SOME RESULTS ON THE NAVIER-STOKES EQUATIONS IN THIN 3D DOMAINS

DRAGOŞ IFTIMIE AND GENEVIÈVE RAUGEL

Dedicated to Jack Hale on the occasion of his 70th birthday

1. Introduction

As is well-known, the Navier-Stokes equations describe the time evolution of solutions of mathematical models of viscous incompressible fluids. From the mathematical point of view, global existence of weak solutions is known to hold in every space dimension. Uniqueness of weak solutions is known in dimension 2 (see [18]). In dimension 3, to obtain global existence and uniqueness, one has to assume additional regularity and smallness assumptions on the initial data and the forcing term. A natural question is how to use the good properties of the 2D Navier-Stokes equations to improve the uniqueness and regularity results for the 3D equations, when the domain is thin. In this paper, we consider the existence and uniqueness of solutions of the Navier-Stokes equations in thin three-dimensional domains $Q_\varepsilon = \Omega \times (0, \varepsilon)$, where Ω is a suitable bounded domain in $\mathbb{R}^2$ and ε is a positive parameter, $0 < \varepsilon \leq 1$. We do a detailed study of this question in the case of two types of boundary conditions: the purely periodic condition (PP) and the periodic-Dirichlet boundary condition (PD), that is, periodic condition in the thin vertical direction and homogeneous Dirichlet conditions on the lateral boundary $\Gamma_l = \partial\Omega \times (0, \varepsilon)$. When (PD) boundary conditions are considered, we assume that Ω is a regular domain in $\mathbb{R}^2$, while, in the case of the (PP) boundary conditions, $\Omega = (0, l_1) \times (0, l_2)$, where l_1, l_2 are positive numbers. Our results also hold for other types of boundary conditions, such as those considered in [27] (See remarks 1.2, 1.3 and 1.4).

The study of the Navier-Stokes equations on thin domains originates in a series of papers of Hale and Raugel ([12], [13], [14]), concerning the reaction-diffusion and damped wave equations on thin domains. In thin three-dimensional domains, inspired by the methods developed in ([12], [13], [14]), Raugel and Sell ([24], [25]) proved global existence of strong solutions for large initial data and forcing terms, in the case of the boundary conditions (PP) and (PD). As in [12], an essential tool in their proof is the vertical mean operator M (see (1.17)), which allows to decompose every function g on Q_ε into the sum of a function Mg which does not depend on the vertical variable and a function $(I - M)g$ with vanishing vertical mean and thus to use more precise Sobolev and Poincaré inequalities. Later, in the case of Dirichlet boundary conditions, Avrin [1] showed global existence of strong solutions of the Navier-Stokes equations on thin three-dimensional domains for large data, by applying a contraction principle argument and carefully analyzing the dependence of the solution on the first eigenvalue of the corresponding Laplace operator. The analysis in the case of Dirichlet boundary conditions in a thin domain is simpler, because the size of the first eigenvalue is of order ε^{-2} and thus the above decomposition is of no use. Next, using the same tools as Raugel and Sell together with improved Agmon inequalities, Temam and Ziane ([27], [28]) generalized the results of [24], [25] to other boundary conditions and, in the case of the free boundary conditions, to thin spherical domains. In the periodic case, Moise, Temam and Ziane [22] proved global existence of strong solutions for initial data, that are larger than in [25]. Also in the (PP) case, using anisotropic spaces, Iftimie [15] showed existence and uniqueness of solutions for less regular initial data and proved that initial data u_0 with larger $(I-M)u_0$ part could be taken. Finally, in the same case, Montgomery-Smith [23] gives global existence results, which are not contained in [22].

In this paper, we improve the previous existence and uniqueness results in two directions, by requiring less regularity on the initial data and by allowing a larger size of the initial data and forcing term. We also emphasize the importance played by the third component of the vertical mean value of the data. For instance, in the (PD) and (PP) cases, we show that, for any real number γ, $0 \le \gamma < 1/2$, there exists a positive constant K_γ such that, for $0 < \varepsilon \le 1$, if the initial data u_0 and forcing term f satisfy

$$\|Mu_0\|_{L^2(Q_\varepsilon)} \le K_\gamma \varepsilon^{1/2}, \quad \|A_\varepsilon^{1/4}(I-M)u_0\|_{L^2(Q_\varepsilon)} \le K_\gamma,$$
$$\sup_t \|MP_\varepsilon f(t)\|_{L^2(Q_\varepsilon)} \le K_\gamma \varepsilon^{1/2}, \quad \sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2(Q_\varepsilon)} \le K_\gamma \varepsilon^{-1/2} |\ln \varepsilon|^\gamma,$$

where A_ε is the Stokes operator and P_ε is the Leray projection, then the Navier-Stokes equations have a global solution $u \in C^0([0,+\infty); (L^2(Q_\varepsilon))^3)$, which is unique in the class of weak Leray solutions. In the purely periodic case, one can also choose $\gamma = 1/2$; furthermore, in this case, assuming that Mu_0 is more regular, we obtain global existence of a solution u in $C^0([0,+\infty); (H^{1/2}(Q_\varepsilon))^3)$, which is unique in the class of weak Leray solutions, if, for instance, u_0 and f satisfy

$$\begin{aligned}
\|A_\varepsilon^{1/2}(Mu_0)\|_{L^2(Q_\varepsilon)} + \sup_t \|MP_\varepsilon f(t)\|_{L^2(Q_\varepsilon)} &\le k_0 \varepsilon^{-\beta} , \\
\|Mu_{03}\|_{L^2(Q_\varepsilon)} + \sup_t \|A_\varepsilon^{-1/2}(MP_\varepsilon f_3(t))\|_{L^2(Q_\varepsilon)} &\le k_0 \varepsilon^{\beta} , \\
\sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2(Q_\varepsilon)} &\le k_0 \varepsilon^{-3/4-\beta/2} , \\
\|A_\varepsilon^{1/4}((I-M)u_0)\|_{L^2(Q_\varepsilon)} &\le k_0 \varepsilon^{1/4-\beta/2} ,
\end{aligned}$$

where $0 \le \beta \le 1/2$. These results are stated more precisely in the theorems 1.1, 1.2 and 1.3 below. To prove these theorems, we at first show sharp estimates of the nonlinear term appearing in the Navier-Stokes equations by working in the anisotropic Sobolev spaces $L^{q,q'}(Q_\varepsilon) = L^q(\Omega; L^{q'}(0,\varepsilon))$, for $q \neq q'$ and also by taking into account commutator properties. In the purely periodic case, like in [25], we use the conservation of enstrophy of the variable $M\widetilde{u}(t) = (Mu_1(t), Mu_2(t), 0)$. But, unlike [25], we work directly in the domain Q_ε, that is, we do not rescale the domain Q_ε to a domain of thickness 1.

We recall that the Navier-Stokes equations in the bounded domain Q_ε are given by

$$\begin{aligned}
&\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f , \\
&\mathrm{div} u = 0 , \\
&u(\cdot, 0) = u_0 ,
\end{aligned} \tag{1.1}$$

where ∇ is the gradient operator, Δ is the Laplace operator, f is a forcing term and $u(x,t) = (u_1, u_2, u_3)(x,t)$, $p(x,t)$ are the velocity vector and the pressure at point $x = (x_1, x_2, x_3)$ and time t respectively. We assume that the viscosity ν is a fixed positive number. Here these equations are mainly supplemented either with the periodic-Dirichlet boundary conditions (PD) or with purely periodic conditions (PP) on ∂Q_ε. In the (PP) case, we require in addition that the data u_0 and f have a vanishing total mean value, that is,

$$\int_{Q_\varepsilon} u_0 dx = \int_{Q_\varepsilon} f dx = 0 \ . \tag{1.2}$$

In order to describe our results more precisely and write the Navier-Stokes equations in an abstract form, we need to introduce some notation. For $m \in \mathbb{N}$, we denote by $H^m(Q_\varepsilon)$ the Hilbert space $\{g \in L^2(Q_\varepsilon); \sum_{0\le j\le m} \int_{Q_\varepsilon} |D^j g|^2 dx < +\infty\}$ equipped with the classical norm $\|\cdot\|_{H^m}$. For $m < s < m+1$, we denote by $H^s(Q_\varepsilon)$ the interpolated Hilbert space $[H^m(Q_\varepsilon), H^{m+1}(Q_\varepsilon)]_\theta$, where $\theta = s - m$ and we endow this space with the standard norm $\|\cdot\|_{H^s}$. As usual, $H^0(Q_\varepsilon)$ is denoted by $L^2(Q_\varepsilon)$ and $\|g\|_{L^2} = (\int_{Q_\varepsilon} g^2 dx)^{1/2}$. Likewise, for $m \ge 0$, we introduce the space $H^m_p(Q_\varepsilon)$, which is the closure in $H^m(Q_\varepsilon)$ of those smooth functions that are periodic in Q_ε, and, for $m < s < m+1$, we introduce the interpolated Hilbert space $H^s_p(Q_\varepsilon) = [H^m_p(Q_\varepsilon), H^{m+1}_p(Q_\varepsilon)]_\theta$, where $\theta = s - m$. We also define the spaces $\dot{H}^s_p(Q_\varepsilon) = \left\{g \in H^s_p(Q_\varepsilon); \int_{Q_\varepsilon} g(x)dx = 0\right\}$. The spaces $H^s_p(Q_\varepsilon)$ and $\dot{H}^s_p(Q_\varepsilon)$ can be described in terms of Fourier series; for k in the integer lattice $\mathbb{Z}^3$, we set $ka \equiv (k_1a_1, k_2a_2, k_3a_3)$, where $a_1 = l_1^{-1}$, $a_2 = l_2^{-1}$, $a_3 = \varepsilon^{-1}$, and we write

$$g(x) = \varepsilon^{-1/2}\sqrt{a_1a_2} \sum_{k\in\mathbb{Z}^3} g_k \exp(2i\pi ka\cdot x) \ , \tag{1.3}$$

where $g_k \in \mathbb{R}, \overline{g}_k = g_{-k}$ and $g_k = \varepsilon^{-1/2}\sqrt{a_1a_2}\int_{Q_\varepsilon} g(x)\exp(-2i\pi ka\cdot x)dx$. Then, $g \in H^s_p(Q_\varepsilon)$ is in the subspace $\dot{H}^s_p(Q_\varepsilon)$ if and only if $g_{(0,0,0)} = 0$. The usual norm $\|g\|_{H^s}$ and semi-norm $|g|_{H^s}$ on $H^s_p(Q_\varepsilon)$ can be expressed as follows

$$\|g\|^2_{H^s} = \sum_{k\in\mathbb{Z}^3}(1+|ka|^2)^s|g_k|^2 \ , \quad |g|^2_{H^s} = \sum_{k\in\mathbb{Z}^3}|ka|^{2s}|g_k|^2 \ , \tag{1.4}$$

and the semi-norm $|\cdot|_{H^s}$ is actually a norm on the subspace $\dot{H}^s_p(Q_\varepsilon)$. We now define the operator $\Delta_p = \Delta$, with domain $D(-\Delta_p) = \dot{H}^2_p(Q_\varepsilon)$. Clearly, for $0 \le s \le 2$, $D((-\Delta_p)^{s/2}) = \dot{H}^s_p(Q_\varepsilon)$ and the semi-norm $\|\cdot\|_s \equiv \|(-\Delta_p)^{s/2})\cdot\|_{L^2}$ is a norm on $\dot{H}^s_p(Q_\varepsilon)$, which is equivalent to the norm $|\cdot|_{H^s}$, with constants independent of ε.

In the (PD) boundary case, we introduce the space $H^1_d(Q_\varepsilon)$, which is the closure in $H^1(Q_\varepsilon)$ of those smooth functions that are periodic, of period ε in the vertical direction and have compact support in $\Omega \times [0,\varepsilon]$. We then define the operator $\Delta_d = \Delta$, with domain $D(-\Delta_d) = \{g \in H^1_d(Q_\varepsilon); \Delta g \in L^2(Q_\varepsilon)\}$. Clearly, $D((-\Delta_d)^{1/2}) = H^1_d(Q_\varepsilon)$. For $0 \le s \le 2$, we thus introduce the space $H^s_d(Q_\varepsilon) \equiv D((-\Delta_d)^{s/2})$ equipped with the graph norm $\|\cdot\|_s \equiv \|(-\Delta_d)^{s/2})\cdot\|_{L^2}$. We recall that, for $0 \le s < 1/2$, $H^s_d(Q_\varepsilon) = H^s(Q_\varepsilon)$ and that, for $1/2 < s \le 2$, $H^s_d(Q_\varepsilon) = \{g \in H^s(Q_\varepsilon); g = 0 \text{ on } \Gamma_l \ ; \ g \text{ periodic in the variable } x_3\}$ (see [11]). The case $s = 1/2$ is more delicate but here we do not need to characterize this space. For details on this question, we refer to [20].

Below, when there is no confusion, we denote by X^s the space $\dot{H}^s_p(Q_\varepsilon)$ or $H^s_d(Q_\varepsilon)$. Using Fourier series in the vertical direction or arguing as in [21], one shows that there exists a positive constant $c_0 \geq 1$, independent of ε, such that, for all $g \in X^2$,

$$c_0^{-1}\|g\|_2 \leq \Big(\sum_{j=0}^{j=2} \|D^j g\|^2_{L^2}\Big)^{1/2} \leq c_0\|g\|_2 \ , \tag{1.5}$$

which implies, by interpolation, that there exists a constant c_1 such that, for all $g \in X^s$ with $0 \leq s \leq 2$,

$$\|g\|_{H^s} \leq c_1\|g\|_s \ , \tag{1.6}$$

Since we are dealing here with vectors, we as well introduce the spaces $(H^s(Q_\varepsilon))^3$, $(H^s_d(Q_\varepsilon))^3$, $(H^s_p(Q_\varepsilon))^3$, etc..., equipped with the corresponding norms and semi-norms. For the abstract setting of the Navier-Stokes equations, we classically consider a Hilbert space H_ε, which is a subspace of $(L^2(Q_\varepsilon))^3$ and depends on the boundary conditions. In the (PP) case, $H_\varepsilon = H_p$ denotes the closure in $(L^2(Q_\varepsilon))^3$ of those smooth vectors u that are periodic in Q_ε and satisfy

$$\int_{Q_\varepsilon} u(x)dx = 0\,, \quad \mathrm{div} u = 0 \ . \tag{1.7}$$

In the (PD) case, $H_\varepsilon = H_d$ denotes the closure in $(L^2(Q_\varepsilon))^3$ of those smooth vectors u that are periodic in the vertical direction, have compact support in $\Omega \times [0,\varepsilon]$ and satisfy $\mathrm{div} u = 0$ in Q_ε. The classical subspaces

$$\begin{aligned} V_\varepsilon &= V_p \equiv H_p \cap (\dot{H}^1_p(Q_\varepsilon))^3 = \Big\{u \in (\dot{H}^1_p(Q_\varepsilon))^3; \mathrm{div} u = 0\Big\} \ , \\ V_\varepsilon &= V_d \equiv H_d \cap (H^1_d(Q_\varepsilon))^3 = \Big\{u \in (H^1_d(Q_\varepsilon))^3; \mathrm{div} u = 0\Big\} \ , \end{aligned}$$

are also useful. If $((\cdot,\cdot))$ denotes the inner product on V_ε, we introduce the Stokes operator A_ε as the isomorphism from V_ε onto the dual V'_ε of V_ε defined by

$$\langle A_\varepsilon u, v\rangle_{V'_\varepsilon, V_\varepsilon} = ((u,v))\,, \quad \forall v \in V_\varepsilon \ .$$

One can also extend A_ε as a linear unbounded operator on H_ε. The domain $D(A_\varepsilon) \equiv \{u \in V_\varepsilon; A_\varepsilon u \in H_\varepsilon\}$ is exactly the space $(H^2(Q_\varepsilon))^3 \cap V_\varepsilon$, in the (PP) and (PD) cases that we consider here. If P_ε denotes the orthogonal (Leray) projection of $(L^2(Q_\varepsilon))^3$ onto H_ε, the Stokes operator A_ε is given by

$$A_\varepsilon u = -P_\varepsilon \Delta u\,, \quad \forall u \in D(A_\varepsilon) \ .$$

Furthermore, in the cases (PP) and (PD), the Cattabriga-Solonnikov inequality holds uniformly in ε, that is, there exist positive constants $c_2 = c_2(\Omega) > 1$ and $c_3 = c_3(\Omega) > 1$, independent of ε, such that, for $0 < \varepsilon \leq 1$, for any $u \in D(A_\varepsilon)$,

$$c_3^{-1}\Big(\sum_{j=0}^{j=2} \|D^j u\|^2_{L^2}\Big)^{1/2} \leq c_2^{-1}\|\Delta u\|_{L^2} \leq \|A_\varepsilon u\|_{L^2} \leq c_2\|\Delta u\|_{L^2} \leq c_3\Big(\sum_{j=0}^{j=2} \|D^j u\|^2_{L^2}\Big)^{1/2} \ . \tag{1.8}$$

In the (PP) case, the property (1.8) directly follows from (1.5), since then $A_\varepsilon u = -\Delta u$, for all $u \in D(A_\varepsilon)$. In the (PD) case, the inequality (1.8) is proved, as in [21], by extending u by periodicity to the domain $Q_1 = \Omega \times [0,1]$ and applying the known Cattabriga-Solonnikov inequality in Q_1.

For $0 \leq s \leq 2$, we denote by V_ε^s the space $D(A_\varepsilon^{s/2})$, equipped with the natural norm $\|\cdot\|_{V_\varepsilon^s} \equiv |\cdot|_s \equiv \|A_\varepsilon^{s/2}\cdot\|_{L^2}$. Arguing as in [9] and using (1.8), one shows that, for $0 \leq s \leq 2$, $D(A_\varepsilon^{s/2}) = (X^s)^3 \cap H_\varepsilon$ and that there exists a constant $c_4 > 1$, independent of ε, such that, for $0 \leq s \leq 2$,

$$c_4^{-1}\|(-\Delta)^{s/2}u\|_{L^2} \;\leq\; \|A_\varepsilon^{s/2}u\|_{L^2} \;\leq\; c_4\|(-\Delta)^{s/2}u\|_{L^2}\,, \quad \forall u \in V_\varepsilon^s\ . \tag{1.9}$$

For $0 \leq s \leq 1$, we also consider the dual space $V_\varepsilon^{-s} \equiv D(A_\varepsilon^{-s/2})$ of $D(A_\varepsilon^{s/2})$, endowed with the dual norm $|u|_{-s} = \sup_{z\in V_\varepsilon^s, z\neq 0}(\langle u,z\rangle_{V_\varepsilon',V_\varepsilon}/|z|_s)$.

Finally, let B_ε be the bilinear form on V_ε defined, for $(u_1,u_2) \in V_\varepsilon \times V_\varepsilon$, by

$$\langle B_\varepsilon(u_1,u_2),u_3\rangle_{V_\varepsilon',V_\varepsilon} \;=\; \int_{Q_\varepsilon}(u_1\cdot\nabla)u_2\cdot u_3\,dx \quad \forall u_3 \in V_\varepsilon\ .$$

For the sake of simplicity, we assume, in the whole paper (except in Theorem 1.2), that the data u_0 and f satisfy the conditions

$$u_0 \in V_\varepsilon^s \quad \text{for some}\, s\ ,\ 0 \leq s \leq 1\ , \quad f \in L^\infty(0,+\infty;H_\varepsilon)\ . \tag{1.10}$$

In Theorem 1.2, we shall suppose that $f \in L^2(0,+\infty;H_\varepsilon)$. The Navier-Stokes equations, supplemented with the boundary conditions (PP) or (PD) can then be written as a differential equation in V_ε':

$$\begin{aligned}\partial_t u + \nu A_\varepsilon u + B_\varepsilon(u,u) \;&=\; P_\varepsilon f\ ,\\ u(\cdot,0) \;&=\; u_0\ .\end{aligned} \tag{1.11}$$

Here $\partial_t u$ denotes the derivative (in the sense of distributions) of u with respect to t.

We now recall three classical existence results of solutions to (1.11) (see [4], [5], [8], [17], [18], [19], [26], [29], ...), which are valid if f belongs to $L^\infty(0,+\infty;H_\varepsilon)$ or to $L^2(0,+\infty;H_\varepsilon)$:

•(P1) For $u_0 \in H_\varepsilon$, there exists a solution u of (1.11) (not necessarily unique), such that

$$u \in L^2_{loc}([0,+\infty);V_\varepsilon) \cap L^\infty(0,+\infty;H_\varepsilon) \cap W^{1,4/3}_{loc}([0,+\infty);V_\varepsilon') \tag{1.12}$$

and, for all $0 \leq t \leq +\infty$,

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u(s)\|_{L^2}^2 ds \;\leq\; \|u_0\|_{L^2}^2 + 2\int_0^t (f(s),u(s))ds\ . \tag{1.13}$$

A solution u of (1.11) satisfying (1.12) and (1.13) is called a weak Leray solution.

•(P2) For $u_0 \in V_\varepsilon$, there exist a time $T_\varepsilon = T_\varepsilon(Q_\varepsilon,\nu,u_0,P_\varepsilon f)$ and a unique solution u of (1.11), such that

$$u \in L^2_{loc}([0,T_\varepsilon;)V_\varepsilon^2) \cap C^0([0,T_\varepsilon);V_\varepsilon)\ . \tag{1.14}$$

Such a solution is usually called a strong solution of (1.11).

•(P3) For $u_0 \in V_\varepsilon^{1/2}$, there exist a time $T_\varepsilon^* = T_\varepsilon^*(Q_\varepsilon,\nu,u_0,P_\varepsilon f)$ and a unique solution u of (1.11), such that

$$u \in L^2_{loc}([0,T_\varepsilon^*);V_\varepsilon^{3/2}) \cap C^0([0,T_\varepsilon^*);V_\varepsilon^{1/2})\ . \tag{1.15}$$

Furthermore, using a classical small data argument like in [24], for instance, one shows that, if

$$\|A_\varepsilon^{1/4} u_0\|_{L^2} + \sup_s \|A_\varepsilon^{-1/4} P_\varepsilon f(s)\|_{L^2} \leq C\varepsilon^{1/2} , \tag{1.16}$$

where C is independent of ε, then the solution u of (1.11) is global in time, that is, $T_\varepsilon^* = +\infty$.

Here, we improve this global existence result as well as those of [15], [22], [23], [24], [25] and [27]. Before giving the precise statements, we need to define the mean value operator M in the vertical direction:

$$(Mf)(x_1, x_2) = \frac{1}{\varepsilon} \int_0^\varepsilon f(x_1, x_2, s) ds \quad \forall f \in L^2(Q_\varepsilon) . \tag{1.17}$$

We extend this operator $M \in \mathcal{L}(L^2(Q_\varepsilon); L^2(Q_\varepsilon))$ to an operator in $\mathcal{L}((L^2(Q_\varepsilon))^3; (L^2(Q_\varepsilon))^3)$ by setting $Mu = (Mu_1, Mu_2, Mu_3)$, for any vector $u \in (L^2(Q_\varepsilon))^3$. Clearly, M and $I - M$ are orthogonal projections in $L^2(Q_\varepsilon)$ and $(L^2(Q_\varepsilon))^3$ and commute with the derivations D_i, for $i = 1, 2, 3$. Moreover, $MH_\varepsilon \subset H_\varepsilon$. Using these properties and the fact that P_ε is an orthogonal projection onto H_ε, one shows that

$$MP_\varepsilon u = P_\varepsilon Mu , \quad \forall u \in (L^2(Q_\varepsilon)^3 , \tag{1.18}$$

which implies that

$$MA_\varepsilon u = A_\varepsilon Mu , \quad \forall u \in D(A_\varepsilon) . \tag{1.19}$$

One directly deduces from (1.19) that M also commutes with the operator A_ε^s, for $s \geq 0$.

The Navier-Stokes equations can now be rewritten as a system of equations for $v \equiv Mu$ and $w \equiv (I - M)u$

$$\begin{aligned} &\partial_t v + \nu A_\varepsilon v + MB_\varepsilon(v, v) + MB_\varepsilon(w, w) = MP_\varepsilon f , \\ &v(\cdot, 0) = Mu_0 , \end{aligned} \tag{1.20}$$

and

$$\begin{aligned} &\partial_t w + \nu A_\varepsilon w + (I - M)(B_\varepsilon(v, w) + B_\varepsilon(w, v) + B_\varepsilon(w, w)) = (I - M)P_\varepsilon f , \\ &w(\cdot, 0) = (I - M)u_0 . \end{aligned} \tag{1.21}$$

For the sake of simplicity, we suppose in the whole paper that $0 < \varepsilon \leq 1$. We could of course replace the upper bound 1 by any positive real number ε_0; in this case, the constants appearing in our results would also depend on ε_0.

In the case of the (PD) and (PP) boundary conditions, we show the following results.

Theorem 1.1. *For any nonnegative numbers α, β, γ, s, satisfying $0 \leq \beta < 1$, $0 \leq \gamma < 1/2$, $\sup(\beta, 2\gamma, 1/2, \alpha/(\alpha + 1 - \beta)) < s < 1$, there exists a positive constant $K_* = K_*(\alpha, \beta, \gamma, s)$ such that, for $0 < \varepsilon \leq 1$, if the initial data $(Mu_0, (I - M)u_0) \in H_\varepsilon \times V_\varepsilon^{1/2}$ and the forcing term $f \in L^\infty(0, \infty; (L^2(Q_\varepsilon))^3)$ satisfy*

$$\begin{aligned} &\|Mu_0\|_{L^2} \leq K_* \varepsilon^{-\alpha + 1/2} , \quad |(I - M)u_0|_{1/2} \leq K_* , \\ &\sup_t \|MP_\varepsilon f(t)\|_{L^2} \leq K_* \varepsilon^{-\beta + 1/2} , \quad \sup_t \|(I - M)P_\varepsilon f(t)\|_{L^2} \leq K_* \varepsilon^{-1/2} |\ln \varepsilon|^\gamma , \end{aligned} \tag{1.22}$$

and the additional condition

(1.23)

$$\Big(\varepsilon^3 \sup_t \|(I-M)P_\varepsilon f(t)\|^2_{L^2} \exp K_*^{-1}(\varepsilon^{-1/s} \sup_t \|MP_\varepsilon f(t)\|^{2/s}_{L^2} + \varepsilon^{1/s} \sup_t \|(I-M)P_\varepsilon f(t)\|^{2/s}_{L^2})$$
$$+ |(I-M)u_0|^2_{1/2}\Big) \times \exp K_*^{-1}(\varepsilon^{-1/2}\|Mu_0\|_{L^2})^{2/s} \leq K_* \ ,$$

then the equations (1.11) admit a global solution $u \in C^0([0,+\infty); H_\varepsilon) \cap L^2_{loc}([0,+\infty); V_\varepsilon) \cap H^1_{loc}([0,+\infty); V'_\varepsilon)$*, which is unique in the class of weak Leray solutions. Moreover,* $(I-M)u \in L^\infty(0,\infty; V_\varepsilon^{1/2}) \cap L^2_{loc}([0,\infty); V_\varepsilon^{3/2})$ *and the estimates (4.24) and (4.25) hold, for* $t \geq 0$*.*

Remarks 1.1. **i)** In the particular case $\alpha = \beta = 0$, the condition (1.23) always holds, provided the constant K_* is small enough. If $\gamma = 0$, (1.23) can be written as

$$\Big(|(I-M)u_0|^2_{1/2} + \varepsilon^3 \sup_t \|(I-M)P_\varepsilon f(t)\|^2_{L^2} \exp K_*^{-1}(\varepsilon^{-1/2} \sup_t \|MP_\varepsilon f(t)\|_{L^2})^{2/s}\Big)$$
$$\times \exp K_*^{-1}(\varepsilon^{-1/2}\|Mu_0\|_{L^2})^{2/s} \leq K_* \ .$$

ii) In the case of periodic boundary conditions, we can set $s = 1$, $\beta = 1$, $\gamma = 1/2$ in the hypotheses (1.22) and (1.23). Moreover, the limitation on $\|Mu_0\|_{L^2}$ disappears, provided that the following condition holds:

(1.24)

$$\Big(\varepsilon^3 \sup_t \|(I-M)P_\varepsilon f(t)\|^2_{L^2} \exp K_*^{-1}(\varepsilon^{-1} \sup_t \|MP_\varepsilon f(t)\|^2_{L^2} + \varepsilon \sup_t \|(I-M)P_\varepsilon f(t)\|^2_{L^2})$$
$$+ |(I-M)u_0|^2_{1/2}\Big) \times \exp K_*^{-1}(\varepsilon^{-1/2}\|Mu_0\|_{L^2})^2 \leq K_* \ ,$$

This improvement will be explained in Remark 4.1.
iii) Applying the Poincaré inequality (2.1) below to the term $(I-M)u_0$, we easily see that the above theorem still holds if, in the conditions (1.22), (1.23), $|(I-M)u_0|_{1/2}$ is replaced by $\varepsilon^{1/2}K_0|(I-M)u_0|_1$.

The above theorem has already been proved in [15], in the frame of anisotropic spaces and Littlewood-Paley theory, in the particular case of periodic boundary conditions and vanishing forcing term f.

Remark 1.2. We also improve the results of [1] in the case of homogeneous Dirichlet boundary conditions, by requiring less regularity on the initial data u_0. In this case, we introduce the Laplace operator $\Delta_{dd} = \Delta$, with domain $D(-\Delta_{dd}) = \{g \in H^1_0(Q_\varepsilon); \Delta g \in L^2(Q_\varepsilon)\}$, where $H^1_0(Q_\varepsilon)$ is the closure in $H^1(Q_\varepsilon)$ of those smooth functions that have compact support in Q_ε. For $0 \leq s \leq 2$, we define the space $X^s \equiv D((-\Delta_{dd})^{s/2})$ equipped with the norm $\|\cdot\|_s \equiv \|(-\Delta_{dd})^{s/2}\cdot\|_{L^2}$. If $H_\varepsilon = \{u \in (L^2(Q_\varepsilon))^3;\ \mathrm{div} u = 0;\ u\cdot\nu_\varepsilon = 0 \,\mathrm{on}\, \partial Q_\varepsilon\}$, $V_\varepsilon = \{u \in H_\varepsilon;\ u = 0 \,\mathrm{on}\, \partial Q_\varepsilon\}$, where ν_ε is the outer normal to the boundary ∂Q_ε, we define the corresponding Stokes operator A_ε with domain $D(A_\varepsilon) \equiv \{u \in V_\varepsilon; A_\varepsilon \in H_\varepsilon\}$. From [7], it follows that $D(A_\varepsilon) = (H^2(Q_\varepsilon))^3 \cap V_\varepsilon$. Arguing as in [27], one shows that the Cattabriga-Solonnikov inequality (1.8) holds uniformly in ε and that the inequalities (1.9) are still true. We remark that the first eigenvalue of $-\Delta_{dd}$ (respectively A_ε) is of order ε^{-2}, which implies that the Poincaré inequalities (2.1) and (2.2) below hold, with $(I-M)$ replaced by I. Hence, the decomposition $u = Mu + (I-M)u$ is of no use. Replacing simply w by u and v by 0 in the proof of Theorem 1.1, one shows that there exists a positive constant K such that, for $0 < \varepsilon \leq 1$, if $u_0 \in V_\varepsilon^{1/2}$ and $f \in L^\infty(0,\infty; L^2(Q_\varepsilon))$ satisfy

$$|u_0|_{1/2} \leq K \ , \quad \sup_t \|P_\varepsilon f(t)\|_{L^2} \leq K\varepsilon^{-3/2} \ , \tag{1.25}$$

then the equations (1.11) admit a global solution $u \in C^0([0,+\infty); H_\varepsilon) \cap L^\infty(0,\infty; V_\varepsilon^{1/2}) \cap L^2_{loc}([0,\infty); V_\varepsilon^{3/2})$,which is unique in the class of weak Leray solutions. Moreover, there exists a positive constant C independent of ε, such that, for $t \geq 0$,

$$|u(t)|_{1/2}^2 \leq \exp(-C\varepsilon^{-2}t)|u_0|_{1/2}^2 + C\varepsilon^3 \sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2}^2 \ . \tag{1.26}$$

Remark 1.3. As in [27], if $\Omega = (0,l_1) \times (0,l_2)$, we can consider the Navier-Stokes equations (1.1), supplemented with the (DP) boundary conditions, that is, homogeneous Dirichlet boundary conditions on $\Gamma_v = (\Omega \times \{x_3 = 0\}) \cup (\Omega \times \{x_3 = \varepsilon\})$ and periodic conditions in the variables x_1, x_2. As before, one defines the corresponding spaces X^s, H_ε, V_ε and the corresponding Stokes operator A_ε. The inequalities (1.8) and (1.9) still hold. Thus, like in Remark 1.2, ones proves that there exists a positive constant K such that, for $0 < \varepsilon \leq 1$, if $u_0 \in V_\varepsilon^{1/2}$ and $f \in L^\infty(0,\infty; L^2(Q_\varepsilon))$ satisfy the conditions (1.25), then the equations (1.11) admit a global solution $u \in C^0([0,+\infty); H_\varepsilon) \cap L^\infty(0,\infty; V_\varepsilon^{1/2}) \cap L^2_{loc}([0,\infty); V_\varepsilon^{3/2})$, which is unique in the class of weak Leray solutions, and the estimate (1.26) holds.

We now assume that the forcing term Pf belongs to $L^2(0,\infty; (L^2(Q_\varepsilon))^3)$, which is a rather strong requirement. But, in this case, we can remove every smallness assumption on the data Mu_0 and $MPf(t)$, provided the data w_0 and $(I-MP)f(t)$ are small enough.

Theorem 1.2. *For any positive number s, $1/2 < s < 1$, there exists a positive constant $\widetilde{K} = \widetilde{K}(s)$ such that, for $0 < \varepsilon \leq 1$, if the initial data $(Mu_0, (I-M)u_0) \in H_\varepsilon \times V_\varepsilon^{1/2}$ and the forcing term $f \in L^2(0,\infty; (L^2(Q_\varepsilon))^3)$ satisfy*

$$\left(|(I-M)u_0|_{1/2}^2 + \varepsilon \int_0^{+\infty} \|(I-M)P_\varepsilon f(\tau)\|_{L^2}^2 d\tau\right) \exp \widetilde{K}^{-1}\left(\varepsilon^{-1}\|Mu_0\|_{L^2}^2 + \varepsilon^{-1}\int_0^{+\infty} \|MP_\varepsilon f(\tau)\|_{L^2}^2 d\tau\right)^{1/s} \leq \widetilde{K} \ , \tag{1.27}$$

then there exists a global solution $u \in C^0([0,+\infty); H_\varepsilon) \cap L^2_{loc}([0,+\infty); V_\varepsilon) \cap H^1_{loc}([0,+\infty); V_\varepsilon')$ of (1.11) which is unique in the class of weak Leray solutions. Moreover, $(I-M)u \in L^\infty(0,\infty; V_\varepsilon^{1/2}) \cap L^2_{loc}([0,\infty); V_\varepsilon^{3/2})$.

If, for instance, in Theorem 1.1, we want to choose Mu_0 of order ε^θ, for $\theta < 1/2$, we need to assume that $(I-M)u_0$ and $(I-M)P_\varepsilon f$ are exponentially small functions of ε. However, in the case of the (PP) boundary conditions, these drastic restrictions become much milder. In the theorem below, we split the vector field $v \equiv Mu$ into two parts

$$Mu = M\widetilde{u} + M(u_3) \equiv (Mu_1, Mu_2, 0) + (0,0,Mu_3) \ , \tag{1.28}$$

and set $\widetilde{v} = M\widetilde{u}$. In the proof, we use the conservation of enstrophy for the vector field $\widetilde{v}$.

Theorem 1.3. *There exist positive constants k_1, k_2, k_3, k_4 and k_5 such that, for $0 < \varepsilon \leq 1$, if the initial data $(Mu_0, (I-M)u_0) \in V_p \times V_p^{1/2}$ and the forcing term $f \in L^\infty(0,\infty;(L^2(Q_\varepsilon))^3)$ satisfy*

$$\begin{aligned} &|Mu_0|_1 \leq k_1\varepsilon^{-1/2}, \quad |(I-M)u_0|_{1/2} \leq k_2 \\ &\sup_t \|MP_\varepsilon f(t)\|_{L^2} \leq k_3\varepsilon^{-1/2}, \quad \sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2} \leq k_4\varepsilon^{-1} , \end{aligned} \tag{1.29}$$

and the additional condition

(1.30)
$$\begin{aligned} \mathcal{A}_0 \equiv \Big(|M\widetilde{u}_0|_1 + \sup_t \|\widetilde{MP_\varepsilon f}(t)\|_{L^2} + \varepsilon^{-1/2}|(I-M)u_0|^2_{1/2} + \varepsilon^{3/2}\sup_t \|(I-M)P_\varepsilon f(t)\|^2_{L^2}\Big) \\ \times \Big(\|M(u_{03})\|_{L^2} + \sup_t |M(P_\varepsilon f)_3(t)|_{-1}\Big) \leq k_5 , \end{aligned}$$

then the equations (1.11) admit a global solution $u(t) \in C^0([0,\infty);V_p^{1/2}) \cap L^\infty(0,\infty;V_p^{1/2}) \cap L^2_{loc}([0,\infty);V_p^{3/2})$, which is unique in the class of weak Leray solutions. Moreover, Mu belongs to the space $C^0([0,\infty);V_p) \cap L^\infty(0,\infty;V_p) \cap L^2_{loc}([0,\infty);V_p^2)$ and the estimates (5.33) and (5.43) hold, for every $t \geq 0$.

Remark 1.4. Similar existence results hold, if one considers the Navier-Stokes equations (1.1), supplemented with the (FP) boundary conditions, that is, with the free boundary condition

$$u_3(x_1,x_2,x_3) = 0 , \quad \partial_{x_3} u_j(x_1,x_2,x_3) = 0 , \quad j=1,2 , \quad x_3 = 0, \varepsilon , \tag{1.31}$$

and periodic conditions in the variables x_1, x_2. As before, one defines the corresponding spaces H_ε, V_ε and the Stokes operator A_ε. In the proofs of Section 2, one also needs to define the spaces X^s, which are now different for u_j, $j=1,2$ and u_3. Since $u_3(x_1,x_2,0) = u_3(x_1,x_2,\varepsilon) = 0$, one introduces the following mean value operator M_{FP} on H_ε,

$$M_{FP}u = (Mu_1, Mu_2, 0) , \quad \forall u \in H_\varepsilon .$$

Then, one easily checks that the theorems 1.1, 1.2 and 1.3 are still true, if the operator M is replaced by the corresponding operator M_{FP}. Remark that, since $M_{FP}(0,0,u_{03}) = M_{FP}(0,0,(P_\varepsilon f)_3) = 0$, the additional condition (1.30) disappears. The proof of Theorem 1.3 in the (FP) case is actually much simpler than in the periodic case, because the term v_3 is zero. Also in the (FP) case, Theorem 1.3 improves the corresponding result of [27].

In Theorem 5.1 of Section 5 , we shall give another global existence and uniqueness result, involving the L^p-norm of Mu_{03}. As a direct consequence of Theorem 1.3 , we obtain the following simple corollary:

Corollary 1.1. *There exists a positive constant k_0, such that, for $0 < \varepsilon \leq 1$, for $0 \leq \beta \leq 1/2$, if the initial data $(Mu_0, (I-M)u_0) \in V_p \times V_p^{1/2}$ and the forcing term $f \in L^\infty(0,\infty;(L^2(Q_\varepsilon))^3)$ satisfy*

(1.32)
$$\begin{aligned} &|M\widetilde{u}_0|_1 + \sup_t \|\widetilde{MP_\varepsilon f}(t)\|_{L^2} \leq k_0\varepsilon^{-\beta} , \quad \sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2} \leq k_0\varepsilon^{-3/4-\beta/2} , \\ &|Mu_{03}|_1 + \sup_t \|MP_\varepsilon f_3(t)\|_{L^2} \leq k_0\varepsilon^{-1/2} , \quad \|Mu_{03}\|_{L^2} + \sup_t |MP_\varepsilon f_3(t)|_{-1} \leq k_0\varepsilon^{\beta} , \end{aligned}$$

and

$$|(I-M)u_0|_{1/2} \le k_0 \varepsilon^{1/4-\beta/2} , \tag{1.33}$$

then the equations (1.11) admit a global solution $u(t) \in C^0([0,\infty); V_p^{1/2}) \cap L^\infty(0,\infty; V_p^{1/2}) \cap L^2_{loc}([0,\infty); V_p^{3/2})$*, which is unique in the class of weak Leray solutions. Moreover,* Mu *belongs to the space* $C^0([0,\infty); V_p) \cap L^\infty(0,\infty; V_p) \cap L^2_{loc}([0,\infty); V_p^2)$ *and the estimates (5.33) and (5.43) hold, for every* $t \ge 0$.

Applying the Poincaré inequality (2.2) to $(I-M)u_0$, we at once get the following global existence result:

Corollary 1.2. *There exists a positive constant* k_0*, such that, for* $0 < \varepsilon \le 1$*, for* $0 \le \beta \le 1/2$*, if the initial data* $u_0 \in V_p$ *and the forcing term* $f \in L^\infty(0,\infty; (L^2(Q_\varepsilon))^3)$ *satisfy the conditions (1.32) and*

$$|(I-M)u_0|_1 \le k_0 \varepsilon^{-1/4-\beta/2} , \tag{1.34}$$

then the equations (1.11) have a unique global strong solution $u(t) \in C^0([0,\infty); V_p) \cap L^2_{loc}([0,\infty); V_p^2)$.

Remark 1.5. In [25], it has been proved, in the (PP) case, that there exists $\varepsilon_1 > 0$ such that, for $0 < \varepsilon \le \varepsilon_1$, the equations (1.11) admit a unique global strong solution $u \in C^0([0,\infty); V_\varepsilon)$, if the data satisfy the following conditions, where δ is a small positive constant,

$$\begin{aligned} &|Mu_0|_1 + \sup_t \|MP_\varepsilon f(t)\|_{L^2} \le C\varepsilon^{\delta+7/24} , \\ &|(I-M)u_0|_1 \le C\varepsilon^{\delta-5/48}, \quad \sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2} \le C\varepsilon^{\delta-1/2} , \end{aligned} \tag{1.35}$$

or

$$\begin{aligned} &|Mu_0|_1 \le C\varepsilon^{\delta-1/32}, \quad \sup_t \|MP_\varepsilon f(t)\|_{L^2} \le C\varepsilon^{\delta-1/16} , \\ &|(I-M)u_0|_1 \le C\varepsilon^{\delta-1/8}, \quad \sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2} \le C\varepsilon^{\delta-1/2} , \\ &\|Mu_{03}\|_{L^2} + \sup_t \|MP_\varepsilon f_3(t)\|_{L^2} \le k_0\varepsilon . \end{aligned} \tag{1.36}$$

In [22], Moise, Temam and Ziane have shown that, in the (PP) case, there exists $\varepsilon_1 > 0$ such that, for $0 < \varepsilon \le \varepsilon_1$, the equations (1.11) admit a unique global strong solution $u \in C^0([0,\infty); V_\varepsilon)$, if the data satisfy the following conditions, where δ is a small positive constant,

$$\begin{aligned} &|Mu_0|_1 + \sup_t \|MP_\varepsilon f(t)\|_{L^2} \le C\varepsilon^{\delta+1/6} , \\ &|(I-M)u_0|_1 + \sup_t \|(I-M)P_\varepsilon f(t)\|_{L^2} \le C\varepsilon^{\delta-1/6} . \end{aligned} \tag{1.37}$$

Choosing $\beta = 0$ in Corollary 1.2, one at once sees that the conditions (1.32) and (1.34) allow larger data than the hypotheses (1.35), (1.36) or (1.37). Finally, Corollary 1.2 improves as well the results of [23], where global existence and uniqueness are proved under the assumption $|u_0|_1 + \sup_t \|P_\varepsilon f(t)\|_{L^2} \le C$, for some constant C.

An outline of the paper is as follows. In order to estimate the quadratic term in (1.11), we prove some auxiliary inequalities in Section 2. Section 3 is devoted to a uniqueness result. In Section 4, we give the proofs of the theorems 1.1 and 1.2. Section 5 contains the proofs of the theorems 1.3 and 5.1.

In the sequel, we shall write P for the projection P_ε. The constants $K, K_1, \dots$ and $C, C_1, \dots$ will always denote positive constants, that are independent of ε. We recall that we denote the spaces $\dot{H}^s_p$ or H^s_d by X^s, when no distinction concerning the boundary conditions is necessary.

2. Auxiliary estimates

In (1.17) we have introduced the mean value operator $M \in \mathcal{L}(L^2(Q_\varepsilon); L^2(Q_\varepsilon))$ and extended it to an operator $M \in \mathcal{L}((L^2(Q_\varepsilon))^3; (L^2(Q_\varepsilon))^3)$, by setting $Mu = (Mu_1, Mu_2, Mu_3)$. This operator M allows to decompose every function $f \in L^2(Q_\varepsilon)$ into $f = Mf + (I-M)f$, where Mf is a function of x_1 and x_2 only and $(I-M)f$ satisfies the following Poincaré inequality

(2.1)
$$\|(I-M)f\|_{L^2} \le \widetilde{K}_0 \varepsilon^s \|(I-M)f\|_{H^s} \le K_0 \varepsilon^s \|(I-M)f\|_s , \quad \forall f \in X^s, \quad 0 \le s \le 2 ,$$

where $\widetilde{K}_0$, K_0 are independent of s, f and ε (see [14], [12], for instance). We notice that the constant K_0 in the inequality (2.1) can be chosen so that

$$\|(I-M)u\|_{L^2} \le K_0 \varepsilon^s |(I-M)u|_s , \quad \forall u \in V^s_\varepsilon, \quad 0 \le s \le 2 . \tag{2.2}$$

These inequalities will be often used below.

We shall also need the following classical Poincaré inequalities, for $0 \le s \le 2$,

$$\|u\|_{L^2} \le \mu_0^s \|u\|_s , \quad \forall u \in X^s , \tag{2.3}$$

and

$$\|u\|_{L^2} \le \mu_0^s |u|_s , \quad \forall u \in V^s_\varepsilon , \tag{2.4}$$

where μ_0 is a positive constant depending only on Ω.

We denote by $L^{q,q'}(Q_\varepsilon) = L^q(\Omega; L^{q'}(0,\varepsilon))$ or simply $L^{q,q'}$ the space of (classes of) functions u such that $\|u\|_{L^{q,q'}} = \| \|u\|_{L^{q'}_{x_3}(0,\varepsilon)} \|_{L^q_{x'}(\Omega)}$ is finite, where $x' = (x_1, x_2)$. Of course, $L^{q,q}$ is the usual space $L^q(Q_\varepsilon)$ and the norm $\|u\|_{L^{q,q}}$ is denoted by $\|u\|_{L^q}$.

The following property of a divergence-free vector field will also be frequently used:

$$\|\nabla u\|^2_{L^2(Q_\varepsilon)} = \sum_i \|\nabla u_i\|^2_{L^2(Q_\varepsilon)} = -\int_{Q_\varepsilon} u \cdot \Delta u = \int_{Q_\varepsilon} u \cdot A_\varepsilon u = \|A_\varepsilon^{1/2} u\|^2_{L^2(Q_\varepsilon)} . \tag{2.5}$$

Lemma 2.1. *There exists a positive constant K_1 so that, for $0 < \varepsilon \le 1$, if $w_i \in X^{s_i}$ are three functions satisfying $Mw_i = 0$, $0 \le s_i < 3/2$, for $i = 1,2,3$, and $s_1 + s_2 + s_3 = 3/2$, then*

$$\left| \int_{Q_\varepsilon} w_1(x) w_2(x) w_3(x)\, dx \right| \le K_1 \|w_1\|_{s_1} \|w_2\|_{s_2} \|w_3\|_{s_3}. \tag{2.6}$$

Furthermore, there exists a positive constant K_2 such that, for $0 < \varepsilon \leq 1$, if $v_1 \in X^{\widetilde{s}_1}$, $0 \leq \widetilde{s}_1 < 1$ is a function independent of x_3, $0 \leq s_i < 1$, for $i = 2, 3$ and $\widetilde{s}_1 + s_2 + s_3 = 1$, then

$$\left| \int_{Q_\varepsilon} v_1(x) w_2(x) w_3(x)\, dx \right| \leq K_2 \varepsilon^{-1/2} \|v_1\|_{\widetilde{s}_1} \|w_2\|_{s_2} \|w_3\|_{s_3} . \tag{2.7}$$

Remark 2.1. It will be clear from the proof below that, if we omit the dependence on ε, Lemma 2.1 still holds for functions without vanishing mean in the thin direction.

Lemma 2.1 is a consequence of the following result.

Lemma 2.2. *Assume that $0 \leq s < 3/2$ and $q, q' \in [2, \infty)$ are such that $\frac{2}{q} + \frac{1}{q'} = \frac{3}{2} - s$. Then the following embedding holds:*

$$X^s \hookrightarrow L^{q,q'}(Q_\varepsilon) .$$

Moreover, there exists a positive constant K_3 such that , for $0 < \varepsilon \leq 1$, for any $w \in X^s$ satisfying $Mw = 0$,

$$\|w\|_{L^{q,q'}} \leq K_3 \|w\|_s . \tag{2.8}$$

Proof of Lemma 2.1. Let us assume that Lemma 2.2 is proved. The particular case $q = q'$ implies the embedding $X^s \hookrightarrow L^q(Q_\varepsilon)$ provided that $1/q = 1/2 - s/3$. Therefore, there exist three positive constants C_1, C_2 and C_3 independent of ε such that

$$\|w_i\|_{L^{q_i}} \leq C_i \|w_i\|_{s_i} , \quad \forall i \in \{1, 2, 3\},$$

where $1/q_i = 1/2 - s_i/3$. Since $1/q_1 + 1/q_2 + 1/q_3 = 1$, Hölder's inequality gives

$$\left| \int_{Q_\varepsilon} w_1(x) w_2(x) w_3(x)\, dx \right| \leq \|w_1\|_{L^{q_1}} \|w_2\|_{L^{q_2}} \|w_3\|_{L^{q_3}} \leq C_1 C_2 C_3 \|w_1\|_{s_1} \|w_2\|_{s_2} \|w_3\|_{s_3} ,$$

which implies (2.6) with $K_1 = C_1 C_2 C_3$.

Now we prove the inequality (2.7). As in the introduction, we define the usual Hilbert spaces $H^s(\Omega)$ by interpolation, when $s \geq 0$ is not an integer. Remarking that, for all $v \in H^j(\Omega)$, $j \in \mathbb{N}$, $\|v\|_{H^j(\Omega)} = \varepsilon^{-1/2} \|v\|_{H^j(Q_\varepsilon)}$, we deduce, by interpolation, that, for $s \geq 0$,

$$\|v\|_{H^s(\Omega)} \leq \varepsilon^{-1/2} \|v\|_{H^s(Q_\varepsilon)} , \quad \forall v \in H^s(\Omega) . \tag{2.9}$$

Due to the two-dimensional Sobolev embedding $H^{\widetilde{s}_1}(\Omega) \hookrightarrow L^{\widetilde{q}_1}(\Omega)$, where $1/\widetilde{q}_1 = (1 - \widetilde{s}_1)/2$, and to the estimates (1.6) and (2.9), we obtain

$$\|v_1\|_{L^{\widetilde{q}_1}(\Omega)} \leq \widetilde{C} \|v_1\|_{H^{\widetilde{s}_1}(\Omega)} \leq \widetilde{C} \varepsilon^{-1/2} \|v_1\|_{H^{\widetilde{s}_1}(Q_\varepsilon)} \leq \widetilde{C}_1 \varepsilon^{-1/2} \|v_1\|_{\widetilde{s}_1} ,$$

where $\widetilde{C}_1 = c_1 \widetilde{C}$ is a positive constant independent of ε. On the other hand, one can apply Lemma 2.2 with $q' = 2$ to get the existence of two constants $\widetilde{C}_2$ and $\widetilde{C}_3$ independent of ε such that, for $i = 2, 3$,

$$\|w_i\|_{L^{\widetilde{q}_i, 2}} \leq \widetilde{C}_i \|w_i\|_{s_i} ,$$

where $1/\widetilde{q}_i = (1-s_i)/2$ for $i=2,3$. Hölder's inequality adapted to the case of anisotropic spaces and the equality $1/\widetilde{q}_1 + 1/\widetilde{q}_2 + 1/\widetilde{q}_3 = 1$ yield

$$\begin{aligned}\left|\int_{Q_\varepsilon} v_1(x)w_2(x)w_3(x)\,dx\right| &\le \|v_1\|_{L^{\tilde{q}_1}(\Omega)}\|w_2\|_{L^{\tilde{q}_2,2}}\|w_3\|_{L^{\tilde{q}_3,2}}\\ &\le \widetilde{C}_1\widetilde{C}_2\widetilde{C}_3\varepsilon^{-1/2}\|v_1\|_{\tilde{s}_1}\|w_2\|_{s_2}\|w_3\|_{s_3},\end{aligned}$$

whence the inequality (2.7) with $K_2 = \widetilde{C}_1\widetilde{C}_2\widetilde{C}_3$. The proof is completed. □

Proof of Lemma 2.2. Let $d \ge 0$. Like in the introduction, we can define the operator $d-\Delta_2 = d - \partial^2_{x_1x_1} - \partial^2_{x_2x_2}$ on Ω, supplemented either with homogeneous Dirichlet boundary conditions in the (PD) case or with periodic boundary conditions in the (PP) case. Let $(\lambda_k,\varphi_k)_{k\ge 0}$ be a sequence of eigenvalues and eigenfunctions of $-\Delta_2$, such that $(\varphi_k)_{k\ge0}$ forms an orthonormal basis in $L^2(\Omega)$ and that $0 \le \lambda_0 < \lambda_1 \le \cdots$. For $0\le\sigma\le 2$, the operator $(d-\Delta_2)^{\sigma/2}$ writes, for any $v \in D((d-\Delta_2)^{\sigma/2})$,

$$(d-\Delta_2)^{\sigma/2}v(x') \;=\; \sum_{k\ge0}(d+\lambda_k)^{\sigma/2}v_k\varphi_k(x')\;,$$

where $v = \sum_{k\ge0} v_k\varphi_k(x')$. We notice that $(\varepsilon^{-1/2}e^{2i\pi n x_3/\varepsilon}\varphi_k(x'))_{n\in\mathbb{Z},k\ge0}$ is an orthonormal basis in $L^2(Q_\varepsilon)$ and the operator $(d-\Delta)^{\sigma/2}$ on Q_ε (where $\Delta=\Delta_d$ or $\Delta=\Delta_p$ according to the boundary conditions) writes, for any $u\in X^\sigma$,

$$(d-\Delta)^{\sigma/2}u(x) \;=\; \varepsilon^{-1/2}\sum_{n\in\mathbb{Z},k\ge0}(d+\lambda_k+(\frac{2\pi n}{\varepsilon})^2)^{\sigma/2}u_{nk}e^{2\pi i n x_3/\varepsilon}\varphi_k(x')\;,$$

where $u = \varepsilon^{-1/2}\sum_{n\in\mathbb{Z},k\ge0}u_{nk}e^{2\pi i n x_3/\varepsilon}\varphi_k(x')$.

Again, like in the introduction, for $0\le\sigma\le2$, we define the Hilbert spaces $H^\sigma_p(0,\varepsilon)$ and $H^\sigma_p(0,1)$ of periodic functions on $(0,\varepsilon)$ and $(0,1)$. Performing a change of variables from $(0,\varepsilon)$ to $(0,1)$ and using the Sobolev embedding in dimension 1, $H^{\sigma'}_p(0,1)\hookrightarrow L^{q'}(0,1)$, where $\sigma' = 1/2-1/q'$, we obtain, for any $g(x_3)\equiv\varepsilon^{-1/2}\sum_{n\in\mathbb{Z}}g_ne^{2\pi i n x_3/\varepsilon}$ in $H^{\sigma'}_p(0,\varepsilon)$,

(2.10)
$$\begin{aligned}\|g\|_{L^{q'}(0,\varepsilon)} &= \varepsilon^{1/q'}\|\varepsilon^{-1/2}\sum_{n\in\mathbb{Z}}g_ne^{2\pi i n y}\|_{L^{q'}(0,1)}\\ &\le C_1\varepsilon^{1/q'}(\|\varepsilon^{-1/2}\sum_{n\in\mathbb{Z}}g_ne^{2\pi i n y}\|_{L^2(0,1)} + \|(-\partial^2_{yy})^{\sigma'/2}\varepsilon^{-1/2}\sum_{n\in\mathbb{Z}}g_ne^{2\pi i n y}\|_{L^2(0,1)})\\ &\le C_2\varepsilon^{1/q'-1/2}(\|g\|_{L^2(0,\varepsilon)} + \varepsilon^{\sigma'}\|(-\partial^2_{x_3x_3})^{\sigma'/2}g\|_{L^2(0,\varepsilon)})\;.\end{aligned}$$

But we have

$$\|g\|_{L^2(0,\varepsilon)} + \varepsilon^{\sigma'}\|(-\partial^2_{x_3x_3})^{\sigma'/2}g\|_{L^2(0,\varepsilon)} \;\le\; C_3\|(1-\partial^2_{x_3x_3})^{\sigma'/2}g\|_{L^2(0,\varepsilon)}\;. \tag{2.11}$$

If $g\in H^{\sigma'}_p(0,\varepsilon)$ satisfies $Mg=0$, then, due to the Poincaré inequality (2.1), we improve the above inequality and obtain

$$\begin{aligned}\|g\|_{L^2(0,\varepsilon)} + \varepsilon^{\sigma'}\|(-\partial^2_{x_3x_3})^{\sigma'/2}g\|_{L^2(0,\varepsilon)} &\le (K_0+1)\varepsilon^{\sigma'}\|(-\partial^2_{x_3x_3})^{\sigma'/2}g\|_{L^2(0,\varepsilon)}\\ &\le C_4\varepsilon^{\sigma'}\|(1-\partial^2_{x_3x_3})^{\sigma'/2}g\|_{L^2(0,\varepsilon)}\;.\end{aligned} \tag{2.12}$$

The estimates (2.10), (2.11) and (2.12) imply that

$$\|g\|_{L^{q'}(0,\varepsilon)} \le \varepsilon^{1/q'-1/2} C_2 C_0(\varepsilon) \|(1-\partial^2_{x_3x_3})^{\sigma'/2} g\|_{L^2(0,\varepsilon)} \,, \tag{2.13}$$

where $C_0(\varepsilon) = C_3$ in the general case and $C_0(\varepsilon) = C_4\varepsilon^{\sigma'}$, when $Mg = 0$. Let now u be a function in X^s and $q, q' \in [2, \infty)$. If $\sigma' = 1/2 - 1/q'$, we deduce from (2.13) that

$$\begin{aligned}\|u\|_{L^{q,q'}} &= \|\|u\|_{L^{q'}_{x_3}}\|_{L^q_{x'}} \le \varepsilon^{1/q'-1/2} C_2 C_0(\varepsilon) \| \, \|(1-\partial^2_{x_3x_3})^{\sigma'/2} u\|_{L^2_{x_3}}\|_{L^q_{x'}} \\ &\le \varepsilon^{1/q'-1/2} C_2 C_0(\varepsilon) \| \, \|(1-\partial^2_{x_3x_3})^{\sigma'/2} u\|_{L^q_{x'}}\|_{L^2_{x_3}} \,,\end{aligned}$$

where we could interchange the order of integrations, since $q \ge 2$. Now, the 2-dimensional Sobolev embedding $D((1-\Delta_2)^{\sigma/2})) \hookrightarrow L^q(\Omega)$ with $1/q = (1-\sigma)/2$ implies that

$$\|u\|_{L^{q,q'}} \le \varepsilon^{1/q'-1/2} C_2 C_0(\varepsilon) C_5 \|(1-\Delta_2)^{\sigma/2}(1-\partial^2_{x_3x_3})^{\sigma'/2} u\|_{L^2} \,. \tag{2.14}$$

But, as $\sigma + \sigma' = s$,

$$\begin{aligned}\|(1-\Delta_2)^{\frac{\sigma}{2}}(1-\partial^2_{x_3x_3})^{\frac{\sigma'}{2}} u\|^2_{L^2} &= \varepsilon^{-1}\| \sum_{n\in\mathbb{Z},k\ge 0} (1+\lambda_k)^{\frac{\sigma}{2}}(1+(\frac{2\pi n}{\varepsilon})^2)^{\frac{\sigma'}{2}} u_{nk} e^{2\pi i n \frac{x_3}{\varepsilon}} \varphi_k(x')\|^2_{L^2} \\ &= \sum_{n\in\mathbb{Z},k\ge 0} (1+\lambda_k)^{\sigma}(1+(\frac{2\pi n}{\varepsilon})^2)^{\sigma'} |u_{nk}|^2 \\ &\le \sum_{n\in\mathbb{Z},k\ge 0} (1+\lambda_k+(\frac{2\pi n}{\varepsilon})^2)^{\sigma+\sigma'} |u_{nk}|^2 \\ &\le \|(1-\Delta)^s u\|^2_{L^2} \,.\end{aligned}$$

Finally , we remark that there exists a positive constant C_6 such that, for $0 < \varepsilon \le 1$, for any $u \in X^s$, $\|(1-\Delta)^s u\|_{L^2} \le C_6\|(-\Delta)^s u\|_{L^2}$. Therefore, we deduce from (2.14) that

$$\|u\|_{L^{q,q'}} \le \varepsilon^{1/q'-1/2} C_2 C_0(\varepsilon) C_5 C_6 \|u\|_s \,, \tag{2.15}$$

which proves the embedding $X^s \hookrightarrow L^{q,q'}(Q_\varepsilon)$. If $w \in X^s$ satisfies $Mw = 0$, then, according to (2.12), the inequality (2.15) becomes

$$\|u\|_{L^{q,q'}} \le \varepsilon^{1/q'-1/2} C_2 C_0(\varepsilon) C_5 C_6 \|u\|_s \le C_2 C_4 C_5 C_6 \|u\|_s \,,$$

and the estimate (2.8) is proved. ◻

In the periodic case, we need an inequality in which the H^1 norm of 2-dimensional functions appears. This estimate cannot be deduced from Lemma 2.1. We shall show it with the help of the following commutator estimate.

Lemma 2.3. *There exists a positive constant K_4 such that, for $0 < \varepsilon \le 1$, for any functions $f \in \dot{H}^1_p(Q_\varepsilon)$ and $g \in \dot{H}^{1/2}_p(Q_\varepsilon)$, where f is independent of x_3 and $Mg = 0$, the following estimate holds,*

$$\|[f, (-\Delta)^{1/4}]g\|_{L^2} \le K_4 \varepsilon^{-1/2} \|f\|_1 \|g\|_{1/2},$$

where $[f, (-\Delta)^{1/4}]g = f(-\Delta)^{1/4}g - (-\Delta)^{1/4}(fg)$.

Proof. As in (1.3), we consider the Fourier series of f and g:

$$f(x) \;=\; \varepsilon^{-1/2}\sqrt{a_1a_2}\sum_{m\in\mathbb{Z}^3, m_3=0} f_m e^{2i\pi m a\cdot x}\;, \quad g(x) \;=\; \varepsilon^{-1/2}\sqrt{a_1a_2}\sum_{n\in\mathbb{Z}^3} g_n e^{2i\pi n a\cdot x}\;,$$

where, for $k\in\mathbb{Z}^3$, $ka \equiv (k_1a_1, k_2a_2, k_3a_3)$ and $a_1 = l_1^{-1}$, $a_2 = l_2^{-1}$, $a_3 = \varepsilon^{-1}$. A straightforward computation gives

$$[f,(-\Delta)^{1/4}]g \;=\; \varepsilon^{-1}a_1a_2\sqrt{2\pi}\sum_{m,n\in\mathbb{Z}^3, m_3=0} f_m g_n(|na|^{1/2} - |(n+m)a|^{1/2})e^{2i\pi(m+n)a\cdot x}\;,$$

where $|ka| = (k_1^2a_1^2 + k_2^2a_2^2 + k_3^2a_3^2)^{1/2}$. Hence,

$$\|[f,(-\Delta)^{1/4}]g\|_{L^2} \;\leq\; \varepsilon^{-1/2}\left[2\pi a_1a_2\sum_{k\in\mathbb{Z}^3}\Big(\sum_{m+n=k, m_3=0}|f_m||g_n|\,\big|\,|(n+m)a|^{1/2} - |na|^{1/2}\big|\Big)^2\right]^{1/2}.$$

Since $m_3 = 0$, there exists a positive constant C_1, independent of ε, m and n, such that

$$\big|\,|(n+m)a|^{\frac12} - |na|^{\frac12}\big| \;\leq\; C_1|m_1^2a_1^2 + m_2^2a_2^2|^{\frac14}\;.$$

The previous two inequalities imply that

$$\|[f,(-\Delta)^{1/4}]g\|_{L^2} \leq C_1\|((-\Delta_2)^{1/4}\check{f})\check{g}\|_{L^2}, \tag{2.16}$$

where

$$\check{f}(x) \;=\; \varepsilon^{-1/2}\sqrt{a_1a_2}\sum_{m\in\mathbb{Z}^3, m_3=0}|f_m|e^{2i\pi m a\cdot x}\;, \quad \check{g}(x) \;=\; \varepsilon^{-1/2}\sqrt{a_1a_2}\sum_{n\in\mathbb{Z}^3}|g_n|e^{2i\pi n a\cdot x}\;,$$

have the same H^s-norms as f and g respectively, where $\check{f}$ is independent of x_3 and where $M\check{g} = 0$. Hölder's anisotropic inequality together with Lemma 2.2 give

$$\begin{aligned}\|((-\Delta_2)^{1/4}\check{f})\check{g}\|_{L^2} \;&\leq\; \|(-\Delta_2)^{1/4}\check{f}\|_{L^4(\Omega)}\|\check{g}\|_{L^{4,2}} \\ &\leq C_2\|(-\Delta_2)^{1/4}\check{f}\|_{L^4(\Omega)}\|\check{g}\|_{1/2} \;\leq\; C_2\|(-\Delta_2)^{1/4}\check{f}\|_{L^4(\Omega)}\|g\|_{1/2}\;.\end{aligned} \tag{2.17}$$

Due to the classical Sobolev embedding $H^{1/2}(\Omega)\hookrightarrow L^4(\Omega)$, we also have

$$\begin{aligned}\|(-\Delta_2)^{1/4}\check{f}\|_{L^4(\Omega)} \;&\leq\; C_3\|(-\Delta_2)^{1/4}\check{f}\|_{H^{1/2}(\Omega)} \;\leq\; C_4\|(-\Delta_2)^{1/2}\check{f}\|_{L^2(\Omega)} \\ &\leq\; C_5\varepsilon^{-1/2}\|\check{f}\|_1 \;=\; C_5\varepsilon^{-1/2}\|f\|_1\;.\end{aligned} \tag{2.18}$$

We deduce from the relations (2.16), (2.17) and (2.18) that

$$\|[f,(-\Delta)^{1/4}]g\|_{L^2} \;\leq\; C_1C_2C_5\varepsilon^{-1/2}\|f\|_1\|g\|_{1/2}.$$

The proof is completed. ☐

As a consequence, we obtain the following lemma:

Lemma 2.4. *There exists a positive constant K_5 such that, for $0<\varepsilon\leq 1$, for any vector fields $v\in(\dot{H}_p^1(Q_\varepsilon))^3$ and $w\in(\dot{H}_p^{3/2}(Q_\varepsilon))^3$, where v is divergence-free and independent of x_3 and where $Mw=0$, the following estimate holds*

$$\Big|\int_{Q_\varepsilon} v(x)\nabla w(x)(-\Delta)^{1/2}w(x)\,dx\Big| \leq\; K_5\varepsilon^{-1/2}\|v\|_1\|w\|_{1/2}\|w\|_{3/2}.$$

Proof. We can write

$$I \equiv \int_{Q_\varepsilon} v(x)\nabla w(x)(-\Delta)^{1/2}w(x)\,dx = \int_{Q_\varepsilon} (-\Delta)^{1/4}\big(v(x)\nabla w(x)\big)(-\Delta)^{1/4}w(x)\,dx$$
$$= \int_{Q_\varepsilon} [(-\Delta)^{1/4}, v]\nabla w(x)(-\Delta)^{1/4}w(x)\,dx + \int_{Q_\varepsilon} v(x)\nabla(-\Delta)^{1/4}w(x)(-\Delta)^{1/4}w(x)\,dx\ .$$

But an integration by parts shows that

$$\int_{Q_\varepsilon} v(x)\nabla(-\Delta)^{1/4}w(x)(-\Delta)^{1/4}w(x)\,dx = -\int_{Q_\varepsilon} \operatorname{div} v(x)|(-\Delta)^{1/4}w(x)|^2\,dx = 0\ .$$

Therefore, we can use Lemma 2.3 to deduce that

$$|I| = \Big|\int_{Q_\varepsilon} [(-\Delta)^{1/4}, v]\nabla w(x)(-\Delta)^{1/4}w(x)\,dx\Big| \le \|[(-\Delta)^{1/4}, v]\nabla w\|_{L^2}\|(-\Delta)^{1/4}w\|_{L^2}$$
$$\le C_1\varepsilon^{-1/2}\|v\|_1\|\nabla w\|_{1/2}\|w\|_{1/2} \le C_2\varepsilon^{-1/2}\|v\|_1\|w\|_{1/2}\|w\|_{3/2}\ .$$

□

Lemma 2.5. *There exists a constant K_6 independent of ε such that, for $\varepsilon > 0$, if $v \in (\dot H^1_p(Q_\varepsilon))^3$ is a divergence-free vector field and $v^* \in \dot H^2_p(Q_\varepsilon)$ is a function, that are independent of x_3, then*

$$\Big|\int_{Q_\varepsilon} v(x)\nabla v^*(x)\Delta v^*(x)\,dx\Big| \le K_6\varepsilon^{-1/2}\|v\|_1\|v^*\|_1\|v^*\|_2. \tag{2.19}$$

Proof. Since v is divergence-free, simple integrations by parts give

$$\int_{Q_\varepsilon} v(x)\nabla v^*(x)\Delta v^*(x)\,dx = -\sum_{i,j=1}^{2}\int_{Q_\varepsilon} \partial_{x_j}v_i(x)\partial_{x_i}v^*(x)\partial_{x_j}v^*(x)\,dx.$$

We deduce from Hölder's inequality and from a two-dimensional Gagliardo-Nirenberg estimate that

$$\Big|\int_{Q_\varepsilon} v(x)\nabla v^*(x)\Delta v^*(x)\,dx\Big| \le \varepsilon\|v\|_{H^1(\Omega)}\|\nabla v^*\|^2_{L^4(\Omega)} \le C_1\varepsilon\|v\|_{H^1(\Omega)}\|v^*\|_{H^1(\Omega)}\|v^*\|_{H^2(\Omega)}\ ,$$

which implies the lemma. □

3. A uniqueness result

The aim of this section is to prove a uniqueness result for weak Leray solutions. In short, this result says that only the "purely 3-dimensional" part of the solution needs to be "strong" in order to obtain uniqueness. In particular, uniqueness of 2D solutions in the class of 3D weak Leray solutions is obtained. Let us note that, in the case of periodic boundary conditions, this particular fact was already proved by Gallagher [10], while uniqueness of 2D solutions in the class of some "strong" solutions was shown by Iftimie [15].

We start with a remark on the regularity of weak Leray solutions.

Remark 3.1. Let u be a weak Leray solution such that $(I - M)u \in L^\infty(0,T;V_\varepsilon^{1/2}) \cap L^2(0,T;V_\varepsilon^{3/2})$. Then $u \in C^0([0,T];H_\varepsilon)$ and $\partial_t u \in L^2(0,T;V'_\varepsilon)$.

We first show that $u\nabla u$ belongs to $L^2(0,T;V_\varepsilon')$. Let $\varphi \in L^2(0,T;V_\varepsilon)$ be a smooth vector in the x variable ($\varphi \in L^2(0,T;V_\varepsilon^2)$ is actually sufficient). We have

$$(3.1) \qquad \int_0^T \int_{Q_\varepsilon} u\nabla u\,\varphi\,dx\,dt \;=\; \int_0^T \int_{Q_\varepsilon} (u\nabla Mu\,\varphi + u\nabla(I-M)u\,\varphi)\,dx\,dt\ .$$

Since φ is a smooth vector in the variable x, a simple integration by parts gives

$$\int_0^T \int_{Q_\varepsilon} u\nabla Mu\,\varphi\,dx\,dt \;=\; -\int_0^T \int_{Q_\varepsilon} u\nabla\varphi\,Mu\,dx\,dt\ .$$

By Lemma 2.1 and Remark 2.1, we thus obtain,

$$\left|\int_0^T \int_{Q_\varepsilon} u\nabla Mu\,\varphi\,dx\,dt\right| \;\le\; C_\varepsilon \left(\int_0^T |u|_{1/2}^2 |Mu|_{1/2}^2\,dt\right)^{1/2} \|\varphi\|_{L^2(0,T;V_\varepsilon)}\ ,$$

and

$$\begin{aligned}\left|\int_0^T \int_{Q_\varepsilon} u\nabla(I-M)u\varphi\,dx\,dt\right| \;&\le\; C_\varepsilon \left(\int_0^T \|u(t)\|_{L^2}^2 |\nabla(I-M)u(t)|_{1/2}^2\,dt\right)^{1/2} \|\varphi\|_{L^2(0,T;V_\varepsilon)} \\ &\le\; C_\varepsilon \left(\int_0^T \|u(t)\|_{L^2}^2 |(I-M)u(t)|_{3/2}^2\,dt\right)^{1/2} \|\varphi\|_{L^2(0,T;V_\varepsilon)}\ .\end{aligned}$$

By a classical density argument, these estimates are still true for any $\varphi \in L^2(0,T;V_\varepsilon)$. We thus conclude that

$$\begin{aligned}\left(\int_0^T \|u\nabla u\|_{V_\varepsilon'}^2\,dt\right)^{1/2} \;\le\; & C_\varepsilon(\|u\|_{L^4(0,T;V_\varepsilon^{1/2})}\|Mu\|_{L^4(0,T;V_\varepsilon^{1/2})} \\ & + \|u\|_{L^\infty(0,T;H_\varepsilon)}\|(I-M)u\|_{L^2(0,T;V_\varepsilon^{3/2})})\ .\end{aligned}$$

As $A_\varepsilon u$ and f also belong to $L^2(0,T;V_\varepsilon')$, we infer from the above inequality that $\partial_t u$ is in the space $L^2(0,T;V_\varepsilon')$. The properties $u \in L^2(0,T;V_\varepsilon)$ and $\partial_t u \in L^2(0,T;V_\varepsilon')$ finally imply that u belongs to $C^0([0,T];H_\varepsilon)$. The proof of the remark is completed.

We can now prove a uniqueness result.

Theorem 3.1. (Uniqueness) *Let u be a weak Leray solution of the Navier-Stokes equations (1.11) such that $(I-M)u \in L^\infty(0,T;V_\varepsilon^{1/2}) \cap L^2(0,T;V_\varepsilon^{3/2})$. Then u is unique in the class of the weak Leray solutions.*

Proof. Let $\widetilde{u}$ be a weak Leray solution with the same initial data as u. The difference $u-\widetilde{u}$ satisfies the following equation in V_ε',

$$(3.2) \qquad \partial_t(u-\widetilde{u}) + \nu A_\varepsilon(u-\widetilde{u}) + B_\varepsilon(u-\widetilde{u},u) + B_\varepsilon(\widetilde{u},u-\widetilde{u}) \;=\; 0\ .$$

We would like to take the inner product in $L^2(Q_\varepsilon)$ of this equation with $u-\widetilde{u}$ and to integrate in space and time. The result would be the inequality (3.10) below. Unfortunately, this is not possible without some additional justification because the integral

$$(3.3) \qquad \int_0^t \int_{Q_\varepsilon} \widetilde{u}\nabla(u-\widetilde{u})(u-\widetilde{u})\,dx\,d\tau,$$

which is supposed to vanish, may not converge. Nevertheless, one can argue as in [26] and [29] (see also [10]). The idea is that, instead of multiplying the equation of $u-\widetilde{u}$ by $u-\widetilde{u}$

which yields regularity problems, one can multiply the equation of u by $\widetilde{u}$, the equation of $\widetilde{u}$ by u and then subtract the two energy inequalities satisfied by u and $\widetilde{u}$; the result is the same. This argument is detailed below.

We saw at the end of the proof of Remark 3.1 that all the terms in the equation of u belong to $L^2(0,T;V_\varepsilon')$. So we can multiply the equation of u by $\widetilde{u} \in L^2(0,T;V_\varepsilon)$ and integrate in space and time to obtain

$$\int_0^t \int_{Q_\varepsilon} (\partial_t u\, \widetilde{u} + \nu \nabla u\, \nabla \widetilde{u} + u \cdot \nabla u\, \widetilde{u})\, dx d\tau \;=\; \int_0^t \int_{Q_\varepsilon} f\, \widetilde{u}\, dx d\tau \;. \tag{3.4}$$

Unfortunately, we cannot directly multiply the equation of $\widetilde{u}$ by u and then integrate in space and time, because $\partial_t \widetilde{u}$ and u are only in $L^{4/3}(0,T;V_\varepsilon')$ and $L^2(0,T;V_\varepsilon)$ respectively. As $u \in C^0([0,T];H_\varepsilon) \cap L^2(0,T;V_\varepsilon) \cap H^1(0,T;V_\varepsilon')$, by a standard smoothing procedure, we can find a sequence of smooth divergence free vector fields $u_n \in V_\varepsilon$, such that u_n converges strongly to u in $C^0([0,T];H_\varepsilon) \cap L^2(0,T;V_\varepsilon) \cap L^4(0,T;V_\varepsilon^{1/2})$, $\partial_t u_n$ converges strongly to $\partial_t u$ in $L^2(0,T;V_\varepsilon')$ and $(I-M)u_n$ converges strongly to $(I-M)u$ in $L^2(0,T;V_\varepsilon^{3/2})$. Multiplying the equation of $\widetilde{u}$ by u_n and integrating by parts yield

$$\int_0^t \int_{Q_\varepsilon} (\partial_t \widetilde{u}\, u_n + \nu \nabla \widetilde{u}\, \nabla u_n - \widetilde{u} \cdot \nabla u_n\, \widetilde{u})\, dx d\tau = \int_0^t \int_{Q_\varepsilon} f\, u_n\, dx d\tau \;. \tag{3.5}$$

We now pass to the limit in n in the above equation. With the regularities and convergences at hand, it is easily seen that

(3.6)
$$\int_0^t \int_{Q_\varepsilon} \nabla \widetilde{u} \nabla u_n\, dx d\tau \to \int_0^t \int_{Q_\varepsilon} \nabla \widetilde{u} \nabla u\, dx d\tau \quad \text{and} \quad \int_0^t \int_{Q_\varepsilon} f u_n\, dx d\tau \to \int_0^t \int_{Q_\varepsilon} f u\, dx d\tau \;.$$

On the other hand, by Lemma 2.1, we have

$$\begin{aligned}
\int_0^t \int_{Q_\varepsilon} \widetilde{u} \cdot \nabla (u - u_n) \widetilde{u}\, dx d\tau \;&=\; \int_0^t \int_{Q_\varepsilon} (\widetilde{u} \cdot \nabla M(u-u_n)\, \widetilde{u} + \widetilde{u} \cdot \nabla (I-M)(u-u_n)\, \widetilde{u})\, dx d\tau \\
&\leq\; C_\varepsilon \int_0^t |\widetilde{u}|_{1/2}^2 (|M(u-u_n)|_1 + |(I-M)(u-u_n)|_{3/2})\, d\tau \\
&\leq C_\varepsilon \|\widetilde{u}\|^2_{L^4(0,T;V_\varepsilon^{1/2})} \big(\|Mu - Mu_n\|_{L^2(0,T;V_\varepsilon)} \\
&\qquad\qquad + \|(I-M)u - (I-M)u_n\|_{L^2(0,T;V_\varepsilon^{3/2})} \big).
\end{aligned}$$

We deduce that

$$\int_0^t \int_{Q_\varepsilon} \widetilde{u} \cdot \nabla u_n\, \widetilde{u}\, dx d\tau \to \int_0^t \int_{Q_\varepsilon} \widetilde{u} \cdot \nabla u\, \widetilde{u}\, dx d\tau \;. \tag{3.7}$$

Finally, we integrate by parts to obtain that

$$\int_0^t \int_{Q_\varepsilon} \partial_t \widetilde{u}\, u_n\, dx d\tau \;=\; -\int_0^t \int_{Q_\varepsilon} \widetilde{u}\, \partial_t u_n\, dx d\tau + \int_{Q_\varepsilon} (\widetilde{u}(t)\, u_n(t) - \widetilde{u}(0)\, u_n(0)) dx \;.$$

As $\partial_t u_n$ and u_n converge to $\partial_t u$ and u in $L^2(0,T;V'_\varepsilon)$ and $C([0,T];H_\varepsilon)$ respectively, we infer from the above equality that,

$$\int_0^t \int_{Q_\varepsilon} \partial_t \widetilde{u}\, u_n \, dx d\tau \to - \int_0^t \int_{Q_\varepsilon} \widetilde{u}\, \partial_t u \, dx d\tau + \int_{Q_\varepsilon} (\widetilde{u}(t)\, u(t) - \widetilde{u}(0)\, u(0))\, dx \ . \tag{3.8}$$

Putting together the properties (3.5), (3.6), (3.7) and (3.8) finally yields

$$\begin{aligned} &\int_0^t \int_{Q_\varepsilon} (-\widetilde{u}\, \partial_t u + \nu \nabla \widetilde{u}\, \nabla u - \widetilde{u} \cdot \nabla u\, \widetilde{u}\,) dx d\tau \\ &\qquad = - \int_{Q_\varepsilon} (\widetilde{u}(t)\, u(t) - \widetilde{u}(0)\, u(0)) dx + \int_0^t \int_{Q_\varepsilon} f\, u\, dx d\tau \ . \end{aligned} \tag{3.9}$$

Since u and $\widetilde{u}$ are weak Leray solutions, the two following energy inequalities hold:

$$\begin{aligned} \frac{1}{2} \|u(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla u\|_{L^2}^2 d\tau \ &\leq \ \frac{1}{2} \|u_0\|_{L^2}^2 + \int_0^t \int_{Q_\varepsilon} f\, u\, dx d\tau \ , \\ \frac{1}{2} \|\widetilde{u}(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla \widetilde{u}\|_{L^2}^2 d\tau \ &\leq \ \frac{1}{2} \|u_0\|_{L^2}^2 + \int_0^t \int_{Q_\varepsilon} f\, \widetilde{u}\, dx d\tau \ . \end{aligned}$$

We now add both energy inequalities and subtract relations (3.4) and (3.9) to obtain

$$\frac{1}{2} \|u(t) - \widetilde{u}(t)\|_{L^2}^2 + \nu \int_0^t \|u - \widetilde{u}\|_1^2 d\tau \ \leq \ \int_0^t \int_{Q_\varepsilon} (-\widetilde{u} \cdot \nabla u\, \widetilde{u} + u \cdot \nabla u\, \widetilde{u}) dx d\tau \ .$$

Arguing as in Remark 3.1, one shows that the integral $\int_0^t \int_{Q_\varepsilon} (-u \cdot \nabla u\, u + \widetilde{u} \cdot \nabla u\, u) dx d\tau$ is absolutely convergent and vanishes. Thus we deduce from the previous inequality that

$$\|u(t) - \widetilde{u}(t)\|_{L^2}^2 + 2\nu \int_0^t \|u - \widetilde{u}\|_1^2 \, d\tau \ \leq \ -2 \int_0^t \int_{Q_\varepsilon} (u - \widetilde{u}) \nabla u (u - \widetilde{u}) dx d\tau \ . \tag{3.10}$$

Writing $\nabla u = \nabla((I-M)u + Mu)$ and applying Lemma 2.1, we get, for any $0 < s < 1$,

$$\begin{aligned} \|u(t) - \widetilde{u}(t)\|_{L^2}^2 + 2\nu \int_0^t \|u - \widetilde{u}\|_1^2 \, d\tau &\leq C_1 \int_0^t \|u - \widetilde{u}\|_{L^2} \|u - \widetilde{u}\|_1 \|(I-M)u\|_{3/2} \, d\tau \\ &\quad + C_2 \int_0^t \|Mu\|_1 \|u - \widetilde{u}\|_s \|u - \widetilde{u}\|_{1-s} \, d\tau \ . \end{aligned}$$

Since the interpolation inequality $\|u - \widetilde{u}\|_{\widetilde{s}} \leq C_3 \|u - \widetilde{u}\|_1^{\widetilde{s}} \|u - \widetilde{u}\|_{L^2}^{1-\widetilde{s}}$ holds, for any $\widetilde{s} \in [0,1]$, we infer from the above inequality that

$$\begin{aligned} \|u(t) - \widetilde{u}(t)\|_{L^2}^2 + 2\nu \int_0^t \|u - \widetilde{u}\|_1^2 \, d\tau \ &\leq \ 2\nu \int_0^t \|u - \widetilde{u}\|_1^2 \, d\tau \\ &\quad + C_4 \int_0^t \|u - \widetilde{u}\|_{L^2}^2 (\|(I-M)u\|_{3/2}^2 + \|Mu\|_1^2) \, d\tau \ , \end{aligned}$$

that is

$$\|u(t) - \widetilde{u}(t)\|_{L^2}^2 \ \leq \ C_4 \int_0^t \|u - \widetilde{u}\|_{L^2}^2 (\|(I-M)u\|_{3/2}^2 + \|Mu\|_1^2) \, d\tau \ .$$

And the result follows from Gronwall's inequality. □

4. The case of mixed boundary conditions

In this section, we shall prove the theorems 1.1 and 1.2.

Proof of Theorem 1.1. The proof is based on a Galerkin approximation, using the first m eigenvectors $\psi_1, \psi_2, \dots, \psi_m$ of the Stokes operator A_ε. Since M and A_ε commute, we can choose these eigenvectors ψ_j so that, either $\psi_j \in MV_\varepsilon^2$ or $\psi_j \in (I-M)V_\varepsilon^2$. Let $\mathcal{P}_m : H_\varepsilon \to H_\varepsilon$ denote the projector onto the space $\mathcal{V}_m$ generated by the first m eigenfunctions. We remark that $\mathcal{P}_m M = M\mathcal{P}_m$. The above properties imply that, for every $s \in [0,2]$ and for every $u \in V_\varepsilon^s$,

$$|\mathcal{P}_m(I-M)u|_s \le |(I-M)u|_s \ , \quad |\mathcal{P}_m Mu|_s \le |Mu|_s \ . \tag{4.1}$$

We know (see [5], Chapter 8, for example or [26]), that, for every $m \in \mathbb{N}$, there exists a global solution $u_m \in C^1([0,+\infty); V_\varepsilon^2 \cap \mathcal{V}_m)$ of the equations (1.11) or also of (1.20) and (1.21), where B_ε is replaced by $\mathcal{P}_m B_\varepsilon$ and $P_\varepsilon f$ by $\mathcal{P}_m P_\varepsilon f$ and where the initial condition is $u_m(0) = \mathcal{P}_m(I-M)u_0 + \mathcal{P}_m Mu_0 \equiv w_{0m} + v_{0m}$. Moreover, for every $\tau > 0$, u_m and $\partial_t u_m$ are uniformly bounded with respect to m in the spaces $L^\infty(0,+\infty;H_\varepsilon) \cap L^2(0,\tau;V_\varepsilon)$ and $L^{4/3}(0,\tau;V_\varepsilon')$ respectively. We want to show that this solution $u_m \equiv w_m + v_m = (I-M)u_m + Mu_m$ satisfies the additional estimates and properties given in Theorem 1.1, which will be preserved, when m goes to $+\infty$. In order to simplify the notation, we shall drop the subscript m in all the a priori estimates below, when there is no confusion. Taking the inner product of the modified equation (1.21) with $A_\varepsilon^{1/2}w$ gives, for $t \ge 0$,

$$\begin{aligned}
&\frac{1}{2}\partial_t|w(t)|_{1/2}^2 + \nu|w(t)|_{3/2}^2 + \int_{Q_\varepsilon}(w\nabla w\,(I-M)A_\varepsilon^{1/2}w)(t,x)dx + \int_{Q_\varepsilon}(v\nabla w\,A_\varepsilon^{1/2}w)(t,x)dx \\
&\qquad + \int_{Q_\varepsilon}(w\nabla v\,A_\varepsilon^{1/2}w)(t,x)dx = \int_{Q_\varepsilon}((I-M)Pf\,A_\varepsilon^{1/2}w)(t,x)dx.
\end{aligned} \tag{4.2}$$

Since $I-M$ commutes with $A_\varepsilon^{1/4}$, we get by Lemma 2.1, for $t \ge 0$,

$$\begin{aligned}
\left|\int_{Q_\varepsilon}(w\nabla w(I-M)A_\varepsilon^{1/2}w)(t,x)dx\right| &\le C|w(t)|_1\|\nabla w(t)\|_{L^2}|(I-M)A_\varepsilon^{1/2}w(t)|_{1/2} \\
&\le C|w(t)|_1^2|w(t)|_{3/2} \ .
\end{aligned}$$

A simple interpolation inequality now yields, for $t \ge 0$,

$$\begin{aligned}
\left|\int_{Q_\varepsilon}(w\nabla w(I-M)A_\varepsilon^{1/2}w)(t,x)dx\right| &\le C|w(t)|_{1/2}|w(t)|_{3/2}^2 \\
&\le C\nu^{-1}|w(t)|_{1/2}^2|w(t)|_{3/2}^2 + \frac{\nu}{8}|w(t)|_{3/2}^2 \ .
\end{aligned} \tag{4.3}$$

The inequality (2.7) of Lemma 2.1 implies, for $s \in [1/2,1)$,

$$\begin{aligned}
\left|\int_{Q_\varepsilon}(v\nabla w\,A_\varepsilon^{1/2}w)(t,x)dx\right| &\le C_s\varepsilon^{-1/2}|v(t)|_s\|\nabla w(t)\|_{L^2}|A_\varepsilon^{1/2}w(t)|_{1-s} \\
&\le C_s\varepsilon^{-1/2}|v(t)|_s|w(t)|_{2-s}|w(t)|_1 \ ,
\end{aligned}$$

where C_s denotes a positive constant depending only s. We find again by interpolation that, for $t \geq 0$,

$$\left| \int_{Q_\varepsilon} (v \nabla w \, A_\varepsilon^{1/2} w)(t,x) dx \right| \leq C_s \varepsilon^{-1/2} |v(t)|_s |w(t)|_{1/2}^s |w(t)|_{3/2}^{2-s} \tag{4.4}$$
$$\leq C_s \, \nu^{1-2/s} \varepsilon^{-1/s} |v(t)|_s^{2/s} |w(t)|_{1/2}^2 + \frac{\nu}{8} |w(t)|_{3/2}^2 \, .$$

Due to the estimate (2.7) of Lemma 2.1, we also have, for $t \geq 0$,

$$\left| \int_{Q_\varepsilon} (w \nabla v \, A_\varepsilon^{1/2} w)(t,x) dx \right| \leq C \varepsilon^{-1/2} |w(t)|_{1/2} \|\nabla v(t)\|_{L^2} |A_\varepsilon^{1/2} w(t)|_{1/2} \tag{4.5}$$
$$\leq C \varepsilon^{-1/2} |v(t)|_1 |w(t)|_{1/2} |w(t)|_{3/2}$$
$$\leq C \, \varepsilon^{-1} \nu^{-1} |v(t)|_1^2 |w(t)|_{1/2}^2 + \frac{\nu}{8} |w(t)|_{3/2}^2 \, .$$

Finally, we obtain, due to (4.1) and the Poincaré inequality (2.2) that, for $t \geq 0$,

$$\left| \int_{Q_\varepsilon} ((I-M)Pf \, A_\varepsilon^{1/2} w)(t,x) dx \right| \leq C \|\mathcal{P}_m (I-M)Pf(t)\|_{L^2} |w(t)|_1 \tag{4.6}$$
$$\leq C \varepsilon^{1/2} \|(I-M)Pf(t)\|_{L^2} |w(t)|_{3/2}$$
$$\leq C \nu^{-1} \varepsilon \|(I-M)Pf(t)\|_{L^2}^2 + \frac{\nu}{8} |w(t)|_{3/2}^2 \, .$$

We now fix the real number $s \in [1/2, 1)$. We recall that, according to the hypotheses of Theorem 1.1, s is chosen so that

$$1 \, > \, s \, > \, \sup(\beta, 2\gamma, \frac{1}{2}, \frac{\alpha}{\alpha + 1 - \beta}) \, . \tag{4.7}$$

The inequalities (4.3), (4.4), (4.5) and (4.6), together with (4.2), imply, for $t \geq 0$,

(4.8)
$$\partial_t |w(t)|_{1/2}^2 + \frac{\nu}{2} |w(t)|_{3/2}^2 \leq C_1 |w(t)|_{1/2}^2 \big(\nu^{1-2/s} \varepsilon^{-1/s} |v(t)|_s^{2/s} + \nu^{-1} \varepsilon^{-1} |v(t)|_1^2 + \nu^{-1} |w(t)|_{3/2}^2 \big)$$
$$+ C_1 \nu^{-1} \varepsilon \|(I-M)Pf(t)\|_{L^2}^2 \, .$$

Due to the property (4.1) and to the hypothesis (1.22) on the initial data, when K_* is small enough, there exists a positive time T such that, for $t \in [0,T)$,

$$|w(t)|_{1/2}^2 < \frac{\nu^2}{4C_1^2} \, , \tag{4.9}$$

and, that, if $T < \infty$,

$$|w(T)|_{1/2}^2 = \frac{\nu^2}{4C_1^2} \, . \tag{4.10}$$

We shall show by contradiction that $T = +\infty$. We derive from (4.8), (4.9), and (4.10), that, for $t \in [0,T]$,

$$\partial_t |w(t)|_{1/2}^2 + \frac{\nu}{4} |w(t)|_{3/2}^2 \leq C_1 |w(t)|_{1/2}^2 \big(\nu^{1-2/s} \varepsilon^{-1/s} |v(t)|_s^{2/s} + \nu^{-1} \varepsilon^{-1} |v(t)|_1^2 \big) \tag{4.11}$$
$$+ C_1 \nu^{-1} \varepsilon \|(I-M)Pf(t)\|_{L^2}^2 \, .$$

which in turn implies

$$(4.12)\quad \partial_t |w(t)|_{1/2}^2 + \frac{\nu\varepsilon^{-2}K_0^{-2}}{4}|w(t)|_{1/2}^2 \le C_1|w(t)|_{1/2}^2\big(\nu^{1-2/s}\varepsilon^{-1/s}|v(t)|_s^{2/s} + \nu^{-1}\varepsilon^{-1}|v(t)|_1^2\big) + C_1\nu^{-1}\varepsilon\|(I-M)Pf(t)\|_{L^2}^2 \ .$$

Set

$$\begin{aligned} h(t) &= C_1\nu^{1-2/s}\varepsilon^{-1/s}\int_0^t |v(\tau)|_s^{2/s}d\tau + C_1\nu^{-1}\varepsilon^{-1}\int_0^t |v(\tau)|_1^2 d\tau - \nu t K_0^{-2}\varepsilon^{-2}/8 \\ h^*(t) &= h(t) - \nu t K_0^{-2}\varepsilon^{-2}/8 \ . \end{aligned}$$

An application of Gronwall's lemma in (4.12) gives, for $0 \le t \le T$,

(4.13)

$$\begin{aligned} |w(t)|_{1/2}^2 &\le \exp(h^*(t))\,|w_0|_{1/2}^2 + C_1\varepsilon\nu^{-1}\int_0^t \|(I-M)Pf(\tau)\|_{L^2}^2 \exp(h^*(t)-h^*(\tau))d\tau \\ &\le \exp(h^*(t))\,|w_0|_{1/2}^2 + 8\nu^{-2}K_0^2C_1\varepsilon^3(\sup_t \|(I-M)Pf(t)\|_{L^2}^2)(\sup_{0\le\tau\le t}\exp(h(t)-h(\tau))) \ . \end{aligned}$$

The estimate of $h(t)-h(\tau)$ is simple and comes from the usual L^2-energy estimates on the velocity u. If we take the inner product of the modified equation (1.11) with u, we obtain, for $t \ge 0$,

(4.14)

$$\begin{aligned} \partial_t\|u(t)\|_{L^2}^2 + 2\nu|u(t)|_1^2 &= 2\int_{Q_\varepsilon}(Pf\cdot u)(t,x)\,dx \\ &\le 2(\|(I-M)Pf(t)\|_{L^2}\|w(t)\|_{L^2} + \|MPf(t)\|_{L^2}\|v(t)\|_{L^2}) \\ &\le \nu^{-1}(K_0\varepsilon\|(I-M)Pf(t)\|_{L^2} + \mu_0\|MPf(t)\|_{L^2})^2 + \nu|u(t)|_1^2 \ . \end{aligned}$$

It follows that, for $t \ge 0$,

$$(4.15)\quad \partial_t\|u(t)\|_{L^2}^2 + \nu\mu_0^{-2}\|u(t)\|_{L^2}^2 \le \partial_t\|u(t)\|_{L^2}^2 + \nu|u(t)|_1^2 \le \nu^{-1}B \ ,$$

where

$$B = 2(\mu_0^2\sup_t\|MPf(t)\|_{L^2}^2 + \varepsilon^2K_0^2\sup_t\|(I-M)Pf(t)\|_{L^2}^2) \ .$$

Gronwall's lemma implies, for $t \ge 0$,

$$(4.16)\quad \|u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2 + \nu^{-2}\mu_0^2B(1-e^{-\nu\mu_0^{-2}t}) \ .$$

Integrating (4.15) from t_0 to t_1, where $0 \le t_0 \le t_1$, one finds

$$(4.17)\quad \begin{aligned} \|u(t_1)\|_{L^2}^2 + \nu\int_{t_0}^{t_1}|u(\tau)|_1^2 d\tau &\le \|u(t_0)\|_{L^2}^2 + \nu^{-1}(t_1-t_0)B \\ &\le \|u_0\|_{L^2}^2 + \nu^{-1}B(\nu^{-1}\mu_0^2(1-e^{-\nu\mu_0^{-2}t_0}) + (t_1-t_0)) \ . \end{aligned}$$

By interpolation, we can write,

$$(4.18)\quad \int_{t_0}^{t_1}|v(\tau)|_s^{2/s}d\tau \le C_2\sup_{t_0\le\tau\le t_1}\|v(\tau)\|_{L^2}^{2(1-s)/s}\int_{t_0}^{t_1}|v(\tau)|_1^2 d\tau \ .$$

Since M is an orthogonal projection in H_ε and V_ε, we infer from (4.16), (4.17) and (4.18) that, for $0 \le t_0 \le t_1$,

$$h(t_1) - h(t_0) \le C_3\Big(\varepsilon^{-1} + \varepsilon^{-1/s}\big(\|u_0\|^2_{L^2} + t_1^* B\big)^{(1-s)/s}\Big) \times \Big(\|u_0\|^2_{L^2} + (t_1 - t_0 + t_0^*)B\Big) - \nu(t_1 - t_0)\varepsilon^{-2}K_0^{-2}/8 \ , \tag{4.19}$$

where $t^* = \min(1,t)$. To simplify, we set $\|v_0\|_{L^2} = K_*\varepsilon^{1/2}g_0(\varepsilon)$ and $\sup_t \|MP_\varepsilon f(t)\|_{L^2} = K_*\varepsilon^{1/2}g_1(\varepsilon)$, where, by the hypotheses (1.22), $0 < g_0 \le \varepsilon^{-\alpha}$ and $0 < g_1 \le \varepsilon^{-\beta}$. We thus can write

$$\begin{aligned} \|u_0\|^2_{L^2} &= \|v_0\|^2_{L^2} + \|w_0\|^2_{L^2} \le K_*^2\varepsilon(K_0^2 + g_0^2) \ , \\ B &\le C_4 K_*^2 \varepsilon(|\ln\varepsilon|^{2\gamma} + g_1^2) \ . \end{aligned} \tag{4.20}$$

We now deduce from (4.19) and (4.20) that, for $0 \le t_0 \le t_1$,

$$\begin{aligned} h(t_1) - h(t_0) \le\ & C_5\left(1 + K_*^{2/s}g_0^{2/s} + (t_1^* + t_0^*)K_*^{2/s}(g_1^{2/s} + |\ln\varepsilon|^{2\gamma/s})\right) \\ & -(t_1 - t_0)\Big(\frac{\nu K_0^{-2}}{8}\varepsilon^{-2} - C_5 K_*^{2/s}(g_1^{2/s} + |\ln\varepsilon|^{2\gamma/s} + (1 + g_0^{\frac{2(1-s)}{s}})(g_1^2 + |\ln\varepsilon|^{2\gamma})\Big) \ . \end{aligned} \tag{4.21}$$

Due to the choice (4.7) of s and the hypotheses (1.22) and (1.23), we infer from (4.21) that, when K_* is small enough, we have, for $0 \le t_0 \le t_1$,

$$h(t_1) - h(t_0) \le C_6(1 + K_*^{2/s}g_0^{2/s} + K_*^{2/s}g_1^{2/s} + K_*^{2/s}|\ln\varepsilon|^{2\gamma/s}) \ . \tag{4.22}$$

Likewise, we derive from (4.21) that, when K_* is small enough, we have, for $t \ge 0$,

$$h(t) \le C_7(1 + K_*^{2/s}g_0^{2/s}) \ . \tag{4.23}$$

Finally, we deduce from (4.22), (4.23), (4.13), (4.7) and the hypotheses (1.22), (1.23), where K_* is small enough, that, for $0 \le t \le T$,

$$\begin{aligned} |w(t)|^2_{1/2} \le\ & C_9\exp(C_8K_*^{2/s}g_0^{2/s})\Big(\exp(-\frac{\nu t K_0^{-2}}{8}\varepsilon^{-2})|w_0|^2_{1/2} \\ & + \varepsilon^3 \sup_t \|(I-M)Pf(t)\|^2_{L^2}\exp(C_8K_*^{2/s}(g_1^{2/s} + |\ln\varepsilon|^{\frac{2\gamma}{s}}))\Big) \\ \le\ & C_{10}K_* \le \frac{\nu^2}{8C_1^2} \ , \end{aligned} \tag{4.24}$$

which contradicts the property (4.10), if $T < +\infty$. It follows that $T = +\infty$. Remark that the estimate (4.16) implies, for $t \ge 0$,

$$\|v(t)\|^2_{L^2} \le \|u_0\|^2_{L^2} + \nu^{-2}\mu_0^2 B \le \|u_0\|^2_{L^2} + C_{11}\varepsilon^{-1} \ . \tag{4.25}$$

We have just proved that, under the hypotheses (1.22) and (1.23), for any $m \in \mathbb{N}$, the solution $u_m \in C^1([0,+\infty);\mathcal{V}_m)$ of the modified Navier-Stokes equations (1.11) with initial data $u_m(0) = \mathcal{P}_m u_{0m}$ satisfies

$$\sup_{t\ge 0} |w_m(t)|^2_{1/2} < C_{12} \ , \tag{4.26}$$

where C_{12} is a positive constant independent of ε and m. Integrating the inequality (4.11) from 0 to t and using the estimates (4.26), (4.17) and (4.18), one also shows that, for any $t \in [0, +\infty)$,

$$\int_0^t |w_m(s)|_{3/2}^2 ds \le C_{13}(\varepsilon) t \ , \tag{4.27}$$

where $C_{13}(\varepsilon)$ is a positive constant independent of m, but depending on ε.

We remark that v_{0m} and w_{0m} converge to Mu_0 and $(I-M)u_0$ in H_ε and $V_\varepsilon^{1/2}$ respectively. Now, a classical argument (see [5], Chapter 8 or [26]) shows that $u = \lim_{m\to+\infty} u_m$ belongs to the space $L^\infty(0,\infty; H_\varepsilon) \cap L^2_{loc}([0,\infty); V_\varepsilon^1)$, is a weak Leray solution of the equations (1.11) with initial data $u(0) = u_0$ and that, due to the properties (4.26) and (4.27), $(I-M)u$ belongs to $L^\infty(0,\infty; V_\varepsilon^{1/2}) \cap L^2_{loc}([0,\infty); V_\varepsilon^{3/2})$. The uniqueness of the solution u follows from Theorem 3.1. Remark 3.1 implies that u belongs to the space $C^0([0,+\infty); H_\varepsilon) \cap H^1_{loc}([0,+\infty); V'_\varepsilon)$. $\square$

Remark 4.1. In the (PP) case, we can apply Lemma 2.4 in order to estimate the term $|\int_{Q_\varepsilon} (v\nabla w A_\varepsilon^{1/2} w)(t,x)dx|$, which gives

$$|\int_{Q_\varepsilon} (v\nabla w A_\varepsilon^{1/2} w)(t,x)dx| \le C\nu^{-1}\varepsilon^{-1}|v(t)|_1^2 |w(t)|_{1/2}^2 + \frac{\nu}{8}|w(t)|_{3/2}^2 \ .$$

In this case, $h(t) = C_1\nu^{-1}\varepsilon^{-1}\int_0^t |v(\tau)|_1^2 d\tau - \nu t K_0^{-2}\varepsilon^{-2}/8$ and the estimate of $h(t_1) - h(t_0)$ becomes

$$\begin{aligned} h(t_1) - h(t_0) \ \le & C_5\left(1 + K_*^2 g_0^2 + (t_1^* + t_0^*)K_*^2(g_1^2 + |\ln\varepsilon|^{2\gamma})\right) \\ & - (t_1 - t_0)\Big(\frac{\nu K_0^{-2}}{8}\varepsilon^{-2} - C_5 K_*^2(g_1^2 + |\ln\varepsilon|^{2\gamma})\Big) \ . \end{aligned} \tag{4.28}$$

Hence, we can choose $\beta = 1$, $\gamma = 1/2$ in the hypothesis (1.22). Moreover, the limitation on Mu_0 disappears, provided that the condition (1.24) holds.

We now prove Theorem 1.2.

Proof of Theorem 1.2. The proof follows the same lines as the proof of Theorem 1.1. So we shall only indicate the main changes in the estimate of $|w(t)|_{1/2}$, for $0 \le t \le T$. Let $s \in [1/2, 1)$ be fixed. Arguing as in (4.15) and (4.16), we deduce from (4.14) that, for $t \ge 0$,

$$\|u(t)\|_{L^2}^2 + \nu \int_0^t |u(\tau)|_1^2 d\tau \le D \ , \tag{4.29}$$

where $D = \|u_0\|_{L^2}^2 + C_{14}\int_0^t (\|MPf(\tau)\|_{L^2}^2 + \varepsilon^2\|(I-M)Pf(\tau)\|_{L^2}^2)d\tau$. The hypothesis (1.27) imply that

$$D \le C_{15}\varepsilon(\widetilde{K} + \varepsilon^{-1}\|Mu_0\|_{L^2}^2 + \varepsilon^{-1}\int_0^t \|MPf(\tau)\|_{L^2}^2 d\tau) \ . \tag{4.30}$$

The application of Gronwall's lemma to (4.12) and the estimate (4.29) give, for $0 \le t \le T$,

(4.31)

$$|w(t)|_{1/2}^2 \le (\exp C_{16}(\varepsilon^{-1}D + (\varepsilon^{-1}D)^{1/s}))(|w_0|_{1/2}^2 + \varepsilon C_1\nu^{-1}\int_0^{+\infty} \|(I-M)Pf(\tau)\|_{L^2}^2 d\tau) \ ,$$

which implies, due to (4.30) and the hypothesis (1.27), where $\widetilde{K}$ is small enough, that, for $0 \le t \le T$,

$$|w(t)|_{1/2}^2 < \frac{\nu^2}{8C_1^2} \ . \tag{4.32}$$

Now we finish the proof by arguing like in the proof of Theorem 1.1. $\square$

5. The case of periodic boundary conditions

In the periodic case, we obtain better results than those described in Theorem 1.1 because we can use the conservation of enstrophy property, which is valid for two-dimensional periodic vector fields. We recall that, for this reason, we split the vector field $v \equiv Mu$ into two parts

$$Mu = M\widetilde{u} + M(u_3) \equiv (Mu_1, Mu_2, 0) + (0, 0, Mu_3) \ ,$$

and set $\widetilde{v} = M\widetilde{u}$. Likewise, we shall split the forcing term as follows

$$M(Pf) = M\widetilde{Pf} + M((Pf)_3) \equiv (M(Pf)_1, M(Pf)_2, 0) + (0, 0, M(Pf)_3) \ .$$

We recall that, in the periodic case, $P\Delta u = \Delta u$ if $u \in V_p^2$. We begin this section by two auxiliary lemmas.

Lemma 5.1. *Let u be a weak solution of the Navier-Stokes equations such that $w = (I - M)u \in L^\infty(0, T; V_p^{1/2}) \cap L^2(0, T; V_p^{3/2})$ and $v = Mu \in L^\infty(0, T; V_p) \cap L^2(0, T; V_p^2)$. Then we have the following estimates, for $0 < \gamma \le \nu/(2\mu_0^2)$ and $0 \le t \le T$,*

$$\begin{aligned} \|v_3(t)\|_{L^2}^2 \le \ & \exp(-\gamma t)\|v_3(0)\|_{L^2}^2 + \frac{2}{\nu}\sup_s \|A_\varepsilon^{-1/2}(M(Pf(s))_3)\|_{L^2}^2 \\ & + \frac{2}{\nu}K_7\varepsilon\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds \ , \end{aligned} \tag{5.1}$$

and, for $2 \le q < +\infty$,

$$\begin{aligned} \|v_3(t)\|_{L^q}^2 \le \ & K_8(q)\Big(\|v_3(0)\|_{L^q}^2 + \varepsilon^{\frac{2}{q}}\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds \\ & + \sup_s \|\nabla(-\Delta_2)^{-1}(M(Pf(s))_3)\|_{L^q}^2 + \varepsilon^{-2+\frac{6}{q}}\int_{(t-1)^+}^t |w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds)\Big) \ , \end{aligned} \tag{5.2}$$

where $(t-1)^+ = \sup(0, t-1)$ and $K_8(q)$ is a positive constant independent of ε, but depending on q.

Proof. The function v_3 satisfies the following linear equation

$$\partial_t v_3 - \nu\Delta v_3 + \widetilde{v}\nabla v_3 + M((w\nabla w)_3) = M((Pf)_3) \ . \tag{5.3}$$

We first take the scalar product in $L^2(Q_\varepsilon)$ of the above equation with v_3. Since $\widetilde{v}$ and w are divergence-free vector fields, we obtain, by integrating by parts, that

$$\int_{Q_\varepsilon} (\widetilde{v}\nabla v_3)v_3 \, dx = \frac{1}{2}\int_{Q_\varepsilon} \widetilde{v}\nabla v_3^2 \, dx = 0 \ , \tag{5.4}$$

and that

$$(5.5) \qquad \int_{Q_\varepsilon} (w\nabla w_3)v_3\,dx \;=\; -\int_{Q_\varepsilon}(w\nabla v_3)w_3\,dx\ .$$

Applying the estimate (2.7) of the lemma 2.1 to the term $|\int_{Q_\varepsilon}(w\nabla v_3)w_3dx|$, we get, for $0\le t\le T$,

$$\frac{1}{2}\partial_t\|v_3(t)\|_{L^2}^2+\frac{\nu}{2}|v_3(t)|_1^2\;\le\;\frac{1}{\nu}\|A_\varepsilon^{-1/2}(M(Pf(t))_3)\|_{L^2}^2+\frac{1}{\nu}C_1\varepsilon^{-1}|w(t)|_{1/2}^4\ ,$$

or also, by (2.3),

$$(5.6) \quad \partial_t\|v_3(t)\|_{L^2}^2+\frac{\nu}{2}|v_3(t)|_1^2+\frac{\nu}{2\mu_0^2}\|v_3(t)\|_{L^2}^2 \le \frac{2}{\nu}\|A_\varepsilon^{-1/2}(M(Pf(t))_3)\|_{L^2}^2+\frac{2}{\nu}C_1K_0^2\varepsilon|w(t)|_{1/2}^2|w(t)|_{3/2}^2\ .$$

Integrating the inequality (5.6) and using the Gronwall lemma, we obtain, for $0<\gamma\le\nu/(2\mu_0^2)$ and for $0\le t\le T$,

$$(5.7) \quad \|v_3(t)\|_{L^2}^2+\frac{\nu}{2}\exp(-\gamma t)\int_0^t\exp(\gamma s)|v_3(s)|_1^2ds\;\le\;\exp(-\gamma t)\|v_3(0)\|_{L^2}^2 + \frac{2}{\nu\gamma}\sup_s\|A_\varepsilon^{-1/2}(M(Pf(s))_3)\|_{L^2}^2 + \frac{2}{\nu}C_1K_0^2\varepsilon\exp(-\gamma t)\int_0^t\exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2ds\ .$$

Integrating now the inequality (5.6) from t_0 to t_1, we deduce from (5.7) that, for $0\le t_0\le t_1$,

$$(5.8) \quad \int_{t_0}^{t_1}|v_3(s)|_1^2ds\;\le\;\frac{2}{\nu}\exp(-\gamma t_0)\|v_3(0)\|_{L^2}^2+\frac{4}{\nu^2}(\frac{1}{\gamma}+t_1-t_0)\sup_s\|A_\varepsilon^{-1/2}(M(Pf)_3(s))\|_{L^2}^2 + \frac{4}{\nu^2}C_1K_0^2\varepsilon\Big(\exp(-\gamma t)\int_0^{t_0}\exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2ds + \int_{t_0}^{t_1}|w(s)|_{1/2}^2|w(s)|_{3/2}^2ds\Big)\ .$$

We now fix a real number $q\ge2$. Multiplying (5.3) by $|v_3|^{q-2}v_3$, integrating over Q_ε and remarking, as in (5.4), that $\int_{Q_\varepsilon}(\widetilde{v}\nabla v_3)|v_3|^{q-2}v_3dx=0$ we obtain, for $0\le t\le T$

$$(5.9) \quad \frac{1}{q}\partial_t\|v_3\|_{L^q}^q+\nu(q-1)\int_{Q_\varepsilon}|v_3|^{q-2}((\partial_{x_1}v_3)^2+(\partial_{x_2}v_3)^2)dx + \int_{Q_\varepsilon}(w\nabla w_3)|v_3|^{q-2}v_3dx\;=\;\int_{Q_\varepsilon}(M(Pf)_3)|v_3|^{q-2}v_3dx\ .$$

Arguing as in (5.5), we remark that

$$(5.10) \qquad \int_{Q_\varepsilon}(w\nabla w_3)|v_3|^{q-2}v_3\,dx\;=\;-(q-1)\int_{Q_\varepsilon}w_3|v_3|^{q-2}(w_1\partial_{x_1}v_3+w_2\partial_{x_2}v_3)\,dx\ .$$

Furthermore, we have

$$(5.11)\qquad \int_{Q_\varepsilon} (M(Pf)_3)|v_3|^{q-2} v_3\, dx \;=\; (q-1)\int_{Q_\varepsilon} |v_3|^{q-2}\nabla((-\Delta_2)^{-1}(M(Pf)_3))\nabla v_3\, dx \ .$$

Using Hölder inequalities, we deduce from the equalities (5.9), (5.10) and (5.11) that, for $0 \le t \le T$,

$$\frac{1}{q}\partial_t \|v_3(t)\|^q_{L^q} + \frac{\nu}{2}(q-1)\int_{Q_\varepsilon} |v_3(t)|^{q-2}((\partial_{x_1} v_3(t))^2 + (\partial_{x_2} v_3(t))^2)dx \;\le\; (q-1)\nu^{-1}\Big(\int_{Q_\varepsilon} |v_3(t)|^{q-2}|\nabla((-\Delta_2)^{-1}(M(Pf(t))_3))|^2 dx + \varepsilon^{-2+2/q}\sum_{i=1}^{2} \|v_3(t)\|^{q-2}_{L^q}\|w_3(t)\|^2_{L^{2q,2}}\|w_i(t)\|^2_{L^{2q,2}}\Big) \ ,$$

or also, due to Lemma 2.2

$$(5.12)\qquad \frac{1}{q}\partial_t \|v_3(t)\|^q_{L^q} \;\le\; (q-1)\nu^{-1}C_2\|v_3(t)\|^{q-2}_{L^q}\big(\|\nabla(-\Delta_2)^{-1}(M(Pf(t))_3)\|^2_{L^q} + \varepsilon^{-2+\frac{2}{q}}|w(t)|^4_{1-\frac{1}{q}}\big) \ .$$

Since $|w|_{1-1/q} \le C_3|w|^{1/2+1/q}_{1/2}|w|^{1/2-1/q}_{3/2}$, we derive from (5.12) and the Poincaré inequality for w that

$$(5.13)\qquad \frac{1}{2}\partial_t \|v_3(t)\|^2_{L^q} \;\le\; (q-1)\nu^{-1}C_4\big(\|\nabla(-\Delta_2)^{-1}(M(Pf(t))_3)\|^2_{L^q} + \varepsilon^{-2+6/q}|w(t)|^2_{1/2}|w(t)|^2_{3/2}\big) \ .$$

Integrating the estimate (5.13), we obtain, for $0 \le t \le \inf(1,T)$,

$$(5.14)\qquad \|v_3(t)\|^2_{L^q} \;\le\; \|v_3(0)\|^2_{L^q} + (q-1)\nu^{-1}C_4\big(\sup_s \|\nabla(-\Delta_2)^{-1}(M(Pf(s))_3)\|^2_{L^q} + \varepsilon^{-2+6/q}\int_0^t |w(s)|^2_{1/2}|w(s)|^2_{3/2}ds\big) \ .$$

Using the uniform Gronwall lemma, we also deduce from (5.13) that, for $1 \le t \le T$,

$$(5.15)\qquad \|v_3(t)\|^2_{L^q} \;\le\; \int_{t-1}^t \|v_3(s)\|^2_{L^q}ds + (q-1)\nu^{-1}C_4\big(\sup_s \|\nabla(-\Delta_2)^{-1}(M(Pf(s))_3)\|^2_{L^q} + \varepsilon^{-2+6/q}\int_{t-1}^t |w(s)|^2_{1/2}|w(s)|^2_{3/2}ds\big) \ .$$

But, from the Sobolev embedding $H^1(\Omega) \subset L^q(\Omega)$, for $1 \le q < +\infty$ and the inequality (5.8), we infer that

$$\begin{aligned}(5.16)\qquad \int_{t-1}^t \|v_3(s)\|^2_{L^q}ds \;&\le\; \varepsilon^{-1+2/q}\int_{t-1}^t |v_3(s)|^2_1 ds \\ &\le\; C_q\Big(\|v_3(0)\|^2_{L^q} + \sup_s \|\nabla(-\Delta_2)^{-1}(M(Pf(s))_3)\|^2_{L^q} + \\ &\quad + \varepsilon^{2/q}\big[\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|^2_{1/2}|w(s)|^2_{3/2}ds + \int_{t-1}^t |w(s)|^2_{1/2}|w(s)|^2_{3/2}ds\big]\Big) \ .\end{aligned}$$

Finally the estimates (5.14), (5.15) and (5.16) imply that, for $0 \le t \le T$,

$$(5.17)\quad \|v_3(t)\|_{L^q}^2 \le \widetilde{C}_q\Big(\|v_3(0)\|_{L^q}^2 + \varepsilon^{\frac{2}{q}}\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds$$
$$+ (q-1)\big(\sup_s \|\nabla(-\Delta_2)^{-1}(M(Pf(s))_3)\|_{L^q}^2 + \varepsilon^{-2+\frac{6}{q}}\int_{(t-1)^+}^t |w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds\big)\Big)\ ,$$

where $(t-1)^+ = \sup(0, t-1)$. □

Lemma 5.2. *Let u be a weak solution of the Navier-Stokes equations such that $w = (I - M)u \in L^\infty(0,T;V_p^{1/2}) \cap L^2(0,T;V_p^{3/2})$ and $v = Mu \in L^\infty(0,T;V_p) \cap L^2(0,T;V_p^2)$. Then we have the following estimate, for $0 < \gamma \le \nu\mu_0^{-2}$ and $0 \le t \le T$,*

$$(5.18)\quad |\widetilde{v}(t)|_1^2 \le |\widetilde{v}(0)|_1^2 + K_9\Big(\sup_s \|M\widetilde{Pf}(s)\|_{L^2}^2$$
$$+ \varepsilon^{-1}\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds\Big)\ .$$

Proof. We first recall the equation satisfied by $\widetilde{v}$, that is,

$$\partial_t\widetilde{v} - \nu\Delta\widetilde{v} + M(\widetilde{B_\varepsilon(v,v)}) + M(\widetilde{B_\varepsilon(w,w)}) = M(\widetilde{Pf})\ .$$

Taking the scalar product in $L^2(Q_\varepsilon)$ of the above equation with $-\Delta\widetilde{v}$ and remarking, as in [25] that

$$\int_{Q_\varepsilon}(v\nabla v)\Delta\widetilde{v}\,dx = \int_{Q_\varepsilon}(\widetilde{v}\nabla\widetilde{v})\Delta\widetilde{v}\,dx = \varepsilon\int_\Omega(\widetilde{v}\nabla\widetilde{v})\Delta\widetilde{v}\,dx_1dx_2 = 0\ ,$$

we obtain the equality

$$(5.19)\qquad \frac{1}{2}\partial_t|\widetilde{v}|_1^2 + \nu|\widetilde{v}|_2^2 + \int_{Q_\varepsilon}(\widetilde{w\nabla w})(-\Delta\widetilde{v})\,dx = \int_{Q_\varepsilon}\widetilde{Pf}(-\Delta\widetilde{v})\,dx\ .$$

Since, by the estimate (2.7) of Lemma 2.1,

$$\int_{Q_\varepsilon}(\widetilde{w\nabla w})(-\Delta\widetilde{v})dx \le C\varepsilon^{-1/2}|w|_{1/2}|w|_{3/2}\|\Delta\widetilde{v}\|_{L^2}\ ,$$

we infer from (5.19), by using also a Young inequality, that, for $0 \le t \le T$,

$$(5.20)\qquad \partial_t|\widetilde{v}(t)|_1^2 + \nu|\widetilde{v}(t)|_2^2 \le 2\nu^{-1}\|M\widetilde{Pf}(t)\|_{L^2}^2 + 2\nu^{-1}C^2\varepsilon^{-1}|w(t)|_{1/2}^2|w(t)|_{3/2}^2\ ,$$

or also

$$(5.21)\qquad \partial_t|\widetilde{v}(t)|_1^2 + \frac{\nu}{\mu_0^2}|\widetilde{v}(t)|_1^2 \le 2\nu^{-1}\|M\widetilde{Pf}(t)\|_{L^2}^2 + 2\nu^{-1}C^2\varepsilon^{-1}|w(t)|_{1/2}^2|w(t)|_{3/2}^2\ ,$$

Integrating the inequality (5.21) and using the Gronwall lemma, we obtain, for $0 < \gamma \le \nu\mu_0^{-2}$ and for $0 \le t \le T$,

$$(5.22)\quad |\widetilde{v}(t)|_1^2 \le \exp(-\gamma t)|\widetilde{v}(0)|_1^2 + \frac{2}{\nu\gamma}\sup_s \|M\widetilde{Pf}(s)\|_{L^2}^2$$
$$+ 2\nu^{-1}C^2\varepsilon^{-1}\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds\ ,$$

which at once implies the estimate (5.18) of Lemma 5.2. $\square$

Now we can prove Theorem 1.3.

Proof. Like in the proof of Theorem 1.1, we consider a Galerkin approximation, using the first m eigenfunctions ψ_1, ψ_2, ...,ψ_m of the Stokes operator A_ε. As in Theorem 1.1, these eigenfunctions ψ_j are chosen so that, either $\psi_j \in MH_p$ or $\psi_j \in (I-M)H_p$. Moreover, if the eigenvector ψ_j is independent of the third variable x_3, it can be chosen so that, either $\psi_j \equiv M\psi_j = (M\psi_{j1}, M\psi_{j2}, 0)$ or $\psi_j = (0,0,M\psi_{j3})$. These properties imply that, if $\mathcal{P}_m : H_p \to H_p$ denotes the projector onto the space $\mathcal{V}_m$ generated by the first m eigenfunctions, then, for every $s \in [0,2]$ and for every $u \in V_p^s$, the inequalities

$$|\mathcal{P}_m M u_3|_s \leq |Mu_3|_s \ , \quad |\mathcal{P}_m \widetilde{Mu}|_s \leq |\widetilde{Mu}|_s \ , \tag{5.23}$$

as well as the inequalities (4.1) hold. We recall that $\mathcal{P}_m(I-M)u_0$ (resp. $\mathcal{P}_m M u_0$) converges to $(I-M)u_0$ (resp. Mu_0) in $V_p^{1/2}$ (resp. V_p), as m goes to $+\infty$.

Like in the proof of Theorem 1.1, we know (see [5], Chapter 8, for example, or [26]) that, for every $m \in \mathbb{N}$, there exists a global solution $u_m \in C^1([0,+\infty); V_p^2 \cap \mathcal{V}_m)$ of the equations (1.11) or also of (1.20) and (1.21), where B_ε is replaced by $\mathcal{P}_m B_\varepsilon$ and Pf by $\mathcal{P}_m Pf$ and where the initial condition is $u_m(0) = \mathcal{P}_m(I-M)u_0 + \mathcal{P}_m M u_0 \equiv w_{0m} + v_{0m}$. Moreover, for every $\tau > 0$, u_m and $\partial_t u_m$ are uniformly bounded with respect to m in the spaces $L^\infty(0,+\infty; H_p) \cap L^2(0,\tau; V_p)$ and $L^{4/3}(0,\tau; V_p')$ respectively. We want to show that this solution u_m satisfies the additional estimates and properties given in Theorem 1.3. In order to simplify the notation, we drop the subscript m, when there is no confusion. Like in the proof of Theorem 1.1, we take the scalar product in $L^2(Q_\varepsilon)$ of the modified equation (1.21) with $A_\varepsilon^{1/2} w = (-\Delta)^{1/2} w$ and obtain the equality (4.2). Applying the inequality (2.6) of Lemma 2.1, we have, for $t \geq 0$,

(5.24)
$$\left| \int_{Q_\varepsilon} (w\nabla w (-\Delta)^{\frac{1}{2}} w)(t,x))dx \right| \leq K_1 |w(t)|_{\frac{1}{2}} |\nabla w(t)|_{\frac{1}{2}} |(-\Delta)^{\frac{1}{2}} w(t)|_{\frac{1}{2}} \leq C |w(t)|_{1/2} |w(t)|_{3/2}^2 \ .$$

In order to estimate the term $\int_{Q_\varepsilon} v\nabla w((-\Delta)^{1/2}w)dx$, we apply Lemma 2.4 and obtain, for $t \geq 0$,

(5.25)
$$\left| \int_{Q_\varepsilon} (v\nabla w (-\Delta)^{1/2} w)(t,x)dx \right| \leq K_5 \varepsilon^{-1/2} |v(t)|_1 |w(t)|_{1/2} |w(t)|_{3/2} \leq C\varepsilon^{1/2} |v(t)|_1 |w(t)|_{3/2}^2 \ .$$

To bound the third nonlinear term $\int_{Q_\varepsilon} w\nabla v((-\Delta)^{1/2}w)dx$, we can use the estimate (2.7) of Lemma 2.1 as follows,

$$\begin{aligned} \left| \int_{Q_\varepsilon} (w\nabla v (-\Delta)^{1/2} w)(t,x)dx \right| &\leq K_2 \varepsilon^{-1/2} |\nabla v(t)|_0 |w(t)|_{1/2} |(-\Delta)^{1/2} w(t)|_{1/2} \\ &\leq C\varepsilon^{1/2} |v(t)|_1 |w(t)|_{3/2}^2 \ . \end{aligned} \tag{5.26}$$

Finally, like in the proof of Theorem 1.1, we write, for $t \geq 0$,

$$\left| \int_{Q_\varepsilon} ((I-M)f(-\Delta)^{1/2} w)(t,x)dx \right| \leq C\nu^{-1}\varepsilon \|(I-M)Pf(t)\|_{L^2}^2 + \frac{\nu}{4} |w(t)|_{3/2}^2 \ . \tag{5.27}$$

Due to the estimates (4.2), (5.24), (5.25), (5.26) and (5.27), we have, for $t \geq 0$,

(5.28)
$$\partial_t |w(t)|^2_{1/2} + 2(\frac{3\nu}{4} - C_1|w(t)|_{1/2} - C_2\varepsilon^{1/2}|v(t)|_1)|w(t)|^2_{3/2} \leq C_3\nu^{-1}\varepsilon\|(I-M)Pf(t)\|^2_{L^2}.$$

Due to the property (4.1) and to the hypothesis (1.29) on the initial conditions, where k_1, k_2 are small enough, there exists a positive time T such that, for $t \in [0,T)$,

$$C_1|w(t)|_{1/2} + C_2\varepsilon^{1/2}|v(t)|_1 \, < \, \frac{\nu}{2} \, . \tag{5.29}$$

and, that, if $T < \infty$,

$$C_1|w(T)|_{1/2} + C_2\varepsilon^{1/2}|v(T)|_1 \, = \, \frac{\nu}{2} \, . \tag{5.30}$$

We shall show by contradiction that $T = +\infty$. To this end, we shall estimate separately the terms $|w(t)|_{1/2}$, $|\widetilde{v}(t)|_1$ and $|v_3(t)|_1$. The estimate of the term $|\widetilde{v}(t)|_1$ will be a consequence of Lemma 5.2.

We derive from the estimates (5.28), (5.29) and (5.30) that, for $t \in [0,T]$,

$$\partial_t |w(t)|^2_{1/2} + \frac{\nu}{2}|w(t)|^2_{3/2} \, \leq \, C_3\nu^{-1}\varepsilon\|(I-M)Pf(t)\|^2_{L^2}, \tag{5.31}$$

which in turn implies that

$$\begin{aligned} \partial_t |w(t)|^2_{1/2} + \frac{\nu}{2}\varepsilon^{-2}K_0^{-2}|w(t)|^2_{1/2} \, &\leq \partial_t |w(t)|^2_{1/2} + \frac{\nu}{4}\varepsilon^{-2}K_0^{-2}|w(t)|^2_{1/2} + \frac{\nu}{4}|w(t)|^2_{3/2} \\ &\leq C_3\nu^{-1}\varepsilon\|(I-M)Pf(t)\|^2_{L^2} \, . \end{aligned} \tag{5.32}$$

The Gronwall lemma then gives, for $t \in [0,T]$,

$$|w(t)|^2_{1/2} \, \leq \, \exp(-\frac{\nu}{2}\varepsilon^{-2}K_0^{-2}t)|w_0|^2_{1/2} + C_4\nu^{-2}\varepsilon^3 \sup_s \|(I-M)Pf(s)\|^2_{L^2} \, . \tag{5.33}$$

On the other hand, integrating the inequalities (5.32), we get, for $0 < \gamma \leq \frac{\nu}{4}\varepsilon^{-2}K_0^{-2}$ and for $0 \leq t_1 < t_2 \leq T$,

$$\begin{aligned} |w(t_2)|^2_{1/2} + \exp(-\gamma t_2)\frac{\nu}{4}\int_{t_1}^{t_2} \exp(\gamma s)|w(s)|^2_{3/2} ds \, &\leq \, \exp(-\gamma(t_2-t_1))|w(t_1)|^2_{1/2} \\ &+ C_3(\gamma\nu)^{-1}(1-\exp(-\gamma(t_2-t_1)))\varepsilon \sup_s \|(I-M)Pf(s)\|^2_{L^2} \, . \end{aligned} \tag{5.34}$$

We now fix a positive number γ, satisfying $0 < \gamma \leq \inf(\frac{\nu}{2\mu_0^2}, \frac{\nu}{4}\varepsilon^{-2}K_0^{-2})$. We deduce from the estimates (5.33) and (5.34) that, for $t \in [0,T]$,

(5.35)
$$\begin{aligned} \exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|^2_{1/2}|w(s)|^2_{3/2} ds \, &\leq \, |w_0|^4_{\frac{1}{2}} + C_3C_4\gamma^{-1}\nu^{-3}\varepsilon^4 \sup_s \|(I-M)Pf(s)\|^4_{L^2} \\ &+ |w_0|^2_{\frac{1}{2}} \sup_s \|(I-M)Pf(s)\|^2_{L^2}[C_4\nu^{-2}\varepsilon^3 \\ &+ C_3(\gamma\nu)^{-1}\varepsilon(1-\exp(-\gamma t))\exp(-\frac{\nu}{2}\varepsilon^{-2}K_0^{-2}t)] \\ &\leq \, C_5|w_0|^4_{\frac{1}{2}} + C_6 \sup_s \|(I-M)Pf(s)\|^4_{L^2}(\varepsilon^4 + \varepsilon^2 E(t)) \, , \end{aligned}$$

where $E(t) = (1-\exp(-\gamma t))^2 \exp(-\nu\varepsilon^{-2}K_0^{-2}t)$. On the one hand, we remark that, for $t \geq t_\varepsilon$, where $t_\varepsilon = -2\varepsilon^2 K_0^2\ \nu^{-1}\ln\varepsilon$, $E(t) \leq \varepsilon^2$. On the other hand, for $t \leq t_\varepsilon$, we notice that $E(t) \leq (1-\exp(-\gamma t))^2 \leq \gamma^2 t_\varepsilon^2 \leq C\varepsilon^2$. From these remarks and from the estimate (5.35), we finally infer that, for $t \in [0,T]$,

$$\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|_{1/2}^2 |w(s)|_{3/2}^2 ds \ \leq\ \varepsilon D_1\ , \tag{5.36}$$

where $D_1 = C_7(\varepsilon^{-1}|w_0|_{1/2}^4 + \varepsilon^3 \sup_s \|(I-M)Pf(s)\|_{L^2}^4)$.

Lemma 5.2, the inequality (5.36) and the property (4.1) imply that, for $t \in [0,T]$,

$$|\widetilde{v}(t)|_1^2 \ \leq\ D_0 + D_1\ , \tag{5.37}$$

where $D_0 = |\widetilde{v}_0|_1^2 + C_8 \sup_s \|M(\widetilde{Pf})(s)\|_{L^2}^2$.

It remains to estimate the term $|v_3(t)|_1$. Taking the scalar product in $L^2(Q_\varepsilon)$ of the modified equation (5.3) with $A_\varepsilon v_3$, applying the estimate (2.7) of Lemma 2.1 as well as the estimate (2.19) of Lemma 2.5, we obtain, for $t \in [0,T]$,

$$\begin{aligned}\partial_t|v_3(t)|_1^2 + 2\nu|v_3(t)|_2^2 \ \leq\ & 2|v_3(t)|_2\big(\|\mathcal{P}_m M(Pf)_3(t)\|_{L^2} \\ & + K_2\varepsilon^{-1/2}|w(t)|_{1/2}|w(t)|_{3/2} + K_6\varepsilon^{-1/2}|\widetilde{v}(t)|_1|v_3(t)|_1\big)\ ,\end{aligned}$$

or also

$$\begin{aligned}\partial_t|v_3(t)|_1^2 + \frac{2\nu}{\mu_0^2}|v_3(t)|_1^2 \ \leq\ & \partial_t|v_3(t)|_1^2 + 2\nu|v_3(t)|_2^2 \ \leq \\ \leq\ & C_9\nu^{-1}\big(\|\mathcal{P}_m M(Pf)_3(t)\|_{L^2}^2 + \varepsilon^{-1}|w(t)|_{1/2}^2|w(t)|_{3/2}^2 + \varepsilon^{-1}|\widetilde{v}(t)|_1^2|v_3(t)|_1^2\big)\ .\end{aligned} \tag{5.38}$$

By integration, it follows from (5.38) and (5.23) that, for $t \in [0,T]$,

$$\begin{aligned}|v_3(t)|_1^2 \ \leq\ & \exp(-\gamma t)|v_3(0)|_1^2 + C_9\nu^{-1}\Big(\gamma^{-1}\sup_s \|M(Pf)_3(s)\|_{L^2}^2 \\ & + \varepsilon^{-1}\exp(-\gamma t)\int_0^t \exp(\gamma s)|w(s)|_{1/2}^2|w(s)|_{3/2}^2 ds \\ & + \varepsilon^{-1}\exp(-\gamma t)\sup_s |\widetilde{v}(s)|_1^2 \int_0^t \exp(\gamma s)|v_3(s)|_1^2 ds\Big)\ .\end{aligned} \tag{5.39}$$

We infer from (5.7), (5.34), (5.36) and (5.39) that, for $t \in [0,T]$,

$$|v_3(t)|_1^2 \ \leq\ |v_3(0)|_1^2 + C_{10}\Big(\sup_s \|M(Pf)_3(s)\|_{L^2}^2 + D_1 + (D_0+D_1)D_2\Big)\ , \tag{5.40}$$

where

$$D_2 \ =\ \varepsilon^{-1}\big(\|v_3(0)\|_{L^2}^2 + \sup_s \|A_\varepsilon^{-1/2}(M(Pf)_3(s))\|_{L^2}^2\big) + \varepsilon D_1\ . \tag{5.41}$$

Finally, the inequalities (5.37) and (5.40) give, for $t \in [0,T]$,

$$|v(t)|_1^2 \ \leq\ |v_3(0)|_1^2 + D_0 + D_1 + C_{10}\Big(\sup_s \|M(Pf)_3(s)\|_{L^2}^2 + D_1 + (D_0+D_1)D_2\Big)\ . \tag{5.42}$$

If k_1, k_2, k_3, k_4, k_5 and k_6 are small enough, the properties (4.1), (5.23), the hypotheses (1.29) and (1.30) together with the estimates (5.33) and (5.42) imply that, for $t \in [0, T]$,

$$C_1|w(T)|_{1/2} + C_2\varepsilon^{1/2}|v(T)|_1 < \frac{\nu}{4} , \tag{5.43}$$

which contradicts the equality (5.30). It follows that $T = +\infty$.

We have just proved that, under the hypotheses (1.29) and (1.30), for any $m \in \mathbb{N}$, the solution $u_m \in C^1([0, +\infty); \mathcal{V}_m)$ of the modified Navier-Stokes equations (1.11) with initial data $u_m(0) = \mathcal{P}_m u_{0m}$ satisfies

$$\sup_{t\geq 0}(|w_m(t)|_{1/2} + \varepsilon^{1/2}|v_m(t)|_1) < C_{11} , \tag{5.44}$$

where C_{11} is a positive constant independent of ε and m. Integrating the inequalities (5.31), (5.20) and (5.38) and using the estimates (5.44), (5.37), (5.6) as well as the hypotheses (1.29) and (1.30), one also shows that, for any $t \in [0, +\infty)$,

$$\int_0^t (|w_m(s)|^2_{3/2} + \varepsilon|v_m(s)|^2_2)ds \leq \varepsilon^{-1}C_{12}t , \tag{5.45}$$

where C_{12} is a positive constant independent of ε and m.

Like in the proof of Theorem 1.1, a classical argument (see [5], Chapter 8 or [26]) together with the estimates (5.44) and (5.45), shows that $u = \lim_{m\to+\infty} u_m$ belongs to the space $L^\infty(0, \infty; V_p^{1/2}) \cap L^2_{loc}([0, \infty); V_p^{3/2})$, is a weak Leray solution of the equations (1.11) with initial data $u(0) = u_0$ and that $Mu \in L^\infty(0, \infty; V_p) \cap L^2_{loc}([0, \infty); V_p^2)$. The uniqueness of the solution u follows from Theorem 3.1. Arguing as in Remark 3.1, we actually show that $\partial_t u$ belongs to $L^2_{loc}([0, \infty); V_p^{-1/2})$. Indeed, we deduce from the equality (3.1), Lemma 2.1 and Remark 2.1 that, for any $t \geq 0$, for any $\varphi \in L^2(0, t; V_p^{1/2})$,

$$\left|\int_0^t \int_{Q_\varepsilon} (u\nabla u)(s, x)\varphi(s, x)\, dxds\right| \leq C_\varepsilon \|u\|_{L^\infty(0,t;V_p^{1/2})}\|u\|_{L^2(0,t;V_p^{3/2})}\|\varphi\|_{L^2(0,t;V_p^{1/2})} ,$$

which implies that $u\nabla u$ belongs to $L^2_{loc}([0, \infty); V_p^{-1/2})$. It follows, since Δu and Pf also belong to this space, that $\partial_t u$ belongs to $L^2_{loc}([0, \infty); V_p^{-1/2})$. As $u \in L^2_{loc}([0, \infty); V_p^{3/2}) \cap H^1_{loc}([0, \infty); V_p^{-1/2})$, u is also in the space $C^0([0, +\infty); V_p^{1/2})$. The vector $v = Mu$ actually lies in the space $C^0([0, +\infty); V_p)$. Indeed, applying the estimate (2.7) of Lemma 2.1 and Remark 2.1, we obtain, for $t \geq 0$ and $\varphi \in L^2(0, t; H_p)$,

$$\begin{aligned}&\left|\int_0^t \int_{Q_\varepsilon} (v\nabla v + w\nabla w)(s, x)M\varphi(x)\, dxds\right| \\ &\quad \leq C_\varepsilon(\|v\|_{L^\infty(0,t;V_p^{1/2})}\|v\|_{L^2(0,t;V_p^{3/2})} + \|w(s)\|_{L^\infty(0,t;V_p^{1/2})}\|w(s)\|_{L^2(0,t;V_p^{3/2})})\|\varphi\|_{L^2(0,t;H_p)} ,\end{aligned}$$

which implies that $MB_\varepsilon(v, v) + MB_\varepsilon(w, w)$ belongs to the space $L^2_{loc}([0, \infty); H_p)$. As $v \in L^2_{loc}([0, \infty); V_p^2)$ and $Pf \in L^2_{loc}([0, \infty); H_p)$, we deduce from the equation (1.20) that $\partial_t v \in L^2_{loc}([0, \infty); H_p)$ and thus that $v \in C^0([0, +\infty); V_p)$. □

In some sense, we can improve the global existence results given in Theorem 1.3, if, in the various estimates, we also take into account the L^q-norm of v_3, where, for instance, $q \geq 3$. The hypotheses in the following theorem are rather involved, but, in the applications, it allows to take larger initial data and forcing terms.

Theorem 5.1. *For any real number $q > 2$, there exist positive constants $k_1(q)$, $k_2(q)$, $k_3(q)$, $k_4(q)$, $k_5(q)$ and $k_6(q)$ such that, for $0 < \varepsilon \le 1$, if the initial data $(Mu_0, (I-M)u_0) \in V_p \times V_p^{1/2}$ and the force $f \in L^\infty(0,\infty;(L^2(Q_\varepsilon))^3)$ satisfy*

$$
\begin{aligned}
&|Mu_0|_1 \le k_1(q)\varepsilon^{-1/2}, \quad |(I-M)u_0|_{1/2} \le k_2(q) \\
&\sup_t \|M\widetilde{Pf}(t)\|_{L^2} \le k_3(q)\varepsilon^{-1/2}, \quad \sup_t \|(I-M)Pf(t)\|_{L^2} \le k_4(q)\varepsilon^{-1} \\
&\|Mu_{03}\|_{L^q} + \sup_t \|\nabla(-\Delta_2)^{-1}(MPf_3)(t)\|_{L^q} \le k_5(q)\varepsilon^{-1+3/q} \, ,
\end{aligned}
\tag{5.46}
$$

and the additional condition

$$
\begin{aligned}
&\Big(\varepsilon^{1-3/q}(\|Mu_{03}\|_{L^q} + \sup_t \|\nabla(-\Delta_2)^{-1}(MPf_3)(t)\|_{L^q}) + |(I-M)u_0|_{1/2}^2 \\
&+ \varepsilon^2 \sup_t \|(I-M)Pf(t)\|_{L^2}^2\Big) \times \Big(\varepsilon^{1/2}(|Mu_{30}|_1 + \sup_t \|M(Pf)_3(t)\|_{L^2}) + \mathcal{A}_0\Big) \le k_6(q) \, ,
\end{aligned}
\tag{5.47}
$$

where $\mathcal{A}_0$ has been defined in Theorem 1.3, then there exists a global solution $u(t) \in C^0([0,\infty);V_p^{1/2}) \cap L^\infty(0,\infty;V_p^{1/2}) \cap L^2_{loc}([0,\infty);V_p^{3/2})$ of (1.11) which is unique in the class of weak Leray solutions. Moreover, $Mu \in C^0([0,\infty);V_p) \cap L^\infty(0,\infty;V_p) \cap L^2_{loc}([0,\infty);V_p^2)$ and $u(t)$ satisfies the estimates (5.33), (5.37), (5.59) and (5.60), for every $t \ge 0$.

Proof. We use the same Galerkin basis as in the proof of Theorem 1.3, so that the properties (4.1) and (5.23) hold. Since $\mathcal{P}_m Mu_0$ converges to Mu_0 in V_p, $\mathcal{P}_m Mu_{03}$ also converges to Mu_{03} in $L^q(\Omega)$. Hence, there exists $m_1 = m_1(\varepsilon, q)$ such that, for $m \ge m_1$, $\|\mathcal{P}_m Mu_{03}\|_{L^q(\Omega)} \le 2\|Mu_{03}\|_{L^q(\Omega)}$ and thus that

$$
\|\mathcal{P}_m Mu_{03}\|_{L^q(Q_\varepsilon)} \le 2\|Mu_{03}\|_{L^q(Q_\varepsilon)} \, . \tag{5.48}
$$

Likewise, for any $t \in [0,+\infty)$, $\mathcal{P}_m \nabla(-\Delta_2)^{-1}(MPf_3)(t)$ converges to $\nabla(-\Delta_2)^{-1}(MPf_3)(t)$ in $L^q(\Omega)$, when m goes to $+\infty$. But, since $\{\nabla(-\Delta_2)^{-1}(MPf)(t) \,|\, t \in [0,+\infty)\}$ is a bounded set in V_p, $\{\nabla(-\Delta_2)^{-1}(MPf_3)(t) \,|\, t \in [0,+\infty)\}$ is a compact set in $L^q(\Omega)$ and thus, there exists $m_2 = m_2(\varepsilon, q)$ such that, for $m \ge m_2$, for $t \in [0,+\infty)$,

$$
\|\mathcal{P}_m \nabla(-\Delta_2)^{-1}(MPf_3)(t)\|_{L^q(\Omega)} \le 2 \sup_t \|\nabla(-\Delta_2)^{-1}(MPf_3)(t)\|_{L^q(\Omega)} \, ,
$$

and

$$
\|\mathcal{P}_m \nabla(-\Delta_2)^{-1}(MPf_3)(t)\|_{L^q(Q_\varepsilon)} \le 2 \sup_t \|\nabla(-\Delta_2)^{-1}(MPf_3)(t)\|_{L^q(Q_\varepsilon)} \, . \tag{5.49}
$$

We set $m_0 = \sup(m_1, m_2)$. Like in the proof of Theorem 1.3, for every $m \ge m_0$, we know that there exists a global solution $u_m \equiv v_m + w_m = Mu_m + (I-M)u_m$ of the equations (1.11), where B_ε is replaced by $\mathcal{P}_m B_\varepsilon$ and Pf by $\mathcal{P}_m Pf$ and where the initial condition is $u_m(0) = \mathcal{P}_m u_0 \equiv w_{0m} + v_{0m}$. We shall prove a priori estimates on the solution u_m. We again drop the subscript m, when there is no confusion. Like in the proof of Theorem 1.3, taking the inner product in $L^2(Q_\varepsilon)$ of the modified equation (1.21) with $A_\varepsilon^{1/2}w$, we are led to estimate $\int_{Q_\varepsilon} w\nabla w((-\Delta)^{1/2}w)dx$, $\int_{Q_\varepsilon} v\nabla w((-\Delta)^{1/2}w)dx$ and $\int_{Q_\varepsilon} w\nabla v((-\Delta)^{1/2}w)dx$. The estimate of the first term does not change and is given in (5.24). Decomposing v into $\widetilde{v} + v_3$

and applying the inequality (5.25) to $\widetilde{v}$, we can write, for $t \geq 0$,

(5.50)

$$\left|\int_{Q_\varepsilon} (v\nabla w\,(-\Delta)^{1/2}w)(t,x)dx\right| \leq C\varepsilon^{1/2}|\widetilde{v}(t)|_1|w(t)|_{3/2}^2 + \left|\int_{Q_\varepsilon} (v_3\partial_{x_3}w(-\Delta)^{1/2}w)(t,x)dx\right| .$$

But an anisotropic Hölder inequality and Lemma 2.2 imply that, for $t \geq 0$,

$$\begin{aligned}\left|\int_{Q_\varepsilon} (v_3\partial_{x_3}w(-\Delta)^{\frac{1}{2}}w)(t,x)dx\right| &\leq C\varepsilon^{-\frac{1}{q}}\|v_3(t)\|_{L^q}\|(-\Delta)^{\frac{1}{2}}w(t)\|_{L^{\frac{2q}{q-1},2}}\|\partial_{x_3}w(t)\|_{L^{\frac{2q}{q-1},2}} \\ &\leq C\varepsilon^{-\frac{1}{q}}\|v_3(t)\|_{L^q}|w(t)|_{1+1/q}^2 ,\end{aligned}$$

or also, due to the Poincaré inequality for w,

$$\left|\int_{Q_\varepsilon} (v_3\partial_{x_3}w\,(-\Delta)^{1/2}w)(t,x)dx\right| \leq C\varepsilon^{1-3/q}\|v_3(t)\|_{L^q}|w(t)|_{3/2}^2 . \tag{5.51}$$

To estimate the third term, we again write v as $\widetilde{v}+v_3$, apply the inequality (5.26) to $\widetilde{v}$ and remark that $\int_{Q_\varepsilon} w\nabla v_3((-\Delta)^{1/2}w_3)dx = -\int_{Q_\varepsilon} v_3 w(\nabla((-\Delta)^{1/2}w_3))dx$, which implies that, for $t \geq 0$,

(5.52)

$$\begin{aligned}\left|\int_{Q_\varepsilon} (w\nabla v(-\Delta)^{1/2}w)(t,x)dx\right| &\leq C\varepsilon^{1/2}|\widetilde{v}(t)|_1|w(t)|_{3/2}^2 + \left|\int_{Q_\varepsilon} (w\nabla v_3(-\Delta)^{1/2}w_3)(t,x)dx\right| \\ &\leq C\Big(\varepsilon^{1/2}|\widetilde{v}(t)|_1|w(t)|_{3/2}^2 + |w_3(t)|_{3/2}\|(-\Delta)^{1/4}(wv_3)(t)\|_{L^2}\Big) .\end{aligned}$$

It remains to bound the term $\|(-\Delta)^{1/4}(wv_3)\|_{L^2}$. A quick computation using Fourier series shows that we have, for any $h \in \dot{H}_p^{1/2}(Q_\varepsilon)$,

$$\|(-\Delta)^{1/4}h\|_{L^2} \leq \|\,\|(-\Delta_2)^{1/4}h\|_{L^2_{x'}(\Omega)}\|_{L^2_{x_3}(0,\varepsilon)} + \|\,\|(-\partial^2_{x_3x_3})^{1/4}h\|_{L^2_{x_3}(0,\varepsilon)}\|_{L^2_{x'}(\Omega)} .$$

Since v_3 is independent of x_3, it follows from the above inequality that

$$\begin{aligned}\|(-\Delta)^{1/4}(wv_3)\|_{L^2} \leq C\Big(&\|\,\|(-\Delta_2)^{1/4}(v_3w)\|_{L^2_{x'}(\Omega)}\|_{L^2_{x_3}(0,\varepsilon)} \\ &+ \|v_3\|(-\partial^2_{x_3x_3})^{1/4}w\|_{L^2_{x_3}(0,\varepsilon)}\|_{L^2_{x'}(\Omega)}\Big) .\end{aligned} \tag{5.53}$$

But, the Hölder inequality, Lemma 2.2 and the Poincaré inequality (2.2) imply that

(5.54)

$$\|v_3\|(-\partial^2_{x_3x_3})^{\frac{1}{4}}w\|_{L^2_{x_3}(0,\varepsilon)}\|_{L^2_{x'}(\Omega)} \leq C\|v_3\|_{L^q(\Omega)}\|(-\partial^2_{x_3x_3})^{\frac{1}{4}}w\|_{L^{2q/(q-2),2}} \leq C\varepsilon^{1-\frac{3}{q}}\|v_3\|_{L^q}|w|_{\frac{3}{2}} .$$

On the other hand, applying the periodic version of the multiplication property given in Theorem 5.1 of [16], we can write,

$$\begin{aligned}\|\,\|(-\Delta_2)^{1/4}(v_3w)\|_{L^2_{x'}(\Omega)}\|^2_{L^2_{x_3}(0,\varepsilon)} \leq C\Big(\int_0^\varepsilon &(\|(-\Delta_2)^{1/4}v_3\|^2_{L^r(\Omega)}\|w\|^2_{L^{2r/(r-2)}_{x'}(\Omega)} \\ &+ \|(-\Delta_2)^{1/4}w\|^2_{L^{2q/(q-2)}_{x'}(\Omega)}\|v_3\|^2_{L^q(\Omega)})dx_3\Big) ,\end{aligned}$$

where $r = \frac{4q}{q+2}$. Using the two-dimensional Sobolev embedding theorems, we infer from the above estimate that

(5.55)
$$\| \|(-\Delta_2)^{1/4}(v_3 w)\|_{L^2_{x'}(\Omega)}\|^2_{L^2_{x_3}(0,\varepsilon)} \leq C(\int_0^{\varepsilon} (\|(-\Delta_2)^{1/4} v_3\|^2_{L^r(\Omega)} \|(-\Delta_2)^{s_1/2} w\|^2_{L^2_{x'}(\Omega)} + \|(-\Delta_2)^{1/4+s_2/2} w\|^2_{L^2_{x'}(\Omega)} \|v_3\|^2_{L^q(\Omega)}) dx_3) \ ,$$

where $s_1 = 2/r$ and $s_2 = 2/q$. Applying then the following Gagliardo-Nirenberg inequality (see [2] or [3])

$$\|(-\Delta_2)^{1/4} v_3\|_{L^r(\Omega)} \leq C_q \|v_3\|^{1/2}_{L^q(\Omega)} \|(-\Delta_2)^{1/2} v_3\|^{1/2}_{L^2(\Omega)} \ ,$$

we deduce from (5.55) that

$$\begin{aligned} \| \|(-\Delta_2)^{1/4}(v_3 w)\|_{L^2_{x'}(\Omega)}\|_{L^2_{x_3}(0,\varepsilon)} &\leq C_q (\|v_3\|^{1/2}_{L^q(\Omega)} \|(-\Delta_2)^{1/2} v_3\|^{1/2}_{L^2(\Omega)} \|(-\Delta)^{s_1/2} w\|_{L^2} \\ &\quad + \|v_3\|_{L^q(\Omega)}) \|(-\Delta)^{\frac{s_2}{2}+\frac{1}{4}} w\|_{L^2}) \\ &\leq C_q \Big(\varepsilon^{\frac{3}{4}-\frac{3}{2q}} \|v_3\|^{\frac{1}{2}}_{L^q} |v_3|^{\frac{1}{2}}_1 + \varepsilon^{1-\frac{3}{q}} \|v_3\|_{L^q} \Big) |w|_{3/2} \ . \end{aligned} \tag{5.56}$$

Finally, due to the estimates (5.24), (5.50), (5.51), (5.52), (5.53), (5.54) and (5.56), the solution $u_m = w_m + v_m$ satisfies the following inequality, for $t \geq 0$,

$$\begin{aligned} \partial_t |w_m(t)|^2_{1/2} + 2\Big(\frac{3\nu}{4} - C_1 |w_m(t)|_{1/2} - C_2 \varepsilon^{1/2} |\widetilde{v}_m(t)|_1 - c_{1q} \varepsilon^{1-3/q} \|v_{m3}(t)\|_{L^q} \\ - c_{2q} \varepsilon^{3/4-3/(2q)} \|v_{m3}(t)\|^{1/2}_{L^q} |v_{m3}(t)|^{1/2}_1 \Big) |w_m(t)|^2_{3/2} \\ \leq C_3 \nu^{-1} \varepsilon \|(I-M)Pf\|^2_{L^2} \ . \end{aligned}$$

Since $u_m(t)$ belongs to the space $C^0([0,\tau); V_p^2 \cap \mathcal{V}_m)$, we infer from the properties (4.1), (5.23), (5.48) and the hypothesis (5.46) on the initial conditions, where $k_1(q)$, $k_2(q)$, $k_5(q)$ are small enough, that there exists a positive time T such that, for $t \in [0,T)$,

(5.57)
$$C_1 |w_m(t)|_{\frac{1}{2}} + C_2 \varepsilon^{\frac{1}{2}} |\widetilde{v}_m(t)|_1 + c_{1q} \varepsilon^{1-\frac{3}{q}} \|v_{m3}(t)\|_{L^q} + c_{2q} \varepsilon^{\frac{3}{4}-\frac{3}{2q}} \|v_{m3}(t)\|^{\frac{1}{2}}_{L^q} |v_{m3}(t)|^{\frac{1}{2}}_1 < \frac{\nu}{2} \ ,$$

and, if $T < +\infty$,

(5.58)
$$C_1 |w_m(T)|_{\frac{1}{2}} + C_2 \varepsilon^{\frac{1}{2}} |\widetilde{v}_m(T)|_1 + c_{1q} \varepsilon^{1-\frac{3}{q}} \|v_{m3}(T)\|_{L^q} + c_{2q} \varepsilon^{\frac{3}{4}-\frac{3}{2q}} \|v_{m3}(T)\|^{\frac{1}{2}}_{L^q} |v_{m3}(T)|^{\frac{1}{2}}_1 = \frac{\nu}{2} \ ,$$

Then, as shown in the proof of Theorem 1.3, $w_m(t)$, $\widetilde{v}_m(t)$ and $v_{m3}(t)$ satisfy the estimates (5.33), (5.34), (5.35), (5.36), (5.37) and (5.40), for $t \in [0,T]$. Moreover, we deduce from the inequalities (5.2), (5.36) and the property (5.49), that, for $t \in [0,T]$,

$$\|v_{m3}(t)\|_{L^q} \leq C(q)(\|v_{03}\|_{L^q} + \sup_s \|\nabla(-\Delta_2)^{-1}(M(Pf(s))_3)\|_{L^q} + \varepsilon^{-1/2+3/q} D_1^{1/2}) \ . \tag{5.59}$$

Using the estimates (5.33), (5.37), (5.40) and (5.59), one shows that, if the hypotheses (5.46) and (5.47) are satisfied for sufficiently small constants $k_1(q)$, $k_2(q)$, $k_3(q)$, $k_4(q)$,

$k_5(q)$ and $k_6(q)$, then, for $t \in [0,T]$,

$$C_1|w_m(t)|_{\frac{1}{2}} + C_2\varepsilon^{\frac{1}{2}}|\widetilde{v}_m(t)|_1 + c_{1q}\varepsilon^{1-\frac{3}{q}}\|v_{m3}(t)\|_{L^q} + c_{2q}\varepsilon^{\frac{3}{4}-\frac{3}{2q}}\|v_{m3}(t)\|_{L^q}^{\frac{1}{2}}|v_{m3}(t)|_1^{\frac{1}{2}} < \frac{\nu}{4} ,$$

which contradicts the equality (5.58). It follows that $T = +\infty$. Thus, we have proved that, under the hypotheses (5.46) and (5.47), for every integer m, $m \geq m_0$, the solution $u_m \in C^1([0,+\infty);\mathcal{V}_m)$ of the modified Navier-Stokes equations (1.11) with initial data $u_m(0) = \mathcal{P}_m u_0$ satisfies

$$\sup_{t\geq 0}\Big(|w_m(t)|_{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}|\widetilde{v}_m(t)|_1 + \varepsilon^{1-\frac{3}{q}}\|v_{m3}(t)\|_{L^q} + \varepsilon^{\frac{3}{4}-\frac{3}{2q}}\|v_{m3}(t)\|_{L^q}^{\frac{1}{2}}|v_{m3}(t)|_1^{\frac{1}{2}}\Big) < C , \tag{5.60}$$

where C is a positive constant independent of ε and m. We now finish the proof, by arguing as in the proof of Theorem 1.3. □

References

[1] J. D. Avrin, *Large-eigenvalue global existence and regularity results for the Navier-Stokes equation*, J. Differential Equations **127** (1996), 365–390.

[2] H. Bahouri and I. Gallagher, *Paraproduit sur le groupe de Heisenberg et applications*, preprint.

[3] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976.

[4] M. Cannone, *Ondelettes, Paraproduits et Navier-Stokes*, Diderot éditeur, Arts et Sciences, 1995.

[5] P. Constantin and C. Foiaş, *Navier-Stokes equations*, University of Chicago Press, Chicago, 1988.

[6] R. Danchin, *Poches de tourbillon visqueuses*, J. Math. Pures Appl. **76** (1997), 609–647.

[7] M. Dauge, *Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. Part I:Linearized equations*, SIAM J. Math. Anal. **20** (1989), 74–97.

[8] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[9] H. Fujita and H. Morimoto, *On fractional powers of the Stokes operator.*, Proc. Japan Acad. **46** (1970), 1141–1143.

[10] I. Gallagher, *The tridimensional Navier-Stokes equations with almost bidimensional data: stability, uniqueness, and life span*, Internat. Math. Res. Notices **18** (1997), 919–935.

[11] P. Grisvard, *Caractérisation de quelques espaces d'interpolation*, Arch. Rational Mech. Anal. **25** (1967), 40–63.

[12] J. K. Hale and G. Raugel *A damped hyperbolic equation on thin domains*, Trans. Amer. Math. Soc. **329** (1992), 185–219.

[13] ——, *Partial differential equations on thin domains*, Differential equations and mathematical physics (Birmingham, AL, 1990), Academic Press, Boston, MA, 1992, pp. 63–97.

[14] ——, *Reaction-diffusion equation on thin domains*, J. Math. Pures Appl. **71** (1992), 33–95.

[15] D. Iftimie, *The 3D Navier-Stokes equations seen as a perturbation of the 2D Navier-Stokes equations*, Bull. Soc. Math. France, to appear.

[16] J. Johnsen, *Pointwise multiplication of Besov and Triebel-Lizorkin spaces*, Math. Nachr. **175** (1995), 85–133.

[17] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach Science Publishers, New York, 1969.

[18] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.

[19] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Gauthier Villars, Paris, 1969.

[20] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 1*, Dunod, Paris, 1968.

[21] J. E. Marsden, T. S. Ratiu, and G. Raugel, *The Euler equations on thin domains*, preprint.

[22] I. Moise, R. Temam, and M. Ziane, *Asymptotic analysis of the Navier-Stokes equations in thin domains*, Topol. Methods Nonlinear Anal. **10** (1997), 249–282.
[23] S. Montgomery-Smith, *Global regularity of the Navier-Stokes equation on thin three dimensional domains with periodic boundary conditions*, Electronic J. Diff. Eqns. **11** (1999), 1–19.
[24] G. Raugel and G. R. Sell, *Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions*, J. Amer. Math. Soc. **6** (1993), 503–568.
[25] ———, *Navier-Stokes equations on thin 3D domains. II. Global regularity of spatially periodic solutions*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI, Longman Sci. Tech., Harlow, 1994, pp. 205–247.
[26] R. Temam, *Navier-Stokes equations*, third ed., North-Holland, Amsterdam, 1984.
[27] R. Temam and M. Ziane, *Navier-Stokes equations in three-dimensional thin domains with various boundary conditions*, Adv. Differential Equations **1** (1996), 499–546.
[28] ———, *Navier-Stokes equations in thin spherical domains*, Contemporary Math. **209** (1997), 281–314.
[29] W. von Wahl, *The equations of Navier-Stokes and abstract parabolic equations*, Friedr. Vieweg & Sohn, Braunschweig, 1985.