of $X_{g,K}$ in $N_{X_{g,K}/X_g}$ to a tubular neighborhood of $X_{g,K}$ in X_g.

Let $P:\mathcal{V}'_{\epsilon_0}\mapsto X_K$ be the obvious projection. Also, recall that the function $\gamma(s)$ was defined in (7.110).

Definition 8.4. Put

$$\begin{aligned}\mu_1&=\langle\mu,K\rangle-\gamma\left(2|Z|/\epsilon_0\right)\langle\mu,K\rangle(P.),\\ \mu_2&=\gamma\left(2|Z|/\epsilon_0\right)\langle\mu,K\rangle(P.).\end{aligned} \tag{8.8}$$

Then the function μ_1 vanishes on X_K. Moreover μ_2 is locally constant on $\mathcal{V}'_{\epsilon_0/4}$. Also

$$\langle \mu, K\rangle = \mu_1 + \mu_2. \tag{8.9}$$

By (8.7), we deduce that near X_K,

$$\mu_1(x,Z) = \mathcal{O}(|Z|^2). \tag{8.10}$$

Clearly

$$\mathrm{Tr_s}\left[\frac{i\langle \mu, K\rangle}{v} g \exp\bigl(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr)\right] = \mathrm{Tr_s}\left[\frac{i\mu_1}{v} g \exp\bigl(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr)\right] + \mathrm{Tr_s}\left[\frac{i\mu_2}{v} g \exp\bigl(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr)\right]. \tag{8.11}$$

Set

$$\widetilde{d}_{v,j} = \frac{-i\mu_j}{v} \exp\left(\frac{\overline{\partial}_K \partial_K}{2\pi v}\langle \mu, K\rangle\right), \; j = 1,2. \tag{8.12}$$

Then

$$\widetilde{d}_v = \widetilde{d}_{v,1} + \widetilde{d}_{v,2}. \tag{8.13}$$

Similarly, we define $\widetilde{e}_{v,1}, \widetilde{e}_{v,2}$ by replacing $\widetilde{d}_v$ by $\widetilde{d}_{v,1}, \widetilde{d}_{v,2}$ in the right-hand side of the first equation in (6.8). Then

$$\widetilde{e}_v = \widetilde{e}_{v,1} + \widetilde{e}_{v,2}. \tag{8.14}$$

By the same arguments as in the proof of Theorem 8.2, one finds easily that given $v \in \mathbf{R}^*_+$, as $t \to 0$,

$$\mathrm{Tr_s}\left[\frac{i\mu_j}{v} g \exp\bigl(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr)\right] \to -\widetilde{e}_{v,j}, \; j = 1,2. \tag{8.15}$$

To establish inequality (6.12) in Theorem 6.6, we will instead establish corresponding statements for the two supertraces appearing in the right-hand side of (8.11).

8.4. **Localization of the problem.**

Definition 8.5. For $t > 0, v \in \mathbf{R}^*_+$, set

$$\mathcal{A}^v_t = tL_K + t^2 D^{X,2}_{\frac{1}{t}-\frac{1}{v}}. \tag{8.16}$$

Comparing with (7.41), we get

$$\mathcal{A}_t = \mathcal{A}^{+\infty}_t. \tag{8.17}$$

Clearly

$$\mathcal{A}^v_t = t^2 D^{X,2} + t\left(L_K + (1-\frac{t}{v})[D^X, \frac{c(K^X)}{2\sqrt{2}}]\right) - \left(1-\frac{t}{v}\right)^2 \frac{|K^X|^2}{8}. \tag{8.18}$$

From (8.18) and comparing with (7.43), it is clear that for $t \in]0,1], v \in [t,1]$, $\mathcal{A}^v_t$ verifies the same estimates as $\mathcal{A}_t$ in Theorem 7.11, with constants which are uniform in $t \in]0,1], v \in [t,1]$.

Now we take $\alpha \in]0, a_X/8]$ as in Section 7.3. Also we use the notation in Section 7. By (7.80),

$$\exp\bigl(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr) = \widetilde{F}_t(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}) + \widetilde{I}_t(\mathcal{A}^v_t). \tag{8.19}$$

Theorem 8.6. *There exist* $\beta > 0, C > 0, C' > 0$ *such that if* $K \in \mathfrak{g}, |K| \leq \beta, t \in]0,1], v \in [t,1]$,

$$\|\widetilde{I}_t(\mathcal{A}_t^v)\|_1 \leq C \exp(-C'/t). \tag{8.20}$$

Proof. The proof of our Theorem is the same as the proof of Theorem 7.15. ☐

Remark 8.7. From (8.19), it follows that to establish inequality (6.12) in Theorem 6.6 for any of the supertraces appearing in the right-hand side of (8.11), we may as well replace $\exp\bigl(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr)$ by $\widetilde{F}_t(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}})$.

As in Section 7.3, $\widetilde{F}_t\bigl(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr)(x,x')$ vanishes if $d^X(x,x') \geq \alpha$, and, as a function of x', depends only on the restriction of the operator $L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}$ to the ball $B^X(x,\alpha)$. By our choice of α, $\widetilde{F}_t\bigl(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\bigr)(g^{-1}x,x)$ is nonzero only if $x \in \mathcal{V}_{\epsilon_0}$.

It follows from the above that our proof of inequality (6.12) in Theorem 6.6 can be localized near X_g.

8.5. A rescaling of the normal coordinate to $X_{g,K}$ in X_g.

In the sequel, we fix $g \in G, K_0 \in \mathfrak{z}(g)$, and we assume that $K = zK_0$, with $z \in \mathbf{R}^*$.

Recall that X_g and $X_{g,K}$ are totally geodesic in X. Given $\epsilon > 0$, let $\mathcal{U}''_\epsilon$ be the ϵ-neighborhood of $X_{g,K}$ in $N_{X_{g,K}/X_g}$. There exists $\epsilon''_0 \in]0, a^X/32]$ such that for $0 < \epsilon \leq 16\epsilon''_0$, the map $(y_0, Z_0) \in N_{X_{g,K}/X_g,\mathbf{R}} \to \exp^{X_g}_{y_0}(Z) \in X_g$ is a diffeomorphism from $\mathcal{U}''_\epsilon$ into the tubular neighborhood $\mathcal{V}''_\epsilon$ of $X_{g,K}$ in X_g. By replacing ϵ_0, ϵ''_0 by $\inf(\epsilon_0, \epsilon''_0)$, we may and we will assume that $\epsilon''_0 = \epsilon_0$.

Since X_g is totally geodesic in X, the connection ∇^{TX} induces the holomorphic Hermitian connection $\nabla^{N_{X_g/X}}$ on $N_{X_g/X}$.

If (y_0, Z_0) is taken as before, we identify $N_{X_g/X,(y_0,Z_0)}$ to $N_{X_g/X,y_0}$ by parallel transport along the geodesic $s \in [0,1] \to Z_0$. If $y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, Z \in N_{X_g/X,\mathbf{R},y_0}, |Z_0|, |Z| \leq 4\epsilon_0$, we identify (y_0, Z_0, Z) to $\exp^X_{\exp^{X_g}_{y_0}(Z_0)}(Z) \in X$. Therefore (y_0, Z_0, Z) defines a coordinate system on X near $X_{g,K}$.

Recall that the function k was defined in (7.89). Also $\widetilde{e}_{v,1}$ was defined as in (6.8), with $\widetilde{d}_v$ replaced by $\widetilde{d}_{v,1}$. Put

$$\beta_{v,1} = \bigl(2\pi\bigr)^{\dim X_g} \Bigl[\mathrm{Td}_{g,K}(TX, h^{TX}) \mathrm{ch}_{g,K}(E, h^E) \widetilde{d}_{v,1}\Bigr]^{\max}. \tag{8.21}$$

For $|z|$ small enough, $\beta_{v,1}$ is a smooth function on X_g. Set

$$\ell'' = \dim(X_{g,K}). \tag{8.22}$$

A first important result in our proof of the estimate (6.12) in Theorem 6.6 is as follows.

Theorem 8.8. *There exist* $c \in]0,1], \gamma \in]0,1]$ *such that for* $p \in \mathbf{N}$, *there is* $C > 0$ *such that if* $z \in \mathbf{R}, |z| \leq c, t \in]0,1], v \in [t,1], y_0 \in X_{g,K}, Z_0 \in \mathbf{N}_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq$

$\frac{\epsilon_0}{\sqrt{v}}$, *then*

(8.23)
$$\left| v^{\dim N_{X_{g,K}/X_g}} \left(\int_{\substack{Z \in N_{X_g/X,\mathbf{R},y_0} \\ |Z| \leq \epsilon_0}} \operatorname{Tr_s} \left[\frac{i\mu_1}{v} g\widetilde{F}_t(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}) \big(g^{-1}(y_0, \sqrt{v}Z_0, Z), \right.\right.\right.$$
$$\left.\left.\left. (y_0, \sqrt{v}Z_0, Z)\big) \right] k(y_0, \sqrt{v}Z_0, Z) \frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} + \beta_{v,1}(y_0, \sqrt{v}Z_0) \right) \right|$$
$$\leq C \frac{(1+|Z_0|)^{2\ell''+1}}{(1+|zZ_0|)^p} \left(\frac{t}{v} \right)^{\gamma}.$$

Proof. Sections 8.6- 8.11 will be devoted to the proof of Theorem 8.8. ☐

8.6. **A local coordinate system near** $X_{g,K}$ **and a Getzler rescaling.** Take $y_0 \in X_{g,K}$. If $Z \in (T_{\mathbf{R}}X_g)_{y_0}, |Z| \leq 4\epsilon_0$, we identify Z to $\exp^{X_g}_{y_0}(Z) \in X_g$. We trivialize $N_{X_g/X}$ along the geodesic $s \in [0,1] \to sZ \in X_g$ by parallel transport with respect to to the connection $\nabla^{N_{X_g/X}}$. Then $(Z, Z') \in (T_{\mathbf{R}}X_g)_{y_0} \times N_{X_g/X,\mathbf{R},y_0} \to \exp^{TX}_{\exp^{X_g}_{y_0}(Z)}(Z') \in X, |Z|, |Z'| \leq 4\epsilon_0$ defines a coordinate system on X near y_0.

Since g preserves geodesics and parallel transport, in the above coordinate system

(8.24)
$$g(Z, Z') = (Z, gZ').$$

If $Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq 4\epsilon_0$, we trivialize TX along the geodesic $s \in [0,1] \to sZ_0 \in X_g$ by parallel transport with respect to ∇^{TX}. Then $Z \in (T_{\mathbf{R}}X)_{y_0}, |Z| \leq 4\epsilon_0 \to \exp_{Z_0}(Z) \in X$ is a coordinate system near $Z_0 \in X_g$. By an abuse of notation, we will often write $Z_0 + Z$ instead of $\exp_{Z_0}(Z)$.

Now we fix $Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq 4\epsilon_0$, and we take $Z \in (T_{\mathbf{R}}X)_{y_0}, |Z| \leq 4\epsilon_0$. The curve $s \in]0,1] \to \exp_{Z_0}(sZ)$ lies in $B^{TX}_{y_0}(0, 5\epsilon_0)$. Moreover we identify $TX_{Z_0+Z}, \Lambda(T^{*(0,1)}X)_{Z_0+Z}, E_{Z_0+Z}$ with $TX_{Z_0}, \Lambda(T^{*(0,1)}X)_{Z_0}, E_{Z_0}$ by parallel transport with respect to the obvious connections $\nabla^{TX}, \nabla^{\Lambda(T^{*(0,1)}X)}, \nabla^E$ along this curve.

When $Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}$ is allowed to vary, we identify $TX_{Z_0}, \Lambda(T^{*(0,1)}X)_{Z_0}, E_{Z_0}$ to $TX_{y_0}, \Lambda(T^{*(0,1)}X)_{y_0}, E_{y_0}$ by parallel transport along $s \in [0,1] \to sZ_0$ with respect to the given connections.

We may and we will assume that ϵ_0 is small enough so that if $|Z_0|, |Z| \leq 4\epsilon_0$, then

(8.25)
$$\frac{1}{2} h^{TX}_{y_0} \leq h^{TX}_{Z_0+Z} \leq \frac{3}{2} h^{TX}_{y_0}.$$

Recall that for $x \in X_g$, the vector space $\mathbf{H}_x$ was defined in Section 7.6. We still define $\rho(Z)$ as in (7.111), (7.112).

We fix $Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}$. The considered trivializations depend explicitly on Z_0. Therefore the action of the operator $L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}$ depends explicitly on Z_0. We denote by $(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}})_{Z_0}$ the action of this operator centered at Z_0, i.e.

(8.26)
$$(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}})_{Z_0} f = (L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}) f(Z_0 + Z).$$

In (8.26), the operator $(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}})_{Z_0}$ acts on $\mathbf{H}_{Z_0}$. Also $\mathbf{H}_{Z_0}$ is identified with $\mathbf{H}_{y_0}$, so that ultimately, $(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}})_{Z_0}$ acts on $\mathbf{H}_{y_0}$.

We define $k'_{(y_0,Z_0)}(Z)$ as in (7.98).

Definition 8.9. Put

$$L^{1,(t,v)}_{Z_0,K} = (1-\rho^2(Z))\big(-\frac{t}{2}\Delta^{TX}\big) + \rho^2(Z)\big(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)_{Z_0}. \tag{8.27}$$

By proceeding as in (7.114),(7.115), and using (8.24), we find that if $Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, Z \in N_{X_g/X,\mathbf{R},y_0}, |Z_0|, |Z| \leq \epsilon_0$,

$$\widetilde{F}_t\left(L_K + C^2_{K,t}\right)\left(g^{-1}(Z_0,Z),(Z_0,Z)\right) k'_{(y_0,Z_0)}(Z) = \widetilde{F}_t\left(L^{1,(t,v)}_{Z_0,K}\right)\left(g^{-1}Z,Z\right). \tag{8.28}$$

We still define H_t as in (7.117). Let $L^{2,(t,v)}_{Z_0,K}$ be the operator obtained from $L^{1,(t,v)}_{Z_0,K}$ as in (7.118).

Let $(e_1,\dots,e_{2\ell''})$, $(e_{2\ell''+1},\dots,e_{2\ell'})$, and $(e_{2\ell'+1},\dots,e_{2\ell})$ be orthonormal oriented bases of $T_{\mathbf{R}}X_{g,K,y_0}$ and $N_{X_{g,K}/X_g,\mathbf{R},y_0}, N_{X_g/X,\mathbf{R},y_0}$. Recall that for $1 \leq j \leq 2\ell'$, $c_t(e_j)$ was defined in (7.120).

Definition 8.10. Let $L^{3,(t,v)}_{Z_0,K}$ be the operator obtained from $L^{2,(t,v)}_{Z_0,K}$ by replacing $c(e_i)$ by $c_t(e_i)$ for $1 \leq i \leq 2\ell''$, by $c_{t/v}(e_i)$ for $2\ell'+1 \leq i \leq 2\ell''$, while leaving unchanged the $c(e_i)$'s for $2\ell'+1 \leq i \leq 2\ell$.

We denote by $\widetilde{F}_t\big(L^{3,(t,v)}_{Z_0,K}\big)(Z,Z')$ the smooth kernel associated to the operator $\widetilde{F}_t\big(L^{3,(t,v)}_{Z_0,K}\big)$ with respect to $dv_{TX}(Z')/(2\pi)^{\dim X}$.

Proposition 8.11. *For* $y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0, Z \in N_{X_g/X,\mathbf{R},y_0}$, $|Z| \leq \epsilon_0/\sqrt{t}$, *the following identity holds,*

$$\begin{aligned} t^{\dim N_{X_g/X}} \operatorname{Tr_s}\Big[\frac{i\mu_1}{v} g\widetilde{F}_t\big(L_K+tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\big(g^{-1}(Z_0,\sqrt{t}Z),(Z_0,\sqrt{t}Z)\big)\Big] k'_{(y_0,Z_0)}(\sqrt{t}Z) = \\ \frac{(-i)^{\ell'}}{v^{\ell'-\ell''}} \operatorname{Tr_s}^{\Lambda(N^{*(0,1)}_{X_g/X})\otimes E}\Big[\frac{i\mu_1(Z_0,\sqrt{t}Z)}{v} g\widetilde{F}_t(L^{3,(t,v)}_{Z_0,K})(g^{-1}Z,Z)\Big]^{\max}. \end{aligned} \tag{8.29}$$

Proof. Using (8.28), the proof is a trivial modification of the proof of Proposition 7.25. ☐

Recall that j denotes the embedding $X_g \to X$.

Definition 8.12. If $x \in X_g$, let $\overline{L}^{3,(0,v)}_{x,K}$ be the operator in $\big(\Lambda(T_{\mathbf{R}}X^*_g) \otimes c(N_{X_g/X}) \otimes \operatorname{End}(E)\big) \otimes \operatorname{Op}_x$,

$$\begin{aligned} \overline{L}^{3,(0,v)}_{x,K} = -\frac{1}{2}\left(\nabla_{e_i} + \frac{1}{2}\langle\big(j^*R^{TX} - \nabla^{TX}_{\cdot}K^X\big)Z,e_i\rangle\right)^2 - \frac{1}{2v}\big(d - i_{K^X}\big)j^*K'^X \\ + j^*R^E - m^E(K) + \frac{1}{2}\operatorname{Tr}[j^*R^{TX} - m^{TX}(K)]. \end{aligned} \tag{8.30}$$

Let $\psi_v \in \operatorname{End}\big(\Lambda(T^*_{\mathbf{R}}X_g)\big)$ be the morphism of exterior algebras such that $\psi_v(e^i) = \sqrt{v}e^i$ for $1 \leq i \leq 2\ell''$, $\psi_v(e^i) = e^i$ for $2\ell''+1 \leq i \leq 2\ell'$. Recall that $(y_0,Z_0) \in X_g$, and that $\Lambda\big(T^*_{\mathbf{R}}X_g\big)_{(y_0,Z_0}$ has been identified to $\Lambda\big(T^*_{\mathbf{R}}(X_g\big)_{y_0}$.

Definition 8.13. Let $L^{3,(0,v)}_{Z_0,K}$ be the operator

$$L^{3,(0,v)}_{Z_0,K} = \psi_v \overline{L}^{3,(0,v)}_{Z_0,K} \psi_v^{-1}. \tag{8.31}$$

Theorem 8.14. *Given* $Z_0 \in \mathbf{N}_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0$, *as* $t \to 0$,

$$L^{3,(t,v)}_{Z_0,K} \to L^{3,(0,v)}_{Z_0,K}. \tag{8.32}$$

Proof. First we assume that $v = 1$, so that our proof will work for any $x \in X_g$. Using (3.1), we find that as $t \to 0$,

$$\begin{aligned} \frac{1 \leq i,j \leq 2\ell \; t}{4} c_t(e_i) c_t(e_j) \langle \nabla^{TX}_{e_i} K^X(\sqrt{t}Z), e_j\rangle \to \\ \sum_{1 \leq i,j \leq 2\ell'} \frac{1}{2} e^i \wedge e^j \langle \nabla^{TX}_{e_i} K^X(x), e_j\rangle = \frac{1}{2} dj^* K'^X. \end{aligned} \tag{8.33}$$

Our Theorem is now a trivial consequence of (8.1), (8.33) and of arguments which were already used in the proof of Proposition 7.27. □

8.7. **A family of norms.** Here we imitate [BL, Section 11 k)] and Section 7.8. For $0 \leq p \leq 2\ell'', 0 \leq q \leq 2(\ell' - \ell'')$, put

$$\Lambda^{(p,q)}(T^*_{\mathbf{R}} X_g)_{y_0} = \Lambda^p(T^*_{\mathbf{R}} X_{g,K})_{y_0} \otimes \Lambda^q(N^*_{X_{g,K}/X_g,\mathbf{R}})_{y_0}. \tag{8.34}$$

The various $\Lambda^{(p,q)}(T^*_{\mathbf{R}}(X_g)$ are mutually orthogonal in $\Lambda(T^*_{\mathbf{R}} X_g)_{y_0}$. Let $\mathbf{I}_{y_0}$ be the vector space of smooth sections of $(\Lambda(T^*_{\mathbf{R}} X_g) \widehat{\otimes} \Lambda(N^{*(0,1)}_{X_g/X}) \otimes E)_{y_0}$ over $(T_{\mathbf{R}} X)_{y_0}$, let $\mathbf{I}_{(p,q),y_0}$ be the vector space of smooth sections of $\big(\Lambda^{(p,q)}(T^*_{\mathbf{R}}(X_g) \otimes \Lambda(N^{*(0,1)}_{X_g/X}) \otimes E\big)_{y_0}$ over $(T_{\mathbf{R}} X)_{y_0}$. Let $\mathbf{I}^0_{y_0}, \mathbf{I}^0_{(p,q),y_0}$ be the corresponding vector spaces of square-integrable sections.

Definition 8.15. For $t \in [0,1], v \in \mathbf{R}^*_+, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0/\sqrt{v}, s \in \mathbf{I}^0_{(p,q),y_0}$, set

$$\begin{aligned} |s|^2_{t,v,Z_0,0} = \int_{(T_{\mathbf{R}} X)_{y_0}} |s|^2 \Big(1 + \big(|Z| + |Z_0|\big)\rho(\sqrt{t}Z/2)\Big)^{2(2\ell''-p)} \\ \Big(1 + \sqrt{v}|Z|\rho(\sqrt{t}Z/2)\Big)^{2(2(\ell'-\ell'')-q)}. \end{aligned} \tag{8.35}$$

Then (8.35) induces a Hermitian product $\langle\ \rangle_{t,v,Z_0,0}$ on $I^0_{(p,q),y_0}$. We equip $\mathbf{I}^0_{y_0} = \bigoplus \mathbf{I}^0_{(p,q),y_0}$ with the direct sum of these Hermitian products.

Now we have a result already proved in [BL, Proposition 11.24].

Proposition 8.16. *For* $t \in]0,1], v \in [t,1], y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0/\sqrt{v}$, *the following families of operators acting on* $\mathbf{I}^0_{y_0}$ *are uniformly bounded,*

$$\begin{aligned} & 1_{|\sqrt{t}Z| \leq 4\epsilon_0} \sqrt{t} c_t(e_j)\,;\ 1_{|\sqrt{t}Z| \leq 4\epsilon_0} |Z| \sqrt{t} c_t(e_j), 1 \leq j \leq 2\ell''; \\ & 1_{|\sqrt{t}Z| \leq 4\epsilon_0} |Z_0| \sqrt{t} c_t(e_j), 1 \leq j \leq 2\ell''; \\ & 1_{|\sqrt{t}Z| \leq 4\epsilon_0} \sqrt{t/v}\, c_{t/v}(e_j),\ 1_{|\sqrt{t}Z| \leq 4\epsilon_0} |Z| \sqrt{t} c_{t/v}(e_j),\ 2\ell'' + 1 \leq j \leq 2\ell'. \end{aligned} \tag{8.36}$$

Proof. We refer to [BL, Proposition 11.24] for a detailed proof, when making $t = u^2, v = \frac{1}{T^2}$. If $\sqrt{t}|Z| \leq 4\epsilon_0$, by (7.112), $\rho\big(\frac{\sqrt{t}Z}{2}\big) = 1$. Then we just observe the key bounds

$$\begin{aligned} & t\big(1+|Z|+|Z_0|\big) \leq C\sqrt{t};\ t|Z|\big(1+|Z|+|Z_0|\big) \leq C; \\ & t|Z_0|\big(1+|Z|+|Z_0|\big) \leq C;\ \frac{t}{v}\big(1+\sqrt{v}|Z|\big) \leq C; \\ & \frac{t}{\sqrt{v}}|Z|\big(1+\sqrt{v}|Z|\big) \leq C,\ \frac{\sqrt{v}|Z|}{1+\sqrt{v}|Z|} \leq 1, \end{aligned} \tag{8.37}$$

from which our Proposition follows easily. □

Definition 8.17. For $t \in [0,1], v \in \mathbf{R}_+^*, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0/\sqrt{v}, s \in \mathbf{I}^0_{(p,q),y_0}$, set

$$\begin{aligned} |s|^2_{t,v,Z_0,1} = |s|^2_{t,v,Z_0,0} + \frac{1}{v}\Big|\rho\big(\sqrt{t}Z\big)\,\big|K^X\big|\,\big(\sqrt{v}Z_0+\sqrt{t}Z\big)s\Big|^2_{t,v,Z_0,0} + \\ + \sum_1^{2\ell} |\nabla_{e_i} s|^2_{t,v,Z_0,0}. \end{aligned} \tag{8.38}$$

Observe that Definition 8.17 is the strict analogue of [BL, Definition 11.25], with $u = \sqrt{t}, T = \frac{1}{\sqrt{v}}$ and V^- replaced by K^X. Also note that $|s|_{t,v,Z_0,1}$ depends explicitly on K. In the sequel, it will be understood that if $z \in \mathbf{R}$, when replacing K by zK_0, in (8.38), K^X is replaced by zK_0^X, so that $|s|_{t,v,Z_0,1}$ will in fact also depend on $z \in \mathbf{R}^*$.

If $Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0, Z \in T_\mathbf{R}X_{y_0}$, if $U \in (T_\mathbf{R}X)_{y_0}$, let $\tau^{Z_0}U(Z) \in T_\mathbf{R}X_{Z_0+Z}$ be the parallel transport of U along the curve $t \to 2tZ_0, 0 \leq t \leq 1/2$, $t \to \exp_{Z_0}\left((2t-1)Z\right)$, $1/2 \leq t \leq 1$ with respect to ∇^{TX}.

Theorem 8.18. *There exist constants* $C_1 > 0, \dots, C_4 > 0$ *such that if* $t \in]0,1], v \in [t,1], n \in \mathbf{N}, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0/\sqrt{v}$, *if* $z \in \mathbf{R}, |z| \leq 1$, *if* $s, s' \in \mathbf{I}_{y_0}$ *have compact support in* $\big\{Z \in (T_\mathbf{R}X)_{y_0}, |Z| \leq n\big\}$, *then*

$$\begin{aligned} & \operatorname{Re}\langle L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0} s, s\rangle_{t,v,Z_0,0} \geq C_1|s|^2_{t,v,Z_0,1} - C_2(1+|nz|^2)|s|^2_{t,v,Z_0,0}, \\ & |\operatorname{Im}\langle L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0} s, s\rangle_{t,v,Z_0,0}| \leq C_3(1+|nz|)|s|_{t,v,Z_0,1}|s|_{t,v,Z_0,0}, \\ & |\langle L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0} s, s'\rangle_{t,v,Z_0,0}| \leq C_4(1+|nz|^2)|s|_{t,v,Z_0,1}|s'|_{t,v,Z_0,1}. \end{aligned} \tag{8.39}$$

Proof. By proceeding as in the proofs of [BL, Theorem 11.26] and of Theorem 7.31, we can handle all the terms in the right-hand side of (8.1) except the "annoying" term

$$-\rho^2(\sqrt{t}Z)\frac{t}{4v}\langle \nabla^{TX}_{\tau^{\sqrt{v}Z_0} e_j(\sqrt{t}Z)} zK^X(\sqrt{v}Z_0+\sqrt{t}Z), \tau^{\sqrt{v}Z_0} e_{j'}(\sqrt{t}Z)\rangle c(e_j)c(e_{j'}). \tag{8.40}$$

Recall that by (2.6), (2.8),

$$\nabla^{TX}\nabla^{TX}K^X + i_{K^X}R^{TX} = 0. \tag{8.41}$$

Take $i, 1 \leq i \leq 2\ell''$. We claim that at $(Z_0, Z) = (0,0)$, $\nabla^{TX}_{\tau^{Z_0} e_i(Z)} K^X$ and its first derivatives in Z_0, Z vanish. The vanishing of this section follows from the fact that $e_i \in (T_\mathbf{R}X_{g,K})_{y_0}$ and that K^X vanishes on $X_{g,K}$. Also $\tau^0 e_i(Z)$ is parallel along

$s \to sZ$. Using (8.41), it is now clear that the first derivative in Z of the above section also vanishes. Finally $\tau^{Z_0} e_i(0)$ is parallel along $s \in [0,1] \to sZ_0$. Using (8.41) again, the derivative of the above section in the variable Z_0 also vanishes. It then follows that for $1 \leq i \leq 2\ell''$,

$$\nabla^{TX}_{\tau^{\sqrt{v}Z_0} e_i(\sqrt{t}Z)} K^X(\sqrt{v}Z_0 + \sqrt{t}Z) = \mathcal{O}\big(\sqrt{v}|Z_0| + \sqrt{t}|Z|\big)^2. \tag{8.42}$$

From (8.42), we deduce that for $1 \leq j \leq 2\ell'', 1 \leq j' \leq 2\ell'', |Z_0| \leq \epsilon_0/\sqrt{v}$,

$$\begin{aligned} \rho^2(\sqrt{t}Z)\frac{t}{4v}\langle \nabla^{TX}_{\tau^{\sqrt{v}Z_0} e_j(\sqrt{t}Z)} zK^X(\sqrt{v}Z_0 + \sqrt{t}Z), \tau^{\sqrt{v}Z_0} e_{j'}(\sqrt{t}Z)\rangle c_t(e_j)c_t(e_{j'}) = \\ \rho^2(\sqrt{t}Z)|z|\mathcal{O}\big(|Z_0| + \sqrt{\frac{t}{v}}|Z|\big)^2 \sqrt{t}c_t(e_j)\sqrt{t}c_t(e_{j'}). \end{aligned} \tag{8.43}$$

Using the fact that $t/v \leq 1$ and also Proposition 8.16, we find that the operator in (8.43) remains uniformly bounded with respect to $|\ |_{t,v,Z_0,0}$. The same arguments show that if $1 \leq j \leq 2\ell'', 2\ell''+1 \leq j' \leq 2\ell'$, when replacing in (8.43) $c_t(e_{j'})$ by $c_{t/v}(e_{j'})$, the corresponding boundedness result still holds. Similarly, if for $2\ell'+1 \leq j' \leq 2\ell$, in (8.43), we replace $c_t(e_{j'})$ by $c(e_{j'})$, we still obtain a uniform boundedness result.

Using again Proposition 8.16, if $2\ell''+1 \leq j, j' \leq 2\ell'$, the operators

$$\frac{t}{v} c_{t/v}(e_j) c_{t/v}(e_{j'}) \tag{8.44}$$

are uniformly bounded. Similarly for $2\ell''+1 \leq j \leq 2\ell', 2\ell'+1 \leq j' \leq 2\ell$, since $t/v \leq 1$, using Proposition 8.16, the operators $\frac{t}{v}c_t(e_j)c(e_{j'})$ are uniformly bounded. Finally for $2\ell'+1 \leq j, j' \leq 2\ell$, the operators $\frac{t}{v}c(e_j)c(e_{j'})$ are also uniformly bounded.

The proof of our Theorem is completed. □

Remark 8.19. It should be pointed out that the term (8.40) could cause trouble. In fact its "size" (after the standard Getzler rescaling, where for $1 \leq j \leq 2\ell'$, $c(e_j)$ is changed into $c_t(e_j)$), is of the order z/v, while the size of the nonnegative term $\frac{1}{v}|zK^X|^2$ is $\frac{|z|^2}{v}$, which is in fact smaller for small $|z|$.

Definition 8.20. Put

$$L^{3,(t,v)}_{Z_0,K,n} = -\Big(1 - \gamma\big(\frac{|Z|}{2(n+2)}\big)\Big)\frac{\Delta^{TX}}{2} + \gamma\big(\frac{|Z|}{2(n+2)}\big)L^{3,(t,v)}_{Z_0,K}. \tag{8.45}$$

Using (8.25) and proceeding as in (7.146), i.e. using finite propagation speed, we get for $Z \in T_{\mathbf{R}}X_{y_0}, |Z| \leq p$,

$$\widetilde{F}_{t,n}(L^{3,(t,v)}_{Z_0,K})(Z,Z') = \widetilde{F}_{t,n}(L^{3,(t,v)}_{Z_0,K,n+p})(Z,Z'). \tag{8.46}$$

Clearly, when replacing in (8.39) $L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}$ by $L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n}$, the estimates (8.39) still hold when assuming only that s, s' have compact support.

Now we use the notation of Section 7.10, while replacing the norms $|\ |_{t,x,0}, |\ |_{t,x,1}$, by the norms $|\ |_{t,v,Z_0,0}, |\ |_{t,v,Z_0,1}$. Then it follows from the above that the obvious analogue of Theorem 7.34 holds for $L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n}$.

8.8. **Regularizing properties of the resolvent of $L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n}$.** Now we proceed as in [BL, Section 11m)] and in Section 7.11. Since X_K is a compact manifold, there exists a finite family of smooth functions $f_1,\dots,f_r : X \to [0,1]$ which have the following properties:

- $$X_K = \bigcap_{j=1}^{r} \{x \in X, f_j(x) = 0.\}$$
- On X_K, $df_1,\dots,df_r$ span $N_{X_{g,K}/X,\mathbf{R}}$.

Definition 8.21. Let $\mathcal{Q}_{t,v,Z_0}$ be the family of operators

$$\mathcal{Q}_{t,v,Z_0} = \Big\{\nabla_{e_i}, 1 \le i \le 2\ell;\ \frac{z}{\sqrt{v}}\rho\big(\sqrt{t}Z\big)f_j\big(\sqrt{v}Z_0 + \sqrt{t}Z\big), 1 \le j \le r\Big\}. \tag{8.47}$$

Let$\mathcal{Q}^j_{t,v,Z_0}$ be the set of operators $Q_1 \dots Q_j$, with $Q_i \in \mathcal{Q}_{t,v,Z_0}$. Now we prove the obvious analogue of [BL, Proposition 11.29] and of Proposition 7.36.

Proposition 8.22. *Take $k \in \mathbf{N}$. There exists $C_k > 0$ such that if $t \in]0,1], v \in [t,1], n \in \mathbf{N}, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \le \epsilon_0/\sqrt{v}, z \in \mathbf{R}, |z| \le 1, Q_1,\dots,Q_k \in \mathcal{Q}_{t,v,Z_0}$, if $s,s' \in C^\infty\Big((T_{\mathbf{R}}X)_{y_0}, (\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X}) \otimes E)_{y_0}\Big)$ have compact support, then*

$$\Big|\langle[Q_1,[\dots Q_k, L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n}]\dots]s,s\rangle_{t,v,Z_0,0}\Big| \le C(1+n^2)|s|_{t,v,Z_0,0}|s|_{t,v,Z_0,1}. \tag{8.48}$$

Proof. We proceed as in the proof of [BL, Proposition 11.29], and in the proof of Propositions 7.36. We concentrate here on the new terms with respect to [BL] and to Proposition 7.36. In fact note that

$$\begin{aligned}\Big[\nabla_{e_i}, \rho^2\big(\sqrt{t}Z\big)\frac{|K^X(\sqrt{v}Z_0+\sqrt{t}Z|^2)}{2v}\Big] = \rho^2\big(\sqrt{t}Z\big)\sqrt{\frac{t}{v}}\langle\nabla^{TX}_{e_i}K^X, \frac{K^X}{\sqrt{v}}\rangle\big(\sqrt{v}Z_0{+}\sqrt{t}Z\big)\\ + \sqrt{\frac{t}{v}}(\nabla_{e_i}\rho)\big(\sqrt{t}Z\big)\frac{\rho\big(\sqrt{t}Z\big)|K^X|^2\big(\sqrt{v}Z_0+\sqrt{t}Z\big)}{\sqrt{v}},\end{aligned} \tag{8.49}$$

and so

$$\Big|[\nabla_{e_i}, \rho^2\big(\sqrt{t}Z\big)\frac{|K^X(\sqrt{v}Z_0+\sqrt{t}Z|^2}{2v}]\Big| \le C\sqrt{\frac{t}{v}}\left(1+\rho^2\big(\sqrt{t}Z\big)\frac{|K^X|^2}{v}\big(\sqrt{v}Z_0+\sqrt{t}Z\big)\right). \tag{8.50}$$

Since $\frac{t}{v} \le 1$, inequality (8.50) is enough to control the commutator in the left-hand side. Higher derivatives are even easier to control, since t/v is uniformly bounded.

Also, as we saw in the proof of Theorem 8.18, for any j,j',

$$\nabla_{\cdot}\langle\nabla^{TX}_{\tau^{Z_0}e_j(Z)}K^X(Z_0+Z), \tau^{Z_0}e_{j'}(Z)\rangle \tag{8.51}$$

vanishes for $Z_0 = 0, Z = 0$, so that

$$\nabla_{\cdot}\langle\nabla^{TX}_{\tau^{Z_0}e_j(Z)}K^X(Z_0+Z), \tau^{Z_0}e_{j'}(Z)\rangle = \mathcal{O}\big(|Z_0|+|Z|\big). \tag{8.52}$$

Using the fact that $t/v \le 1$ and also Proposition 8.16, we find that the commutator corresponding to (8.52) is uniformly bounded with respect to $|\ |_{t,v,Z_0,0}$. Higher commutators are harmless, again because $t/v \le 1$ and also by Proposition 8.16.

The commutators with the $\frac{1}{\sqrt{v}}\rho\big(\sqrt{t}Z\big)f_j\big(\sqrt{v}Z_0+\sqrt{t}Z\big)$'s and the mixed commutators can be easily controlled. In fact

(8.53)
$$\Big[\nabla_{e_i}, \frac{1}{\sqrt{v}}\rho\big(\sqrt{t}Z\big)f_j\big(\sqrt{v}Z_0+\sqrt{t}Z\big)\Big] = \sqrt{\frac{t}{v}}\Big(\nabla_{e_i}\rho\big(\sqrt{t}Z\big)f_j\big(\sqrt{v}Z_0+\sqrt{t}Z\big) + \\ + \rho\big(\sqrt{t}Z\big)\nabla_{e_i}f_j\big(\sqrt{v}Z_0+\sqrt{t}Z\big)\Big).$$

Since t/v is bounded, the commutator corresponding to (8.53) is again harmless.

The proof of our Proposition is completed. ☐

If $s \in C^\infty\Big((T_{\mathbf{R}}X)_{y_0}, (\Lambda(T^*_{\mathbf{R}}X_g)\widehat{\otimes}\Lambda(N^{*(0,1)}_{X_g/X})\otimes E)_{y_0}\Big)$ has compact support, put

$$\|s\|^2_{t,v,Z_0,k} = \sum_{j=0}^{k}\sum_{Q\in\mathcal{Q}^j_{t,v,Z_0}} |Qs|^2_{t,v,Z_0,0}. \tag{8.54}$$

We claim there is $C>0$ such that for any $z\in\mathbf{R}, |z|\le 1$,

$$\|s\|_{t,v,Z_0,1} \le C|s|_{t,v,Z_0,1}. \tag{8.55}$$

Since f_j vanishes on X_K and since $\nabla^{TX}_{\cdot}K^X|_{X_K}$ acts as an invertible operator on $N_{X_K/X}$,there exists $C>0$ such that for $1\le j\le r$, for $x\in X$,

$$|f_j(x)| \le C|K^X(x)|, \tag{8.56}$$

so that (8.55) holds. Then the obvious analogue of [BL, Theorem 11.30] and of Theorem 7.37 for $L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n}$ holds. In fact, using the analogue of Theorem 7.34 (which we just proved in Section 8.7), Proposition 8.22, (8.55) and proceeding as in the proof of [BL, Theorem 11.30], we obtain the analogue of Theorem 7.37.

8.9. Uniform estimates on the kernel of $\widetilde{F}_{t,n}(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0})$.

Now we proceed as in [BL, Section 11 n)] and in Section 7.12.

Theorem 8.23. *There exist $C'>0, C''>0, C'''>0$ such that for $\eta\in]0,1]$ small enough, $z\in\mathbf{R}^*, |z|\le c_\eta$, for $m,m'\in\mathbf{N}$, there is $C>0, r\in\mathbf{N}$ such that for $t\in]0,1], v\in[t,1], n\in\mathbf{N}, y_0\in X_{g,K}, Z_0\in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0|\le\epsilon_0/\sqrt{v}, Z, Z'\in(T_{\mathbf{R}}X)_{y_0}, |Z|,|Z'|\le\epsilon_0/\sqrt{t}$,*

$$\big(1+|zZ_0|\big)^m \sup_{|\alpha|,|\alpha'|\le m}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\widetilde{F}_{t,n}\big(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}\big)(Z,Z')\Big| \le \\ C\big(1+|Z|+|Z'|\big)^r\big(1+|Z_0|\big)^{2\ell''} \\ \exp\Big(-C'n^2/4+2C''\eta^2\sup(|Z|^2,|Z'|^2)-C'''|Z-Z'|^2\Big). \tag{8.57}$$

Proof. We proceed as in the proof of [BL, Theorem 11.31], [B13, Theorem 11.14], and of Theorem 7.38. Then the obvious analogue of (7.179) holds, with $Q\in\mathcal{Q}^k_{t,v,Z_0}, Q'\in\mathcal{Q}^{k'}_{t,v,Z_0}, k+k'\le q$.

We still define $J^0_{y_0,p}$ as in the proof of Theorem 7.38. The obvious analogue of (7.181) is now

$$|s|_0 \le |s|_{t,v,Z_0,0} \le C\big(1+p\big)^{2\ell'}\big(1+|Z_0|\big)^{2\ell''}|s|_0. \tag{8.58}$$

Therefore the analogue of (7.182) is now

$$\|Q\widetilde{F}_{t,n}(L^{3,t,v}_{\sqrt{v}Z_0,zK_0,n+p})Q'\|_{\infty,p} \le C\big(1+p\big)^{(2\ell'+m_q)}\big(1+|Z_0|\big)^{2\ell''} \exp\big(-C'n^2/4+2C''\eta^2p^2\big). \tag{8.59}$$

Using Sobolev inequalities as in (7.183), we deduce from (8.59) that there exists $r \in \mathbf{N}$ such that given α, α',

$$\sup_{|Z|\le p,|Z'|\le p}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\widetilde{F}_{t,n}\big(L^{3,t,v}_{\sqrt{v}Z_0,zK_0,n+p}\big)(Z,Z')\Big| \le C\big(1+p\big)^{(2\ell'+r)}\big(1+|Z_0|\big)^{2\ell''} \exp\big(-C'n^2/4+2C''\eta^2p^2\big). \tag{8.60}$$

Now we exploit the fact that the operators $\rho\big(\sqrt{t}Z\big)\frac{z}{\sqrt{v}}f_j(\sqrt{v}Z_0+\sqrt{t}Z)$ also lie in $\mathcal{Q}_{t,v,Z_0}$. In fact since the $df_j|_{X_K}$'s span $N_{X_K/X,\mathbf{R}}$, there is $C>0$ such that on X,

$$\sum_{j=1}^{r} f_j^2(x) \ge C d^{X,2}(x,X_K). \tag{8.61}$$

By (7.112), if $|Z|\le 2\epsilon_0/\sqrt{t}$, then $\rho\big(\sqrt{t}Z\big)=1$. Using (8.59) and proceeding as in the proof of (8.60), we find that given $k'\in\mathbf{N}$ and α,α',

$$\sup_{\substack{|Z|,|Z'|\le \\ \inf(p,\epsilon_0/\sqrt{t})}}\Big|\Big\{\Big[\frac{z^2}{v}d^{X,2}\big(\sqrt{v}Z_0+\sqrt{t}Z,X_K\big)\Big]^{k'}\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\widetilde{F}_{t,n}\big(L^{3,t,v}_{\sqrt{v}Z_0,zK_0,n+p}\big)(Z,Z')\Big\}\Big| \le C\big(1+p\big)^{(2\ell'+r)}\big(1+|Z_0|\big)^{2\ell''}\exp\big(-C'n^2/4+2C''\eta^2p^2\big). \tag{8.62}$$

If $|Z_0|\le\epsilon_0/\sqrt{v}, |Z|\le\inf(p,\epsilon_0/\sqrt{t})$, if $t/v\le 1$,

$$\begin{aligned}&\frac{1}{\sqrt{v}}d^X(\sqrt{v}Z_0,X_K)\le\frac{1}{\sqrt{v}}d^X(\sqrt{v}Z_0+\sqrt{t}Z,X_K)+Cp,\\&\frac{1}{\sqrt{v}}d^X(\sqrt{v}Z_0,X_K)=|Z_0|.\end{aligned} \tag{8.63}$$

Using (8.62), (8.63), we find that given $k'\in\mathbf{N}$, and α,α', there exist $C>0, r\in\mathbf{N}$ such that

$$\sup_{\substack{|Z|,|Z'|\le \\ \inf(p,\epsilon_0/\sqrt{t})}}\Big||zZ_0|^{k'}\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\widetilde{F}_{t,n}\big(L^{3,t,v}_{\sqrt{v}Z_0,zK_0,n+p}\big)(Z,Z')\Big| \le C\big(1+p\big)^{(2\ell'+r)}\big(1+|Z_0|\big)^{2\ell''}\exp\big(-C'n^2/4+2C''\eta^2p^2\big). \tag{8.64}$$

By proceeding as in the proof of Theorem 7.38, i.e. by replacing $\widetilde{F}_{t,n}$ by $\widetilde{F}^h_{t,n}$ and using finite propagation speed, we obtain in the right-hand side of (8.64) the extra factor $\exp\big(-C'''|Z-Z'|^2\big)$, i.e. we have established (8.57).

The proof of our Theorem is completed. □

8.10. **An estimate on the kernel of $\widetilde{F}_t(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0})$.** We will proceed formally as in Section 7.13. We extend the definition of $|s|_{t,v,Z_0,0}, |s|_{t,v,Z_0,1}$ to the case where $t=0$. Now we prove the analogue of [BL, Theorem 11.35] and of Theorem 7.39.

Theorem 8.24. *There exists $C>0, r\in\mathbf{N}$ such that for $t\in]0,1], v\in[t,1], z\in\mathbf{R}, |z|\leq 1$, $n\in\mathbf{N}, y_0\in X_{g,K}, Z_0\in N_{X_{g,K}/X_g,\mathbf{R},y_0}$, $|Z_0|\leq\epsilon_0/\sqrt{v}$, then*

$$\Big|\big(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n}-L^{3,(0,v)}_{\sqrt{v}Z_0,zK_0,n}\big)s\Big|_{t,v,Z_0,-1}\leq \\ C\big(1+n^r\big)\sqrt{\frac{t}{v}}\big(1+|Z_0|\big)|s|_{0,v,Z_0,1}. \tag{8.65}$$

Proof. We need to establish an analogue of (7.193). We only consider the terms which do not appear in the proof of [BL, Theorem 11.35] and of Theorem 7.39. Clearly

$$\frac{1}{v}\Big(|K^X(\sqrt{v}Z_0+\sqrt{t}Z)|^2-|K^X(\sqrt{v}Z_0)|^2\Big)= \\ \frac{1}{v}\langle K^X(\sqrt{v}Z_0+\sqrt{t}Z)-K^X(\sqrt{v}Z_0),K^X(\sqrt{v}Z_0+\sqrt{t}Z)+K^X(\sqrt{v}Z_0)\rangle. \tag{8.66}$$

From (8.66) and from the fact that K^X vanishes on $X_{g,K}$, so that for $(Z_0,Z)=(0,0)$, then $K^X(\sqrt{v}Z_0+\sqrt{t}Z)=0$, we get

$$\frac{1}{v}\Big(|K^X(\sqrt{v}Z_0+\sqrt{t}Z)|^2-|K^X(\sqrt{v}Z_0)|^2\Big)=\frac{1}{v}\mathcal{O}\big(|\sqrt{t}Z|(|\sqrt{v}Z_0|+|\sqrt{t}Z|)\big)= \\ \mathcal{O}\Big(\sqrt{t/v}|Z||Z_0|+(t/v)|Z|^2\Big). \tag{8.67}$$

From (8.67), we deduce that for $t\in]0,1], v\in[t,1], |Z|\leq n$,

$$\left|\frac{1}{v}\Big(|K^X(\sqrt{v}Z_0+\sqrt{t}Z|^2)-|K^X(\sqrt{v}Z_0)|^2\Big)\right|\leq \\ C\sqrt{\frac{t}{v}}(1+n^2)\big(1+|Z_0|\big). \tag{8.68}$$

Now we have to consider the terms containing $\nabla^{TX}_{\cdot}K^X$. By (8.41), we get for $1\leq j,j'\leq 2\ell$,

$$\Big\langle\nabla^{TX}_{\tau^{\sqrt{v}Z_0}e_j(\sqrt{t}Z)}K^X\big(\sqrt{v}Z_0+\sqrt{t}Z\big),\tau^{\sqrt{v}Z_0}e_{j'}(\sqrt{t}Z)\Big\rangle- \\ \Big\langle\nabla^{TX}_{\tau^{\sqrt{v}Z_0}e_j(0)}K^X\big(\sqrt{v}Z_0\big),\tau^{\sqrt{v}Z_0}e_{j'}(0)\Big\rangle=\mathcal{O}\big(\sqrt{t}|Z|\big)\mathcal{O}\big(\sqrt{v}|Z_0|+\sqrt{t}|Z|\big). \tag{8.69}$$

Using Proposition 8.16 and (8.69), we get (8.65) easily. The proof of our Theorem is completed. □

Now we establish an analogue of [BL, Theorem 11.36] and of Theorem 7.40.

Theorem 8.25. *Given $\eta\in]0,1]$, there exist $c_\eta\in]0,1], d>0, r\in\mathbf{N}$ such that if $z\in\mathbf{R}, |z|\leq c_\eta, t\in]0,1], v\in[t,1], n\in\mathbf{N}, Z_0\in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0|\leq\epsilon_0/\sqrt{v}, \lambda\in$*

$U_{\eta n+d}$, *if s has compact support, then*

$$(8.70)\quad \Big|\big((\lambda - L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n})^{-1} - (\lambda - L^{3,(0,v)}_{\sqrt{v}Z_0,zK_0,n})^{-1}\big)s\Big|_{t,v,Z_0,0} \leq C\sqrt{\frac{t}{v}}(1+n^r)\big(1+|Z_0|\big)(1+|\lambda|^r)|s|_{0,v,Z_0,0}.$$

Proof. We use Theorem 8.24, and we proceed as in the proof of Theorem 7.40. The proof of our Theorem is completed. □

Now we establish an analogue of Theorem 7.41.

Theorem 8.26. *For $\eta \in]0,1]$ small enough, there exist $C>0, r \in \mathbf{N}$ such that if $t \in]0,1], v \in [t,1], n \in \mathbf{N}, z \in \mathbf{R}, |z| \leq c_\eta, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0/\sqrt{v}, p \in \mathbf{N}$, then*

$$(8.71)\quad \|\widetilde{F}_{t,n}\big(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}\big) - \widetilde{F}_{0,n}\big(L^{3,(0,v)}_{\sqrt{v}Z_0,zK_0}\big)\|_{\infty,p} \leq C\sqrt{\frac{t}{v}}\big(1+|Z_0|\big)^{2\ell''+1}(1+p)^r\exp\big(-C'n^2/4 + 2C''\eta^2p^2\big).$$

Proof. Using (8.58) and Theorem 8.25, the proof of our Theorem is the same as the proof of Theorem 7.41. □

Theorem 8.27. *There exist $C'>0, C''>0, C'''>0$ such that for $\eta \in]0,1]$ small enough, there is $c_\eta \in]0,1]$ such that given $m \in \mathbf{N}$, there is $C>0$ for which if $t \in]0,1], v \in [t,1], z \in \mathbf{R}, |z| \leq c_\eta, n \in \mathbf{N}, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0/\sqrt{v}, Z, Z' \in (T_{\mathbf{R}}X)_{y_0}, |Z|, |Z'| \leq \epsilon_0/\sqrt{t}$,*

$$(8.72)\quad \Big|\big(\widetilde{F}_{t,n}(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}) - \widetilde{F}_{0,n}(L^{3,(0,v)}_{\sqrt{v}Z_0,zK_0})\big)(Z,Z'\Big| \leq C\Big(\frac{t}{v}\Big)^{1/4(2\ell+1)}\frac{\big(1+|Z_0|\big)^{2\ell''+1}}{\big((1+|zZ_0|\big)^m}\big(1+|Z|+|Z'|\big)^r \exp\big(-C'n^2/4 + 2C''\eta^2\sup(|Z|^2,|Z'|^2) - \frac{C'''}{2}|Z-Z'|^2\big).$$

Proof. Using Theorems 8.23 and 8.27, the proof of our Theorem is the same as the proof of Theorem 7.42. □

Theorem 8.28. *There exist $C'>0, C''>0, C'''>0$ such that for $\eta \in]0,1]$ small enough, there exists $c_\eta \in]0,1]$ such that given $m \in \mathbf{N}$, there exists $C>0$ such that if $t \in]0,1], v \in [t,1], z \in \mathbf{R}, |z| \leq c_\eta, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0| \leq \epsilon_0/\sqrt{v} Z, Z' \in (T_{\mathbf{R}}X)_{y_0}, |Z|, |Z'| \leq \epsilon_0/\sqrt{t}$,*

$$(8.73)\quad \Big|\big(\widetilde{F}_t(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}) - \exp\big(-L^{3,(0,v)}_{\sqrt{v}Z_0,zK_0}\big)(Z,Z')\Big| \leq C\left(\frac{t}{v}\right)^{1/4(2\ell+1)}\frac{\big(1+|Z_0|\big)^{2\ell''+1}}{\big(1+|zZ_0|\big)^m}\big(1+|Z|+|Z'|\big)^r \exp\left(2C''\eta^2\sup(|Z|^2,|Z'|^2) - \frac{C'''}{2}|Z-Z'|^2\right).$$

Proof. Our Theorem follows from (7.142) and from Theorem 8.27. □

8.11. **A proof of Theorem 8.8.** Now we take $c > 0$ as in (7.213). By (8.73), there exists $C''''' > 0$ such that if $Z \in N_{X_g/X,\mathbf{R},y_0}, |Z| \leq \epsilon_0/\sqrt{t}$,

$$\tag{8.74} \left|\left(\widetilde{F}_t(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}) - \exp\big(-L^{3,(0,v)}_{\sqrt{v}Z_0,zK_0}\big)\right)(g^{-1}Z,Z)\right| \leq \\ C\left(\frac{t}{v}\right)^{1/4(2\ell+1)} \frac{\big(1+|Z_0|\big)^{2\ell''+1}}{\big(1+|zZ_0|\big)^m}\big(1+|Z|\big)^r \exp\big(4C''\eta^2|Z|^2 - C'''''|Z|^2\big).$$

For $\eta \in]0,1]$ small enough,

$$\tag{8.75} 4C''\eta^2 - C''''' \leq -C'''''/2.$$

Using (8.74), (8.75), for $Z \in N_{X_g/X,\mathbf{R},y_0}, |Z| \leq \epsilon_0/\sqrt{t}$,

$$\tag{8.76} \left|\left(\widetilde{F}_t(L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}) - \exp\big(-L^{3,(0,v)}_{\sqrt{v}Z_0,zK_0}\big)\right)(g^{-1}Z,Z)\right| \leq \\ C\left(\frac{t}{v}\right)^{1/4(2\ell+1)} \frac{\big(1+|Z_0|\big)^{2\ell''+1}}{\big(1+|zZ_0|\big)^m} \exp\left(-\frac{C'''''}{4}|Z|^2\right).$$

Now using (8.7), we get

$$\tag{8.77} \frac{\mu_1\big(\sqrt{v}Z_0+\sqrt{t}Z\big)-\mu_1\big(\sqrt{v}Z_0\big)}{v} = \frac{\mathcal{O}\big(\sqrt{t}|Z|\big)\mathcal{O}\big(\sqrt{t}|Z|+\sqrt{v}|Z_0|\big)}{v},$$

so that

$$\tag{8.78} \Big|\frac{\mu_1\big(\sqrt{v}Z_0+\sqrt{t}Z\big)-\mu_1\big(\sqrt{v}Z_0\big)}{v}\Big| \leq C\Big(\frac{t}{v}|Z|^2 + \left(\frac{t}{v}\right)^{1/2}|Z||Z_0|\Big).$$

Also by (8.10),

$$\tag{8.79} \Big|\frac{\mu_1\big(\sqrt{v}Z_0\big)}{v}\Big| \leq C|Z_0|^2.$$

Using (7.224), (8.21), (8.29),(8.30), (8.57), (8.76), (8.78), (8.79), we get (8.23). The proof of Theorem 8.8 is completed.

8.12. **The supertrace containing μ_1.**

Theorem 8.29. *There exist $c > 0, r \in \mathbf{N}, C > 0, \gamma \in]0,1]$ such that if $z \in \mathbf{R}^*, |z| \leq c$, then*

$$\tag{8.80} |z|^r\left|\mathrm{Tr_s}\left[\frac{i\mu_1}{v}\exp\left(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\right)\right] + \widetilde{e}_{v,1}\right| \leq C\left(\frac{t}{v}\right)^{\gamma}.$$

Proof. Let $\overline{k}(y_0,Z_0)$ be the function defined on $X_g \cap \mathcal{U}_{\epsilon'_0}$ by the relation

$$\tag{8.81} dv_{X_g}(y_0,Z_0) = \overline{k}(y_0,Z_0)dv_{X_{g,K}}(y_0)dv_{N_{X_{g,K}/X_g}}(Z_0).$$

Then

$$\tag{8.82} \overline{k}|_{X_{g,K}} = 1.$$

Recall that $\widetilde{F}_t\big(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)(g^{-1}x,x)$ vanishes on $X \setminus \mathcal{U}_{\epsilon_0}$. Using (7.89), (8.81), we get

(8.83)
$$\begin{gathered}\int_{\mathcal{U}'_{\epsilon_0}} \mathrm{Tr_s}\left[\frac{i\mu_1}{v} g\widetilde{F}_t\big(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)(g^{-1}x,x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}} + \int_{X_g\cap\mathcal{U}'_{\epsilon_0}} \beta_{v,1} \frac{dv_{X_g}}{(2\pi)^{\dim X_g}} = \\ \int_{X_{g,K}} v^{\dim N_{X_{g,K}/X_g}} \Bigg[\int_{\substack{Z_0\in N_{X_{g,K}/X_g,\mathbf{R},y_0}, |Z_0|\le\epsilon_0/\sqrt{v} \\ Z\in N_{X_g/X,\mathbf{R},y_0}, |Z|\le\epsilon_0}} \mathrm{Tr_s}\Big[\frac{i\mu_1}{v} g\widetilde{F}_t\big(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big) \\ \big(g^{-1}(y_0,\sqrt{v}Z_0,Z),(y_0,\sqrt{v}Z_0,Z)\big)\Big] k(y_0,\sqrt{v}Z_0,Z)\frac{dv_{N_{X_g/X}}(Z)}{(2\pi)^{\dim N_{X_g/X}}} + \beta_{v,1}(y_0,\sqrt{v}Z_0)\Bigg] \\ \overline{k}(y_0,\sqrt{v}Z_0)\frac{dv_{N_{X_{g,K}/X_g}}(Z_0)}{(2\pi)^{\dim N_{X_{g,K}/X_g}}}\frac{dv_{X_{g,K}}(y_0)}{(2\pi)^{\dim X_{g,K}}}.\end{gathered}$$

Using Theorem 8.8 and (8.83), we find that there exists $C > 0$ such that for $z \in \mathbf{R}^*, |z| \le c$,

$$\begin{gathered}|z|^{2\ell''+1}\left|\int_{\mathcal{U}'_{\epsilon_0}} \mathrm{Tr_s}\left[\frac{i\mu_1}{v} g\widetilde{F}_t\big(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)(g^{-1}x,x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}} + \right. \\ \left. \int_{X_g\cap\mathcal{U}'_{\epsilon_0}} \beta_{v,1} \frac{dv_{X_g}}{(2\pi)^{\dim X_g}}\right| \le C\left(\frac{t}{v}\right)^{\gamma}\end{gathered} \tag{8.84}$$

.

Now we claim similar estimates can be very easily obtained for

$$\begin{gathered}\left|\int_{X\setminus\mathcal{U}'_{\epsilon_0}} \mathrm{Tr_s}\left[\frac{i\mu_1}{v} g\widetilde{F}_t\big(L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)(g^{-1}x,x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}} + \right. \\ \left. \int_{X_g\setminus\mathcal{U}'_{\epsilon_0}} \beta_{v,1} \frac{dv_{X_g}}{(2\pi)^{\dim X_g}}\right| .\end{gathered} \tag{8.85}$$

In fact, on $X \setminus \mathcal{U}'_{\epsilon_0}$, we observe that $\frac{1}{2v}|K^X|^2$ has a positive lower bound. We then adapt the above techniques to the case where $X_{g,K} = \emptyset$. In particular for $1 \le i \le 2\ell'$, the Clifford variables $c(e_i)$ are now replaced by $c_{t/v}(e_i)$, while being unchanged for $2\ell' + 1 \le i \le 2\ell$. The potentially annoying term $\frac{i\mu_1}{v}$ is more than killed by the presence of the term $\frac{1}{2v}|K^X|^2$. Details are left to the reader.

The proof of Theorem 8.29 is completed. □

8.13. **Control of the term containing μ_2.** The purpose of this Section is to prove the following second key result.

Theorem 8.30. *There exists $c \in]0,1], C > 0, \gamma \in]0,1]$ such that for $z \in \mathbf{R}, |z| \le c, t \in]0,1], v \in [t,1]$*

$$|z|^2\left|\mathrm{Tr_s}\left[\frac{i\mu_2}{v}\exp\big(-L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\right] + \widetilde{e}_{v,2}\right| \le C\left(\frac{t}{v}\right)^{\gamma}. \tag{8.86}$$

Proof. The remainder of the Section is devoted to the proof of this result. □

We take odd Grassmann variables $da, \overline{da}$ as in Sections 4.5 and 7.14, and we use the notation in (7.230). Set

$$\mathcal{B}_{t,v,} = L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}} + \overline{da}\sqrt{t}D^X_{-\frac{1}{t}-\frac{1}{v}}. \tag{8.87}$$

Proposition 8.31. *The following identity holds,*

$$\begin{aligned}\mathcal{B}_{t,v} = &-\frac{t}{2}\left(\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{e_I} - \left(\frac{1}{t}+\frac{1}{v}\right)\frac{\langle K^X, e_i\rangle}{2} - \frac{\overline{da}c(e_i)}{\sqrt{2t}}\right)^2 \\ &+ \frac{|K^X|^2}{2v} + \frac{t}{8}H + \frac{t}{4}c(e_i)c(e_j)\left(\left(R^E + \frac{1}{2}\mathrm{Tr}[R^{TX}]\right)(e_i,e_j) - \frac{1}{v}\langle\nabla^{TX}K^X, e_j\rangle\right) \\ &- \left(m^E(K) + \frac{1}{2}\mathrm{Tr}[m^{TX}(K)\right).\end{aligned} \tag{8.88}$$

Proof. Our identity follows from easily from (8.1). ☐

Proposition 8.32. *The following identity holds,*

$$\begin{aligned}t\frac{\partial}{\partial t}\mathrm{Tr_s}&\left[\frac{i\mu_2}{v}g\exp\left(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\right)\right] = \\ &\frac{1}{v}\mathrm{Tr_s}\left[\sqrt{\frac{t}{2}}c\big(i\nabla_{.}\mu_2\big)g\exp\left(-\mathcal{B}_{t,v}\right)\right]^{\overline{da}}.\end{aligned} \tag{8.89}$$

Proof. Clearly,

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Tr_s}&\left[\frac{i\mu_2}{v}g\exp\left(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\right)\right] = \\ &\frac{\partial}{\partial b}\mathrm{Tr_s}\left[\frac{i\mu_2}{v}g\exp\left(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}} - b[\sqrt{t}D^X_{\frac{1}{t}-\frac{1}{v}}, \frac{\partial}{\partial t}(\sqrt{t}D^X_{\frac{1}{t}-\frac{1}{v}})]\right)\right]_{b=0} \\ &= -\frac{\partial}{\partial b}\mathrm{Tr_s}\left[[\sqrt{t}D^X_{\frac{1}{t}-\frac{1}{v}}, \frac{i\mu_2}{v}]g\exp\left(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}} - b\frac{\partial}{\partial t}(\sqrt{t}D^X_{\frac{1}{t}-\frac{1}{v}})\right)\right].\end{aligned} \tag{8.90}$$

Clearly,

$$\begin{aligned}[\sqrt{t}D^X_{\frac{1}{t}-\frac{1}{v}}, \frac{i\mu_2}{v}] &= \frac{1}{v}\sqrt{\frac{t}{2}}c(\nabla_{.}i\mu_2), \\ t\frac{\partial}{\partial t}(\sqrt{t}D^X_{\frac{1}{t}-\frac{1}{v}}) &= \frac{\sqrt{t}}{2}D^X_{-\frac{1}{t}-\frac{1}{v}}.\end{aligned} \tag{8.91}$$

From (8.90)-(8.91), we get (8.89). ☐

Remark 8.33. Incidentally note that

$$\frac{1}{v}\mathrm{Tr_s}\left[\sqrt{\frac{t}{2}}c\big(i\nabla_{.}\mu_2\big)g\exp\left(-\mathcal{B}_{t,v}\right)\right] = \frac{1}{v}\mathrm{Tr_s}\left[\sqrt{\frac{t}{2}}c\big(i\nabla_{.}\mu_2\big)g\exp\left(-\mathcal{B}_{t,v}\right)\right]^{\overline{da}}, \tag{8.92}$$

because, if we make $\overline{da} = 0$, in the left-hand side of (8.92), we evaluate the supertrace of an odd operator, which is 0.

By (7.80),

$$\exp\left(-\mathcal{B}_{t,v}\right) = \widetilde{F}_t\big(\mathcal{B}_{t,v}\big) + \widetilde{I}_t\big(t\mathcal{B}_{t,v}\big). \tag{8.93}$$

Now

$$t\mathcal{B}_{t,v} = t^2 D^{X,2} + t\left(L_K + \big(1-\frac{t}{v}\big)[D^X, \frac{c(K^X)}{2\sqrt{2}}]\right) - \left(1-\frac{t}{v}\right)^2 \frac{|K^X|^2}{8} + \overline{da}\big(t^{3/2}D^X - t^{1/2}\big(1+\frac{t}{v}\big)\frac{c(K^X)}{2\sqrt{2}}\big). \tag{8.94}$$

By (8.94), it is clear that for $t \in]0,1], v \in [t,1]$ and $|K|$ small enough, the operator $t\mathcal{B}_{t,v}$ verifies estimates similar to $\mathcal{A}_t$ in Theorem 7.11. One then finds easily that for $|K|$ small enough, $t \in]0,1], v \in [t,1]$,

$$\|\widetilde{I}_t(\mathcal{B}_{t,v})\|_1 \leq C\exp\big(-C'/t\big). \tag{8.95}$$

In particular, by (8.95), we get for $t \in]0,1], v \in [t,1]$,

$$\Big|\frac{1}{v}\operatorname{Tr_s}\left[\sqrt{\frac{t}{2}}c\big(i\nabla_{.}\mu_2\big)g\widetilde{I}_t(t\mathcal{B}_{t,v})\right]^{\overline{da}}\Big| \leq C\exp\big(-C'/t\big). \tag{8.96}$$

By (8.88), and proceeding as in [BGS1, II, proof of Theorem 2.16] [BeGeV, Chapter 10] , we find that as $t \to 0$,

$$\operatorname{Tr_s}\left[\sqrt{\frac{t}{2}}c\big(i\nabla_{.}\mu_2\big)\widetilde{F}_t\big(-\mathcal{B}_{t,v}\big)\right] \to 0. \tag{8.97}$$

Incidentally, note that because of the Grassmann variable $\overline{da}$, the relevant techniques to establish (8.97) are either probabilistic, or use techniques developed in [BeGeV] to establish the local families index theorem of [B4]. Note here that the relevant computations in a more complicate context have already been done in [B14, Section 12] (see in particular [B14, eq. (12.44) and (12.48)]). Also (8.97) fits nicely with (8.89).

Now we will exploit the essential fact that $\nabla_{.}\mu_2$ vanishes near X_K. In fact using (8.97) and proceeding as in Sections 8.4 -8.12, and in particular as in the end of the proof of Theorem 8.29, we find that given $m \in \mathbf{N}, t \in]0,1], v \in [t,1]$,

$$\Big|\frac{z^2}{v}\Big|^m\Big|\operatorname{Tr_s}\left[\sqrt{\frac{t}{2}}c\big(i\nabla_{.}\mu_2\big)\widetilde{F}_t\big(\mathcal{B}_{t,v}\big)\right]\Big| \leq C\left(\frac{t}{v}\right)^\gamma. \tag{8.98}$$

In particular, by Proposition 8.32 and by (8.96), (8.98), for $t \in]0,1], v \in [t,1]$,

$$|z|^2\Big|\frac{\partial}{\partial t}\operatorname{Tr_s}\Big[i\frac{\mu_2}{v}\exp\big(-L_K + tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\Big]\Big| \leq \frac{C}{t}\left(\frac{t}{v}\right)^\gamma. \tag{8.99}$$

By (8.97) and by integration of (8.99) in the variable t , we get

$$|z|\Big|\operatorname{Tr_s}\Big[i\frac{\mu_2}{v}\exp\big(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\big)\Big] + \widetilde{e}_{v,2}\Big| \leq C\left(\frac{t}{v}\right)^\gamma. \tag{8.100}$$

From (8.96), (8.100), we get (8.86), i.e. we have completed the proof of Theorem 8.30.

8.14. **A proof of the last part of Theorem 6.6.** When $v \in [1,+\infty[$, $\frac{1}{v}$ remains bounded. By using the methods of Section 7 and of the present Section, one sees easily that for $z \in \mathbf{R}$, and $|z|$ small enough, there exists $C > 0$ such that for $t \in]0,1], v \in [1,+\infty[$,

$$\left|\operatorname{Tr_s}\left[i\langle\mu, K\rangle g \exp\left(-L_K - tD^{X,2}_{\frac{1}{t}-\frac{1}{v}}\right)\right]\right| \leq C, \tag{8.101}$$

which is equivalent to (6.13). The proof of Theorem 6.6 is completed.

9. A proof of Theorem 6.7

The purpose of this Section is to establish Theorem 6.7. We use techniques which are closely related to [BL, Section 12], and also to Sections 7 and 8. The main difficulty with respect to [BL] is still to obtain the adequate control of the considered operators for $|z|$ small enough.

This Section is organized as follows. In Section 9.1, we establish an algebraic identity on supertraces, which we will need to obtain the two terms in the asymptotic expansion in Theorem 6.7. In Section 9.2, we state our main convergence result. Sections 9.3-9.6 are devoted to the proof of this result. In Section 9.3, we establish a Lichnerowicz formula. In Section 9.4, we make a local change of coordinates near $X_{g,K}$ and introduce a simple Getzler rescaling on the Clifford algebra. This coordinate system turns out to be too elementary, and does not allow us to obtain the adequate control of the local supertraces. In Section 9.5, a more complicate coordinate system is introduced near $X_{g,K}$, which is compatible with what we did in Sections 7 and 8. This will allow us to establish the right estimates, and prove the result announced in Section 9.2. Finally, in Section 9.7, we prove Theorem 6.7.

In this Section, we use the notation in Sections 7-9. Also we fix $g \in G, K_0 \in \mathfrak{z}(g)$, and we assume that $K = zK_0$, with $z \in \mathbf{R}^*$.

9.1. **A simple identity of supertraces.** Let $da, \overline{da}$ be odd anticommuting Grassmann variables as in Sections 7.14 and 8.13. Then $da, \overline{da}$ span $\Lambda^1(\mathbf{R}^2) \otimes_{\mathbf{R}} \mathbf{C}$.

Definition 9.1. Put

$$\mathcal{C}_{t,v} = L_K + tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})} - \frac{da}{2v\sqrt{t}} c(iJ^{TX}K^X) + \frac{\overline{da}\sqrt{t}}{\sqrt{2}} D^X_{-\frac{1}{t}(1-\frac{1}{v})}. \tag{9.1}$$

Proposition 9.2. *The following identity holds,*

$$\frac{\partial}{\partial t} \mathrm{Tr_s} \left[\frac{i\langle \mu, K\rangle}{v} g \exp\big(-L_K - tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})}\big)\right] = \mathrm{Tr_s}\left[g \exp\big(-\mathcal{C}_{t,v}\big)\right]^{da\overline{da}}. \tag{9.2}$$

Proof. Clearly,

$$\begin{aligned} &\frac{\partial}{\partial t} \mathrm{Tr_s} \left[\frac{i\langle \mu, K\rangle}{v} g \exp\left(-L_K - tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v}}\right)\right] \\ &= \frac{\partial}{\partial b} \mathrm{Tr_s} \left[\frac{i\langle \mu, K\rangle}{v} g \exp\left(-L_K - tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})} - b[\sqrt{t}D^X_{\frac{1}{t}(1-\frac{1}{v})}, \frac{\partial}{\partial t}\sqrt{t}D^X_{\frac{1}{t}(1-\frac{1}{v})}]\right)\right] |_{b=0} \\ &= -\frac{\partial}{\partial b} \mathrm{Tr_s} \left[[\sqrt{t}D^X_{\frac{1}{t}(1-\frac{1}{v})}, \frac{i\langle \mu, K\rangle}{v}] \exp\left(-L_K - tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})} - b\frac{\partial}{\partial t}\sqrt{t}D^X_{\frac{1}{t}(1-\frac{1}{v})}\right)\right] |_{b=0}. \end{aligned} \tag{9.3}$$

By (1.17), (2.4),

$$[\sqrt{t}D^X_{\frac{1}{t}(1-\frac{1}{v})}, \frac{i\langle \mu, K\rangle}{v}] = \frac{\sqrt{t}}{\sqrt{2}v} c(i\nabla_{\cdot}\langle \mu, K\rangle) = \frac{\sqrt{t}}{\sqrt{2}v} c(iJ^{TX}K^X). \tag{9.4}$$

Also

$$t\frac{\partial}{\partial t}\sqrt{t}D^X_{\frac{1}{t}(1-\frac{1}{v})} = \frac{\sqrt{t}}{2} D^X_{-\frac{1}{t}(1-\frac{1}{v})}. \tag{9.5}$$

So by (9.3)-(9.5), we get (9.2). □

Remark 9.3. Put

$$\mathcal{C}'_{t,v} \;=\; L_{(1-\frac{1}{v})K} \;+\; tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})} \;-\; \frac{da}{2v\sqrt{t}}c(iJ^{TX}K^X) \;+\; \frac{\overline{da}\sqrt{t}}{\sqrt{2}}D^X_{-\frac{1}{t}(1-\frac{1}{v})}. \tag{9.6}$$

Then

$$\mathrm{Tr_s}\left[g\exp\left(-\mathcal{C}_{t,v}\right)\right] = \mathrm{Tr_s}\left[ge^{K/v}\exp\left(-\mathcal{C}'_{t,v}\right)\right]. \tag{9.7}$$

We make temporarily $da = 0, \overline{da} = 0$. Observe that

$$\mathcal{C}'_{t,v} = L_{(1-\frac{1}{v})K} + tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})}$$

is just the operator

$$L_K + tD^{X,2}_{\frac{1}{t}},$$

with K replaced by $(1-\frac{1}{v})K$. By (9.7), $\mathrm{Tr_s}[g\exp(-\mathcal{C}_{t,v})]$ is just $\mathrm{Tr_s}[g'\exp(-L_K - tD^{X,2}_{\frac{1}{t}})$, with K replaced by $(1-\frac{1}{v})K$, and g' replaced by $ge^{K/v}$. Incidentally note that by (2.41), this quantity is just $L(ge^K)$. So the considerations which follow are relevant only when $da, \overline{da}$ are nonzero. Still, to make our discussion simpler, we concentrate on the trivial case $da = 0, \overline{da} = 0$, by discussing the behaviour as $t \to 0$ of the associated local supertraces.

By using the techniques of Section 7, if $g' \in G$, if $K_0 \in \mathfrak{z}(g')$, if for $z \in \mathbf{R}$, $K = zK_0$, for $|z|$ small enough, as $t \to 0$, the local supertrace of $\mathrm{Tr_s}[g'\exp(-L_{(1-\frac{1}{v}K} - tD^{X,2}_{\frac{1}{t}(1-\frac{1}{v})})]$ converges to an an explicit current over $X_{g'}$. Still, by (7.212)-(7.215), the size of $|z|$ is explicitly related to g'. Here $g' = ge^{K/v}$ depends on z. This already indicates that the techniques of Section 7 cannot be used directly to establish a convergence result for $\mathrm{Tr_s}\left[g\exp\left(-\mathcal{C}_{t,v}\right)\right]^{da\overline{da}}$ as $t \to 0$.

9.2. **A convergence result.**

Theorem 9.4. *For $z \in \mathbf{R}^*$, and $|z|$ small enough, for any $v \in [1, +\infty[$, as $t \to 0$,*

$$\mathrm{Tr_s}\left[g\exp\left(-\mathcal{C}_{t,v}\right)\right]^{da\overline{da}} \to \int_{X_{g,K}} \mathrm{Td}_{ge^K}(TX)\Phi_{1/v,g,K}(N_{X_K/X})\mathrm{ch}_{ge^K}(E). \tag{9.8}$$

Proof. The purpose of Sections 9.3-9.6 is to establish our Theorem. □

9.3. **A Lichnerowicz formula.**

Theorem 9.5. *The following identity holds,*

$$\begin{aligned}\mathcal{C}_{t,v} = -\frac{t}{2}\Big(\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{e_i} - \frac{1}{2t}(1+\frac{1}{v})\langle K^X, e_i\rangle - \frac{\overline{da}}{2\sqrt{t}}c(e_i)\Big)^2 \\ - \frac{da}{2v\sqrt{t}}c(iJ^{TX}K^X) + \frac{\overline{da}}{2v\sqrt{t}}c(K^X) + \frac{|K^X|^2}{2vt} + \frac{t}{8}H \\ + \frac{t}{4}c(e_i)c(e_j)\big(R^E + \frac{1}{2}\mathrm{Tr}[R^{TX}]\big)(e_i, e_j) - \frac{1}{4v}\langle\nabla^{TX}_{e_i}K^X, e_j\rangle c(e_i)c(e_j) \\ - \big(m^E(K) + \frac{1}{2}\mathrm{Tr}[m^{TX}(K)]\big).\end{aligned} \tag{9.9}$$

Proof. When $da = 0, \overline{da} = 0$, (9.9) is just (8.1), with v replaced by tv. It is then easy to verify that (9.9) extends to the general case. □

Again, we write

$$\exp\big(-\mathcal{C}_{t,v}\big) = \widetilde{F}_t\big(\mathcal{C}_{t,v}\big) + \widetilde{I}_t\big(t\mathcal{C}_{t,v}\big). \tag{9.10}$$

Then

$$\begin{aligned} t\mathcal{C}_{t,v} = {} & t^2 D^{X,2} + \frac{t^{3/2}}{\sqrt{2}}\overline{da}D^X + t\Big(L_K + \big(1-\frac{1}{v}\big)[D^X, \frac{c(K^X)}{2\sqrt{2}}]\Big) \\ & + \sqrt{t}\left(-\frac{da}{2v}c(iJ^{TX}K^X) - \overline{da}(1-\frac{1}{v})\frac{c(K^X)}{4}\right) - \left(1-\frac{1}{v}\right)^2 \frac{|K^X|^2}{8}. \end{aligned} \tag{9.11}$$

By proceeding as in Sections 7.3 and 8.4, we find that for $t \in]0,1], v \in [1,+\infty[, |z|$ small enough,

$$\|\widetilde{I}_{\sqrt{t}}\big(t\mathcal{C}_{t,v}\big)\|_1 \leq C\exp\big(-C'/t\big). \tag{9.12}$$

As in Sections 7.3 and (8.6, the problem of evaluating the limit as $t \to 0$ of $\mathrm{Tr_s}[g\exp\big(-\mathcal{C}_{t,v}\big)]^{da\overline{da}}$ is now localized on X_g.

Clearly, there exists $\overline{\epsilon}_0 \in]0, a^X/8]$ such that if $\epsilon \in]0, 8\overline{\epsilon}_0]$, the map $(y_0, Z) \in N_{X_{g,K}/X,\mathbf{R}} \to \exp^X_{y_0}(Z) \in X$ induces a diffeomorphism from the ϵ-neighborhood $\overline{\mathcal{U}}_\epsilon$ of $X_{g,K}$ in $N_{X_{g,K}/X}$ on the tubular neighborhood $\overline{\mathcal{V}}_\epsilon$ of $X_{g,K}$ in X.

As explained in Section 8.3, we may as well assume that $\epsilon_0 = \epsilon_0' = \epsilon_0'' = \overline{\epsilon}_0$. So we now use the notation ϵ_0 instead of $\overline{\epsilon}_0$.

Definition 9.6. Let ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}}$ be the connection on $\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2) \otimes_{\mathbf{R}} \mathbf{C}$,

$${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}} = \nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{\cdot} - \big(1+\frac{1}{v}\big)\frac{K'^X}{2} - \frac{\overline{da}}{2}c(.). \tag{9.13}$$

Now we will evaluate the curvature of ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}}$.

Proposition 9.7. *The following identity holds,*

$$\begin{aligned} {}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},2} = {} & \frac{1}{4}\langle R^{TX}e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\mathrm{Tr}[R^{TX}] + R^E \\ & - \big(1+\frac{1}{v}\big)\frac{dK'^X}{2}. \end{aligned} \tag{9.14}$$

Proof. This follows from (9.13). ☐

Let A, B be smooth sections of $T_{\mathbf{R}}X$. By (9.13),

$${}^1\nabla_A^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}} c(B) = c(\nabla^{TX}_A B) + \overline{da}\langle A, B\rangle. \tag{9.15}$$

Let $c^1(T_{\mathbf{R}}X) \simeq T_{\mathbf{R}}X$ be the set of elements of $c(T_{\mathbf{R}}X)$ which have length 1. By (9.15), it follows that parallel transport with respect to ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}}$ maps $c^1(T_{\mathbf{R}}X)$ into $c^1(T_{\mathbf{R}}X) \oplus \mathbf{C}(\overline{da})$.

Definition 9.8. For $t > 0$, put

$${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},t} = \nabla^{\Lambda(T^{*(0,1)}X)\otimes E} - \big(1+\frac{1}{v}\big)\frac{K'^X}{2t} - \frac{\overline{da}}{2\sqrt{t}}c(.). \tag{9.16}$$

Clearly ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},t}$ is obtained from ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}}$ by scaling K into K/t, and $\overline{da}$ into $\overline{da}/\sqrt{t}$. Then

$$
(9.17)\quad {}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},t,2}=\frac{1}{4}\langle R^{TX}e_i,e_j\rangle c(e_i)c(e_j)+\frac{1}{2}\mathrm{Tr}[R^{TX}]+R^E \\ -\big(1+\frac{1}{v}\big)\frac{dK'^X}{2t}.
$$

Finally the connections ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}}$ and ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},t}$ are g-invariant.

9.4. **A change of coordinates and a Getzler rescaling: a first simple approach.** Here, we follow again the strategy of [BL, Sections 11 and 12] and of Section 8. Namely we will concentrate on the proof of estimates near $X_{g,K}$, while in principle we should also prove corresponding estimates near X_g. However the proof of the estimates near X_g and far from $X_{g,K}$ are in fact much easier.

Take $y_0\in X_{g,K}$. As in Section 8.6, if $Z\in(T_{\mathbf{R}}X)_{y_0}, |Z|\le 4\epsilon_0$, we identify Z to $\exp_{y_0}(Z)\in X$. If $Z\in\big(T_{\mathbf{R}}X\big)_{y_0}, |Z|\le 4\epsilon_0$, we identify $\big(\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}\big)_Z$ to $\big(\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}\big)_{y_0}$ by parallel transport with respect to ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},t}$ along the curve $t\in[0,1]\to tZ\in X$.

We still define $\rho(Z)$ as in (7.111), so that (7.112) holds.

Definition 9.9. Let $L^{1,(t,v)}_{y_0,K}$ be the operator

$$
(9.18)\qquad L^{1,(t,v)}_{y_0,K}=\big(1-\rho^2(Z)\big)\big(-\frac{t}{2}\Delta^{TX}\big)+\rho^2(Z)\mathcal{C}_{t,v}.
$$

Then

$$
(9.19)\qquad L^{1,(t,v)}_{y_0,K}\in\big(c(T_{\mathbf{R}}X)\otimes\mathrm{End}(E)\big)_{y_0}\widehat{\otimes}\Lambda(\mathbf{R}^2\otimes_{\mathbf{R}}\mathbf{C})\otimes\mathrm{Op}_{y_0}.
$$

We still define H_t as in (7.117). Let $L^{2,(t,v)}_{y_0,K}$ be obtained from $L^{1,(t,v)}_{y_0,K}$ as in (7.118).

Let $(e_1,\dots e_{2\ell})$ be an orthonormal oriented basis of $(T_{\mathbf{R}}X)_{y_0}$ taken as after Definition 8.9.

Definition 9.10. Let $L^{3,(t,v)}_{y_0,K}$ be the operator obtained from $L^{2,(t,v)}_{y_0,K}$ by replacing $c(e_i)$ by $c_t(e_i)$ for $1\le i\le 2\ell''$, while leaving the $c(e_i)$'s unchanged for $2\ell''+1\le i\le 2\ell$.

Let $j':X_{g,K}\to X$ be the obvious embedding. The tensors which are considered in the following definition are evaluated at y_0.

Definition 9.11. Let $L^{3,(0,v)}_{y_0}$ be the operator

$$
(9.20)\quad L^{3,(0,v)}_{y_0,K}=-\frac{1}{2}\left(\nabla_{e_i}+\frac{1}{2}\langle\big(j'^*R^{TX}-\big(1+\frac{1}{v}\big)\nabla^{TX}_{\cdot}K^X\big)Z,e_i\rangle\right)^2 \\ +\frac{1}{2v}\left|\nabla^{TX}_ZK^X\right|^2-\frac{1}{4v}\langle\nabla^{TX}_{e_i}K^X,e_j\rangle c(e_i)c(e_j)+\frac{1}{2v}\langle j'^*R^{TX}\nabla^{TX}_ZK^X,Z\rangle \\ -i\frac{da}{2v}c\big(J^{TX}\nabla^{TX}_ZK^X\big)-i\frac{da\overline{da}}{2v}\langle J^{TX}\nabla^{TX}_ZK^X,Z\rangle+j'^*\big(R^E+\frac{1}{2}\mathrm{Tr}[R^{TX}]\big) \\ -\big(m^E(K)+\frac{1}{2}\mathrm{Tr}[\nabla^{TX}_{\cdot}K^X]\big).
$$

Theorem 9.12. *For $v > 0$, as $t \to 0$,*

$$L^{3,(t,v)}_{y_0,K} \to L^{3,(0,v)}_{y_0,K}. \tag{9.21}$$

Proof. If A is a differential operator, we denote by $[A]^{(3)}_t$ the differential operator obtained from A by using the trivialization with respect to ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},t}$, the rescaling of the coordinate Z and the Getzler rescaling of the Clifford variables which was given in Definition 9.10. We start from (9.9). Using (7.106), (7.108), (9.14) and proceeding as in Proposition 7.27, we find that as $t \to 0$,

$$\begin{aligned}\left[-\frac{t}{2}\left(\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{e_i} - \frac{1}{2t}(1+\frac{1}{v})\langle K^X, e_i\rangle - \frac{\overline{da}}{2\sqrt{t}}c(e_i)\right)^2\right]^{(3)}_t \to \\ -\frac{1}{2}\left(\nabla_{e_i} + \frac{1}{2}\langle\big(j'^{*}R^{TX} - (1+\frac{1}{v})\nabla^{TX}K^X\big)Z, e_i\rangle\right)^2.\end{aligned} \tag{9.22}$$

Let $\tau e_j(Z)$ be the parallel transport of e_j along the curve $t \in [0,1] \to tZ$ with respect to the connection ∇^{TX}. Let $\mathcal{O}_1\big(|Z|^2\big)$ be any object in $\big(c(T_{\mathbf{R}}X_{g,K})\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}\big)_{y_0}$ which is of length at most 1 and is also $\mathcal{O}(|Z|^2)$. By (9.15), in the trivialization associated to ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}}$,

$$c\big(\tau e_j(Z)\big) = c(e_j) + \overline{da}\langle Z, e_j\rangle + \mathcal{O}_1\big(|Z|^2\big). \tag{9.23}$$

From (9.23), we deduce that for $1 \leq j \leq 2\ell''$,

$$\Big[\sqrt{t}c\big(\tau e_j(\sqrt{t}Z)\big)\Big]^{(3)}_t = \sqrt{2}e^j\wedge + \mathcal{O}\big(\sqrt{t}|Z|\big), \tag{9.24}$$

and that for $2\ell''+1 \leq j \leq 2\ell$,

$$\Big[c(\tau e_j)(\sqrt{t}Z)\Big]^{(3)}_t = c(e_j) + \overline{da}\langle Z, e_j\rangle + \mathcal{O}\big(\sqrt{t}|Z|^2\big). \tag{9.25}$$

By (8.41),

$$\langle iJ^{TX}K^X(Z), \tau e_j(Z)\rangle = \langle iJ^{TX}\nabla^{TX}_Z K^X_{y_0}, e_j\rangle + \mathcal{O}\big(|Z|^3\big). \tag{9.26}$$

Moreover

$$c(K^X)(Z) = c(\tau e_j)(Z)\langle K^X(Z), \tau e_j(Z)\rangle. \tag{9.27}$$

A similar identity holds for $c(iJ^{TX}K^X)(Z)$. By (9.24)-(9.27), we find that

$$\begin{aligned}\left[-\frac{da}{2v\sqrt{t}}c(iJ^{TX}K^X)(\sqrt{t}Z)\right]^{(3)}_t &= -\frac{da}{2v}\Big(c(iJ^{TX}\nabla^{TX}_Z K^X_{y_0}) + \\ &\overline{da}\langle iJ^{TX}\nabla^{TX}_Z K^X_{y_0}, Z\rangle\Big) + \mathcal{O}(\sqrt{t}|Z|^2), \\ \left[\frac{\overline{da}}{2v\sqrt{t}}c(K^X)(\sqrt{t}Z)\right]^{(3)}_t &= \frac{\overline{da}}{2v}c(\nabla^{TX}_Z K^X_{y_0}) + \mathcal{O}(\sqrt{t}|Z|^2).\end{aligned} \tag{9.28}$$

Also

$$\frac{|K^X|^2}{2vt}(\sqrt{t}Z) \to \frac{1}{2v}|\nabla^{TX}_Z K^X|^2. \tag{9.29}$$

From (8.41), we get

$$\langle \nabla^{TX}_{\tau e_i} K^X, \tau e_j\rangle(Z) = \langle \nabla^{TX}_{e_i} K^X_{y_0}, e_j\rangle - \frac{1}{2}\langle R^{TX}(\nabla^{TX}_Z K^X_{y_0}, Z)e_i, e_j\rangle + \mathcal{O}(|Z|^2). \tag{9.30}$$

By (9.24), (9.25), (9.30), and using the $(2,2)$ symmetry property of the curvature R^{TX} of the Levi-Civita connection ∇^{TX}, we find that as $t \to 0$,

$$\begin{aligned}\frac{1}{4}\Big[\langle \nabla^{TX}_{\tau e_i} K^X, \tau e_j\rangle c(\tau e_i)c(\tau e_j)(\sqrt{t}Z)\Big]^{(3)}_t = \frac{1}{4}\sum_{2\ell''+1\le i,j,\le 2\ell} \langle \nabla^{TX}_{e_i} K^X_{y_0}, e_j\rangle c(e_i)c(e_j) \\ - \frac{1}{2}\langle j'^{*} R^{TX}_{y_0} \nabla^{TX}_Z K^X_{y_0}, Z\rangle + \frac{\overline{da}}{2} c(\nabla^{TX}_Z K^X_{y_0}).\end{aligned} \tag{9.31}$$

From (9.9), (9.22)-(9.31), we get (9.21). The proof of our Theorem is completed. □

Remark 9.13. Consider the exact sequence of holomorphic Hermitian vector bundles on $X_{g,K}$,

$$\mathcal{E} : 0 \to TX_{g,K} \to TX|_{X_{g,K}} \to N_{X_{g,K}/X} \to 0. \tag{9.32}$$

Then the exact sequence (9.32) verifies the assumptions of Section 4.5. Namely g acts on $\mathcal{E}$ as a parallel isometry, $B = \nabla^{TX}_{\cdot} K^X \in \operatorname{End}(\mathcal{E})$ is skew-adjoint, parallel, and commutes with g, and finally (4.39) holds. Using the notation in (4.16),

$$N^{+}_{X_{g,K}/X} = N_{X_K/X}. \tag{9.33}$$

Also using the notation in Definition 4.11, then

$$L^{3,(0,v)}_{y_0,K} = L^{1/v}_0 + R^E - m^E(K). \tag{9.34}$$

Now we use the notation in (4.41)-(4.44). By (4.44),

$$P^{(0)} = \left(1 - \frac{1}{v}\right)^2 (\nabla_{\cdot} K^X)^2, \tag{9.35}$$

so that $P^{(0)} \le 0$. This is again an ominous indication of difficulties to come. In fact, as explained in Section 4.4, the operator in (4.28) is not lower bounded.

To make our estimates easier, in the next Section, we will use a slightly different trivialization, which will be compatible with what was done in Section 8.

9.5. A new trivialization.

Take $Z_0 \in N_{X_{g,K}/X_{g,\mathbf{R}},y_0}, |Z_0| \le \epsilon_0$. If $Z \in (T_{\mathbf{R}}X)_{y_0}$, $|Z| \le 4\epsilon_0$, we identify $\big(\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}\big)_Z$ to $\big(\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}\big)_{Z_0}$ by parallel transport along the curve $s \in [0,1] \to \exp_{Z_0(sZ)}$ with respect to the connection ${}^1\nabla^{\Lambda(T^{*(0,1)}X)\otimes E\otimes\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C},t}$. Also we identify $\big(\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}^2)\otimes_{\mathbf{R}}\mathbf{C}\big)_{Z_0}$ to $\big(\Lambda(T^{*(0,1)}X)\otimes E\widehat{\otimes}\Lambda(\mathbf{R}2)\otimes\mathbf{C}\big)_{y_0}$ by parallel transport along the curve $s \in [0,1] \to sZ_0$ with respect to the connection $\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$.

We may and we will assume that ϵ_0 is small enough so that if $|Z_0| \le \epsilon_0, |Z| \le 4\epsilon_0$,

$$\frac{1}{2}h^{TX}_{y_0} \le h^{TX}_{Z_0+Z} \le \frac{3}{2}h^{TX}_{y_0}. \tag{9.36}$$

We still use the notation in (8.26).

Definition 9.14. Let $L^{'1,(t,v)}_{Z_0,K}$ be the operator

$$L^{'1,(t,v)}_{Z_0,K} = \big(1-\rho^2(Z)\big)\big(-\frac{t}{2}\Delta^{TX}\big) + \rho^2(Z)\big(\mathcal{C}_{t,v}\big)_{Z_0}. \tag{9.37}$$

We still obtain $L^{'2,(t,v)}_{Z_0,K}, L^{'3,(t,v)}_{Z_0,K}$ from $L^{'1,(t,v)}_{Z_0,K}$ as before. Recall that $k'_{y_0,Z_0}(Z)$ was defined in (7.98).

Proposition 9.15. *The following identity holds for* $Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, Z \in N_{X_g/X,\mathbf{R},y_0}$,
$|Z_0| \le \epsilon_0, |Z| \le \epsilon_0/\sqrt{t}$,

$$\begin{aligned} t^{\dim N_{X_{g,K}/X}}\operatorname{Tr_s}\Big[g\widetilde{F}_t\big(\mathcal{C}_{t,v}\big)\big(g^{-1}(Z_0,\sqrt{t}Z),(Z_0,\sqrt{t}Z)\big)\Big]\, k'_{(y_0,Z_0)}(\sqrt{t}Z) \\ = (-i)^{\dim X_{g,K}}\operatorname{Tr_s}\Big[g\widetilde{F}_t\big(L^{'3,(t,v)}_{y_0,Z_0,K}\big)\big(g^{-1}Z,Z\big)\Big]^{\max}. \end{aligned} \tag{9.38}$$

Proof. The proof of (9.38) is the same as the proof of (7.115) and of Proposition 7.25. ☐

Definition 9.16. Let $L^{'3,(0,v)}_{y_0,Z_0,K}$ be the operator

$$\begin{aligned} L^{'3,(0,v)}_{y_0,Z_0,K} = -\frac{1}{2}\left(\nabla_{e_i} + \frac{1}{2}\langle\left(R^{TX} - (1+\frac{1}{v})\nabla^{TX}_{\cdot}K^X\right)_{y_0} Z, e_i\rangle\right)^2 \\ + \frac{1}{2v}|\nabla_{Z+Z_0}K^X|^2 - \frac{1}{4v}\langle\nabla_{e_i}K^X, e_j\rangle c(e_i)c(e_j) + \frac{1}{2v}\langle R^{TX}\nabla^{TX}_{Z+Z_0}K^X, Z+Z_0\rangle \\ -i\frac{da}{2v}c\big(J^{TX}\nabla_{Z+Z_0}K^X_{y_0}\big) - i\frac{da\overline{da}}{2v}\langle J^{TX}\nabla^{TX}_{Z+Z_0}K^X_{y_0}, Z+Z_0\rangle + i^*\big(R^E + \frac{1}{2}\operatorname{Tr}[R^{TX}]\big) - \\ \big(m^E(K) + \frac{1}{2}\operatorname{Tr}[\nabla^{TX}_{\cdot}K^X]\big)_{y_0}. \end{aligned} \tag{9.39}$$

Remark 9.17. Observe that

$$\begin{aligned} L^{'3,(0,v)}_{y_0,Z_0,K} = \exp\left(\frac{1}{2}\langle\big(R^{TX} - (1+\frac{1}{v})\nabla_{\cdot}K^X\big)Z_0, Z\rangle\right)\left[L^{3,(0,v)}_{y_0,K}\right]_{Z+Z_0} \\ \exp\left(-\frac{1}{2}\langle\big(R^{TX} - (1+\frac{1}{v})\nabla_{\cdot}K^X)\big)Z_0, Z\rangle\right). \end{aligned} \tag{9.40}$$

Theorem 9.18. *As* $t \to 0$,

$$L^{'3,(t,v)}_{y_0,\sqrt{t}Z_0,K} \to L^{'3,(0,v)}_{y_0,Z_0,K}. \tag{9.41}$$

Proof. The proof of (9.41) uses the same arguments as the proof of Theorem 9.12. ☐

9.6. A proof of Theorem 9.4.

Theorem 9.19. *There exist* $C' > 0, C'' > 0, C''' > 0$ *such that for* $\eta \in]0,1]$ *small enough, there exists* $c_\eta \in]0,1]$ *for which if* $z \in \mathbf{R}, |z| \le c_\eta$, *given* $m, m' \in \mathbf{N}$, *there is* $C > 0, r \in \mathbf{N}$ *such that for* $t \in]0,1], v \in [1,+\infty[, y_0 \in X_{g,K}, Z_0 \in N_{X_{g,K}/X_g,\mathbf{R},y_0}, Z, Z' \in (T_{\mathbf{R}}X)_{y_0}, |Z_0|, |Z|, |Z'| \le \epsilon_0/\sqrt{t}$,

$$\begin{aligned} \big(1+|\frac{z}{\sqrt{v}}Z_0|\big)^m \sup_{|\alpha|,|\alpha'|\le m'}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\widetilde{F}_t\big(L^{'3,(t,v)}_{y_0,\sqrt{t}Z_0,zK_0}\big)(Z,Z')\right| \\ C\big(1+|Z|+|Z'|\big)^r\big(1+|Z_0|\big)^{2(\ell''+1)+1}\exp\big(2C''\eta^2\sup\big(|Z|^2,|Z'|^2\big) - C'''|Z-Z'|^2\big). \end{aligned} \tag{9.42}$$

Proof. First assume that $v = 1$. We apply Theorem 8.23 to the case $v = t$ and we use (7.142). Note here that $\Lambda\big(T^*_{\mathbf{R}}X_{g,K}\big)$ is now replaced by $\Lambda\big(T^*_{\mathbf{R}}X_{g,K}\big)\widehat{\otimes}\Lambda(\mathbf{C})$, which explains why ℓ'' is now changed in $\ell''+1$. The proof extends easily to the general case. □

Let $\mathbf{K}_{y_0}$ be the vector space of smooth sections of $\big(\Lambda(T^*_{\mathbf{R}}X_{g,K}\oplus\mathbf{R}^2)\widehat{\otimes}\Lambda(N^{*,(0,1)}_{X_{g,K}/X,\mathbf{R}})\otimes E\big)_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$, let $\mathbf{K}_{p,y_0}$ be the vector space of smooth sections of $\big(\Lambda^p(T^*_{\mathbf{R}}X_{g,K}\oplus\mathbf{R}^2)\widehat{\otimes}\Lambda(N^{*,(0,1)}_{X_{g,K}/X,\mathbf{R}}\otimes E\big)_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$. We denote by $\mathbf{K}^0_{y_0}, \mathbf{K}^0_{p,y_0}$ the corresponding vector spaces of square integrable sections.

Definition 9.20. For $t\in]0,1], y_0\in X_{g,K}, Z_0\in N_{X_{g,K}/X_g,\mathbf{R},y_0}, s\in\mathbf{K}_{p,y_0}$, put

$$(9.43)\quad |s|^2_{t,Z_0,0}=\int_{(T_{\mathbf{R}}X)_{y_0}}|s|^2\Big(1+\big(|Z|+|Z_0|\big)\rho\big(\sqrt{t}Z/2\big)\Big)^{2\left(2(\ell''+1)-p\right)}dv_{TX}(Z).$$

Then (9.43) induces an Hermitian product $\langle\ \rangle_{t,Z_0,0}$ on $\mathbf{K}^0_{p,y_0}$. We equip $\mathbf{K}^0_{y_0}$ with the Hermitian product which is the direct sum of the Hermitian products on the $\mathbf{K}^0_{p,y_0}$. Incidentally observe that if $v\in[1,+\infty[$, by replacing v by tv in (8.35), we get a norm which is equivalent to the norm (9.43).

Definition 9.21. For $t\in]0,1], v\in[1,+\infty[, y_0\in X_{g,K}, Z_0\in N_{X_{g,K}/X_g,\mathbf{R},y_0}$, and $s\in\mathbf{K}_{y_0}$ with compact support, put

(9.44)

$$|s|^2_{t,v,Z_0,1}=|s|^2_{t,Z_0,0}+\frac{1}{tv}\Big|\rho\big(\sqrt{t}Z\big)K^X\big(\sqrt{t}(Z_0+Z)\big)\Big|^2_{t,Z_0,0}+\sum_{i=1}^{2\ell}|\nabla_{e_i}s|^2_{t,Z_0,0}.$$

Again the norm $|\ |_{t,v,Z_0,1}$ is equivalent to the obvious modification of the norm in (8.38), with v replaced by tv.

Note that the norm $|\ |_{t,v,Z_0,1}$ depends explicitly on K. It will be understood that when K is replaced by zK_0, the norm $|\ |_{t,v,Z_0,1}$ is modified correspondingly. We define $L'^{3,(t,v)}_{y_0,Z_0,K,n}$ as in (8.45).Using (9.36), and proceeding as in (7.146) , we find that if $Z\in(T_{\mathbf{R}}X)_{y_0}, |Z|, |Z'|\le p$,

$$(9.45)\qquad \widetilde{F}_{t,n}\big(L'^{3,(t,v)}_{y_0,Z_0,K}\big)(Z,Z')=\widetilde{F}_{t,n}\big(L'^{3,(t,v)}_{y_0,Z_0,K,n+p}\big)(Z,Z').$$

One verifies easily that the estimates in Theorem 8.18 still hold with respect to the new norms, with constants which are uniform in $t\in]0,1], v\in[1,+\infty[$. In particular, the obvious analogue of Theorem 7.34 still holds with uniform constants.

Now we will prove an analogue of [BL, Theorem 12.16]. Here for $\lambda\in U_{\eta n+d}$, $\big(\lambda-L'^{3,(t,v)}_{y_0,\sqrt{t}Z_0,zK_0,n}\big)^{-1}), \big(\lambda-L'^{3,(0,v)}_{Z_0,zK_0,n}\big)^{-1}$ will be considered as distributions on $\big(T_{\mathbf{R}}X\big)_{y_0}\times\big(T_{\mathbf{R}}X\big)_{y_0}$.

Theorem 9.22. *Given $\eta>0$, there exists $c_\eta\in]0,1]$ such that if $t\in]0,1], v\in[1,+\infty[, n\in\mathbf{N}, y_0\in X_{g,K}, |Z_0|\le\epsilon_0/\sqrt{t}, z\in\mathbf{R}^*, |z|\le c_\eta$, if $\lambda\in U_{\eta n+d}$,*

$$(9.46)\quad \big(\lambda-L'^{3,(t,v)}_{y_0,\sqrt{t}Z_0,zK_0,n}\big)^{-1}\to\big(\lambda-L'^{3,(0,v)}_{Z_0,zK_0,n}\big)^{-1}$$

in the sense of distributions.

Proof. Clearly

$$(9.47)\quad \big(\lambda - L'^{3,(t,v)}_{y_0,\sqrt{t}Z_0,zK_0,n}\big)^{-1} - \big(\lambda - L'^{3,(0,v)}_{Z_0,zK_0,n}\big)^{-1} = \\ \big(\lambda - L'^{3,(t,v)}_{y_0,\sqrt{t}Z_0,zK_0,n}\big)^{-1}\big(L'^{3,(t,v)}_{y_0,\sqrt{t}Z_0,zK_0,n} - L'^{3,(0,v)}_{Z_0,zK_0,n}\big)\big(\lambda - L'^{3,(0,v)}_{Z_0,zK_0,n}\big)^{-1}.$$

Then we use (9.47), and we proceed as in the proof of [BL, Theorem 12.16]. In fact given $z \in \mathbf{R}^*$ such that $|z|$ is small enough, the situation is the same as in [BL]. ☐

yΓFn(B) By the analogue of Theorem 7.34, if $z \in \mathbf{R}, |z| \leq c_\eta$,

$$\widetilde{F}_p$$

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