Diophantine approximation for negatively curved manifolds, I

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Abstract

Let M be a geometrically finite pinched negatively curved Riemannian manifold with at least one cusp. Inspired by the theory of diophantine approximation of a real (or complex) number by rational ones, we develop a theory of approximation of geodesic lines starting from a given cusp by ones returning to it. We define a new invariant for M, the *Hurwitz constant* of M. It measures how well all geodesic lines starting from the cusp are approximated by ones returning to it. In the case of constant curvature, we express the Hurwitz constant in terms of lengths of closed geodesics and their depths outside the cusp neighborhood. Using the cut locus of the cusp, we define an explicit approximation sequence for geodesic lines starting from the cusp and explore its properties. We prove that the modular once-punctured hyperbolic torus has the minimum Hurwitz constant in its moduli space.[1]

1 Introduction

Let M be a geometrically finite pinched negatively curved Riemannian manifold with cusps. In this paper we study fine properties of the geodesic flow of M arising from the presence of cusps. There are deep connections of diophantine approximation problems, for real or complex numbers by elements of quadratic number fields, to hyperbolic geometry (see [Dan, For1, For2, HV, HS, Pat, Ser3, Schmi, Sul] and the references therein). The first purpose of this paper is to extend the existing theory beyond the arithmetic case. We even allow the curvature to be non constant. To simplify the statements in the introduction, we assume that M has finite volume and only one cusp e.

A geodesic line starting from e either converges to e, or accumulates inside M. We say that the geodesic line is *rational* in the first case, and *irrational* otherwise. Let $\mathbb{H}^2$ denote the upper half space model of the hyperbolic plane. Let M be the orbifold $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$. In the rational (irrational) case, the lift starting from ∞ of the geodesic line ends at a rational (irrational) point on the real line.

For a rational line r, we introduce a new notion of complexity (Definition 2.8), called the *depth* of r. The depth $D(r)$ is the length of r between the first and last meeting point with the boundary of the maximal Margulis neighborhood of the cusp. The set of depths of rational lines is a discrete subset of $\mathbb{R}$, whose asymptotic distribution will be exploited in a future paper. There is a natural distance-like map d (see section 2.1) on the space of geodesic lines starting from the cusp. In constant curvature, it is given by the Euclidean metric on the boundary of the maximal Margulis neighborhood of the cusp. In general, it

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is a slight modification of the quotient of Hamenstädt's metric [Ham] on the horosphere covering that boundary.

The first goal of this paper is to give an analogue to the classical Dirichlet theorem. When specialized to constant negative curvature, it coincides with a known result for geometrically finite Kleinian groups [Pat, Ser1, HV].

Theorem 1.1 *There exists a positive constant K such that for any irrational line ξ starting from e, there exist infinitely many rational lines r with*

$$d(\xi, r) \leq K e^{-D(r)}.$$

We call the infimum of such K the *Hurwitz constant* of e, and denote it by $K_{M,e}$. In the case that M is the hyperbolic orbifold $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$, or the quotient of the hyperbolic 3-space $\mathbb{H}^3$ by some *Bianchi group*, $K_{M,e}$ corresponds to the classical Hurwitz constant for the approximation of real numbers by rational ones, or the approximation of complex numbers by elements of an imaginary quadratic number field.

The distribution (as a subset of $\mathbb{R}$) of the Hurwitz constants of one-cusped hyperbolic 3-manifolds or orbifolds is unknown, as is their infimum. Among the one-cusped Bianchi 3-orbifolds, the smallest Hurwitz constant is obtained by $\mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O}_{-3})$, and afterwards seems to increase with the discriminant.

The second purpose of this paper is to give a geometric interpretation of the Hurwitz constant. Using as height function a *Busemann function* (giving the "distance" to the cusp, see Section 2), normalized to have value 0 on the boundary of the maximal Margulis neigborhood (and to converge to $+\infty$ in the cusp), we prove:

Theorem 1.2 *The lower bound of the maximal heights of the closed geodesics in M is $-\log 2K_M$.*

In particular, there is a positive lower bound for the Hurwitz constant of manifolds M with a given upperbound on the length of the shortest geodesic (since the length of a closed geodesic is at least twice its height).

Let $\mathcal{M}_{g,1}$ be the moduli space of one-cusped hyperbolic metrics on a connected oriented closed surface with genus $g \geq 1$ and one puncture. We prove (see Section 4) that the Hurwitz constant defines a proper continuous map from $\mathcal{M}_{g,1}$ to $[0, +\infty[$. For higher genus, the one-cusped hyperbolic surfaces having the smallest Hurwitz constant in their moduli space are unknown.

Theorem 1.3 *The Hurwitz constant is a proper real-analytic map on the orbifold $\mathcal{M}_{1,1}$, whose minimum $1/\sqrt{5}$ is attained precisely on the modular one-cusped hyperbolic torus.*

If Γ is a geometrically finite Kleinian group which has ∞ as a parabolic fixed point in the upper halfspace model, and is normalized so that the boundary of the maximal Margulis neighborhood of the cusp is covered by the horizontal plane through $(0, 0, 1)$, one has the following result, where $c(\gamma)$ is the lower left entry of the matrix of γ in Γ. The formula depends on the normalization.

Theorem 1.4 *The Hurwitz constant of $\mathbb{H}^3/\Gamma$ is*

$$\inf_{\{\gamma \in \Gamma \text{ hyperbolic}\}} \frac{\sqrt{|\mathrm{tr}^2\gamma - 4|}}{\min_{\{\gamma' \in \Gamma \text{ is conjugate to } \gamma\}} |2c(\gamma')|}.$$

Generalizing the definition in [EP] for geometrically finite hyperbolic manifolds with one cusp, we define the *cut locus of the cusp* of M. This is the subset Σ of points in M from which start at least two (globally) minimizing geodesic rays converging to the cusp. Let L_e be the boundary of any standard Margulis neighborhood of the cusp. It follows that $M - \Sigma$ retracts in a canonical way onto L_e. Under some technical assumptions (see section 5) which are satisfied in the constant curvature case, and that will be assumed throughout the introduction, Σ has a natural smooth locally finite stratification.

The third purpose of this paper is to construct an explicit good approximation to any irrational line ξ starting from e by rational ones. Using the cut locus of the cusp Σ, we define in section 6 an explicit sequence of rational lines r_n converging to ξ. To simplify the definition, assume here that ξ is transverse to the stratification of Σ. In particular, ξ cuts Σ in a sequence of points $(x_n)_{n\in\mathbb{N}}$ with x_n belonging to an open top-dimensional cell σ_n of Σ. For each $n \in \mathbb{N}$, let c_n be the path consisting of the subsegment of ξ between e and x_n, followed by the minimizing geodesic ray from x_n to the cusp e, starting on the other side of σ_n. Let r_n be the unique geodesic line starting from e, properly homotopic to c_n.

Theorem 1.5 *The geodesic line r_n is rational and there exists a constant $c > 0$ (independant of ξ) such that for every n in $\mathbb{N}$,*

$$d(r_n, \xi) \leq c\, e^{-D(r_n)}.$$

For instance, if γ is a closed geodesic transverse to the stratification of Σ, then the geodesic line ξ starting from the cusp and spiraling around γ has an eventually periodic sequence $(r_n)_{n\in\mathbb{N}}$. Conversely, if a geodesic line ξ, starting from the cusp and transverse to the stratification of Σ, has an eventually periodic sequence $(r_n)_{n\in\mathbb{N}}$, then ξ spirals around a closed geodesic in M.

The fourth purpose of this paper is to express an endpoint of an irrational line by means of geometrical data obtained from the manifold. The cut locus of the cusp Σ allows us to parametrize the geodesic lines starting from the cusp by sequences $(a_n)_{n\in\mathbb{N}}$, where the a_n's belong to a countable alphabet (see section 7). This alphabet is the set $\pi_1(L_e, \mathcal{R})$ of homotopy classes of paths in L_e (relative boundary) between points of a geometrically defined finite subset $\mathcal{R}$ of L_e. An irrational line ξ which is transverse to Σ travels from one point x_n of Σ to its next intersection point x_{n+1} with Σ. By the property of Σ, the subpath of ξ between x_n and x_{n+1} is homotopic to a path lying on L_e, which is a_{n+1} (see section 7).

We prove (Theorem 7.2) that ξ is uniquely determined by the sequence $(a_n)_{n\in\mathbb{N}}$. In constant curvature and dimension 3, where L_e is a 2-torus, we give an explicit formula (Theorem 7.5) giving the r_n's in terms of the a_n's, analogous to the expression giving the n-th convergent of an irrational real number in terms of its continued fraction developement. Our search for an explicit formula expressing the endpoint of an irrational line by geometrical data was inspired by [Ser3, Theorem A] (the $\mathrm{SL}_2(\mathbb{Z})$ case) and [For1, Part II] (the $\mathrm{SL}_2(\mathbb{Z}[i])$ case). But, even in the special cases above, our formula gives new information and calls for further study, among others of the growth of the depths of rational lines in the good approximation sequence of an irrational line. One should also consult the work in [Schmi, chapters 2-5], giving a continued fraction expansion for some complex numbers. Schmidt's construction is completely different from ours.

Two more papers in this series are under preparation. In the second we will give a coding of the geodesic flow in M, using cutting sequences of all geodesic lines with the dual tessalation of the cut locus of the cusp, by a subshift of finite type on a countable alphabet. In the third we give an analogue of the Khinchine-Sullivan theorem, and an estimate of the Hausdorff dimension, in terms of s and the bounds on the curvature, of the set of geodesic lines ξ starting from a cusp e for which there exists infinitely many rational lines r with $d(r,\xi) \leq e^{-sD(r)}$.

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2 Rational and irrational lines starting from a cusp

Let M be a (smooth) complete Riemannian n-manifold with pinched negative sectional curvature $-b^2 \leq K \leq -a^2 < 0$. Fix a universal cover $\widetilde{M}$ of M, with covering group Γ.

A *geodesic segment, ray, line* in a M is a locally isometric map from a compact interval, a minorated unbounded interval, $\mathbb{R}$ respectively, into M. Note that any geodesic segment, ray, line in $\widetilde{M}$ is (globally) minimizing, but that it is not always the case in M.

The boundary $\partial\widetilde{M}$ of $\widetilde{M}$ is the space of asymptotic classes of geodesic rays in $\widetilde{M}$. Endowed with the cone topology, the space $\widetilde{M} \cup \partial\widetilde{M}$ is homeomorphic to the closed unit ball in $\mathbb{R}^n$ (see for instance [BGS, Section 3.2]). The *limit set* $\Lambda(\Gamma)$ is the set $\overline{\Gamma x} \cap \partial\widetilde{M}$, for any x in $\widetilde{M}$. See [Bow] for the following definitions.

Definition 2.1 *With Γ and M as above:*

1. *A point ξ in $\Lambda(\Gamma)$ is a* conical limit point *of Γ if it is the endpoint of a geodesic ray in $\widetilde{M}$ which projects to a geodesic in M that is recurrent in some compact subset.*

2. *A point ξ in $\Lambda(\Gamma)$ is a* bounded parabolic point *if it is fixed by some parabolic element in Γ, and if the quotient $(\Lambda(\Gamma) - \{\xi\})/\Gamma_\xi$ is compact, where Γ_ξ is the stabilizer of ξ.*

3. *The group Γ and the manifold M are called* geometrically finite *if every limit point of Γ is conical or bounded parabolic.*

We assume in this paper that M is geometrically finite and *non elementary*, i.e. that the limit set contains at least 3 (hence uncountably many) points. In that case, the limit set is the smallest non empty invariant closed subset of $\partial\widetilde{M}$. The *convex core* $C(M)$ of M is the image by the covering map $\widetilde{M} \to M$ of the convex hull of the limit set of Γ. For instance, if M has finite volume, then M is geometrically finite and $C(M) = M$.

A *cusp* in M is an asymptotic class of minimizing geodesic rays in M along which the injectivity radius goes to 0. If M has finite volume, the cusps are in one-one corespondance with the ends of M. A geodesic ray (line) *converges to* the cusp if some positive subray is asymptotic to a ray in the equivalence class of the cusp. A geodesic ray converges to some cusp if and only if some (any) lift in $\widetilde{M}$ ends in a parabolic fixed point. In all that follows, we fix a cusp e.

Given any minimizing geodesic ray r, recall (see [BGS, Section 3.3]) that the Buseman function $\beta_r : M \to \mathbb{R}$ is the 1-Lipschitz map defined by the limit (which exists for all $x \in M$)

$$\beta_r(x) = \lim_{t\to\infty} (t - d_M(x, r(t))).$$

Fix a minimizing ray r converging to the cusp e. Let $\widetilde{r}$ be any lift of r to $\widetilde{M}$. The following facts follow from the Margulis Lemma (see for instance [BK] and [BGS, Sections 9-10]) and since M is non elementary. There exists $\eta_0 = \eta_0(r)$ in $\mathbb{R}$ such that, given t in $\mathbb{R}$, the quotient of the horosphere $\beta_{\widetilde{r}}^{-1}(t)$ by the stabilizer in Γ of the endpoint $\widetilde{r}(+\infty)$ of $\widetilde{r}$ embeds in M under the covering map $\widetilde{M} \to M$ if and only if $t > \eta_0$. So that for $\eta > \eta' > \eta_0$, the level set $\beta_r^{-1}(\eta)$ identifies with the quotient of an horosphere $\beta_{\widetilde{r}}^{-1}(\eta)$ by the stabilizer $\Gamma_{\widetilde{r}(+\infty)}$. There is a unique minimizing geodesic ray starting perpendicularly to $\beta_r^{-1}(\eta')$ at a given point $x \in \beta_r^{-1}(\eta')$, and entering $\beta_r^{-1}(]\eta', +\infty[)$. It converges to e and meets $\beta_r^{-1}(\eta)$ perpendicularly in exactly one point $\phi(x)$. The distance $d_M(x, \phi(x))$ is the constant $\eta - \eta'$. If the curvature is constant -1, then the homeomorphism $\phi : \beta_r^{-1}(\eta') \to \beta_r^{-1}(\eta)$ induces a contraction of the induced length metrics of ratio $e^{\eta' - \eta} < 1$.

Define $\beta_e(x) = \beta_r(x) - \eta_0$. Since Busemann functions of asymptotic minimizing rays differ by an additive constant, the map $\beta_e : M \to \mathbb{R}$ does not depend on r.

Definition 2.2 *(***Busemann function of the cusp***). The map $\beta_e : M \to \mathbb{R}$ is called the Busemann function of e, and $\beta_e^{-1}(]0, +\infty[)$ the maximal Margulis neighborhood of e*

Consider the set of geodesic lines $c :]-\infty, +\infty[\to M$ whose negative subrays converge to e and that are recurrent in some compact subset or ending in some cusp. Identify two of them if they differ by a translation of the time. An equivalence class will be called a *geodesic line starting from e*. The set of equivalence classes will be called the *link of e in $C(M)$*, and denoted by $Lk(e, C(M))$. It is in one-to-one correspondance with a closed subset of the intersection $\beta_e^{-1}(\eta) \cap C(M)$ of the convex core and a level set of the Busemann function of e, for any $\eta > 0$, by the map which associates to c its first intersection point with $\beta_e^{-1}(\eta)$.

Definition 2.3 *(***Rational and irrational geodesic lines***). A geodesic line starting from e will be called* rational *if it converges to e. A geodesic line starting from e which is not rational and which does not converge into another cusp will be called* irrational.

Being a rational line is equivalent to requiring that the line meets perpendicularly a second time a level set $\beta_e^{-1}(\eta)$, for any $\eta > 0$ (it is contained after that time in $\beta_e^{-1}([\eta, +\infty[)$ and converges to e).

2.1 The distance on the link of the cusp

For our approximation purpose, we will need to measure how close are two points in the link of e in $C(M)$. For that we will define a "distance-like" map $d : Lk(e, C(M)) \times Lk(e, C(M)) \to [0 + \infty[$ as follows.

Let H be any horosphere in $\widetilde{M}$ centered at $a \in \partial\widetilde{M}$. We first define a map $d_H : H \times H \to [0 + \infty[$. Let x be a point on H, let L_x be the geodesic line through x starting at a and let x' be its endpoint (see Figure 1), so that L_x is oriented from x towards x'. For $r > 0$, let H_r be the horosphere centered at the endpoint of L_x, meeting L_x at a point u at signed distance $-\log 2r$ of x along L_x. For every x, y in H, define $d_H(x, y)$ to be the infimum of all $r > 0$ such that H_r meets L_y.

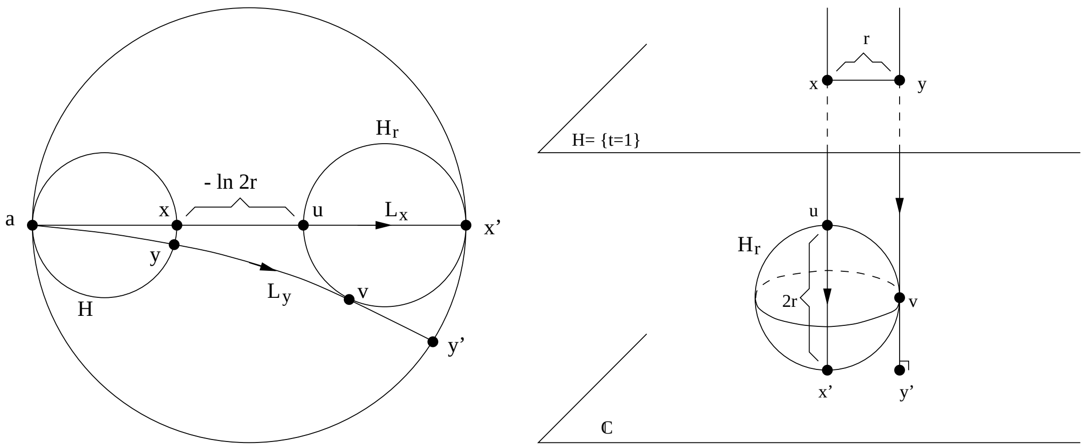

Figure 1 : The "metric" on the link of a cusp.

This map d_H is a priori not symmetric nor transitive (see [HP, Appendix] for a related actual distance). In constant curvature, it coincides with the induced Riemannian metric (which is flat) on the horosphere H. To prove that (see Figure 1), by naturality of the construction, one may assume that a is the point at infinity in the upper half-space model with curvature -1, and H is the horizontal horosphere $t = 1$. Let r be the Euclidean distance between x and y. Let H_r be the horosphere which is tangent to the horizontal coordinate hyperplane at the vertical projection x' of u, and has Euclidean radius r. It bounds a horoball which is the smallest one meeting the vertical line through y. An easy computation shows that the hyperbolic distance between x and u is $-\log 2r$.

If for any two points on an horosphere H in $\widetilde{M}$, there exists an isometry in $\widetilde{M}$ preserving the horoball bounded by H which exchanges the two points, then d_H is symmetric. This is the case if $\widetilde{M}$ is a symmetric space (of non compact type) of rank one.

From the topological point of view, the map d_H is as good as a distance:

Proposition 2.4 *For every x in H, let $B_{d_H}(x, \epsilon) = \{y \in H \,|\, d_H(x, y) < \epsilon\}$. Then $\{B_{d_H}(x, \epsilon) \,|\, \epsilon > 0\}$ is a fundamental system of neighborhoods at x.*

Proof. Since $\widetilde{M}$ is a negatively curved Riemannian manifold, the map $H \to \partial\widetilde{M} - \{a\}$ which sends x to x' is a homeomorphism. So we only have to prove that $\{B'(x', \epsilon) \,|\, \epsilon > 0\}$ is a fundamental system of neighborhoods at x', with $B'(x', \epsilon) = \{y' \in \partial\widetilde{M} - \{a\} \,|\, d_H(x, y) < \epsilon\}$. Assume that $y' \neq x'$, let HB' be the smallest horoball centered at x' and meeting L_y. Let H' be the boundary of HB', then H' is tangent at L_y in a point v and meets L_x in a point u (see Figure 1). Let p be the perpendicular projection of v on L_x. By convexity of the horoballs, the point p belongs to HB', so that x, u, p are in this order on L_x. By the existence of a negative upper bound on the curvature, there exists (see for instance [GH]) a constant $C > 0$ (depending only on the bound) and a map from $L_x \cup L_y$ into a tree T which is an isometry on L_x and on L_y, and preserves the distances up to the additive constant C. Let $\overline{z}$ be the image in T of any $z \in L_x \cup L_y$, and $\overline{a}, \overline{x'}, \overline{y'}$ the ends of T corresponding to a, x', y'. Take $\overline{x}$ as basepoint in T. Since L_x, L_y are asymptotic in a, their images $\overline{L_x}, \overline{L_y}$ meet in a subray from P to the end $\overline{a}$, where P is the point in T which is the projection of $\overline{y'}$ on $\overline{L_x}$. The map $z \mapsto \overline{z}$ preserves the distance (and hence also the Busemann functions) up to the additive constant C. By the properties of the Busemann functions in trees, since u, v lie on the same horosphere in $\widetilde{M}$, the points $\bar{u}, \bar{v}$ are on horospheres centered at $\bar{x}$ at distance at most $2C$. So that if $\overline{v}$ lies between P and $\overline{y'}$, then

$$|d_T(\overline{u}, P) - d(\overline{v}, P)| \leq 2C$$

and if $\overline{v}$ does not lie between P and $\overline{y'}$, then both $\overline{u}$ and $\overline{v}$ lie on $\overline{L_x}$, hence

$$d_T(\overline{u}, \overline{v}) \leq 2C.$$

Since u lies on the smallest horosphere centered at x' and meeting L_y, and since the smallest horoball in T centered at $\overline{x'}$ which contains a point of $\overline{L_y}$ has its horosphere passing through P, it follows that $d_T(\overline{u}, P) \leq 2C$. Hence if $\overline{v}$ lies between P and $\overline{y'}$, one has

$$d_T(\overline{u}, \overline{v}) = d_T(\overline{u}, P) + d(\overline{v}, P) \leq 6C.$$

Therefore the points u and v are at distance in $\widetilde{M}$ at most $7C$. Hence x' and y' are close on $\partial\widetilde{M}$ if and only if $\overline{x'}, \overline{y'}$ are close on ∂T, which is equivalent to $\overline{x}, P, \overline{x'}$ being in this order on $\overline{L_x}$ and $d_T(\overline{x}, P)$ is big, which occurs if and only if x, u, x' are in this order on L_x and $d_{\widetilde{M}}(x, u)$ is big. This proves the result. $\boxed{\textbf{2.4}}$

We now define a map d on $\mathrm{Lk}(e, C(M))$ by taking quotients.

Definition 2.5 *For any $\eta > 0$, let H_η be any horosphere in $\widetilde{M}$ covering $\beta_e^{-1}(\eta)$, and let $\pi : \widetilde{M} \to M$ be the covering map. For any two points x, y in $Lk(e, C(M))$, identified as above with a closed subset of $\beta^{-1}(\eta)$, define $d(x, y)$ as the infinum of $e^\eta d_{H_\eta}(\tilde{x}, \tilde{y})$ for all the preimages $\tilde{x}, \tilde{y}$ of x, y respectively.*

It is easy to show that the map d on $\mathrm{Lk}(e, C(M))\times$ $\mathrm{Lk}(e, C(M))$ does not depend on the choice of the horosphere H_η (by equivariance), nor on $\eta > 0$. This is because if H, H' are horospheres in $\widetilde{M}$ centered at the same point at infinity, with say H' contained in the horoball bounded by H, with Δ the (constant) distance between H' and H, then $d_{H'} = e^{-\Delta} d_H$.

In general, d is not a distance. It is a distance if M is *locally symmetric*. If furthermore the curvature is constant -1, then d coincides with the (flat) induced length metric on $\beta^{-1}(\eta)$ normalized by e^η.

For X a complete simply connected Riemannian manifold with pinched negative curvature with the upper bound on the curvature being -1, Hamenstadt defined (see [Ham]) a metric on the space at infinity minus a point a, depending on an horosphere H centered at a, as follows. The distance between x, y in $\partial X - \{a\}$ is given by

$$\delta(b, c) = \lim_{t \to \infty} e^{-t + \frac{1}{2} d_X(b_t, c_t)}$$

where $t \mapsto b_t, t \mapsto c_t$ are geodesic lines starting from a, ending in b, c and passing through H at time 0. In [HP, Appendix], we proved that this limit exists and defines a metric for any CAT(-1) space X.

By identifying $\partial X - \{a\}$ with the horosphere H as usual (sending $b \neq a$ to the point of intersection with H of the geodesic line starting at a and ending at b), one gets a metric δ_H on H. Our metric d_H is equivalent to δ_H:

Remark 2.6 *Assume that the upper bound of the curvature of M is -1. Then there exists a constant $c > 0$ such that for every x, y in H*

$$\frac{1}{2}\delta_H(x,y) \leq d_H(x,y) \leq c\delta_H(x,y).$$

Proof. Keeping the notations of the definition of d_H (and Figure 1), we have, since u (resp. v) lies on the segment between x (resp. y) and x_t (resp. y_t) for t big enough, by the triangle inequality, since the minimum of the distances between a point of H and a point of H_r is attained by $d_X(x,u)$, and since u, v lie on the same horosphere centered at the point to which x_t converges as $t \to \infty$, if ϵ is small enough, then for t big enough,

$$\begin{aligned} -2t + d_X(x_t, y_t) &= -[d_X(x,u) + d_X(u,x_t)] - [d_X(y,v) + d_X(v,y_t)] + d_X(x_t,y_t) \\ &\leq -d_X(x,u) - d_X(u,x_t) - d_X(y,v) - d_X(v,y_t) + d_X(x_t,v) + d_X(v,y_t) \\ &\leq -2d_X(x,u) + [d_X(v,x_t) - d_X(x_t,u)] \\ &\leq -2d_X(x,u) + \epsilon = 2\log(2d_H(x,y)) + \epsilon \end{aligned}$$

so that

$$\delta_H(x,y) \leq 2d_H(x,y).$$

By the technique of approximation by trees, one may also show that there is an explicitable universal constant c such that

$$d_H(x,y) \leq c\delta_H(x,y).$$

□

Since δ_H is a distance (inducing the right topology), this gives another proof of proposition 2.4.

2.2 The depth of rational geodesic lines

In this subsection we define a notion of complexity for rational geodesic lines. First we present a connection between rational geodesic lines, which were defined in geometric terms and a set of double cosets in $\pi_1(M)$. This will allow us to perform computations in the constant negative curvature case.

Choose a base point on the level set $L = \beta_e^{-1}(1)$. (The first intersection point of the geodesic line starting from e passing through the base point with any level set $\beta_e^{-1}(\eta)$ for $\eta > 0$ gives a base point on that level set and we will use the subsegments of that geodesic line to identify the fundamental groups of M based at the base points on different level sets.) By the Margulis Lemma, the inclusion $i : L \to M$ induces an injection between fundamental groups $i_* : \pi_1 L \to \pi_1 M$ (use the choosen base point both for L and M). We identify $\pi_1 L$ with its image.

Lemma 2.7 *The set of rational lines is in one-to-one correspondence with the set of double cosets $\pi_1 L \backslash \pi_1 M / \pi_1 L$.*

Proof. If r is a rational line, and c is its subpath between the two succesive perpendicular intersection points x, y with L, then choosing a path on L between the base point and x, and from y to the base point defines an element in $\pi_1 M$ whose double coset is uniquely defined. Conversely, the straightening process (inside a given homotopy class of paths in $M - \beta_e^{-1}(]1, +\infty[)$ with endpoint staying in L) associates a rational line to each double coset. More precisely, to any path c in $M - \beta_e^{-1}(]1, +\infty[)$ with endpoints on L, let $\widetilde{c}$ be a lift of c to $\widetilde{M}$. Its endpoints belongs to two lifts $\widetilde{L}_1, \widetilde{L}_2$ which are disjoint, unless c is homotopic into L. The horoballs bounded by $\widetilde{L}_1, \widetilde{L}_2$ are closed convex subsets of $\widetilde{M}$ with no common point at infinity. Since the curvature is non positive, there exists a unique common perpendicular segment s. The projection of s to M gives a geodesic segment in M homotopic to c by an homotopy moving the endpoints along L, which is a subsegment of a rational line. Clearly, the above two maps are inverse one of the other. $\boxed{\mathbf{2.7}}$

We measure the complexity of a rational line r by the length $\ell_\eta(r)$ of its subsegment between the two perpendicular intersection points with any level set $\beta_e^{-1}(\eta)$ for $\eta > 0$, suitably normalized not to depend on η.

Definition 2.8 (**Depth of a rational line**). *The depth of a rational line r is $D(r) = \ell_\eta(r) - 2\eta$.*

By the properties of the level sets $\beta_e^{-1}(\eta)$, the depth $D(r)$ is independant of η.

Remark 2.9 *The set of depths of rational lines starting from e is a discrete subset of $\mathbb{R}$, with finite multiplicities.*

Indeed, the set of intersections with the preimage of $C(M)$ of the horospheres covering $\beta_e^{-1}(1)$ is locally finite in the universal covering of M, since the group acts discretely, and the stabilizer of each such intersection acts cocompactly on it.

2.3 The constant curvature case

In the case of constant negative curvature, we have the following precise description, which also extends to the orbifold case. We restrict to the dimension 3, though everything is valid in higher dimensions using the Vahlen matrices (see for instance [Ah]).

Let M be a connected orientable geometrically finite complete hyperbolic 3-orbifold. Then, according to the description of the thin part of M, each cusp e of M has a neighborhood N isometric (for the induced length metric) to $(\mathcal{T} \times [a, +\infty[, ds^2)$ where $a = a(N) \in \mathbb{R}$ is some constant, $\mathcal{T} = \mathcal{T}(e)$ is a connected orientable flat 2-orbifold with metric dx^2, and $ds^2 = e^{-2t}dx^2 + dt^2$. In all what follows, the cusp e is fixed.

We will call such an N a *standard cusp neighborhood* (of e). The union of all standard cusps neighborhood is isometric to $\mathcal{T} \times]a_0, +\infty[$ endowed with the metric $ds^2 = e^{-2t}dx^2 + dt^2$ for some $a_0 > 0$, and we fix such an isometry. Choose a base point $*$ on $\mathcal{T}$. This gives a choice of base-point as above by considering the geodesic ray corresponding to $\{*\} \times]a_0, +\infty[$. Using orbifold fundamental groups, the set of rational lines is in one-to-one correspondence with the set of double cosets $\pi_1^{\mathrm{orb}} \partial N \backslash \pi_1^{\mathrm{orb}} M / \pi_1^{\mathrm{orb}} \partial N$.

We note that for a geodesic c in an orbifold that meets the singular locus of the orbifold M (which is a finite metric graph with possibly some points removed, corresponding to some ends) at a point x, then ingoing and outgoing tangent vectors of c at x make an angle strictly less than π (unless c is locally contained in the singular locus).

Using a suitable uniformization of M, the depth of a rational line can be computed in algebraic terms. We will use the upper half-space model $\{(z,t) \mid z \in \mathbb{C}, t > 0\}$ for the real hyperbolic 3-space $\mathbb{H}^3$, with the metric

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2}.$$

We fix an isometry between M and $\mathbb{H}^3/\Gamma$ with Γ a subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$, in the following way.

By the Cartan-Hadamard theorem, there exists such an isometry, which induces an isomorphism between the orbifold fundamental group $\pi_1^{\mathrm{orb}} M$ of M and Γ. Up to conjugating Γ, we may assume that the stabilizer of the point at infinity in the upper half space model, Γ_∞ corresponds to $\pi_1^{\mathrm{orb}}(\partial N)$, for N any standard cusp neighborhood of e. For any $h > 0$, let $H_\infty(h)$ be the *horizontal horosphere*, defined by the equation $t = h$ in the upper half-space. Note that Γ_∞ is a discrete group of isometries of each $H_\infty(h)$ in $\mathbb{H}^3$.

We may assume, up to conjugating Γ by a dilatation, that $H_\infty(h)/\Gamma_\infty$ injects into M if and only if $h > 1$ (this is coherent with the previous normalisation). So that each standard cusp neighborhood of e is of the form $H_\infty^+(h)/\Gamma_\infty$ for some $h > 1$, where $H_\infty^+(h)$ is the set of points (z,t) with $t > h$. The uniformization $\mathbb{H}^3 \to M$ is well defined only up to precomposition by an isometry of the form $z \mapsto az + b$ with a, b complex and $|a| = 1$, but the subsequent constructions will not depend on the choice of the uniformization.

In all what follows, we will identify the link of e in M with $\mathbb{C}/\Gamma_\infty$, by the map which sends a geodesic line r starting from e to the endpoint on the horizontal coordinate plane of any lift $\widetilde{r}$ starting at ∞ of r, modulo Γ_∞. The distance d on $Lk(e, M)$ defined in Definition 2.5 is exactly the Euclidean distance on $\mathbb{C}/\Gamma_\infty$, since the map $(z,1) \mapsto z$ from $H_\infty(1)$ (endowed with the induced length metric) to $\mathbb{C}$ (with the Euclidean metric) is an isometry.

Since $\mathrm{Isom}_+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm 1\}$, we will write each $\gamma \in \Gamma$ as

$$\gamma = \pm \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix},$$

which acts on $\mathbb{C} \cup \{\infty\}$ as usual by the Möbius transformation $z \mapsto \dfrac{a(\gamma)z + b(\gamma)}{c(\gamma)z + d(\gamma)}$.

Using these identifications we obtain an explicit expression for the endpoint (on the complex plane) of any rational ray r.

Lemma 2.10 *Let $\Gamma_\infty \gamma \Gamma_\infty$ be the double coset associated to a rational line r. Then $r = \frac{a(\gamma)}{c(\gamma)} \bmod \Gamma_\infty$ and $D(r) = 2\log|c(\gamma)|$.*

Proof. Let $\widetilde{r}$ be any lift of r to $\mathbb{H}^3$ starting from ∞, which is a descending vertical geodesic. Any two such lifts differ by the action on the left of an element of Γ_∞. By the definition of γ, there exists an element α in Γ_∞ such that the endpoint of $\widetilde{r}$ on the horizontal coordinate plane is $\alpha\gamma(\infty)$. Since γ does not fix ∞, $c(\gamma)$ is different from zero. For any α in Γ_∞ we have $\alpha\gamma(\infty) = \alpha(\frac{a(\gamma)}{c(\gamma)})$. By taking the image in $\mathbb{C}/\Gamma_\infty$ the first assertion of the lemma follows.

The horizontal horosphere $H_\infty(1)$ centered at ∞ is mapped by γ to the horosphere centered at $\gamma(\infty)$ of Euclidean diameter $\frac{1}{|c(\gamma)|^2}$. By the definition of the depth, this implies the second assertion of the lemma (which is independant of the choice of the representative of the double coset). $\boxed{\textbf{2.10}}$

3 The approximation constant

In this section we study the approximation of irrational rays by rational ones, in pinched negatively curved Riemannian manifolds. We keep the same notation as in the beginning of the previous section, in particular for the distance d on $Lk(e, C(M))$.

Theorem 3.1 establishes an analogue of the classical approximation theorem by Dirichlet. Theorem 3.2 gives an analogue of the classical Hurwitz constant. These two results were known in constant curvature (see for example [HS, HV, Vul1, Vul2]). Theorem 3.4 gives a relation between the Hurwitz constant and the *heights* of closed and non-cusp converging geodesic lines, which is new even for Fuschian or Kleinian groups. Corollary 3.6 expresses the Hurwitz constant for a Kleinian group Γ in algebraic terms, once a natural normalization of Γ has been made. At the end of this section we present some known values of the Hurwitz constant of the Bianchi groups.

Theorem 3.1 *Let M be a pinched negatively curved geometrically finite Riemannian manifold. For any irrational line ξ starting from a cusp e, there exists a constant $K > 0$ such that for infinitely many rational lines r, one has*

$$d(\xi, r) \leq K e^{-D(r)}.$$

For a given ξ, the infimum over all such K's is denoted by $K(\xi)$, and is called the *Hurwitz constant* of ξ. The supremum of $K(\xi)$ over all irrational lines ξ will be called the *Hurwitz constant* of e, and will be denoted by $K_{M,e}$.

Theorem 3.2 *Let M be a pinched negatively curved geometrically finite Riemannian manifold. For every cusp e of M, we have*

$$0 < K_{M,e} < \infty.$$

The maximum of $K_{M,e}$ on the finitely many cusps e of M is a new invariant of the geometrically finite pinched negatively curved manifold M, that we call the *Hurwitz constant of M*. Theorem 1.1 of the introduction follows from Theorems 3.1 and 3.2

Let P be a non empty closed subset of $C(M)$ which does not meet some neighborhood of e. Define the *height of P with respect to the cusp e* to be the maximum of the normalized Busemann function of e on P, that is

$$ht(P) = \max_{x \in P} \beta_e(x).$$

To prove that the maximum is attained, set $t = \sup_{x \in P} \beta_e(x) \in]0, +\infty[$. Since M is geometrically finite and P is contained in M, the subset $P \cap (\beta_e^{-1}([t-1, +\infty[) - \beta_e^{-1}(]t+1, +\infty[))$ is compact. The maximum of β_e on P is attained in that compact set.

Definition 3.3 *With the above notations,*

- *let $h_{M,e}$ be the infimum of all h in $\mathbb{R}$ such that there exists an irrational line starting from e eventually avoiding the Busemann level set $\beta_e^{-1}([h, +\infty[)$,*
- *let $h'_{M,e}$ be the infimum of the heights of the closure of the geodesic lines contained in $C(M)$ that neither positively nor negatively converge into a cusp, and*

- *let $h''_{M,e}$ be the infimum of the heights of the closed geodesics.*

The *height spectrum* (i.e. the subset of $\mathbb{R}$ consisting of the heights of the closed geodesic), as well as in the surface case its restriction to simple closed geodesic, is worth more study (see for instance [Haa, LS]). We give in the next section examples were $h''_{M,e}$ is attained, and examples where it is not attained (Proposition 4.1).

The following result relates the Hurwitz constant to heights of closed and non-cusp converging geodesic lines. Partial cases of the second equality were known (see the work of Humbert and Ford (see [For2]) for $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}[i])$, and [HS] in the case that $\widetilde{M}$ is isometric to $\mathbb{H}^2$).

Theorem 3.4 *Let M be a non elementary geometrically finite pinched negatively curved Riemannian manifold, and let e be a cusp of M. Then*

$$\frac{1}{2K_{M,e}} = \exp h_{M,e} = \exp h'_{M,e} = \exp h''_{M,e}.$$

Proof. Choose (arbitrarily) one of the parabolic fixed points in $\partial\widetilde{M}$ corresponding to the cusp e, and call it ∞. The other parabolic fixed points which project to e are of the form $\gamma(\infty)$ for $\gamma \in \Gamma$. For h in $\mathbb{R}$ and $\gamma \in \Gamma$, let $H_{\gamma(\infty)}(h) = \gamma H_\infty(h)$ be the horosphere centered at $\gamma(\infty)$ which is a lift of the level set $\beta_e^{-1}(h)$. Let $\widetilde{r}$ be the geodesic line from ∞ to $\gamma(\infty)$, and r be the rational line, which is the projection of $\widetilde{r}$ in the link of e in $C(M)$. By the definition 2.8 of the depth, the (signed) distance between the intersection points of $\widetilde{r}$ with respectively $H_\infty(1)$ and $H_{\gamma(\infty)}(h)$ is $D(r) + h$. If ξ is a point in $Lk(e, C(M))$ close enough to r, let L_ξ be the (unique) geodesic line starting from ∞ which is the closest to $\widetilde{r}$ of the lifts of ξ in $\widetilde{M}$ starting from ∞. Then, by the definition 2.5 of the distance d on the link, the geodesic line L_ξ meets $H_{\gamma(\infty)}(h)$ if and only if $d(r,\xi) \leq \frac{1}{2}e^{-D(r)-h} = \frac{1}{2e^h}e^{-D(r)}$. The first equality follows.

Let $H^+_{\gamma(\infty)}(h)$ be the horoball in $\widetilde{M}$ whose boundary is $H_{\gamma(\infty)}(h)$. Let $\widetilde{C(M)}$ be the convex hull of the limit set $\Lambda(\Gamma)$. It is immediate that $h'_{M,e}$ is the infimum of all h in $\mathbb{R}$ such that $\widetilde{C(M)} - \bigcup_{\gamma\in\Gamma} H^+_{\gamma(\infty)}(h)$ contains a geodesic line. Also, $h''_{M,e}$ is the infimum of all h such that $\widetilde{C(M)} - \bigcup_{\gamma\in\Gamma} H^+_{\gamma(\infty)}(h)$ contains a periodic geodesic line (i.e. one whose stabilizer acts cocompactly on it).

The inequality $h'_{M,e} \leq h''_{M,e}$ is clear. Since M is geometrically finite and non elementary, there exists at least one closed geodesic in M. Hence $h''_{M,e}$ is different $+\infty$, and so is $h'_{M,e}$.

Let us prove that $h_{M,e} \leq h'_{M,e}$. Assume first that $h'_{M,e}$ is not $-\infty$. For every $\epsilon > 0$, let $h \in \mathbb{R}$ such that

$$h'_{M,e} < h \leq h'_{M,e} + \epsilon.$$

By the definition of $h'_{M,e}$, there exists a geodesic line c in $\widetilde{C(M)}$ avoiding $\bigcup_{\gamma\in\Gamma} H^+_{\gamma(\infty)}(h)$, so in particular its endpoints are not in the orbit of ∞. Let L_ξ be the geodesic line starting at ∞ and ending at one of the endpoints of c. The geodesic line L_ξ is contained in $\widetilde{C(M)}$. By the existence of a negative upperbound on the sectional curvature, L_ξ and c are asymptotic at their common endpoint. That is, there exists a positive subray of L_ξ

which is contained in an ϵ-neighborhood of c. In particular this subray does not meet $\bigcup_{\gamma\in\Gamma} H^+_{\gamma(\infty)}(h+\epsilon)$, since the horospheres $H_{\gamma(\infty)}(h)$ and $H_{\gamma(\infty)}(h+\epsilon)$ are at distance ϵ one from the other, with $H^+_{\gamma(\infty)}(h+\epsilon) \subset H^+_{\gamma(\infty)}(h)$. Hence L_ξ projects into M to an irrational line meeting only finitely many projections of horoballs $H^+_{\gamma(\infty)}(h+\epsilon)$. Therefore $h_{M,e} \leq h+\epsilon \leq h'_{M,e}+2\epsilon$. The assertion follows. In particular, $h_{M,e}$ is not $+\infty$. An analogous proof shows that if $h'_{M,e} = -\infty$, then $h_{M,e} = -\infty$.

Let us prove that $h''_{M,e} \leq h_{M,e}$. Assume first that $h_{M,e}$ is not $-\infty$. For every $\epsilon > 0$, let $h > 0$ such that

$$h_{M,e} < h \leq h_{M,e} + \epsilon.$$

By the definition of $h_{M,e}$, there exists a geodesic line L_ξ starting at ∞, ending in a point of $\Lambda(\Gamma)$ which is not a parabolic fixed point, and which meets only finitely many horoballs $H^+_{\gamma(\infty)}(h)$. Let R be a positive subray of L_ξ avoiding these horoballs. Since the endpoint of R is a conical limit point, the image r in M of the ray R is recurent in a compact subset of M. Moreover, it is recurent in a compact subset of T^1M. Therefore r comes arbitrarily close to itself in T^1M. By the closing lemma (see for instance [Ano]), there exists a closed geodesic which is contained in the ϵ-neighborhood of r. Any pre-image of this closed geodesic is a periodic geodesic line, which avoids every horoball $H^+_{\gamma(\infty)}(h+\epsilon)$. Hence $h''_{M,e} \leq h+\epsilon \leq h_{M,e}+2\epsilon$. The inequality follows. An analogous proof shows that if $h_{M,e} = -\infty$, then $h''_{M,e} = -\infty$. This completes the proof of the theorem. $\boxed{\textbf{3.4}}$

Proof of Theorems 3.1 and 3.2: Since M is geometrically finite and non elementary, there exists at least one closed geodesic in M. Furthermore, every closed geodesic meets a fixed compact subset of M, obtained by removing a cusp neigborhood of each end from $C(M)$. Therefore $h''_{M,e}$ is finite. It hence follows that $K_{M,e}$ is positive and finite. $\boxed{\textbf{3.1, 3.2}}$

Till the end of this section, we assume that M is a geometrically finite hyperbolic 3-orbifold, uniformized as in the end of subsection 2.3. For any non parabolic element $\gamma \in \Gamma$, define the *height* of γ to be

$$ht(\gamma) = e^{-\frac{D(r_\gamma)}{2}} \left| \sinh \frac{\ell(\gamma)}{2} \right|$$

where r_γ is the rational line starting from ∞ corresponding to the double coset $\Gamma_\infty\gamma\Gamma_\infty$, and $\ell(\gamma)$ is the complex translation length of γ.

Lemma 3.5 *The height of γ is*

$$ht(\gamma) = \left| \frac{\sqrt{\mathrm{tr}^2\gamma - 4}}{2c(\gamma)} \right|,$$

it is the euclidean vertical coordinate of the highest point on the translation or rotation axis of γ.

Proof. Let $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-parabolic element in Γ. The first claim follows from Lemma 2.10 and the equality $\cosh \frac{\ell(\gamma)}{2} = \frac{\mathrm{tr}\gamma}{2}$.

The second part is well known (see [HS]). The two endpoints γ^-, γ^+ of its translation or rotation axis are the solutions of the equation $\frac{az+b}{cz+d} = z$. Hence $\gamma^\pm = \frac{a-d\pm\sqrt{\mathrm{tr}^2\gamma-4}}{2c}$ (since γ is not parabolic, one has $c \neq 0$). The highest point on the translation axis of γ has vertical coordinate $\frac{1}{2}|\gamma^+ - \gamma^-|$. The result follows. $\boxed{\mathbf{3.5}}$

Corollary 3.6 *If $K_{M,e}$ is the Hurwitz constant of a cusp e of a geometrically finite hyperbolic 3-orbifold M, then*

$$\frac{1}{2K_{M,e}} = \inf_{\{\gamma\in\Gamma \mid \ \mathrm{Re}\ \ell(\gamma)>0\}} \max_{\delta\in\Gamma} ht(\delta\gamma\delta^{-1}).$$

Proof. The exponential of the height of the closed geodesic representing the conjugacy class of an element γ of Γ is the supremum of the Euclidean vertical coordinates of the points on the lifts to $\mathbb{H}^3$ of the closed geodesic, hence is exactly $\sup_{\delta\in\Gamma} ht(\delta\gamma\delta^{-1})$. To see that this upper bound is attained, as the translation length is a conjugacy invariant, one only has to apply Remark 2.9 and Lemma 2.10, which imply that the $|c(\gamma)|$'s, for γ moving the point ∞, form a discrete subset of $\mathbb{R}$ with finite multiplicities, and have a positive lower bound. $\boxed{\mathbf{3.6}}$

Remark. The right handside of the equation may seem to depend on conjugation of Γ, but the map ht has been defined by suitably choosing some conjugate of Γ. The formula may be used to calculate on computers the Hurwitz constants of Fuschian or Kleinian groups, as the Hurwitz constants of the mod p congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Remark. If $M_\mathbb{Z}$ is $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$, then $M_\mathbb{Z}$ has only one end and (the first equality being due to Hurwitz)

$$\frac{1}{2K_{M_\mathbb{Z}}} = \exp\frac{\sqrt{5}}{2} = \exp ht\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is proved in [HS] (as well as analogous statements for the case of Hecke groups) that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ realizes the maximum height in its conjugacy class.

If d is a squarefree positive integer, if $\mathcal{O}_{-d}$ is the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, and M_d is the hyperbolic 3-orbifold quotient of $\mathbb{H}^3$ by the *Bianchi group* $\mathrm{PSL}_2(\mathcal{O}_{-d})$, then (see for instance [Swa]) M_d has one and only one cusp if and only if $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$, in which case the known values and estimates on the Hurwitz constant K_{M_d} are given by the following table, up to our knowledge:

d	1	2	3	7	11	19	43	67	163
K_{M_d}	$\frac{1}{\sqrt{3}}$ [For2]	$\frac{1}{\sqrt{2}}$ [Per2]	$\frac{1}{\sqrt[4]{13}}$ [Per1]	$\frac{1}{\sqrt[4]{8}}$ [Hof]	$\frac{2}{\sqrt{5}}$ [DP]	1 [Poi]	$\sqrt{\frac{11}{5}} \leq K_{M_{43}} \leq \sqrt{\frac{13}{3}}$ [Poi] [Vul1]	?	?

The only d for which we know an element of $PSL_2(\mathcal{O}_{-d})$ realizing the min-max in the expression of $\frac{1}{2K_{M_d}}$ given by Corollary 3.6 is $d = 1$ (see [For2] between the lines), for which the following element works:

$$\begin{pmatrix} 2-i & 2i \\ -2i & 2+i \end{pmatrix}$$

Note that in particular, the formula of Corollary 3.6 explains the general shape of the above results, since in these cases, the Hurwitz constant is an inf-max of numbers of the form $\frac{2\sqrt{N(u)}}{\sqrt[4]{N(v)}}$ with u, v algebraic integers in $\mathcal{O}_{-d}$, hence having integral norms $N(u), N(v)$.

4 The Hurwitz constant of once-punctured hyperbolic tori

In this section, we study the Hurwitz constant for cusped hyperbolic surfaces M with a choosen cusp e, computing it in the case of once-punctured hyperbolic tori. Recall that $h''_{M,e}$ is defined as the infimum of the heights (with respect to e) of the closed geodesics in M

Proposition 4.1 *Let M be the (unique up to isometry) complete hyperbolic thrice punctured sphere, and e_1, e_2, e_3 its cusps. The infimum defining h''_{M,e_1} is not attained on a closed geodesic, but is attained on a (simple) geodesic line starting from e_2 and converging to e_3.*

Proof. The manifold M is obtained by doubling along its boundary an ideal hyperbolic triangle τ. Fix a lift of one of these two triangles in a universal cover $\widetilde{M}$ of M, and identify it with τ (by the covering map). Let ℓ be the side of τ opposite to its vertex (mapping to the cusp) e_1. The height of ℓ is easily seen to be exactly 0. Since M has finite volume, the pairs of endpoints of lifts of closed geodesic are dense in $\partial\widetilde{M} \times \partial\widetilde{M}$. Therefore there exists in $\widetilde{M}$ a sequence of translation axes of covering group elements, whose endpoints converge to the endpoints of ℓ. Projecting to M, one obtains a sequence of closed geodesics which converges, for the uniform Hausdorff distance on compact subsets of M, to (the image in M of) ℓ. We conclude that the heights of these closed geodesics converge to 0. Consider the open subset U of points of M with height strictly less than 0. It is the disjoint union of two open half-cylinders whose fundamental groups are parabolic. Since there are no closed geodesic entirely lying in U, this proves the result. $\boxed{\textbf{4.1}}$

In the same way, there exists an hyperbolic torus M with two cusps e_+, e_- such that the infimum defining h''_{M,e_+} is not attained on a closed geodesic, but is attained on a geodesic line starting from e_- and converging to e_-. One may for instance take the double along the boundary of the hyperbolic pants side lengths $0, a, a$, i.e. one cusp and two totally geodesic boundary of the same length $a > 0$. The difference between the infimum of the heights of simple closed geodesic with the one on all closed curve can go to $+\infty$, as can be seen with the previous example by letting a goes to 0.

Let $\mathcal{M}_{g,1}$ be the moduli space of one-cusped hyperbolic metrics on a closed, connected and oriented surface S with genus $g \geq 1$ and having one puncture.

Proposition 4.2 *Let $h'' : \mathcal{M}_{g,1} \to]-\infty, +\infty[$ be the map, which associates to (the isometry class of) a one-cusped hyperbolic metric on S, the infimum of the heights with respect to its cusp of its closed geodesics. Then h'' is continuous and proper.*

Proof. Let σ, σ_0 be two one-cusped hyperbolic metric on a punctured torus S. If σ is close to σ_0 in the moduli space, then there exists a smooth diffeomorphism f of S such that $f^*\sigma, \sigma_0$ coincide outside a compact subset, and are sufficiently close on that compact subset. In particular, their height functions are close. Every closed geodesic γ for one metric is then a closed curve with geodesic curvature at most ϵ for the other metric. But such a closed curve c is at uniform distance at most d_ϵ from a genuine closed geodesic c', with d_ϵ depending only on ϵ tending to 0 as $\epsilon \to 0$. Hence the height of c and c' are very close. This proves that h'' is continuous.

Let us prove that $h''(u)$ converges to $-\infty$ when u exits every compact subset of $\mathcal{M}_{g,1}$. By the Mumford Lemma, hyperbolic surfaces converge to infinity in their moduli space if and only if they develop a short closed geodesic. By the Margulis Lemma, the height of a short geodesic is low, so the result follows. $\boxed{\mathbf{4.2}}$

This continuity and properness result holds for a much larger class of Riemannian manifolds. For any a, b, v in $]0, +\infty[$, let $\mathcal{M}_{a,b,v,n}$ be the set of (isometry classes of) Riemannian n-manifolds (M, σ) with sectional curvature K satisfying $-b^2 \leq K \leq -a^2$, with volume at most v, and with one cusp e. Endow it with the Lipschitz topology on compact subsets, i.e. (M, σ) is close to (M', σ') if there exist big compact submanifolds K of M and K_0 in M_0, and a smooth diffeomorphism f from K to K' such that $f^*\sigma, \sigma'$ are uniformally close on K'. Recall that by an easy adaptation to the finite volume with one cusp case of the compactness theorem of Cheeger-Gromov, for very $i > 0$ the subset of points of $\mathcal{M}_{a,b,v,n,i}$ whose non-peripheral injectivity radius is at least i is compact. This theorem gives the analog of the Mumford lemma of the previous situation.

We now give the explicit computation of the map $h'' : \mathcal{M}_{1,1} \to]-\infty, +\infty[$ in terms of the Fenchel-Nielsen coordinates. In particular we prove that h'' is real-analytic, that the infimum defining $h''([\sigma])$ is attained on (one of) the shortest simple closed geodesic for $[\sigma] \in \mathcal{M}_{1,1}$. We also compute the maximum of h'' on the moduli space, as well as on which points it is attained.

We start with a few easy geometrical lemmae, whose proofs are either omitted or sketched.

Lemma 4.3 *In the Euclidean plane, consider two circles c, d of radii r, s, bounding disjoint discs, tangent to a line, with distance t between the tangency points P, Q. If $t \geq r + s$ and $r \geq s$, then the radius R of the half-circle C orthogonal to the line, tangent to c, d and which, when starting from $[P, Q]$, meets first d, is bigger than the radius S of the one D first meeting c.*

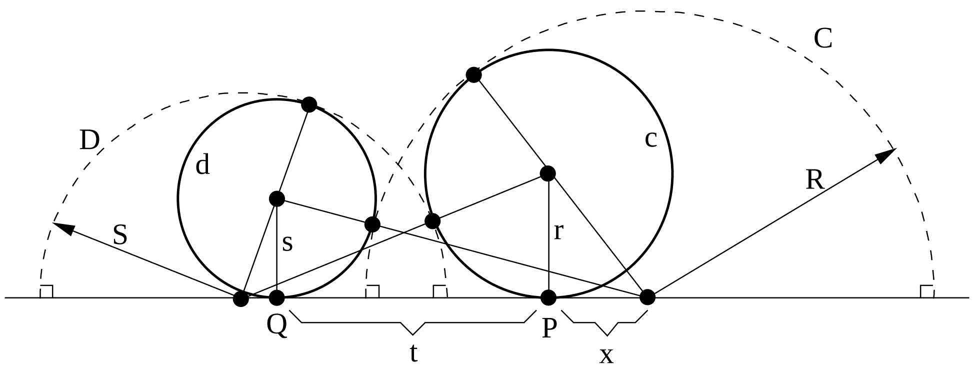

Figure 2 : Heights of common tangent circles.

Proof. By considering the right-angled triangle with vertices at the center of d, the center of C, and Q, we obtain

$$(R+s)^2 = s^2 + (t+x)^2,$$

with x the (signed) distance between the center of C and P. By considering the right-angled triangle with vertices the center of c, the center of C and P, one gets

$$(R-r)^2 = r^2 + x^2.$$

Eliminating x, this implies that

$$R^2(t^2-(r+s)^2) + Rt^2(s-r) - \frac{t^2}{4} = 0.$$

Similarly, one gets

$$S^2(t^2-(r+s)^2) + St^2(r-s) - \frac{t^2}{4} = 0.$$

By substracting the second equation from the first, one gets

$$(R-S)(R+S)(t^2-(s+r)^2) + (s-r)t^2(S+R) = 0$$

so that if $t \geq r+s$ and $r \geq s$, then $R \geq S$. $\boxed{\mathbf{4.3}}$

Lemma 4.4 *For every $\alpha \leq \frac{\pi}{2}$, in the upper-halfspace model of the hyperbolic plane, the distance between points of angle α and of angle $\frac{\pi}{2}$ on a non vertical hyperbolic geodesic is* $\log\,\cot\frac{\alpha}{2}$. □

Let S be the (smooth) once-punctured torus, and γ an essential (i.e. homotopic neither to a point nor to the puncture) simple closed curve on S. The real-analytic Fenchel-Nielsen coordinates (see for instance [FLP]) for the Teichmüller space $\mathcal{T}_{1,1}$ of the marked hyperbolic metrics on S are the length $\ell \in]0,+\infty[$ of (the closed geodesic for the marked hyperbolic metric which is freely homotopic to) γ and the twist parameter $\theta \in \mathbb{R}$ around it.

We define $h''(\ell,\theta)$ as the infimum of the heights of closed geodesics on the image by the canonical map $\mathcal{T}_{1,1} \to \mathcal{M}_{1,1}$ of the point with Fenchel-Nielsen coordinates $\ell \in]0,+\infty]$ and $\theta \in [-\infty,+\infty]$.

We will use the following constants. Let $\ell_{\min} = 2\ln(1+\sqrt{2})$, $\ell_{\max} = 2\ln\frac{3+\sqrt{5}}{2}$, and

$$\theta_{\min}(\ell) = \begin{cases} \frac{4\pi}{\ell}\cosh^{-1}(\sinh\frac{\ell}{2}) & \text{if } \ell \geq \ell_{\min} \\ 0 & \text{otherwise} \end{cases}.$$

The following is a well-known range reduction for the study of $h''(\ell,\theta)$.

Proposition 4.5 *Every point the Teichmüller space $\mathcal{T}_{1,1}$ is equivalent under the mapping class group of S to a point having Fenchel-Nielsen coordinates (ℓ,θ) satisfying $\ell \in]0,\ell_{\max}]$, and $\theta \in [\theta_{\min}(\ell),\pi]$.*

Proof. Recall that the diffeomorphism group of S acts transitively on the essential simple closed curves on S. Hence any marked hyperbolic structure on S is equivalent under the mapping class group to a new one for which (the closed geodesic which is freely homotopic to) γ is one of the shortest closed geodesics. Let us prove that the Fenchel-Nielsen coordinates of the new point in $\mathcal{T}_{1,1}$ satisfy the above requirements.

Step 1: Range of ℓ.

It is well known (see for instance [Schmu]) that the unique (up to isometry) once-punctured hyperbolic torus, such that its *systole* (i.e. the length of its shortest closed geodesic) is maximum, is the modular one $T_{\rm mod}$ (i.e. $T_{\rm mod} = \mathbb{H}^2/\Gamma$ where Γ is the commutator subgroup of $\mathrm{PSL}_2(\mathbb{Z})$). Geometrically, $T_{\rm mod}$ is obtained by gluing isometrically opposite faces of an hyperbolic hexagon with a dihedral symmetry group of order 6, and angles alternatively 0 and $2\pi/3$. The closed geodesics, whose lengths are the smallest, are exactly the three simple geodesics obtained by taking the common perpendicular to opposite edges of the hexagon.

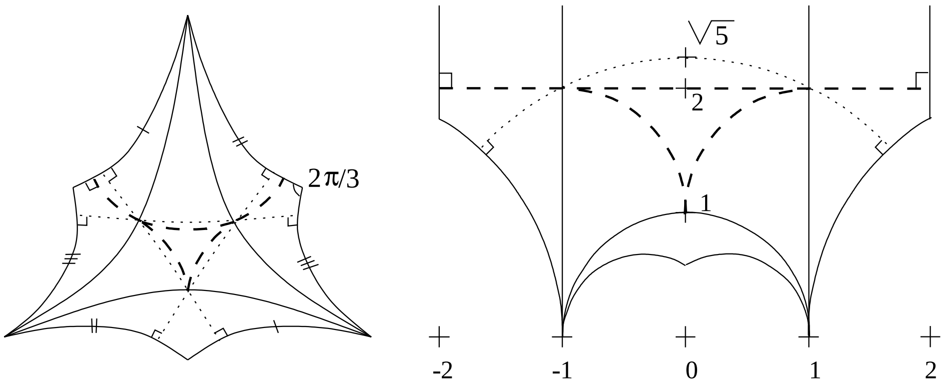

Figure 3 : The modular once-punctured torus.

Using Lemma 4.4, it is easy to see that the Fenchel-Nielsen ℓ-coordinate of $T_{\rm mod}$ (with the obvious marking) is

$$\ell = 4 \log \cot \arctan 2 = \ell_{\max}.$$

(It follows by uniqueness and the Proposition 4.5 that, θ belonging to $[\theta_{\min}(\ell), \pi]$ and $\theta_{\min}(\ell_{\max})$ being π, the Fenchel-Nielsen θ-coordinate of $T_{\rm mod}$ is π, but we will not need this.)

For future reference, the height of γ is (see Figure 3)

$$h''_{\max} = \int_2^{\sqrt{5}} \frac{dt}{t} = \log \frac{\sqrt{5}}{2}. \tag{1}$$

Step 2: Range of θ.

Since Dehn twists of angle multiple of 2π around γ define elements of the mapping class group, and since each (complete finite volume) hyperbolic metric on S has an elliptic involution, we need only to consider the twist angles $\theta \in [0, \pi]$.

Cutting open along γ, one obtains an hyperbolic pair of pants with side lengths $(\ell, \ell, 0)$. It is well-known (see for instance [FLP]) that such a pair of pants is isometric to the double of a right-angled hyperbolic pentagon P with one ideal vertex, along the sides adjacent and opposite to the ideal vertex, the two other sides having length $\frac{\ell}{2}$. We work in the upper halfplane model, with the ideal point at infinity, and P contained in the first quadrant, meeting the vertical axis in $[1, +\infty[$. Let s be the highest point on the side of P opposite to the vertex at infinity.

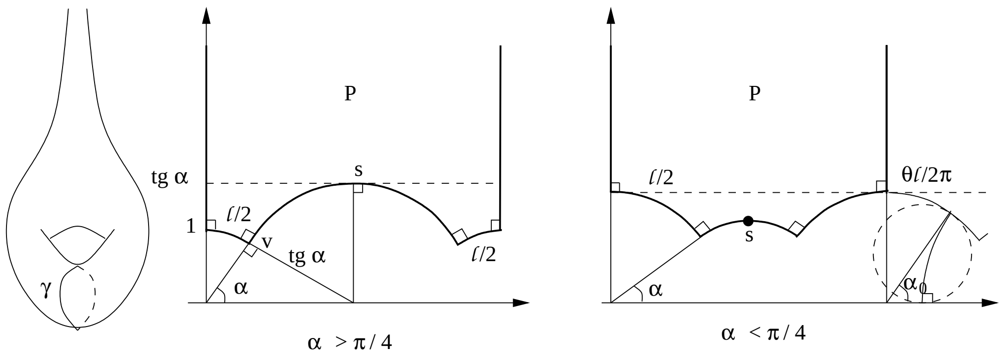

Figure 4 : Once-punctured hyperbolic tori.

Let u be the finite vertex of P on the vertical axis (with vertical coordinate 1), and v be the finite vertex adjacent to u. By Lemma 4.4, the angle α of v at the origin satisfies

$$\tan \frac{\alpha}{2} = e^{-\frac{\ell}{2}}. \tag{2}$$

If ℓ is small enough, then the trace on P of the Busemann level set $\beta_e^{-1}(0)$ is exactly the horizontal segment through s. The infimum of the heights of closed geodesic is attained exactly on γ. A direct computation gives

$$h''(\ell, \theta) = -\log \tan \alpha = \log\ \sinh \frac{\ell}{2}.$$

This formula is valid for ℓ small until s and u are at the same height, that is until $\ell = 2\log(1+\sqrt{2}) = \ell_{\min}$ (for $\alpha = \frac{\pi}{4}$). For future reference, we state this result as a proposition.

Proposition 4.6 *If* $(\ell, \theta) \in \,]0, \ell_{\min}] \times [0, \pi]$, *then* $h''(\ell, \theta) = \log\ \sinh \frac{\ell}{2}$. □

Note that $h''(\ell, \theta)$ is analytic on (ℓ, θ), increasing in ℓ and does not depend on θ on this range.

The once-punctured hyperbolic torus with $\ell = \ell_{min}$ and $\theta = 0$ is the (unique up to isometry) once-punctured hyperbolic torus with an order 4 symmetry group, obtained by identifying (without gliding) the opposite sides of a regular ideal hyperbolic quadrangle. There are exactly two closed geodesics whose heights are minimal. They are obtained by taking the common perpendicular to the opposite sides of the quadrangle.

As ℓ increases starting from $\ell_{\min}$, the length ℓ' of the minimizing segment between the two boundary components of the hyperbolic torus split open along γ decreases, and becomes shorter than the (common) length of the boundary components. An easy computation, using Lemma 4.4, shows that

$$\ell' = 2\log\ \cot(\frac{\pi}{4} - \frac{\alpha}{2}) = 2\log\ \coth \frac{\ell}{4}.$$

In order for γ to remain (one of) the shortest closed geodesic, we need to twist by some angle θ around γ. Let γ' be the closed curve, obtained by following the path which first, in the torus split open along γ, is the shortest common perpendicular between the two boundary curves, and then is the shortest of the two subpaths of γ back to its origin. By Theorem 7.3.6 of [Bea, page 183], recall that the translation length $\ell(gh)$ of the product of two hyperbolic isometries g, h of the hyperbolic plane, of translation lengths $\ell(g), \ell(h)$, with perpendicular translation axes, is given by:

$$\cosh \frac{\ell(gh)}{2} = \cosh \frac{\ell(g)}{2} \cosh \frac{\ell(h)}{2}$$

The closed geodesic freely homotopic to γ' has the same length as γ exactly when

$$\cosh \frac{\ell}{2} = \cosh \frac{\theta\ell}{4\pi} \cosh \frac{\ell'}{2},$$

that is when

$$\theta = \frac{4\pi}{\ell} \cosh^{-1}(\sinh \frac{\ell}{2}) = \theta_{\min}(\ell).$$

Note that $\theta_{\min}(\ell)$ is equal to π exactly when ℓ is the length $\ell_{\max}$ of γ for the modular once-punctured hyperbolic torus. This proves the proposition about the range restriction. $\boxed{\mathbf{4.5}}$

Theorem 4.7 *If* $\ell \in [0, \ell_{\max}]$ *and* $\theta \in [\theta_{\min}(\ell), \pi]$, *then* $h''(\ell, \theta) = \log \sinh \frac{\ell}{2}$. *In particular,* h'' *is real-analytic on* $\mathcal{M}_{1,1}$. *It is the height of one of any of the shortest closed geodesic, which is simple. Its maximum on* $\mathcal{M}_{1,1}$ *is* $h_{\max} = \log \frac{\sqrt{5}}{2}$, *which is attained uniquely on the modular one-cusped hyperbolic torus.*

Proof. By Proposition 4.6, we only have to consider the case $\ell \in [\ell_{\min}, \ell_{\max}]$ and $\theta \in [\theta_{\min}(\ell), \pi]$.

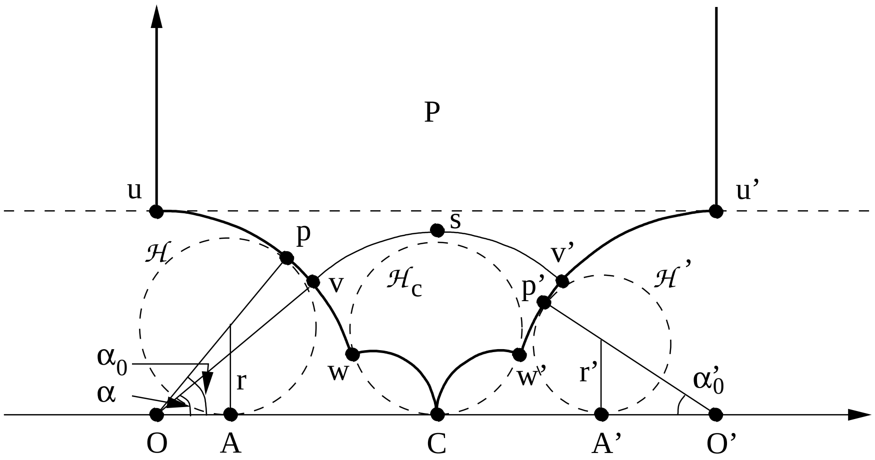

Figure 5 : Computing the heights.

Take as a fundamental domain the union of the pentagon P and its image by the hyperbolic reflection along the side of P opposite to the vertex at infinity of P (see figure 5). Let H_u be the horoball in the upper half space of points of vertical coordinates at least one, so that its intersection points with the boundary of the fondamental domain are u, u'. Let $\widetilde{\gamma}, \widetilde{\gamma}'$ be the two lifts of γ containing u, u' respectively. The horosphere ∂H_u is tangent to $\widetilde{\gamma}, \widetilde{\gamma}'$ respectively at u, u'. Define H_c to be the image of H_u by the reflection along the side of P opposite to the vertex at infinity of P. The horosphere ∂H_c is tangent to $\widetilde{\gamma}, \widetilde{\gamma}'$ at the vertices w, w' respectively of the fondamental domain.

Recall that v, v' were the vertices of P opposite to the point at infinity, with v the closer to u, and α is the angle of v at the origin. Let g be the element of the covering group that is the translation along the geodesic line through v, v' sending v to v', composed with the translation along $\widetilde{\gamma}'$ of length $\frac{\theta\ell}{2\pi}$ (and moving v' towards u'). Let $\mathcal{H}$ be the preimage of H_u by g, and $\mathcal{H}'$ be the image of H_c by g. Denote by A, C, A' the points at infinity of the horoballs $\mathcal{H}, H_c, \mathcal{H}'$ respectively (see Figure 5). Note that $\mathcal{H}$ is tangent to $\widetilde{\gamma}$ at the point p at distance $\frac{\theta\ell}{2\pi}$ from u on the side $[u, v]$ of P. Similarly $\mathcal{H}'$ is tangent to $\widetilde{\gamma}'$ at the point p' at distance $\ell - \frac{\theta\ell}{2\pi}$ from u' on the side $[w', u']$ of the fundamental domain.

We start with a few easy computations that will be needed in the proof.

Lemma 4.8 *The Euclidean radius of the disc H_c is $r_c = 1/(2\sinh^2 \frac{\ell}{2})$.*

Proof. We have already seen (see the discussion after Equation (2)) that if s is the highest point on the side of P opposite to infinity, then the hyperbolic distance between s and the horizontal horosphere ∂H_u is $|\log\sinh\frac{\ell}{2}|$. Since s is the midpoint of the common perpendicular segment to the horospheres $\partial H_u, \partial H_c$, the Euclidean diameter of the disc H_c is $1/\sinh^2\frac{\ell}{2}$. The result follows. $\boxed{\textbf{4.8}}$

Lemma 4.9 *The Euclidean radius of the disc $\mathcal{H}$ is $r = 1/(2\cosh^2\frac{\theta\ell}{4\pi})$.*

Proof. If α_0 is the angle of the center of the disc $\mathcal{H}$ at the origin, then by Lemma 4.4, one has $\tan\frac{\alpha_0}{2} = e^{-\frac{\theta\ell}{2\pi}}$. Since $\partial\mathcal{H}$ is tangent to $\widetilde{\gamma}$ at p, one has $\sin\alpha_0 = \frac{r}{1-r}$. The result follows. $\boxed{\textbf{4.9}}$

Lemma 4.10 *The Euclidean distance between A and C is* $\coth\frac{\ell}{2} - \tanh\frac{\theta\ell}{4\pi}$.

Proof. Let O be the origin in the plane. One has $d(A, C) = d(C, 0) - d(0, A)$ which gives $d(A, C) = \frac{1}{\cos\alpha} - \sqrt{(1-r)^2 - r^2}$. Using Equation (2) and Lemma 4.9, the result follows. $\boxed{\textbf{4.10}}$

Step 1: A computation of the Busemann level $\beta_e^{-1}(0)$.

It is easy to see that the only horospheres in the orbit of H_u, that meet the fundamental domain, are $H_u, \mathcal{H}, \mathcal{H}_c, \mathcal{H}'$ (unless $\theta = 0$, where we have two more horospheres, the translations of $\mathcal{H}, \mathcal{H}'$ along $\widetilde{\gamma}, \widetilde{\gamma}'$ of a distance ℓ, that meet the fundamental domain in w, u' respectively.) Note that since $\theta \geq \theta_{\min}(\ell)$, the Euclidean radius of $\mathcal{H}$, which is $r = 1/(2\cosh^2\frac{\theta\ell}{4\pi})$ by Lemma 4.9, is less than the Euclidean radius of H_c, which is $1/(2\sinh^2\frac{\ell}{2})$ by Lemma 4.8.

Hence the Busemann level set $\beta_e^{-1}(0)$ has a lift which is the horizontal horosphere through s. In particular the height of the closed geodesic γ is $\log\sinh\frac{\ell}{2}$ since no lift of γ enters the interior of the horosphere H_u.

Step 2: Computation of the minimal height of a closed geodesic.

One only has to prove that there is no geodesic line L meeting the fundamental domain and avoiding the horoballs $H_u, \mathcal{H}, H_c, \mathcal{H}'$. By absurd, assume that such a line L exists. The boundary of the fundamental domain, from which one removes the points lying in one of these horoballs, is the disjoint union of 4 geodesic segments $I_1 =]u, p[, I_2 =]p, w[, I_3 =]w', p'[, I_4 =]p', u'[$. The geodesic line enters the fundamental domain through one of I_1, I_2 and exits it through one of I_3, I_4.

Let us first prove that L cannot enter through I_1 and exit through I_4. The diameter of the Euclidean halfcircle L perpendicular to the real axis would be at least the distance between the tangent points of $\mathcal{H}$ and $\mathcal{H}'$ to the real line, plus the Euclidean radius of the discs $\mathcal{H}$ and $\mathcal{H}'$. By Lemma 4.9 and 4.10, one has

$$r = \frac{1}{2\cosh^2\frac{\theta\ell}{4\pi}} \quad \text{and} \quad d(A, C) = \coth\frac{\ell}{2} - \tanh\frac{\theta\ell}{4\pi}.$$

Similarly, with r' the Euclidean radius of $\mathcal{H}'$, one gets

$$r' = \frac{1}{2\cosh^2(\frac{\ell}{2} - \frac{\theta\ell}{4\pi})} \quad \text{and} \quad d(A', C) = \coth\frac{\ell}{2} - \tanh(\frac{\ell}{2} - \frac{\theta\ell}{4\pi}).$$

So that, with $f(\ell, \theta) = r + d(A, A') + r'$, we have

$$f(\ell,\theta) = 2\coth\frac{\ell}{2} + (\frac{1}{2\cosh^2\frac{\theta\ell}{4\pi}} + \frac{1}{2\cosh^2(\frac{\ell}{2} - \frac{\theta\ell}{4\pi})}) - (\tanh(\frac{\theta\ell}{4\pi} + \tanh\frac{\frac{\ell}{2} - \theta\ell}{4\pi})).$$

It is easy to see that $f(\ell, \theta)$ is decreasing in ℓ since $1 - \frac{\theta}{2\pi} > 0$. Since $\cosh x = \cosh(-x)$, and since by taking the derivative, the map $t \mapsto \tanh t + \tanh(x - t)$ is increasing in t for $t \leq 2x$, the function $f(\ell, \theta)$ is also decreasing in θ. Hence

$$f(\ell,\theta) \geq f(\ell_{\max}, \pi) = 2\coth\frac{\ell_{\max}}{2} + 1/\cosh^2\frac{\ell_{\max}}{4} - 2\tanh\frac{\ell_{\max}}{4}$$

which is about 2.58885438, hence strictly more than 2. In particular, the Euclidean radius of L would be strictly more than one, which contradicts the fact that L does no enter H_u.

Let us prove that L cannot enter through I_2. One only has to show that an Euclidean halfcircle L centered on the real axis, bounding an halfdisc D that contains H_c and does not contain $\mathcal{H}$ has radius at least one. The Euclidean radius of L is at least as big as the one of the halfcircle L' which is tangent to both ∂H_c and $\partial\mathcal{H}$, starts from the segment between the tangency points A, C and first meet $\partial\mathcal{H}$ and then ∂H_c. But recall that $\widetilde{\gamma}$ is tangent to both $\partial\mathcal{H}, \partial H_c$ and has radius 1, and that the Euclidean radius of H_c is at least the Euclidean radius of $\mathcal{H}$. By Lemma 4.3, to prove that L' hence L has radius at least 1, one only has to prove that $t = d(A, C) \geq r + r_c$ is positive.

By Lemmae 4.8, 4.9 and 4.10, we have

$$t = \coth\frac{\ell}{2} - \tanh\frac{\theta\ell}{4\pi} - \frac{1}{2\cosh^2\frac{\theta\ell}{4\pi}} - \frac{1}{2\sinh^2\frac{\ell}{2}} = \frac{\sinh(\ell) - 1}{2\sinh^2\frac{\ell}{2}} - \frac{\sinh\frac{\theta\ell}{2\pi} + 1}{2\cosh^2\frac{\theta\ell}{4\pi}}.$$

This is an increasing function in θ, hence its values are greater than or equal to its value at $\theta = \theta_{\min}(\ell)$. So

$$t \geq \frac{\sinh(\ell) - 1}{2\sinh^2\frac{\ell}{2}} - \frac{2\sinh\frac{\ell}{2}\sqrt{\sinh^2\frac{\ell}{2} - 1} + 1}{2\sinh^2\frac{\ell}{2}} = \frac{\sinh\frac{\ell}{2}(\cosh\frac{\ell}{2} - \sqrt{\cosh^2\frac{\ell}{2} - 2}) - 1}{\sinh^2\frac{\ell}{2}}.$$

The numerator is, by an easy derivative computation, a decreasing function of ℓ on $[\ell_{\min}, \ell_{\max}]$, whose value at $\ell = \ell_{\max}$ is about 0.118, hence is positive. This proves that t is indeed positive.

Similarly, one proves that L cannot exit through I_3, which proves the claim. Theorem 4.7 now follows. The maximum of h'' been reached uniquely when $\ell = \ell_{\max}, \theta = \pi$, that is for the modular once-punctured torus, its value $h''_{\max}$ has been computed in Equation (1). This ends the proof. $\boxed{\mathbf{4.7}}$

Theorem 1.3 of the introduction follows from Theorem 4.7.

5 The cut locus of a cusp

We keep the notations of the beginning of section 2. The following definition is due to [EP, section 4] in the case that $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$ and M has one cusp.

Definition 5.1 *(***Cut locus of a cusp***). The cut locus of the cusp e in M is the subset $\Sigma = \Sigma(e)$ of points x in M from which start at least two (globally) minimizing geodesic rays converging to e.*

(In the case M is an hyperbolic 3-orbifold and x is a singular point, we have to count the geodesic rays with multiplicities.)

The definition implies that for any $\eta > 0$, such geodesic rays meet perpendicularly the level set $\beta_e^{-1}(\eta)$ in one and only one point, and the lengths of their subsegment between x and $\beta_e^{-1}(\eta)$ are equal.

There is a *canonical retraction* from $M - \Sigma(e)$ into $\beta_e^{-1}(1)$, which associates to a point x in $M - \Sigma(e)$ the unique intersection point with $\beta_e^{-1}(1)$ of the unique minimizing geodesic ray starting from x and converging to e, if x does not lie in $\beta_e^{-1}([1, +\infty[)$, or the obvious projection to $\beta_e^{-1}(1)$ otherwise. (Note that this retraction is a strong deformation retract).

Let $N : M \to \mathbb{N} - \{0\}$ be the map which assigns to each $x \in M$ the number $N(x)$ of minimizing geodesic rays starting from x and converging to e. (This number is finite since the curvature is negative.) The cut locus of e is by definition $N^{-1}([2, +\infty[)$. It is immediate that N is upper semicontinuous.

In particular, Σ is closed. The set M has a natural partition by the connected components of $N^{-1}(\{k\})$ for k in $\mathbb{N} - \{0\}$. By the same proof as in [Sug, Theorem A], it is easy to show that $N^{-1}(2)$ is a codimension one submanifold of M, which is open and dense in Σ. We will denote $N^{-1}(2)$ by Σ_0.

Let H and H' be horospheres in $\widetilde{M}$ whose horoballs are disjoint, the *equidistant subspace* of H and H' is by definition, the set of points in $\widetilde{M}$ which are at the same distance from H and H'.

Let $x \in \Sigma_0$ and r_1, r_2 be the two minimizing rays starting from x and converging to e. Let $\widetilde{x}$ be a lift of x in $\widetilde{M}$, and $\widetilde{r}_1, \widetilde{r}_2$ be the lifts of r_1, r_2 starting from x. Let H_1, H_2 be the (disjoint) horospheres centered at the points $\widetilde{r}_1(\infty), \widetilde{r}_2(\infty)$ respectively, and covering $\beta_e^{-1}(1)$. The component of Σ_0 which contains x is the image by the covering map $\widetilde{M} \to M$ of an open subset of the equidistant subspace of H_1 and H_2.

In particular, if the curvature is constant, the equidistant subspace of two horospheres H, H' bounding disjoint horoballs is a hyperbolic hyperplane, hence is totally geodesic. So that each component of Σ_0 is (locally) totally geodesic. Furthermore, the equidistant subspace is the unique hyperbolic hyperplane orthogonal to the geodesic line L between the points at infinity of H, H' and passing through the point of L which is equidistant from H and H'. Using the transitivity properties of $\mathrm{Isom}_+(\mathbb{H}^n_\mathbb{R})$ this can be easily seen.

A *stratification* of a smooth manifold M is a partition of M into connected smooth submanifolds called *strata*, such that the closure of each stratum locally meets only finitely many strata. (See [Tro1] for a general survey about topological stratifications. R. Thom's definition (see [Tho] or [Tro1, page 234]) required only the existence of finitely many strata in the closure of any stratum, but a local such assumption is sufficient when dealing with local properties.)

Let X, Y with Y contained in the closure of X be any strata. The stratification is called *(a)-regular* (in the Whitney's sense, see [Whi], with applications to analytic varieties, or [Tro1, page 235]) if for every $x_i \in X$ converging to $y \in Y$ such that the tangent subspace $T_{x_i}X$ converges to some tangent subspace τ, it follows that T_yY is contained in τ.

This condition is precisely the one needed for the transversality of a submanifold to the stratification to be stable in the smooth topology (see [Fel] or [Tro1, page 237]). We now define the technical assumptions refered to in the introduction that we will need on the metric.

Definition 5.2 **(Cute cut locus)**. *The cut locus Σ of the cusp e will be called cute if*

- *the partition of M by components of $N^{-1}(x), x \in M$ is a locally finite (a)-regular stratification of M.*
- *each component σ of Σ_0 is simply connected, and has a unique locally highest point $\widehat{\sigma}$, that belongs to σ.*
- *Σ_0 has only finitely many components.*

The union of codimension 0 strata is exactly $N^{-1}(2) = \Sigma_0$. Since any component of Σ_0 is simply connected, it has a transverse orientation, uniquely determined by the orientation of any transversal subspace to the tangent subspace at any point.

It follows from the fact that the highest point $\widehat{\sigma}$ is unique and in the interior of σ that $\widehat{\sigma}$ is the unique point of σ from which start perpendicularly to σ at least two (and indeed exactly two) minimizing geodesic rays converging to e.

We will call $\widehat{\sigma}$ the *summit* of σ. A *summit* of Σ is the summit of some component of Σ_0. By definition, the cut locus Σ of e does not meet the neighborhood $\beta_e^{-1}(]0, +\infty[)$ of e.

Since M is geometrically finite, if there is only one cusp, if it satisfies the first condition of Definition 5.2, then the number of components of Σ_0 is finite (that is the third condition of Definition 5.2 is automatically satisfied).

Definition 5.3 **(Integral geodesic)**. *A geodesic line starting from e whose first hitting point with Σ is a summit of Σ will be called an integral line.*

Note that an integral line is rational, since the union of two minimizing geodesic rays starting from a summit $\widehat{\sigma}$ of Σ gives by the remark following Definition 5.2 (the image of) a geodesic line converging both positively and negatively to e. Since Σ_0 has only finitely many components, and since there are two integral lines per summit of Σ, the set $\mathcal{R} = \mathcal{R}(e)$ of rational rays is a finite subset of $Lk(e, C(M))$.

Let us describe interesting examples when the cut locus of the cusp is cute. This should also be the case for other rank one symmetric spaces of non compact type.

Proposition 5.4 *(i) If the curvature is constant, then Σ is cute.*

(ii) If the curvature is constant, if we have only one end, and if the codimension of each component of $N^{-1}(\{k\})$ is $k-1$, then for any small enough perturbation with compact support of the metric (in the C^∞ topology), the cut locus of the cusp is cute.

Proof. (i) Since being a locally finite (a)-regular stratification is a local property, one can work in the universal cover $\widetilde{M}$ of M. Take the projective Klein model for $\widetilde{M}$. Consider the union A of all equidistant subspaces of pairs of horospheres covering $\beta_e^{-1}(1)$. This is a locally finite union of analytic submanifolds (linear ones), hence the dimension stratification of Whitney is a locally finite (a)-regular stratification (see [Whi]). Since the lift $\widetilde{\Sigma}$ of Σ is a closed saturated subset of A, the same thing holds for $\widetilde{\Sigma}$.

Let x be a point in a component σ of Σ_0, and r_1, r_2 the two minimizing rays starting from x and converging to e. Let $\widetilde{x}$ be a lift of x in $\widetilde{M}$, and $\widetilde{r}_1, \widetilde{r}_2$ be the lifts of r_1, r_2 starting from x. Let H_1, H_2 be the (disjoint) horospheres centered at the points at infinity a_1, a_2 of $\widetilde{r}_1, \widetilde{r}_2$ and covering $\beta_e^{-1}(1)$. Let z be the intersection point of the equidistant subspace of H_1, H_2 with the geodesic between a_1, a_2.

Let us prove the claim that z belongs to the lift $\widetilde{\sigma}$ of σ containing $\widetilde{x}$, and its image in M is the unique summit of σ.

Let ℓ be the Busemann distance between $\widetilde{x}$ and a_1, and m be the Busemann distance between z and a_1. Let a_3 in $\partial\widetilde{M}$ be any point mapping onto e other than a_1, a_2, and ℓ' (resp. m') be the Busemann distance between $\widetilde{x}$ and a_3 (resp. z and a_3). Note that by assumption $\ell' > \ell$. Since the hyperbolic triangle with vertices $\widetilde{x}, z, a_1$ is rectangle at z, it follows from hyperbolic triangle formulae that $m' > m$. Hence z belongs to the lift of Σ_0 and is hence a summit. By convexity, the geodesic segment between z and $\widetilde{x}$ is contained in the lift of Σ_0. Since the Buseman function is strictly increasing along the geodesic segment from $\widetilde{x}$ to z, it follows that x cannot be a summit. Hence the claim is proved.

Let us prove now that σ is simply connected. By absurd, assume that there is a closed path c based at x which is not homotopic to 0 in σ. The closure of σ is locally convex, by convexity of the Busemann functions. Hence the lift of c starting at $\widetilde{x}$ ends in $\gamma\widetilde{x}$ for some non trivial element γ in the covering group. By analyticity, Γ preserves the equidistant subspace E of H_1, H_2. By convexity, the geodesic between $\widetilde{x}$ and $\gamma\widetilde{x}$ is contained in $\widetilde{\sigma}$, and by the previous claim, so is the geodesic segment between z and γz. But this is a contradiction, since the midpoint m of $[z, \gamma z]$ is contained in the equidistant plane of the horospheres H_1 and γH_1.

Now the third property of Definition 5.2 follows from the fact that the parabolic points are bounded, and that the equidistant subspaces are (in the Klein model of the hyperbolic space) affine subspaces.

(ii) Under the assumptions of (ii), the third property is automatically satisfied. The first property of a cute cut locus follows by transversality arguments. Since the cut locus in constant negative curvature is locally finite, and since the deformation is supported on a compact subset, if the deformation is small enough, the components of Σ_0 will remain simply connected, and by strict convexity of the horospheres, there will remain one and only one new summit close to each old summit, and no new one will be created far from the old ones. $\boxed{\textbf{5.4}}$

The assumption in (ii) that the codimension of $N^{-1}(\{k\})$ is $k-1$ cannot be omitted, since if some cut locus in dimension 3 has a vertex of degree 4, then by small perturbation of the metric, the cut locus can be made not locally finite, as B. Bowditch showed us.

Till the end of this section, we assume that M is a non elementary geometrically finite hyperbolic 3-orbifold, uniformized as in subsection 2.3. All the statements below extend easily to $\mathbb{H}^n_{\mathbb{R}}$.

It follows from Proposition 5.4, and the fact that there cannot be any summit close to another cusp, that Σ is a finite piecewise hyperbolic cell 2-complex, whose 2-cells are compact or finite volume hyperbolic polygons, with one vertex at infinity for each end besides e.

The *summit* of an edge τ of Σ is the unique point $\widehat{\tau}$ (contained in the interior of the edge) from which start perpendicularly to the edge at least three minimizing geodesic segments ending in e.

The following is another description of Σ. It is due to Ford [For1] for some arithmetical cases, extended by Swan [Swa] for all Bianchi groups, and to [EP, section 4] when $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$ and M has one cusp.

Let $\gamma \in \Gamma - \Gamma_\infty$, the *isometric sphere* S_γ of γ is the hyperbolic plane in $\mathbb{H}^3$, defined as the intersection of the upper halfspace with the Euclidean sphere centered at $\gamma^{-1}(\infty)$ and of radius $\frac{1}{|c(\gamma)|}$. It follows that γ maps the horizontal horosphere $H_\infty(h)$ defined by the equation $t = h$ centered at ∞ to the horosphere $H_{\gamma(\infty)}(h)$ centered at $\gamma(\infty)$, which is the Euclidean sphere in the upper halfspace, tangent to $\mathbb{C}$ at $\gamma(\infty)$ and with diameter $\frac{1}{h|c(\gamma)|^2}$. We denote by $H^+_{\gamma(\infty)}(h)$ the horoball bounded by $H_{\gamma(\infty)}(h)$.

The key point is the following fact.

Lemma 5.5 *For any $\gamma \in \Gamma - \Gamma_\infty$ and $h > 0$ big enough, the isometric sphere S_γ is the equidistant subspace of the horospheres $H_\infty(h)$ and $H_{\gamma^{-1}(\infty)}(h)$.*

Proof. Let x, y be the points on a vertical geodesic line L at vertical coordinates $t = h$ and $t = \frac{1}{h|c(\gamma^{-1})|^2}$ respectively. Then the midpoint of $[x, y]$ is the point on L at height $t = \frac{1}{|c(\gamma^{-1})|}$ for any $h > 0$. The hyperbolic plane S_γ is orthogonal to the (vertical) geodesic line between ∞ and $\gamma^{-1}(\infty)$. Hence, by uniqueness, S_γ is the equidistant subspace of $H_\infty(h)$ and $H_{\gamma^{-1}(\infty)}(h)$. $\boxed{\textbf{5.5}}$

Let S^-_γ be the half ball bounded by S_γ. It is easy to see that γ maps $\mathbb{H}^3 - S^-_\gamma$ to $S^-_{\gamma^{-1}}$. Define

$$B_\infty = \mathbb{H}^3 - \bigcup_{\gamma \in \Gamma - \Gamma_\infty} S^-_\gamma,$$

that we will call the *basin of center* ∞. It follows that

1. B_∞ is invariant by Γ_∞,
2. no element of $\Gamma - \Gamma_\infty$ maps an interior point of B_∞ to another interior point of B_∞, and
3. the images of B_∞ by Γ cover $\mathbb{H}^3$.

Therefore B_∞/Γ_∞ is an orbifold whose boundary is a piecewise hyperbolic 2-orbifold, and M is obtained by pairing faces of B_∞/Γ_∞. Since M is geometrically finite, the boundary of B_∞ has finitely many 2-cells up to the action of Γ_∞. For $\gamma \in \Gamma$, define the *basin of center* $\gamma(\infty)$ to be $B_{\gamma(\infty)} = \gamma B_\infty$.

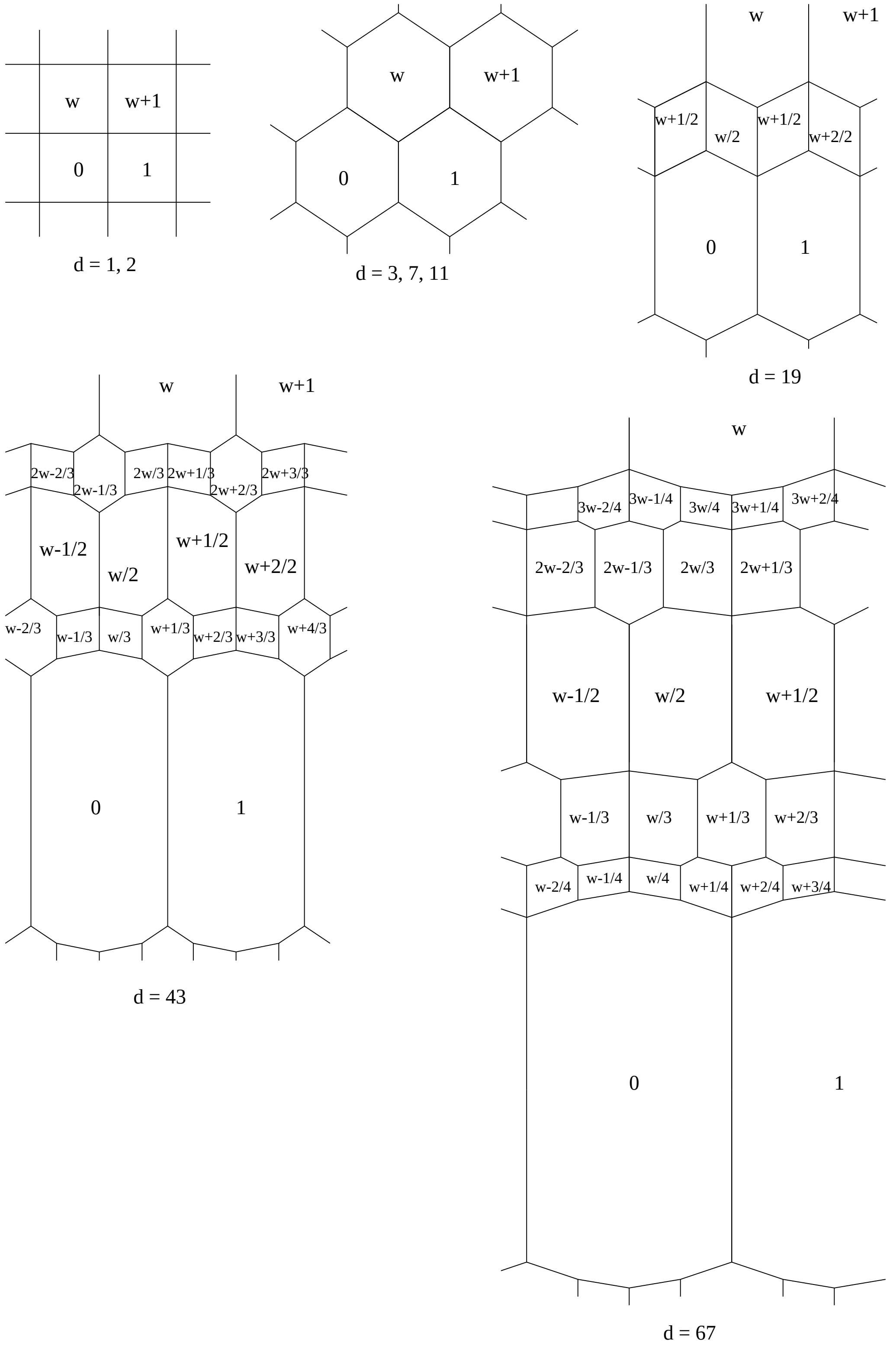

Figure 6 : Cell decomposition of ∂B_∞ for $\mathbb{H}^3 / \mathrm{PSL}_2(\mathcal{O}_{-d})$.

Above is a combinatorial picture of the vertical projection to $\mathbb{C}$ of the cell decomposition of the boundary of the basin at infinity for $\mathbb{H}^3 / \mathrm{PSL}_2(\mathcal{O}_{-d})$. Up to $d = 43$, these drawings are due to [Hat]. The complex number w is $i\sqrt{d}$ if $d \not\equiv -1[\mathrm{mod}\ 4]$ and $\frac{1}{2}(1+i\sqrt{d})$ otherwise.

Using Lemma 5.5, the points in ∂B_∞ are exactly the points from which starts a geodesic ray converging to ∞ and a geodesic ray converging to a point different form ∞ in the orbit of ∞ by Γ. Hence the cut locus Σ is the image of ∂B_∞ by the canonical projection $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$. Furthermore, the summits of the 2-cells of Σ are the images in M of the highest points of the isometric spheres S_γ containing a 2-cell of ∂B_∞, and the summits of the edges of Σ are the image in M of highest points of the halfcircle geodesic lines containing an edge of ∂B_∞. Let $\widetilde{\mathcal{R}}$ be the set of endpoints of the lifts to $\mathbb{H}^3$ starting from ∞ of the integral lines. If P is the set of elements in $\Gamma - \Gamma_\infty$ whose isometric sphere carries a 2-cell of the boundary of the basin at infinity, then $\widetilde{\mathcal{R}}$ is exactly the discrete subset $P(\infty)$.

In the case of the Bianchi orbifolds $\mathrm{PSL}_2(\mathcal{O}_{-d})\backslash\mathbb{H}^3$, the uniformization chosen in section 2 is precisely the natural one $\mathbb{H}^3 \to \mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O}_{-d})$. Though $\widetilde{\mathcal{R}}$ consists of the integer points in the case $d = 1, 2, 3, 7, 11$ for instance, it contains in general rational points with finitely many possible denominators ($q = \pm 1, \pm 2, \pm 3$ for $d = 43$, for instance, see [Poi] for other lists), not all rational lines with these denominators being in $\widetilde{\mathcal{R}}$ (see Figure 6).

6 Good approximating sequence of an irrational line

We keep the same notations as in the beginning of section 2. In section 3, we proved that any irrational line starting from a given cusp can be well approximated by rational lines. The aim of this section is to give an explicit sequence of such rational lines.

Assume that the cut locus of the cusp e is cute. We say that a geodesic line starting from e is *totally irrational* if it recurrent in a compact subset of M (in particular and equivalently in the finite volume case, converges into no cusp), and is transverse to the stratification of the cusp, i.e. it does not meet the singular locus $\Sigma - \Sigma_0$ and is transverse to Σ_0. This last assumption is implied by the previous one in the case of constant curvature, since the components of Σ_0 are (locally) totally geodesic. The set of totally irrational lines is a dense G_δ set in the link of e.

Consider a totally irrational line ξ. Note that Σ is cute and ξ is transverse to Σ, hence the intersection of Σ and ξ has no accumulation point. Since ξ is recurrent in a compact subset, for every $\epsilon > 0$, the ϵ-neighborhood of any positive subray of ξ contains a closed geodesic (by the closing lemma), hence since no closed geodesic is contained in $\mathcal{M} - \Sigma$ (which retracts onto $\beta_e^{-1}(1)$), there are infinitely many intersection points of ξ and Σ. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence of intersection points of ξ with $(\sigma_n)_{n\in\mathbb{N}}$, the sequence of connected components of Σ_0 consecutively passed through by ξ. For each $n \in \mathbb{N}$, let c_n be the path consisting of the subsegment of ξ between e and x_n, followed by the minimizing geodesic ray from x_n to the cusp e, starting on the other side of Σ_n than the one on which ξ arrives at x_n. Let r_n be the unique geodesic line starting from e, properly homotopic to c_n. Since r_n converges to e, it is a rational line.

Definition 6.1 *The sequence $(r_n)_{n\in\mathbb{N}}$ will be called the good approximating sequence of ξ, and r_n the n-th good approximant.*

This definition can be extended to the other irrational lines, since by the (a)-regularity of the stratification of the cut locus, the transversality to the stratification is stable, at the expense of loosing the uniqueness of the sequence, and allowing, for rational lines and lines ending in other cusps, each sequence to be finite.

Remark. When $M = \mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$, this sequence coincides with the usual best approximation sequence in the following sense: let $\widetilde{\xi}$ be the endpoint on the real axis of the lift of ξ starting from ∞ ending in $[0,1[$. Note that ξ is totally irrational if and only if $\widetilde{\xi}$ is an irrational real number. Then the endpoint on the real axis of the lift of r_n starting from ∞ and ending in $[0,1[$ is p_n/q_n, where p_n/q_n is the n-th convergent of the irrational real number $\widetilde{\xi}$. In the case of $M = \mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O}_{-d})$, we don't know if our good approximation sequence coincides with Poitou's best approximation sequence in [Poi].

The good approximating sequence nicely approximates an irrational line. We prove this for totally irrational lines, but the result could be extended for general irrational lines.

Theorem 6.2 *Let M be a non elementary geometrically finite pinched negatively curved Riemannian manifold, with only one cusp e, having a cute cut locus. There exists a constant $c > 0$ such that, for every totally irrational line ξ starting from e with good approximating sequence $(r_n)_{n\in\mathbb{N}}$, for every n in $\mathbb{N}$,*

$$d(r_n, \xi) \leq c\, e^{-D(r_n)}.$$

Proof. Let

$$c = \frac{1}{2}\max_{x\in\Sigma} e^{-\beta_e(x)}.$$

Since Σ is in the complement of $\beta_e^{-1}(]0,+\infty[)$, $\sup_{x\in\Sigma} -\beta_e(x)$ is strictly bigger than 0. The supremum is a maximum, since Σ is non empty and compact in the one cusped case.

Let ξ be a totally irrational line starting from e. Fix a lift $\widetilde{\xi}$ of ξ to the universal cover $\widetilde{M}$ of M, intersecting the preimages of the cut locus in successive points $\widetilde{x_n}$. Let $\widetilde{\sigma_n}$ be the lift of some (simply connected by 5.4) component of Σ_0 containing $\widetilde{x_n}$, and $\widetilde{r_n}$ the lift of r_n starting from the same point at infinity a as $\widetilde{\xi}$.

Since r_n starts and ends in the cusp, then $\widetilde{r_n}$ ends in $\gamma_n a$ for some γ_n in the covering group of $\widetilde{M} \to M$. By the definition of r_n, the geodesic ray from $\widetilde{x_n}$ to $\gamma_n a$ projects to some minimizing ray in M.

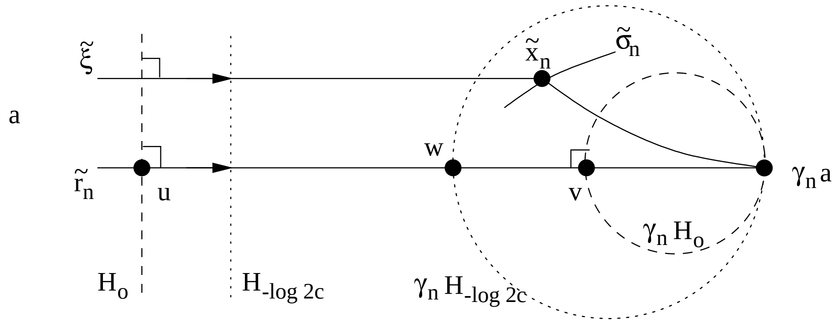

Figure 7 : Convergence properties of the good approximating sequence.

Let $(H_t)_{t\in\mathbb{R}}$ be the unique family of horospheres centered at a such that H_0 maps onto $\beta_e^{-1}(0)$, the distance between H_t and $H_{t'}$ is $|t-t'|$, and H_t converges to a as t goes to

$+\infty$. Since $-\log 2c = \min_{x\in\Sigma}\beta_e(x)$, each component of Σ_0 has a lift contained between the horospheres H_0 and $H_{-\log 2c}$. Hence $\widetilde{\sigma_n}$ is contained between $\gamma_n H_0$ and $\gamma_n H_{-\log 2c}$ (see Figure 7).

Let u, w, v be the intersection points of $\widetilde{r_n}$ with $H_0, \gamma_n H_{-\log 2c}, \gamma_n H_0$, respectively. So that $d_{\widetilde{M}}(u,v) = D(r_n)$ by definition of the depth, and $d_{\widetilde{M}}(w,v) = |-\log 2c| = \log 2c$, since $\widetilde{r_n}$ is perpendicular to $\gamma_n H_0$ and $\gamma_n H_{-\log 2c}$. Since $\widetilde{\xi}$ meets $\widetilde{\sigma_n}$, the horosphere $\gamma_n H_{-\log 2c}$ which is centered at $\gamma_n a$ meets ξ. Therefore, by the definition of the distance d on $Lk(M,e)$, with $\epsilon = +1$ if w lies between u and v, and -1 otherwise,

$$d(r_n, \xi) \leq \frac{1}{2} e^{-\epsilon d_{\widetilde{M}}(u,w)} = \frac{1}{2} e^{-d_{\widetilde{M}}(u,v) + d_{\widetilde{M}}(w,v)} = \frac{1}{2} e^{\log 2c} e^{-D(r_n)}.$$

This proves the result. $\boxed{\mathbf{6.2}}$

Corollary 6.3 *Let M be a non elementary geometrically finite pinched negatively curved Riemannian manifold, with only one cusp e, having a cute cut locus. The good approximating sequence $(r_n)_{n\in\mathbb{N}}$ of any totally irrational line ξ starting from e converges to ξ.*

Proof. This follows from the previous theorem, and from the fact that the depths $D(r_n)$ converge to $+\infty$. We prove this last claim by absurd. By Remark 2.9, the sequence $(r_n)_{n\in\mathbb{N}}$ would have a constant subsequence $(r_{n_k})_{k\in\mathbb{N}}$. Let a be the starting point of $\widetilde{\xi}$ (with the notations of the beginning of the previous proof), hence of $\widetilde{r_{n_k}}$. Let $\Gamma_a \gamma_n \Gamma_a$ be the double coset associated to $(r_n)_{n\in\mathbb{N}}$, with $(\gamma_{n_k})_{k\in\mathbb{N}}$ constant.

Since ξ is irrational, and by strict convexity of the horospheres, the sequence $\widetilde{\sigma_n}$ goes out of every horosphere centered at $\gamma_{n_k} a$. This contradicts the fact that $\widetilde{x_{n_k}}$ lies for each k between the horospheres $\gamma_{n_k} H_0$ and $\gamma_{n_k} H_{-\log 2c}$ (see Figure 7). $\boxed{\mathbf{6.3}}$

Till the end of this section, we assume that M is a non elementary geometrically finite hyperbolic 3-orbifold, uniformized as in subsection 2.3. Again, this extends easily to $\mathbb{H}^n_{\mathbb{R}}$. We know a bit more on the depths of the good approximants r_n, generalizing what Ford did in the case of the Bianchi orbifold $\mathbb{H}^3/PSL(2,\mathbb{Z}[i])$.

If ξ is a totally irrational line starting from the cusp e, let $\widetilde{\xi}$ be a lift of ξ in $\mathbb{H}^3$ starting from ∞. It is a vertical geodesic in the upper halfspace $\mathbb{H}^3$, that cuts, while going downwards, a sequence $(\widetilde{\sigma_n})_{n\in\mathbb{N}}$ of cells of the preimage of Σ_0 in $\mathbb{H}^3$. The n-th cell $\widetilde{\sigma_n}$ passed through by $\widetilde{\xi}$, is contained in the boundary of the basins of $\gamma_n(\infty)$ and $\gamma_{n+1}(\infty)$ for some γ_n, γ_{n+1} in the covering group Γ. By definition of the good approximation sequence, the n-th good approximant r_n is the rational line associated to the double coset $\Gamma_\infty \gamma_n \Gamma_\infty$.

Proposition 6.4 *If $(r_n)_{n\in\mathbb{N}}$ is any good approximation sequence of a totally irrational line ξ, then for all n in $\mathbb{N}$,*

$$D(r_n) < D(r_{n+1}).$$

Proof. With the notations above, by the definition of the summit of $\widetilde{\sigma_n}$, there exist horospheres H_n and H_{n+1} centered respectively at $\gamma_n(\infty)$ and $\gamma_{n+1}(\infty)$ that are tangent at the summit $\widehat{\widetilde{\sigma_n}}$ of $\widetilde{\sigma_n}$. The Euclidean line through $\widehat{\widetilde{\sigma_n}}$ perpendicular to $\widetilde{\sigma_n}$ hence goes through the Euclidean center of the hyperbolic plane P containing $\widetilde{\sigma_n}$, and the Euclidean centers of H_n and H_{n+1}. Hence (see the Figure below), the Euclidean radius of H_{n+1}, (which is contained in the half-ball bounded by P since $\widetilde{\xi}$ is first meeting $B_{\gamma_n(\infty)}$ and then $B_{\gamma_{n+1}(\infty)}$), is strictly smaller than the one of H_n.

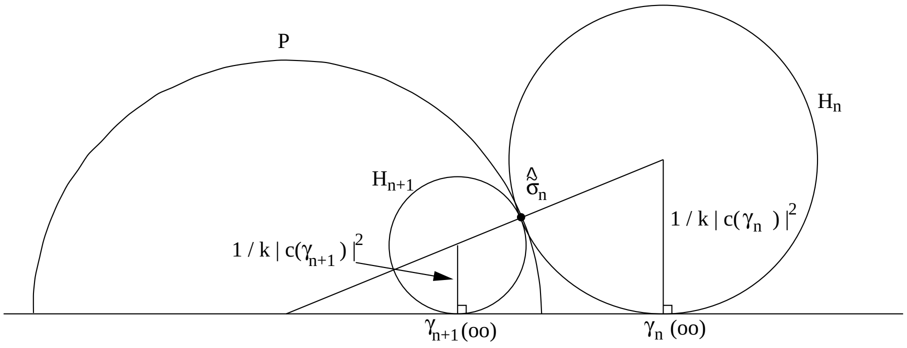

Figure 8 : Depths of convergents are increasing.

Since the point $\widehat{\sigma_n}$ is at the same (hyperbolic) distance from the horospheres $H_{\gamma_n(\infty)}(1)$ and $H_{\gamma_{n+1}(\infty)}(1)$ centered at $\gamma_n(\infty)$ and $\gamma_{n+1}(\infty)$ of Euclidean radii $\frac{1}{2|c(\gamma_n)|^2}$ and $\frac{1}{2|c(\gamma_{n+1})|^2}$, there is a constant $k > 0$ such that the Euclidean radii of H_n and H_{n+1} are respectively $\frac{1}{2k|c(\gamma_n)|^2}$ and $\frac{1}{2k|c(\gamma_{n+1})|^2}$. This proves the result, using Lemma 2.10. $\boxed{\mathbf{6.4}}$

Let $\mathcal{D}$ be the set of real numbers $e^{\frac{D(r)}{2}}$ for r an integral line. Since there are only finitely many integral lines, the subset $\mathcal{D}$ of $[1, +\infty[$ is finite. It consists of $\{1\}$ if there is only one integral line (counted without multiplicity and without orientation), as in the case of the Bianchi orbifolds $\mathbb{H}^3 / \operatorname{PSL}(2, \mathcal{O}_d)$ for $d = 1, 2, 3, 7, 11$.

For g, h elements of $\operatorname{PSL}(2, \mathbb{C})$, define

$$\Delta(g, h) = |\begin{vmatrix} a(g) & a(h) \\ c(g) & c(h) \end{vmatrix}|.$$

This nonnegative real number does not depend on the choosen lifts to $\operatorname{SL}(2, \mathbb{C})$, nor on the representatives of the left cosets of g, h by the stabilizer of any horizontal horosphere. In particular, by Lemma 2.10,

$$\Delta(1, h) = |c(h)| = e^{\frac{D(\Gamma_\infty h \Gamma_\infty)}{2}}$$

if $h \notin \Gamma_\infty$.

Proposition 6.5 *If r_n, r_{n+1} are consecutive good approximants of an irrational line ξ, then $d(r_n, r_{n+1}) \exp(\frac{D(r_n)+D(r_{n+1})}{2})$ belongs to the finite set $\mathcal{D}$. In particular, if there is only one integral line (counted without multiplicity and without orientation), then*

$$d(r_n, r_{n+1}) = e^{-(\frac{D(r_n)+D(r_{n+1})}{2})}$$

Proof. We may assume that ξ is totally irrational. Choose γ_n, γ_{n+1} representatives of the double cosets associated to r_n, r_{n+1} such that the boundaries of the basins $B_{\gamma_n(\infty)}$ and $B_{\gamma_{n+1}(\infty)}$ contain the n-th cell of the preimage of Σ_0 passed through by $\widetilde{\xi}$. By Lemma 2.10, one has

$$\Delta(\gamma_n, \gamma_{n+1}) = |\frac{a(\gamma_n)}{c(\gamma_n)} - \frac{a(\gamma_{n+1})}{c(\gamma_{n+1})}||c(\gamma_n)||c(\gamma_{n+1})| = d(r_n, r_{n+1}) e^{\frac{D(r_n)+D(r_{n+1})}{2}}.$$

Since $\mathrm{SL}_2(\mathbb{C})$ preserves the area in $\mathbb{C}^2$, $\Delta(\gamma_n, \gamma_{n+1}) = \Delta(1, \gamma_n^{-1}\gamma_{n+1})$. Since the closures of the basins $B_{\gamma_n(\infty)}$ and $B_{\gamma_{n+1}(\infty)}$ meet in a cell of the preimage of Σ_0, the rational line associated to the double coset of $\gamma_n^{-1}\gamma_{n+1}$ is an integral line, and the result follows. $\boxed{\mathbf{6.5}}$

Remark. When $M = \mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$, it is well known that $\Delta(\gamma_n, \gamma_{n+1})$ is always 1. This proposition gives a geometric understanding of why this is not always the case, and a geometric interpretation of the possible values, for instance for the Poitou's best approximation sequence when $M = \mathbb{H}^3/\mathrm{PSL}_2(\mathcal{O}_{-\lceil})$ for d large enough. See [Poi] for the list of possible values of $\begin{vmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{vmatrix}$ when $d = 19$ for example.

7 Continued sequence of a totally irrational line

We keep the notations of the beginning of section 2. We assume that the cut locus of the cusp e is cute. The aim of this section is to associate to a totally irrational line ξ starting from e a sequence $(a_n)_{n\in\mathbb{N}}$ in some countable alphabet which will determine it.

Let $\mathcal{R}$ be the finite set which consists of all the first intersection points of integral lines starting from e with $L_e = \beta_e^{-1}(1)$. Let $\pi_1(L_e, \mathcal{R})$ denote the set of homotopy classes relative to endpoints of paths in L_e with endpoints in $\mathcal{R}$. Note that unless there is only one point in $\mathcal{R}$, $\pi_1(L_e, \mathcal{R})$ is not a group, but it is a groupoid for the composition of paths. There is a natural involution on $\mathcal{R}$. It associates to the first intersection point λ, of an integral line r with L_e the second (and last) intersection point, that we will denote by λ^{-1}. It is clear that λ^{-1} is also the first intersection point of the integral line which is r with the opposite orientation. Since L_e is compact and $\mathcal{R}$ finite, the groupoid $\pi_1(L_e, \mathcal{R})$ is countable.

Let $(r_n)_{n\in\mathbb{N}}$ be the good approximation sequence of ξ, with $(x_n)_{n\in\mathbb{N}}$ the successive intersection points of ξ with the cut locus and σ_n the cell of Σ_0 containing x_n. Recall that ξ is oriented and totally irrational, hence for each n, the tangent space to ξ at x_n is oriented and transverse to the tangent subspace to σ_n at x_n. For each n, endow σ_n with the transverse orientation given by the oriented tangent space to ξ at x_n.

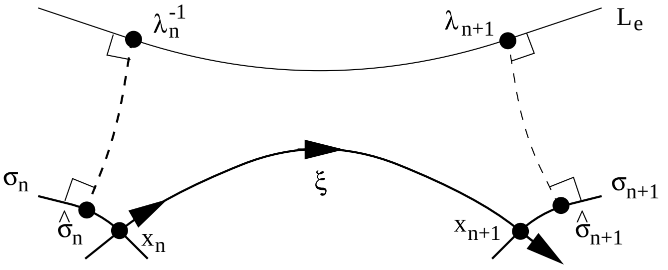

Figure 9 : The continued sequence of an irrational ray.

Note that r_0 is an integral line. Let λ_0 be its first intersection point with L_e. Define $a_0 \in \pi_1(L_e, \mathcal{R})$ to be the class of the constant path at λ_0. We define the sequences $(a_n)_{n\in\mathbb{N}}, (\lambda_n)_{n\in\mathbb{N}}$ by induction. Assume a_n, λ_n are defined. Consider the path c starting at λ_n^{-1}, following the subpath of an integral line from λ_n^{-1} to the summit $\widehat{\sigma_n}$, then any path c_1 in σ_n from $\widehat{\sigma_n}$ to x_n, then the subpath of ξ from x_n to x_{n+1}, then any path c_2 in σ_{n+1} from x_{n+1} to $\widehat{\sigma_{n+1}}$, then the subpath of an integral line starting from $\widehat{\sigma_{n+1}}$ on the negative side of σ_{n+1}, that ends in the point $\lambda_{n+1} \in \mathcal{R}$. By pushing a little bit c_1 on the positive side of σ_n, and c_2 on the negative side of σ_{n+1}, one gets a path contained in the complement of the cut locus in M. Recall that this complement canonically retracts onto L_e. Let a_{n+1} be the class of the path between λ_n^{-1} and λ_{n+1} obtained by retracting c onto L_e.

Definition 7.1 *The sequence $(a_n)_{n\in\mathbb{N}}$ is called the continued sequence of the totally irrational ray ξ.*

Theorem 7.2 *Let M be a non elementary geometrically finite pinched negatively curved Riemannian manifold, with only one cusp e, having a cute cut locus. A totally irrational line is uniquely determined by its continued sequence.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be the continued sequence of the totally irrational ray ξ. Let λ_n^{-1} be the initial point of the path a_n. Consider the path v in M, depending only on the continued sequence, which is obtained by following the minimizing geodesic ray starting from e arriving perpendicularly on λ_0, then the subpath of the integral line from λ_0 to λ_0^{-1}, then a_1, then the subpath of the integral line from λ_1 to λ_1^{-1}, then a_2, etc. It is clear by the construction that v is homotopic to ξ, by an homotopy which is proper on each negative subray (though not necessarily globally proper). Since Σ is compact, the sequence of points $(\lambda_n)_{n\in\mathbb{N}}$ on v stays at uniformely bounded distance from ξ. If ξ' is another totally irrational line having the same continued sequence, lift to the universal cover the homotopy between v and ξ and the one between v and ξ', so that they coincide on some lift of v. Then the lifts of ξ, ξ' are two geodesic lines in $\widetilde{M}$ that have the same origin at infinity, and a sequence of point converging to their endpoint at infinity that stay at uniformely bounded distance. Hence the geodesic lines have the same endpoint at infinity, therefore they coincide. By projecting to M, one gets that ξ, ξ' are equal. $\boxed{\textbf{7.2}}$

Till the end of this section, we assume that M is a non elementary geometrically finite hyperbolic 3-orbifold, uniformized as in subsection 2.3. The orbifold universal cover of L_e is the horizontal horosphere in $\mathbb{H}^3$ which is mapped onto L_e by the choice of the orbifold universal cover $\mathbb{H}^3 \to M$. We will identify that horosphere with $\mathbb{C}$ by vertical projection. Let $\widetilde{\mathcal{R}}$ be the subset of $\mathbb{C}$ corresponding to the lift of $\mathcal{R}$, which is a discrete subset of $\mathbb{C}$. It is invariant by the group Γ_∞ (and it is reduced to one orbit if there is only one integral line (counted without multiplicity and orientation)).

Assume for simplicity that the group Γ_∞ is a covering group of L_e, hence a group of translations of $\mathbb{C}$. Since any path between two points in $\widetilde{\mathcal{R}}$ is homotopic relative to endpoints to the segment between the endpoints, one can naturally identify $\pi_1(L_e, \mathcal{R})$ with the set $\widetilde{\mathcal{R}} - \widetilde{\mathcal{R}}$ of differences of two elements of $\widetilde{\mathcal{R}}$. Indeed, any path between two points of $\mathcal{R}$ has a unique lift, once a preimage of the starting point is choosen, between two points in $\mathbb{C}$, and we associate to the path the difference of the endpoints of the lift. This does not depend on the choosen preimage of the starting point, since any two of them differ by an element of Γ_∞, which acts by translation.

Let ξ be a totally irrational line, and $(r_n)_{n\in\mathbb{N}}$ be the good approximating sequence for ξ. Let $\widetilde{\xi}$ be a lift of ξ starting from ∞, and $\widetilde{r_n}$ the lift of r_n obtained by lifting the homotopy between r_n and ξ_n (the subpath of ξ up to the n-th intersection point x_n with the cut locus, followed by the minimizing geodesic ray from x_n to e on the opposite side). Let $\Gamma_\infty\gamma_n\Gamma_\infty$ be the double coset associated to r_n, with γ_n a representative so that the endpoint of $\widetilde{r_n}$ is $z_n = \gamma_n(\infty)$ (this determines the left coset $\gamma_n\Gamma_\infty$). Define by convention $\gamma_{-1} = id$, so that $z_{-1} = \infty$.

The next result explains how the continued fraction can be explicitely computed in terms of the good approximation sequence.

Proposition 7.3 *Under the above identification of $\pi_1(L_e, \mathcal{R})$ with $\widetilde{\mathcal{R}} - \widetilde{\mathcal{R}}$, one has $a_0 = 0$ and for $n \geq 0$,*

$$a_{n+1} = \gamma_n^{-1}(z_{n+1}) - \gamma_n^{-1}(z_{n-1}).$$

Note that the right hand side does not depend on the left coset of γ_n.

Proof. Let $\widetilde{x_n}, \widetilde{x_{n+1}}$ be consecutive intersection points of $\widetilde{\xi}$ with the preimage of the cut locus. Then $\widetilde{\xi}$ passes at $\widetilde{x_n}$ from the basin of $\gamma_{n-1}(\infty)$ to the basin of $\gamma_n(\infty)$, stays inside the basin of $\gamma_n(\infty)$ between $\widetilde{x_n}$ and $\widetilde{x_{n+1}}$, then passes at $\widetilde{x_{n+1}}$ into the basin of $\gamma_{n+1}(\infty)$. Consider the action of γ_n^{-1} on $\mathbb{H}^3$. It maps z_n to ∞ and $z_{n\pm1}$ to $\gamma_n^{-1}(z_{n\pm1})$. It preserves the set of lifts of any integral line. Since the geodesic lines starting from ∞ are vertical (half-)lines, the summits of the cells $\gamma_n^{-1}(\widetilde{\sigma_n})$ and $\gamma_n^{-1}(\widetilde{\sigma_{n+1}})$ projects vertically to $\gamma_n^{-1}(z_{n-1})$ and $\gamma_n^{-1}(z_{n+1})$ respectively. The result follows. $\boxed{7.3}$

Corollary 7.4 *Assume that there is only one integral line (counted without multiplicity). Then*

$$|a_{n+1}| = \Delta(\gamma_{n+1}, \gamma_{n-1}).$$

Proof.

$$\Delta(\gamma_{n+1}, \gamma_{n-1}) = \Delta(\gamma_n^{-1}\gamma_{n+1}, \gamma_n^{-1}\gamma_{n-1}) = |\gamma_n^{-1}\gamma_{n+1}(\infty) - \gamma_n^{-1}\gamma_{n-1}(\infty)| = |a_{n+1}|.$$

$\boxed{7.4}$

The next result proves that the good approximation sequence can be recovered from the continued sequence. To get r_n, one only has to compute the value of the endpoint $z_n \in \mathbb{C}$ of the lift $\widetilde{r_n}$. We prove that z_n can be expressed in an explicit summation formula in terms of the a_i's with $i \leq n$ and the denominators of the z_i's with $i < n$.

Theorem 7.5 *If ξ is a totally irrational line, then with the above notations and $q_i = c(\gamma_i)$ for $i \geq -1$, one has*

$$z_n = z_0 + \sum_{k=1}^{n} \frac{1}{\sum_{i=0}^{k} (-1)^i q_{i-1}^2 a_i}.$$

Proof. By Proposition 7.3, one has

$$z_{n+1} = \gamma_n(a_{n+1} + \gamma_n^{-1} z_{n-1}).$$

Writing for simplicity $\gamma_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one gets, since $ad - bc = 1$, for $n \geq 1$

$$z_{n+1} = \frac{a\left(a_{n+1} + \frac{dz_{n-1}-b}{-cz_{n-1}+a}\right) + b}{c\left(a_{n+1} + \frac{dz_{n-1}-b}{-cz_{n-1}+a}\right) + d}$$

$$= \frac{z_{n-1} + a^2 a_{n+1} - ac a_{n+1} z_{n-1}}{ac a_{n+1} - c^2 a_{n+1} z_{n-1} + 1}$$

$$= \frac{z_{n-1} + z_n c^2 a_{n+1}(z_n - z_{n-1})}{1 + c^2 a_{n+1}(z_n - z_{n-1})},$$

since $z_n = \gamma_n(\infty) = \frac{a}{c}$. Hence denoting $q_n = c(\gamma_n)$, which depends (up to sign) only on r_n, one gets

$$z_{n+1} = \frac{z_{n-1} + z_n q_n^2 a_{n+1}(z_n - z_{n-1})}{1 + q_n^2 a_{n+1}(z_n - z_{n-1})}.$$

Hence (upon adding and subtracting z_n in the numerator) we have

$$z_{n+1} - z_n = -\frac{z_n - z_{n-1}}{1 + q_n^2 a_{n+1}(z_n - z_{n-1})}.$$

Let $x_p = \frac{1}{z_p - z_{p-1}}$ for $p \geq 1$ and $x_0 = 0$. Then for $n \geq 0$

$$x_{n+1} = -(x_n + q_n^2 a_{n+1}).$$

Since by convention $q_{-1} = 0$, one gets $x_n = \sum_{k=0}^{n}(-1)^k q_{k-1}^2 a_k$. The equality of the theorem now follows from the fact that $z_{n+1} - z_n = \frac{1}{x_{n+1}}$. $\boxed{\textbf{7.5}}$

In the case of $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$, our continued sequence $(a_n)_{n\in\mathbb{N}}$ of ξ slightly differs from the classical continued fraction expansion

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\cdots}}}$$

of the endpoint on the real axis of the lift of ξ starting at ∞ and ending in $]-1,1[$: one has $|a_n - b_n| \leq 1$ (and the difference may be a non recursive function). This is due to working with the cut locus rather than with its dual cell decomposition, but we will come back to that point in another paper.

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