Distribution of Resonances and Decay Rate of the Local Energy for the Elastic Wave Equation

by

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Abstract

We study resonances (scattering poles) associated to the elasticity operator in the exterior of an arbitrary obstacle with Neumann or Dirichlet boundary conditions. We prove that there exists an exponentially small neighborhood of the real axis free of resonances. Consequently we prove that for regular data, the energy for the elastic wave equation decays at least as fast as the inverse of the logarithm of time. According to STEPHANOV-VODEV ([SV1], [SV2]), our results are optimal in the case of Neumann boundary condition, even when the obstacle is a ball of $\mathbb{R}^3$.

Key words. Resonances, energy decay, Elastic wave equation.
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0. Introduction and Main Results

Let $\Omega = \mathbb{R}^3 \backslash \mathcal{O}$ be an exterior domain in $\mathbb{R}^3$ with C^∞ and compact boundary $\partial\Omega$, $g = (g_{ij})$ a C^∞ positive definite symmetric matrix equal to the identity matrix outside a compact set. We consider the elastic wave equation with Neumann or Dirichlet boundary condition

$$(0.1.B) \qquad \begin{cases} \partial_t^2 u - A_e(x, D_x)u = 0 & in \quad \mathbb{R} \times \Omega \\ B(x, D)u = 0 & on \quad \mathbb{R} \times \partial\Omega \\ u(0, x) = u_0(x);\ \partial_t u(0, x) = u_1(x) & in \quad \Omega. \end{cases}$$

Here, $A_e(x, D_x)$ and $B(x, D_x)$ are of the forms

$$(0.2) \qquad A_e(x, D_x) = \mu\Delta + (\mu + \lambda)\nabla(\mathrm{div}),$$

where $\mu > 0$, $3\lambda + 2\mu > 0$ and Δ, ∇ and div are the Laplacian, gradian and divergence operators defined by $\Delta u = \sum_{ij} \partial_i(g_{ij}\partial_j u)$; $\nabla u = (\sum_j (\sqrt{g})_{ij}\partial_j u)_i$ and $\operatorname{div} u = trace(\nabla u)$.

$$(0.3) \qquad B(x,D)u = \begin{cases} u & (Dirichlet\ condition) \\ N(x,D)u = \sigma(u)\cdot n & (Neumann\ condition) \end{cases}$$

where n(x) is the unit outer-normal vector to Ω at $x \in \partial\Omega$. In the above

$$(0.4) \qquad \sigma(u) = \lambda(\operatorname{div} u)I + 2\mu\varepsilon(u)$$

is the stress tensor, where $\varepsilon(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ is the strain tensor. The aim of this work is to study the decay rate of the local energy and the distribution of resonances. For $R > 0$ we define the local energy $E(u,R,t)$ for the solution of (0.1) in $\Omega_R = \Omega \cap B(0,R)$ as

$$(0.5) \quad E(u,R,t) = \frac{1}{2}\int_{\Omega_R}\Big(\lambda|\operatorname{div} u(t,x)|^2 + \mu\sum_{ij}|\varepsilon_{ij}(u)|^2 + \Big|\frac{\partial u}{\partial t}(t,x)\Big|^2\Big)dx.$$

We say that problem (0.1) has the uniform local energy property when, for any $R > 0$ there exists a continuous function $f(t)$ satisfying $Lim f(t) = 0$ as $t \to \infty$ such that $E(u,R,t) \leq f(t)E(u_0,u_1)$ holds for any $t \geq 0$. Shibata and Soga [SS] formulate the scattering theory for the elastic wave equation which is analogous to the theory of Lax and Phillips [LP]. In particular they proved for any initial data u_0, u_1 the local energy for the solution of (0.1) in Ω_R decays as t tends to infinity. Thus it is interesting for us whether the problem (0.1) has uniform rate of the local energy decay.

In the case of the scalar-valued wave equation with the Dirichlet or the Neumann boundary condition or the elastic wave equation with Dirichlet boundary condition, if the obstacle satisfies a non trapping condition in some sense, have the uniform local energy decay property. Furthermore, we can take the rate $f(t)$ as $e^{-\alpha t}$ (see Melrose [Me], Melrose-Sjöstrand [MS], Morawetz [Mo], Iwashita-Shibata [IS] and Yamamoto [Y]). In the case when the obstacle is trapping the Ralston's example (see [Ra]) prove that we cannot generally expect the local energy decays uniformly.

For the elastic wave equation with the Neumann condition, however, there is an interesting phenomenon. It is the existence of the Rayleigh surface wave which seems to propagate along the boundary. Taylor [T1] gives a rigorous treatment of the singularity and he prove there are three types of rays that carry singularities. The first two types are classical rays reflecting at the boundary according to the laws of geometrical optics and the singularities propagate along them with speeds $c_1 = \sqrt{\mu}$, $c_2 = \sqrt{2\mu + \lambda}$. The third type of rays lie on the boundary and singularities propagate along them with a slower propagation speed $c_R > 0$ (the Rayleigh speed). Thus any obstacle is trapping for the problem (0.1.N) (even a ball of $\mathbb{R}^3$) from the point of view of propagation of singularities. Consequently Ikeheta and Nakamura

[IN] and Stephanov-Vodev [SV1] show that the problem with Neumann condition does not have the uniform local energy decay property if the obstacle is a ball in $\mathbb{R}^3$ and the result are extended for any non trapping obstacle by Kawashita [K1], [K2]. But for regular data, we can set

$$p_{m,R}(t) = \sup\left\{\frac{E(u,R,t)}{||(u_0,u_1)||^2_{D(A^m)}}; (u_0,u_1) \neq 0\right\} \tag{0.6}$$

where

$$A = \begin{pmatrix} 0 & -iId \\ -iA_e & 0 \end{pmatrix} \tag{0.7}$$

and as for the rate $p_{m,R}(t)$ with $m > 0$, however, we can show $Lim p_{m,R}(t) = 0$ by the method of Walker [Wa]. Indeed, his proof is based on the Rellich theorem and the local energy decay property, that is, $Lim_{t\to\infty} E(u,R,t) = 0$. An important problem in this direction is to know how fast $p_{m,R}(t)$ converges to zero as $t \to \infty$. Ikeheta and Nakamura [IN] show that for any $\alpha > 0$ we can not get the estimate of the form $p_{m,R}(t) \leq Ce^{-\alpha t}$ even if $\partial\Omega$ is a unit sphere in $\mathbb{R}^3$ and Kawashita [K] shows that $Lim_{t\to\infty} t^\gamma p_{m,R}(t) = +\infty$ hold for any $\gamma > 0$. And in the second important problem is to know how fast $p_{m,R}(t)$ converges to zero if the obstacle is trapping.

A surprising new discovery concerning the decay of the local energy for the scalar-valued wave equation in an exterior obstacle was mode a few years ago by Burq [Bu] who proved that for regular data, the energy decays at least as fast the inverse of the logarithm of time of any arbitrary obstacle.
In this present paper, we show similar result for the solution of (0.1.B) independent of the geometries of the obstacle, precisely we prove $p_{m,R}(t) \leq \frac{C}{Ln(2+t)^{2m}}$.

Theorem 0.1 *For any $R_1, R_2 > 0$ and $m > 0$ there exists $C > 0$ such that for any data $(u_0,u_1) \in D(A^m)$ supported in $\overline{\Omega}_{R_1}$ we have*

$$E(u,R_2,t) \leq \frac{C}{(Ln(2+t))^{2m}} ||(u_0,u_1)||^2_{D(A^m)} \tag{0.8}$$

where u solve (0.1.B).

We have the previous result as a consequence of the existence of an exponentially small neighborhood of the real axis free from the resonances (scattering poles). More precisely the purpose of this paper is to give some information about the location of the poles of the outgoing resolvent $R(z)$ defined as the solution operator of

$$\begin{cases} (\Delta_e + z^2)v(x,z) = f & in \quad \Omega \\ Bv(x,z) = 0 & on \quad \partial\Omega \\ v(x,z) & is\, outgoing \end{cases} \tag{0.9}$$

we say that the function v is outgoing, if $v(x,z)$ is the $L^2(\Omega)-$solution in $Imz < 0$ and the analytic continuation of an L^2-solution in $Imz < 0$ if $Imz \geq 0$.

It is known that the resolvent $R(z)$ acting on functions $v \in L^2_{comp}(\Omega)$ in H^2 is holomorph in $Imz < 0$ and can be extended as a meromorphic function from $Imz < 0$ to the whole complex plan $\mathbb{C}$ with possible poles in $Imz > 0$. Let $\chi_{1,2} \in C_0^\infty$ being a cutoff functions equal to 1 near $\partial\Omega$ the poles of $\chi_1 R(z)\chi_2$ are called resonances.

In the case of the Laplacian with Dirichlet or Neumann boundary condition, then it is well known that (see Melrose and Sjöstrand [MS]) for non-trapping obstacles the resonances lie above logarithmic curves of the type $Imz = \alpha Ln(|Rez|) - \beta$, $\alpha > 0$. For trapping obstacles Burq [Bu] shows the existence of an exponentially small neighborhood of the real axis containing no resonances. Stephanov and Vodev [SV] prove that for the elasticity operator with Neumann boundary condition there exists a sequence of resonances tending exponentially to the real axis for an arbitrary obstacle. In this paper we show that :

Theorem 0.2 *There exist $C_1, C_2, C_3 > 0$ such that the outgoing resolvent is holomorphic in the region*

$$U = \{z \in \mathbb{C}; Imz < C_1 e^{-C_2|Rez|}\} \cap \{|z| > C_3\}. \tag{0.10}$$

Moreover, there exist $C > 0$ and $\delta > 0$, χ_j, $j = 1, 2$ such that in the region U we have the estimate of $R(z)$

$$||\chi_1 R(z)\chi_2||_{\mathcal{L}(L^2,H^2)} \le Ce^{\delta|Rez|}. \tag{0.11}$$

To prove theorem 0.2, we make us of an idea due to Lebeau and Robbiano [LR2] which has been adapted by Burq [Bu] for this kind of problem, it consists to use Carleman estimates to obtain information about the resolvent in a bounded domain. The cost is to use phases functions satisfying Hörmander's assumption and thus growing fast, far from the obstacle.
The point is to show estimations for the solution of $(A_e + \tau^2)u = f$ where τ is a real parameter and f is compactly supported.

Up to our knowledge, only few literature on the system problem is available, even without additional conditions on the boundary. Indeed, no general method is available to solve such problem, excepting multiply the system by the cofactors matrix and then we use the machinery of scalar Carleman estimates (see Hörmander [H1]) for the determinant. Unfortunately this method needs height regularities conditions on the coefficients and specially in the case of the boundary problem it increase the multiplicity of real characteristics near the boundary. And hence the Lopatinskii condition is not easily satisfied. Our approach is different, consisting to diagonalize the system near boundary it is the main technical part of this work.
Let Ω_0 be an bounded domain with smooth boundary $\partial\Omega_0$ and let $A(x, D) = -A_e(x, D) - \tau^2$ with principal symbol $A(x, \xi) = -A_e(x, \xi) - \tau^2$. Let $\varphi(x) \in C^\infty(\mathbb{R}^3)$ we define $A(x, D, \tau) = e^{\tau\varphi} A(x, D_x) e^{-\tau\varphi}$ and denote by $a_\gamma(x, \xi) = {}^t\xi g\xi - \frac{\tau^2}{\gamma}$ for $\gamma \in \{\mu, 2\mu + \lambda\}$. We assume that φ satisfy Hörmander's assumption $H_\varphi : \exists C > 0$ such that $\{Rea_\gamma(x, \xi + i\tau\varphi') : Ima_\gamma(x, \xi + i\tau\varphi')\} > C$ whenever $a_\gamma(x, \xi + i\tau\varphi') = 0$ and $\gamma \in \{\mu, 2\mu + \lambda\}$ and we assume that $\frac{\partial\varphi}{\partial n}|_{\partial\Omega_0} \neq 0$; $\nabla\varphi \neq 0$ in $\overline{\Omega}_0$.

Theorem 0.3 *Assume that* $\frac{\partial\varphi}{\partial n} < -C_0$, *(where* $C_0 > 0$ *is large enough) on* $\Sigma \subset \partial\Omega_0$, *then there exists* $C > 0$ *such that for any* $u \in C^\infty(\overline{\Omega})$ *and* $Bu = 0$ *on* Σ *we have*

$$(0.12)\int_{\Omega_0} |A(x,D,\tau)u|^2 + \tau\int_{\partial\Omega_0\backslash\Sigma}(\tau^2|u|^2 + |\nabla u|^2)d\sigma \geq C\tau\int_{\Omega_0}(\tau^2|u|^2 + |\nabla u|^2).$$

for large enough τ.

Remark 0.1

i) According to Ikeheta-Nakamura *[IN] and* Stephanov-Vodev *[SV1] theorem 0.1 and theorem 0.2, with Neumann boundary condition, are optimal even when the obstacle is ball of* $\mathbb{R}^3$.

ii) Theorems 0.1 follows from Theorem 0.2, this is proved in the general case by Burq *(see [Bu])*

The paper is organized as follow. In section 1 we study the outgoing solution far from obstacle (outside a fixed ball $B(0,R_1)$ contain the obstacle) by using a harmonic spheric decomposition. The construction of the phases satisfies Hörmander's condition and the proof of theorem 0.2 is in section 2. The Carleman estimates is proved in sections 3, 4, 5 and 6.

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1 Harmonic Spherics

Our purpose in this section is to give some informations about the outgoing solutions of $(\Delta_e + z^2)u = 0$ outside a ball $B(0,R_1)$ (where $\Delta_e = A_e$ in $B(0,R_1)^c$). Our analyses here is to close to that of Ikeheta-Nakamura [IN] and Stephanov-Vodev [SV1] and we will keep some of notations from there (see also Morse-Feshbach [MF])

1.1 Preliminaries

Let (r,θ,φ) be the polar coordinates in $\mathbb{R}^3$. Denote by $P_m^n(z), n, m \in \mathbb{Z}_+, m \leq n$ Ferrer's function defined by

$$(1.1) \qquad P_m^n(z) = (2^n n!)^{-1}\Big(1 - z^2\Big)^{\frac{m}{2}}\Big(\frac{\partial^{m+n}}{\partial z}\Big)\Big[(z^2-1)^n\Big].$$

Set $Y_{mn}(\theta,\varphi) = e^{im\varphi}P_m^n(cos\theta)$ and define the vector spherical harmonics P_{mn}, B_{mn}, C_{mn} by

$$(1.2) P_{mn} = Y_{mn}(\theta,\varphi)\omega,\; C_{mn} = (n(n+1))^{-\frac{1}{2}}curl(\omega r Y_{mn}(\theta,\varphi)), B_{mn} = \omega \times C_{mn}$$

where the symbol $(\times)$ denotes the exterior product and $\omega = \frac{x}{R}$ for $x \in S^2(0,R)$.
It is known (see Morse-Feshbach [MF] pp 1898) that $\left\{P_{mn}, B_{mn}, C_{mn}\right\}$ form an orthogonal basis in $L^2(S^2)$. Denote by $h_n(z) = (\frac{\pi}{2z})^{\frac{1}{2}} H_{n+\frac{1}{2}}(z)$ the spherical Hankel function of first order where

$$H_\gamma(z) = \int_{-\infty}^{+\infty - i\pi} e^{z sinh t - \gamma t}\, dt. \tag{1.3}$$

We have the following Lemma for the properties of $H_\gamma(z)$ (see Lemmas 2.3 and 2.4 of Burq [Bu]).

Lemma 1.1

i) For any $a < 1$, there exists $C > 0, \tau_0 > 0$ such that for any $|\tau| > \tau_0$ and $\gamma < |\tau| ar$ where $\tau = Rek > \tau_0, \quad |Imk| \leq 1$ we have

$$-Im\left(\frac{H_\gamma'(kr)}{H_\gamma(kr)}\right) \geq C, \qquad and \qquad \left|Re\left(\frac{H_\gamma'(kr)}{H_\gamma(kr)}\right)\right| \leq \frac{C}{\sqrt{|\tau|}} \tag{1.4}$$

ii) For any $a < 1$ and $0 < R_1 < R_2$, there exists $C > 0, \varepsilon > 0, \tau_0 > 0$ such that for any $|k| > \tau_0$ and $\gamma > \tau a^{-1} R_1, \quad |Imk| \leq 1$ we have

$$\begin{cases} |H_\gamma(kR_2)| \leq Ce^{-\varepsilon\gamma} |H_\gamma(kR_1)| \\ |H_\gamma'(kR_2)| \leq Ce^{-\varepsilon\gamma} |H_\gamma(kR_1)| \end{cases} \tag{1.5}$$

Define now $L_{mn}(r;k), M_{mn}(r;k), N_{mn}(r;k)$ as follows

$$\begin{cases} L_{mn}(r;k) = k^{-1} grad(Y_{mn}(\theta,\varphi) h_n(kr)), \\ M_{mn}(r;k) = curl(r\omega Y_{mn}(\theta,\varphi) h_n(kr)), \\ N_{mn}(r;k) = k^{-1} curl(M_{mn}(r;k)). \end{cases} \tag{1.6}$$

Then it can be easily seen that the following Lemma holds (see Morse-Feshbach [MF])

Lemma 1.2

i) $L_{mn}(r;k), M_{mn}(r;k), N_{mn}(r;k)$ are the eigenfunction of the vector Helmholtz equation

$$(\Delta + k^2)U = 0 \qquad in\ \mathbb{R}^3 \tag{1.7}$$

ii) Let $k_1 = \frac{z}{\mu}, k_2 = \frac{z}{2\mu+\lambda}$. Then $L_{mn}(r;k_1), M_{mn}(r;k_2), N_{mn}(r;k_2)$ are the eigenfunctions of the equation

$$(\Delta_e + z^2)U = 0 \qquad in\ \mathbb{R}^3 \tag{1.8}$$

iii) $Y_{mn}(\theta,\varphi)h_n(kr)$ are the eigenfunctions of the scalar Helmholtz equation $(\Delta + k^2)\phi = 0$ in $\mathbb{R}^3$ and we have

$$\left[k^{-2}\frac{d^2}{dr^2} + \frac{2}{kr}\frac{d}{dr} + \Big(1 - \frac{n(n+1)}{k^2r^2}\Big)\right]h_n(kr) = 0 \tag{1.9}$$

iv) We have the following decomposition of $L_{mn}(r;k), M_{mn}(r;k), N_{mn}(r;k)$ by a linear combination of P_{mn}, B_{mn} and C_{mn} (see Morse-Feshbach [MF] pp 1865)

$$\begin{cases} L_{mn}(r;k) = \frac{1}{k}\frac{d}{dr}(h_n(kr))P_{mn} + \sqrt{n(n+1)}\frac{1}{kr}h_n(kr)B_{mn} \\ \\ M_{mn}(r;k) = \sqrt{n(n+1)}h_n(kr)C_{mn} \\ \\ N_{mn}(r;k) = n(n+1)\frac{1}{kr}h_n(kr)P_{mn} + \sqrt{n(n+1)}\frac{1}{kr}\frac{d}{dr}(rh_n(kr))B_{mn}. \end{cases} \tag{1.10}$$

Let $u(x,z)$ solve the following problem $(\Delta_e + z^2)u = 0$ in the exterior of the ball $B(0,R_1)$ and u is outgoing. The solution space can be decoupled into subspaces, which consist of longitudinal and transverse fields each satisfying the appropriate Helmholtz equation (see Kupradz [Ku]; Ikeheta-Nakamura [I.N])

$$(\Delta + k_2^2)u^{(p)} = 0, \quad (\Delta + k_1^2)u^{(s)} = 0 \tag{1.11}$$

where $u = u^{(p)} + u^{(s)}$ and

$$\begin{cases} u^{(p)} = grad\phi, & \phi \ satisfy \quad (\Delta + k_2^2)\phi = 0 \\ u^{(s)} = curlA, & A \ satisfy \quad (\Delta + k_1^2)A = 0 \\ with \ divA = 0 \end{cases} \tag{1.12}$$

1.2 Study of the outgoing solutions far from the obstacle

Since here we have two sound speeds, we have to consider the following five regions

i) Hyperbolic region: $n(n+1) < \tau_2^2r^2$

ii) Glancing region (I): $n(n+1) = \tau_2^2r^2$

iii) Mixed region: $\tau_2^2r^2 < n(n+1) < \tau_1^2r^2$

iv) Glancing region (II): $n(n+1) = \tau_1^2r^2$

v) Elliptic region: $n(n+1) > \tau_1^2r^2$

Now, we state the main result in this section

Proposition 1.1 *For any $R_2 > R_1 > 0$, there exists C_1, C_2, ε and τ_0, such that for any $z \in \mathbb{C}$; $|Imz| \leq 1$, $|Rez| = |\tau| > \tau_0$ the outgoing solution for $(\Delta_e + z^2)u = 0$ outside a ball $B(0,R_1)$ satisfy*

$$\begin{aligned} \int_{r=R_2}(-ImNu\overline{u})d\sigma \ \geq \ & |\tau|C_1\int_{r=R_2}(|u|^2 + |\tau^{-1}\nabla u|^2)d\sigma \\ & - C_2e^{-\varepsilon|\tau|}\int_{r=R_1}(|u|^2 + |\tau^{-1}\nabla u|^2)d\sigma. \end{aligned} \tag{1.13}$$

Remark 1.1 *We only prove the estimate (1.13) in the hyperbolic, mixed and elliptic regions the glancing regions (I) (II) can be easily incorporated respectively in mixed and elliptic regions (see lemma 1.3, 1.4 and 1.5)*

The next calculus is useful for expanding the outgoing solution for $(\Delta_e + z^2)u = 0$ and Nu in terms of L_{mn}, M_{mn}, N_{mn}.
Let ϕ and A solves (1.12), then ϕ can be rewritten as follows

$$\phi = \sum k_2^{-1}\phi_{mn}(Y_{mn}(\theta,\varphi)h_n(k_2 r)) \tag{1.14}$$

Then we obtain by (1.10)

$$u^{(p)} = grad\phi = \sum \phi_{mn}\Big[h_n'(k_2 r)P_{mn} + \frac{\sqrt{n(n+1)}}{k_2 r}h_n(k_2 r)B_{mn}\Big] \tag{1.15}$$

Since we have represented the boundary condition for $u^{(p)}$ in the form

$$\begin{aligned} Nu^{(p)} &= 2\mu\partial_r u^{(p)} + \lambda\omega div u^{(p)} \\ &= 2\mu\partial_r u^{(p)} - \lambda k_2^2\phi\omega \end{aligned} \tag{1.16}$$

We obtain from (1.15)

$$\begin{aligned} Nu^{(p)} &= \sum \phi_{mn}\Big[\Big(-4\mu\frac{h_n'(k_2 r)}{r} + 2\mu n(n+1)\frac{h_n(k_2 r)}{k_2 r^2} \\ &\quad - (2\mu+\lambda)k_2 h_n(k_2 r)\Big)P_{mn} \\ &\quad + 2\mu\sqrt{n(n+1)}\Big(\frac{h_n'(k_2 r)}{r} - \frac{h_n(k_2 r)}{k_2 r^2}\Big)B_{mn}\Big]. \end{aligned} \tag{1.17}$$

Similarly, let $u^{(s)} = curlA$, where

$$A = k_1^{-1}\sum(A_{mn}^1 M_{mn} + A_{mn}^2 N_{mn}). \tag{1.18}$$

Here we will omit the calculations which are tedious but elementary and will present only the final results. We get by (1.8) and (1.10)

$$\begin{aligned} u^{(s)} &= \sum A_{mn}^1\Big[n(n+1)\frac{h_n(k_1 r)}{k_1 r}P_{mn} + \sqrt{n(n+1)}\Big(\frac{h_n(k_1 r)}{k_1 r} \\ &+ h_n'(k_1 r)\Big)B_{mn}\Big] + A_{mn}^2\sqrt{n(n+1)}h_n(k_1 r)C_{mn}. \end{aligned} \tag{1.19}$$

On the other hand

$$\begin{aligned} Nu^{(s)} &= 2\mu\partial_r u^{(s)} + \mu\omega \times curlu^{(s)} \\ &= 2\mu\partial_r u^{(s)} + \mu k_1^2\omega \times A. \end{aligned} \tag{1.20}$$

Hence, we have

$$
\begin{aligned}
(1.21) \qquad N u^{(s)} &= \mu \sum A^1_{mn} \Big[2n(n+1)\Big(\frac{h'_n(k_1 r)}{r} - \frac{h_n(k_1 r)}{k_1 r^2}\Big) P_{mn} + \\
&\sqrt{n(n+1)}\Big(-2\frac{h'_n(k_1 r)}{r} - k_1 h_n(k_1 r) + \\
&+ 2\frac{n(n+1)}{k_1 r^2} h_n(k_1 r) - 2\frac{h_n(k_1 r)}{k_1 r^2}\Big) B_{mn}\Big] \\
&+ A^2_{mn}\Big[\sqrt{n(n+1)}\Big(k_1 h'_n(k_1 r) - \frac{h_n(k_1 r)}{r}\Big)\Big] C_{mn}
\end{aligned}
$$

Taking into account (1.15), (1.17), (1.19) and (1.21) we obtain

$$
(1.22) \qquad -Im(N u^{(p)} \overline{u^{(p)}}) = (2\mu + \lambda) \sum |\phi_{mn}|^2 Im(k_2 h_n(k_2 r) \overline{h'_n(k_2 r)})
$$

and

$$
\begin{aligned}
(1.23) \quad -Im(N u^{(s)} \overline{u^{(s)}}) = \mu \sum n(n+1) \Big[|A^1_{mn}|^2 Im\Big(k_1 h_n(k_1 r) \overline{h'_n(k_1 r)}\Big) + \\
+ |A^2_{mn}|^2 Im\Big(\overline{k}_1 h_n(k_1 r) \overline{h'_n(k_1 r)}\Big)\Big] + \mathcal{O}\Big(\frac{n(n+1)}{|k_1|} |h_n(k_1 r)|^2\Big)
\end{aligned}
$$

Finally

$$
\begin{aligned}
(1.24) \quad -Im(N u^{(p)} \overline{u^{(s)}} + N u^{(s)} \overline{u^{(p)}}) = \ - \ & \sqrt{\mu(2\mu + \lambda)} \sum n(n+1) Im(\frac{z}{\overline{z}}) \\
& Re\Big(A^1_{mn} \phi_{mn} h_n(k_1 r) \overline{h_n(k_2 r)}\Big)
\end{aligned}
$$

Now we define $g_n^{(p)}, g_n^{(s)}, g_n, \alpha_n$ by $g_n = g_n^{(p)} + g_n^{(s)} + \alpha_n$ where

$$
(1.25) \qquad \left\{ \begin{array}{l}
-Im(N u^{(p)} \overline{u^{(p)}}) = \sum g_n^{(p)}(k_2 r) \\
-Im(N u^{(s)} \overline{u^{(s)}}) = \sum g_n^{(s)}(k_1 r) \\
-Im(N u^{(s)} \overline{u^{(p)}} + N u^{(p)} \overline{u^{(s)}}) = \sum \alpha_n(k_1 r, k_2 r)
\end{array} \right.
$$

and $|u|^2 = \sum |u_n|^2 = \sum \Big(|u_n^{(p)}|^2 + |u_n^{(s)}|^2 + 2Re(u_n^{(p)} \overline{u_n^{(s)}})\Big)$. The rest of this section is devoted to prove Proposition 1.1. Then we need the following three Lemmas.

Lemma 1.3 *(Hyperbolic region). For any $a < 1$, there exist $C > 0, \tau_0 > 0$ such that for any $|\tau| > \tau_0$ and $n(n+1) \leq \tau_2^2 R_2^2 a^2$ we have*

$$
(1.26) \qquad g_n(k_1 R_2, k_2 R_2) \geq C\tau |u_n(R_2)|^2
$$

Proof .
By Lemma 1.1 we have for any $n(n+1) \leq \tau_2^2 R_2^2 a^2$ and $\sigma \in \{k_1, k_2\}$

$$
(1.27) \qquad \left\{ \begin{array}{l}
Im(\sigma h_n(\sigma R_2) \overline{h'_n(\sigma R_2)}) \geq C\tau |h_n(R_2)|^2 \\
Im(\sigma h_n(\sigma R_2) \overline{h'_n(\sigma R_2)}) \geq C\tau |h'_n(R_2)|^2
\end{array} \right.
$$

Then we have

$$(1.28)\qquad \begin{cases} g_n^{(p)}(k_2R_2) \geq C\tau |u_n^{(p)}(R_2)|^2 \\ g_n^{(s)}(k_2R_2) \geq C\tau |u_n^{(s)}(R_2)|^2 \end{cases}$$

Using this and

$$(1.29)\qquad |\alpha_n(k_1r, k_2r)| \leq \frac{C}{\tau}(g_n^{(p)} + g_n^{(s)})$$

we obtain (1.26) •

Now using the Lemma 1.1 can prove the two following Lemma which consist of elliptic region and mixed region.

Lemma 1.4 *(Elliptic region). For any $a < 1$, there exist $C > 0, \tau_0 > 0$ and $\varepsilon > 0$ such that for any $|\tau| > \tau_0$ and $n(n+1) > \tau_1^2 R_1^2 a^{-2}$ we have*

$$(1.30)\qquad |g_n(k_1R_2, k_2R_2)| \leq Ce^{-\varepsilon\sqrt{n(n+1)}}|u_n(R_1)|^2.$$

Lemma 1.5 *(Mixed region). For any $a < 1$, there exist C_1, C_2, $\tau_0 > 0$ and $\varepsilon > 0$ such that for any $|\tau| > \tau_0$ and $\tau_2^2 R_1^2 a^{-2} < n(n+1) \leq \tau_1^2 R_2^2 a^2$ we have*

$$(1.31)\qquad \begin{aligned} g_n(k_1R_2, k_2R_2) &\geq C_1|u_n^{(s)}(k_1R_2)|^2 - C_2e^{-\varepsilon\sqrt{n(n+1)}}|u_n^{(p)}(k_2R_1)|^2 \\ &\geq C_1|u_n(R_2)|^2 - C_2e^{-\varepsilon\sqrt{n(n+1)}}|u_n(R_1)|^2 \end{aligned}$$

1.2.1 Proof of Proposition 1.1

Using Lemma (1.3) we have

$$(1.32)\int_{r=R_2} -Im(Nu\bar{u})d\sigma \;=\; \sum g_n(k_1R_2, k_2R_2) \geq C\tau \sum_{n(n+1)\leq a^2\tau_2^2R_2^2} |u_n(R_2)|^2$$
$$+ \sum_{a^2\tau_2^2R_2^2<n(n+1)\leq a^2\tau_1^2R_1^2} g_n(k_1R_2, k_2R_2)$$
$$- \tau \sum_{n(n+1)\geq a^2\tau_1^2R_2^2} |u_n(R_2)|^2 + |\tau^{-1}\widetilde{g}_n(R_2)|^2.$$

where $\widetilde{g}_n$ define by $|Nu|^2 = \sum |\widetilde{g}_n|^2$. We set $a^2 = \frac{R_1}{R_2}$ then $a^2R_2^2 = a^{-2}R_1^2$. Using Lemma (1.5) and (1.4) we obtain

$$(1.33)\;\int_{r=R_2} -Im(Nu\overline{u}) \geq C_1\tau \sum_{n(n+1)\leq a^2\tau_1^2R_2^2} |u_n(R_2)|^2$$
$$-\sum e^{-\varepsilon\sqrt{n(n+1)}}|u_n(R_1)|^2 - C_2|\tau| \sum_{n(n+1)\geq a^2\tau_1^2R_2^2} |u_n(R_2)|^2 + |\tau^{-1}\widetilde{g}_n(R_2)|^2$$

$$\geq -C'|\tau| \sum_{n(n+1)\geq a^2\tau_1^2R_2^2} \Big(1+\frac{n(n+1)}{\tau^2}\Big)|u_n(R_2)|^2 + |\tau^{-1}\widetilde{g}_n(R_2)|^2$$

$$+C\tau\sum\Big(1+\frac{n(n+1)}{\tau^2}\Big)|u_n(R_2)|^2 - \sum e^{-\varepsilon\sqrt{n(n+1)}}|u_n(R_1)|^2$$

As in the proof of of Lemma 1.4, with (1.17) and (1.21) we obtain for $n(n+1) > a^2\tau_1^2R_2^2$ the following estimate

$$|\tau^{-1}\widetilde{g}_n(R_2)|^2 \leq Ce^{-\varepsilon\sqrt{n(n+1)}}|u_n(R_1)|^2 \tag{1.34}$$

Thus we get from (1.33) and Lemma 1.4

$$\begin{aligned}\int_{r=R_2}(-ImNu\overline{u})d\sigma \ &\geq \ |\tau|C_1\int_{r=R_2}(|u|^2+|\tau^{-1}\nabla u|^2)d\sigma \\ &\quad - C_2e^{-\varepsilon|\tau|}\int_{r=R_1}(|u|^2+|\tau^{-1}\nabla u|^2)d\sigma\end{aligned} \tag{1.35}$$

2 Proof of theorem 0.2

2.1 Construction and properties of the phase function

Our analysis here is close and widely inspired from that of Burq (see [Bu]). The purpose of this section is to construct two phases φ_1, φ_2 which satisfies the Hörmander's conditions, excepting in a finite number of balls, such that on a ball where one of them do not satisfies these conditions the second does and is strictly greater. Moreover far from obstacle the functions φ_1, φ_2 coïncide are radial and satisfy $\varphi'(r) = \kappa$, for $\kappa > 0$ arbitrary small.

2.1.1 Construction in bounded region

Let $R > 0$, such that $B(0,R)$ contain space perturbation and operator perturbation i.e. the obstacle $\mathcal{O} \subset B(0,R)$ and $g = Id$ outside $B(0,R)$.

Proposition 2.1 (see Burq [Bu]). *There exist two functions $\psi_1, \psi_2 \in C^\infty(\overline{\Omega}_R)$ satisfying $\frac{\partial\psi_i}{\partial n}|_{\partial\Omega} < 0$ (∂_n stand for the unit outer-normal vector field at $\partial\Omega$), having only no degenerate critical points such that on $S(0,R)$ we have $\nabla\psi_i\cdot x > 0$ and when $\nabla\psi_i = 0$ then $\nabla\psi_{i+1} \neq 0$ and $\psi_{i+1} > \psi_i$ ($\psi_3 = \psi_1$). Finally ψ_i are radiales and $\psi_1 = \psi_2$ in a neighborhood of $S(0,R)$.*

Then there exists a finite number of points $x_{ij} \in \Omega$, $i = 1,2$, $j = 1,\ldots,N_i$ and $\varepsilon > 0$, such that $B(x_{ij}, 2\varepsilon) \subset \Omega_R$, $B(x_{1,j}, 2\varepsilon) \cap B(x_{2,j}, 2\varepsilon) = \emptyset$ and $\psi_i < \psi_{i+1}$ in $B(x_{ij}, 2\varepsilon)$

(where $\psi_3 = \psi_1$). Denote $\Omega_i = \Omega_R \cap (\bigcup\limits_j B(x_{ij}, \varepsilon))^c$. Let us search φ_i in the form $e^{\beta\psi_i}$ for each large enough β. Taking from $\gamma \in \{\mu, 2\mu + \lambda\}$ and $g = (g_{ij})$

$$\widetilde{a}_\gamma(x, \xi) = {}^t\xi g \xi - \frac{\tau^2}{\gamma} \tag{2.1}$$

and by

$$a_\gamma(x, \xi, \tau) = \widetilde{a}_\gamma(x, \xi + i\tau\varphi') = {}^t\xi g\xi - \tau^{2t}\varphi' g\varphi' + 2i\tau{}^t\xi g\varphi' - \frac{\tau^2}{\gamma}. \tag{2.2}$$

If $a_\gamma(x, \xi, \tau) = 0$ by a simple computation gives

$${}^t\xi g\varphi' = 0 \quad \textit{and} \quad {}^t\xi g\xi = \tau^{2t}\varphi' g\varphi' + \frac{\tau^2}{\gamma}. \tag{2.3}$$

Taking into account (2.3) we obtain

$$\{Rea_\gamma, Ima_\gamma\}(x, \xi, \tau) = 4\tau e^{3\beta\psi}(\beta^4({}^t\psi' g\psi')^2 + O(\beta^3)). \tag{2.4}$$

Hence $\psi' \neq 0$ and $g \geq CId$, then we have

$$\det A_e(x, \xi, \tau) = 0 \quad \textit{implies} \quad \{Rea_\gamma, Ima_\gamma\}(x, \xi, \tau) \geq C. \tag{2.5}$$

2.1.2 Construction of the phase outside a ball

Let us now construct φ in $B(0, R)^c \cap B(0, R_1)$, for fixed $R_1 > R$ such that φ_1, φ_2 are radial and coïncide in a small neighborhood of each x, $R - \alpha < |x| < R$ let us search extend φ in a radial function. We are going to polar coordinates in $\mathbb{R}^3$ $(r, \rho; \theta, \eta)$ where $\theta \in S^2$ the symbol $\widetilde{a}_\gamma$ can be written

$$\widetilde{a}_\gamma = \rho^2 + \frac{\eta^2}{r^2} - \frac{\tau^2}{\gamma} \quad \textit{and} \quad a_\gamma = \rho^2 - \tau^2\varphi'^2 + \frac{\eta^2}{r^2} - \frac{\tau^2}{\gamma} + 2i\tau\varphi'\rho. \tag{2.6}$$

It is easy to see that if $a_\gamma = 0$ then we have

$$\rho = 0 \text{ and } \frac{\eta^2}{r^2} = \tau^2\Big(\varphi'^2 + \frac{1}{\gamma}\Big). \tag{2.7}$$

Taking into account (2.7) we obtain $a_\gamma = 0$ implies

$$\{Rea_\gamma, Ima_\gamma\} = 4\tau^3\varphi'\Big(\varphi''\varphi' + \frac{\frac{1}{\gamma} + \varphi'^2}{r}\Big). \tag{2.8}$$

Let $y_\gamma(r)$ solve the following linear ordinary differential equation

$$\begin{cases} y'y + \frac{1}{\gamma r} = 0 \\ y(R) = \varphi'(R) \end{cases} \tag{2.9}$$

Then we obtain

$$y_\gamma(r) = \Big[\varphi'(R)^2 - \frac{2}{\gamma} Ln\Big(\frac{r}{R}\Big)\Big]^{1/2}. \tag{2.10}$$

Let now $R_1 = Re^{\varphi'(R)^2/2}$ then the constant R depends only the value of φ_i in $r = R$ for a small parameter η and κ such that $0 < \kappa < \frac{\eta}{2}$ we take

$$\widetilde{\varphi}'(r) = \begin{cases} \varphi'(R) & in\ R - \alpha < r < R \\ y_{2\mu+\lambda}(r) & in\ R \leq r < R_1\, e^{-\kappa^2/2} \\ \kappa & in\ R_1\, e^{-\kappa^2/2} \leq r \leq R_2 \end{cases} \tag{2.11}$$

and by the argument in Burq [Bu] we can extend $\widetilde{\varphi}'$ in a smooth phases φ'.

2.2 End of the proof of theorem 0.2

Let $u(x,z)$ solve the following problem

$$\begin{cases} (A_e + z^2)u = f & in\ \Omega \\ Bu = 0 & on\ \partial\Omega \\ u - outgoing & \end{cases} \tag{2.12}$$

where f supported in $\Omega_{R_1} = \Omega \cap B(0,R_1)$. Let χ_i, $i = 1,2$ two cutoff functions equal to 1 in $(\underset{j}{\cup} B(x_{ij}, 2\varepsilon)^c)$ and supported in $(\underset{j}{\cup} B(x_{ij}, \varepsilon)^c)$. We applique theorem 0.3 for the function $e^{\tau\varphi_i}\chi_i u$ in a domain $\Omega_i = (\Omega_{R_2} \cap (\underset{j}{\cup} B(x_{ij}, \varepsilon)^c)$, we get for $\tau^2 = Re(z^2)$ and $g_i = [A_e, \chi_i]u + \chi_i f - iIm(z^2)\chi_i u$

$$\int_{\Omega_i} e^{2\tau\varphi_i}|g_i|^2 + \tau \int_{r=R_2} (\tau^2|u|^2 \ + \ |\nabla u|^2)e^{2\tau\varphi(R_2)} \geq \tag{2.13}$$

$$\geq C\tau \int_{\Omega_i} (\tau^2|\chi_i u|^2 + |\nabla \chi_i u|^2)e^{2\tau\varphi_i}.$$

Using the properties of the phases $\varphi_i > \varphi_{i+1}$ in $\underset{j}{\cup} B(x_{ij+1}, 2\varepsilon)$ then for any large enough τ we have

$$\int_{\Omega_{R_2}} (e^{2\tau\varphi_1} + e^{2\tau\varphi_2})(|f|^2 + |Im(z^2)u|^2) + \tau \int_{r=R_2} (e^{2\tau\varphi_1} + e^{2\tau\varphi_2})(\tau^2|u|^2 + \tag{2.14}$$

$$+|\nabla u|^2)d\sigma \geq C\tau \int_{\Omega_{R_2}} (e^{2\tau\varphi_1} + e^{2\tau\varphi_2})(\tau^2|u|^2 + |\nabla u|^2)$$

Combining the previous estimate with proposition 1.1, we obtain

$$\int_{\Omega_{R_2}} (e^{2\tau\varphi_1} + e^{2\tau\varphi_2})|f|^2 + \tau^2 \int_{r=R_2} (e^{2\tau\varphi_1} + e^{2\tau\varphi_2})(Im(Bu \cdot \overline{u}))d\sigma + \tag{2.15}$$

$$+\int_{\Omega_{R_2}}(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})|Im(z^2)u|^2) \geq C\tau\int_{\Omega_{R_2}}(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})(\tau^2|u|^2+|\nabla u|^2)-$$

$$-(e^{\tau(2\varphi_1(R_2)-\varepsilon)}+e^{\tau(2\varphi_2(R_2)-\varepsilon)})\int_{r=R_1}(\tau^2|u|^2+|\nabla u|^2)d\sigma.$$

By the trace formula we get (see Burq [Bu] (3.6))

$$(2.16)\quad \int_{\Omega_{R_2}}(\tau^2|u|^2+|\nabla u|^2)(e^{2\tau\varphi_1}+e^{2\tau\varphi_2}) \geq \int_{r=R_1}|u|^2(e^{2\tau\varphi_1(R_1)}+e^{2\tau\varphi_2(R_1)}).$$

Taking into account $\Delta_e u = -z^2 u$ in a neighborhood of $S(0,R_1)$ we can get

$$(2.17)\quad \int_{\Omega_{R_2}}(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})(\tau^2|u|^2+|\nabla u|^2) \geq$$

$$\int_{r=R_1}\tau^{-2}|\nabla u|^2(e^{2\tau\varphi_1(R_1)}+e^{2\tau\varphi_2(R_2)})d\sigma.$$

Then we have

$$(2.18)\quad \int_{\Omega_{R_2}}(\tau^2|u|^2+|\nabla u|^2)(e^{2\tau\varphi_1}+e^{2\tau\varphi_2}) \geq$$

$$\tau^{-2}\int_{r=R_1}(\tau^2|u|^2+|\nabla u|^2)(e^{2\tau\varphi_1}+e^{2\tau\varphi_2}).$$

Hence by the fact that $\kappa < \frac{\varepsilon}{2}$ and by (2.15), (2.17) we obtain

$$(2.19)\quad \int_{\Omega_{R_2}}(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})(|f|^2+|Im(z^2)|^2|u|^2)+$$

$$\tau^2\int_{r=R_2}(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})|Im(Bu\cdot\overline{u})| \geq C\tau\int_{\Omega_{R_2}}(e^{2\tau\varphi_1}+e^{2\tau\varphi_2})(\tau^2|u|^2+|\nabla u|^2).$$

On the other hand we get

$$(2.20)\int_{\Omega_{R_2}} f\cdot\overline{u} = \int_{\Omega_{R_2}}(A_e+z^2)u\cdot\overline{u} = \int_{r=R_2}(Bu\cdot\overline{u})d\sigma + \int_{\Omega_{R_2}} z^2|u|^2 - E(u,\overline{u})$$

Then we have

$$(2.21)\qquad Im\int_{r=R_2}(Bu\cdot\overline{u}) = \int_{\Omega_{R_2}} Im(f\cdot\overline{u}) - Im(z^2)|u|^2,$$

combining (2.21) with (2.19) we obtain

$$(2.22)\, e^{C\tau}\Big[\int_{\Omega_{R_2}}|f|^2+\tau^2|Im(z^2)|^2|u|^2+\tau^2|f\cdot\overline{u}|dx\Big] \geq C\tau\int_{\Omega_{R_2}}\tau^2|u|^2+|\nabla u|^2$$

Now assume that $e^{C\tau}|Im(z^2)|^2 \leq C_1$ this is further equivalent to $Im(z) < C_1 e^{-CRez/2}$ then the term $|Im(z^2)|^2|u|^2$ which can be easily incorporated in the R.H.S in (2.22) for large τ. Finally we get if $Imz < e^{-C|Rez|}$ the following estimate

$$e^{C'\tau}\int_{\Omega_{R_2}} |f|^2 \geq C\int_{\Omega_{R_2}} (\tau^2|u|^2 + |\nabla u|^2) \tag{2.23}$$

This completes the proof of theorem 0.2. •

3 Proof of Carleman Estimates

This section is devoted to prove estimates of Carleman's type near the boundary for solutions to boundary value problem of the form

$$\begin{cases} A(x,D)u = f & in\ \Omega_0 \\ B(x,D)u = g & on\ \partial\Omega_0 \end{cases} \tag{3.1}$$

Where $A(x,D)$ a partial differential operator with principal symbol given by

$$A(x,\xi) = \mu^t\xi\xi + (\mu+\lambda)\xi^t\xi - \tau^2 Id \tag{3.2}$$

and $B(x,D)u$ defined by (0.3). The proof is based on scalar Carleman estimates proved essentially in the papers of Lebeau and Robbiano [LR1], [LR2]

3.1 Reduction of the Problems.

3.1.1 Reduction of the Laplacian.

Let Ω_0 be a bounded smooth domain of $\mathbb{R}^n$ with boundary $\partial\Omega_0$ of class C^∞. In a neighborhood of a given $x_0 \in \partial\Omega_0$ we denote by $x = (x', x_n)$ the system of normal geodesic coordinates where $x' \in \partial\Omega_0$ and $x_n \in \mathbb{R}$ are characterized by

$$|x_n| = dist(x,\partial\Omega_0); \quad \Omega_0 = \{x_n > 0\}; \quad dist(x', x) = dist(x, \partial\Omega_0).$$

In this system of coordinates the principal symbol of Laplace operator takes the form

$$\tilde{a}(x,\xi) = {}^t\ell(x,\xi)\ell(x,\xi) = \xi_n^2 + r(x,\xi') \tag{3.3}$$

where $r(x,\xi')$ is a quadratic form, such that there exist $C > 0$

$$r(x,\xi') \geq C|\xi'|^2, \qquad for\ any\ x \in K,\ \ \xi' \in T^*(\partial\Omega_0) \tag{3.4}$$

We set

$$\ell(x,\xi) = \ell_0(x)\xi_n + \ell_1(x,\xi') \tag{3.5}$$

then we have

$$ {}^{t}\ell_1\ell_1 = r(x,\xi'), \quad {}^{t}\ell_0\ell_1 = 0 \tag{3.6} $$

Denote $K = \Big\{x \in \mathbb{R}^n_+;\ |x| \le r_0\Big\}$ Let $\varphi(x)$ be a $C^\infty(\mathbb{R}^n)$ function with values in $\mathbb{R}$, defined in a neighborhood of K. We define the operator

$$ a(x,D,\tau) = e^{\tau\varphi}\widetilde{a}(x,D)e^{-\tau\varphi} := op(a) \tag{3.7} $$

Denote by

$$ a(x,\xi,\tau) = a(x,\xi + i\tau\varphi'). \tag{3.8} $$

the principal symbol of the operator, and we set

$$ op(\widetilde{q}_2) = \frac{1}{2}(op(a) + op(a)^*); \qquad op(\widetilde{q}_1) = \frac{1}{2i}(op(a) - op(a)^*), \tag{3.9} $$

its real and imaginary part. Then we have

$$ \left\{ \begin{array}{l} op(a) = op(\widetilde{q}_2) + iop(\widetilde{q}_1) \\ \widetilde{q}_2 = \xi_n^2 + q_2(x,\xi',\tau); \qquad \widetilde{q}_1 = 2i\tau\xi_n\varphi'_{x_n} + 2q_1(x,\xi',\tau). \end{array} \right. \tag{3.10} $$

Where q_1 and q_2 a tangential symbols of order respectively 1 and 2 given by

$$ \left\{ \begin{array}{l} q_2(x,\xi',\tau) = r(x,\xi') - \tau^2{\varphi'_{x_n}}^2 - \tau^2 r(x,\varphi'_{x'}) \\ q_1(x,\xi',\tau) = \widetilde{r}(x,\xi',\tau\varphi'_{x'}) \end{array} \right. \tag{3.11} $$

and $\widetilde{r}(x,\xi',\eta')$ the bilinear form attached to the quadratic form $r(x,\xi')$.

3.1.2 Reduction of the elasticity System

In the system of normal geodesic coordinates the principal symbol of elasticity operator can be written as

$$ A_e(x,\xi) = \mu\,{}^{t}\ell(x,\xi)\ell(x,\xi)Id + (\lambda+\mu)\ell(x,\xi)\,{}^{t}\ell(x,\xi) \tag{3.12} $$

where $\ell(x,\xi)$ defined by (3.5) and $\ell(x,\xi)\,{}^{t}\ell(x,\xi)$ is the orthogonal projection onto the space spanned by $\ell(x,\xi)$. We set

$$ A(x,D,\tau) = e^{\tau\varphi}A_e(x,D)e^{-\tau\varphi} - \tau^2 Id \tag{3.13} $$

The principal symbol of $A(x,D,\tau)$ is given by

$$ A(x,\xi,\tau) = \sum_{j=0}^{2} A_{2-j}(x,\xi',\tau)\xi_n^j \tag{3.14} $$

where $A_j(x,\xi',\tau)$ a tangential symbols in $S^j(\mathbb{R}^n \times \mathbb{R}^{n-1})$ and defined by

$$(3.15)\quad \begin{cases} A_0 = \mu Id + (\mu+\lambda)\ell_0^t\ell_0, \\ \\ A_1 = 2i\tau\varphi'_{x_n}A_0 + (\mu+\lambda)(\ell_0^t\ell_1 + \ell_1^t\ell_0), \\ \\ A_2 = \mu({}^t\ell_1\ell_1)Id + i\tau\varphi'_{x_n}A_1(x,\xi) - \tau^2\varphi'^2_{x_n}A_0 + (\mu+\lambda)\ell_1^t\ell_1 - \tau^2 Id. \end{cases}$$

For fixed $(x,\xi') \in T^*(\partial\Omega_0)$ let $\alpha(x,\xi',\tau) \in \mathbb{C}$ such that

$$(3.16)\qquad a(x,\xi,\tau) = (\xi_n + i\tau\varphi'_{x_n} + i\alpha)(\xi_n + i\tau\varphi'_{x_n} - i\alpha)$$

then we have also by (3.6)

$$(3.17)\qquad r(x,\xi' + i\tau\varphi'_{x'}) = -(i\alpha)^2 = {}^t\ell_1(x,\xi' + i\tau\varphi'_{x'})\ell_1(x,\xi' + i\tau\varphi'_{x'}).$$

Here assume that $\alpha(x,\xi',\tau) \neq 0$, let $(\omega_1,...,\omega_{n-2})$ the basis of $\Big\{\ell_0,(i\alpha)^{-1}\ell_1\Big\}^{\perp}$. We can define the smooth matrix $H = (\omega_1,...,\omega_{n-2},\ell_0,(i\alpha)^{-1}\ell_1)$ homogeneous of degree zero in (ξ',τ), such that, on a conic neighborhood of (x,ξ',τ) we get
(3.18)$\hat{A} = H^{-1}AH =$

$$\begin{pmatrix} (\mu a - \tau^2)Id_{n-2} & 0 & 0 \\ 0 & (2\mu+\lambda)a - \tau^2 + (\mu+\lambda)(i\alpha)^2 & -(\mu+\lambda)(i\alpha)(\xi_n + i\tau\varphi'_{x_n}) \\ 0 & (\mu+\lambda)(i\alpha)(\xi_n + i\tau\varphi'_{x_n}) & \mu a - \tau^2 - (\mu+\lambda)(i\alpha)^2 \end{pmatrix}.$$

We set now

$$(3.19)\qquad \begin{cases} \ell(x,\xi,\tau) = (\xi_n + i\tau\varphi'_{x_n})\ell_0 + \ell_1(x,\xi',\tau) \\ \widetilde{\ell}(x,\xi,\tau) = (\xi_n + i\tau\varphi'_{x_n})(i\alpha)^{-1}\ell_1 + (i\alpha)\ell_0 \end{cases}$$

then we have ${}^t\ell\widetilde{\ell} = 0$ and we can decouple the system by $P = (\omega_1,...,\omega_{n-2},\widetilde{\ell},\ell)$ we obtain

$$(3.20)\qquad \widetilde{A} = P^{-1}AP = \begin{pmatrix} (\mu a - \tau^2)Id_{n-1} & 0 \\ 0 & (2\mu+\lambda)a - \tau^2 \end{pmatrix}.$$

therefor, we obtain

$$(3.21)\qquad det(A(x,\xi,\tau)) = \mu^{n-1}(2\mu+\lambda)(a_\mu(x,\xi,\tau))^{n-1}a_{2\mu+\lambda}(x,\xi,\tau).$$

where $a_\gamma(x,\xi,\tau) = a(x,\xi,\tau) - \frac{\tau^2}{\gamma}$, and $\gamma \in \{\mu, 2\mu+\lambda\}$

3.1.3 Study of the eigenvalues

The proof of Carleman's estimate rely on a cutout argument based on the nature of the roots with respect ξ_n of $a_\gamma(x,\xi',\xi_n,\tau)$.
Let use now introduce the following micro-local regions:

$$(i)\qquad \mathcal{E}^+ = \left\{(x,\xi',\tau) \in K \times S^{n-1};\ q_2 - \frac{\tau^2}{\mu} + \frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2} > 0\right\},$$

(ii) $$\mathcal{Z}_\gamma = \left\{(x,\xi',\tau)\in K\times S^{n-1}; q_2-\frac{\tau^2}{\gamma}+\frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2}=0\right\},$$

(iii) $$\mathcal{E}^- = \left\{(x,\xi',\tau)\in K\times S^{n-1};\ q_2-\frac{\tau^2}{2\mu+\lambda}+\frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2}<0\right\},$$

(iv) $$\mathcal{M} = \left\{(x,\xi',\tau)\in K\times S^{n-1};\ q_2-\frac{\tau^2}{\mu}+\frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2}<0<q_2-\frac{\tau^2}{2\mu+\lambda}+\frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2}\right\}.$$

And for fixed (x,ξ',τ) let $\alpha_\gamma(x,\xi',\tau)\in\mathbb{C}$ such that

(3.22) $$\begin{aligned} a_\gamma(x,\xi,\tau) &= a(x,\xi,\tau)-\frac{\tau^2}{\gamma} \\ &= (\xi_n+i\tau\varphi'_{x_n}+i\alpha_\gamma)(\xi_n+i\tau\varphi'_{x_n}-i\alpha_\gamma). \end{aligned}$$

Taking into account (3.16) (3.17) we have

(3.23) $$(i\alpha_\gamma)^2=(i\alpha)^2+\frac{\tau^2}{\gamma}.$$

then we get by (3.12) and (3.17)

(3.24) $$\alpha_\gamma^2=\tau^2{\varphi'_{x_n}}^2+q_2-\frac{\tau^2}{\gamma}+2i\tau q_1$$

For a fixed (x,ξ',τ) decompose $a_\gamma(x,\xi,\tau)$ as polynomial in ξ_n then we have the following Lemma

Lemma 3.1

1) For any $(x,\xi',\tau)\in\mathcal{E}^+$ the roots of a_μ and $a_{2\mu+\lambda}$ denote by $z_1^\pm, z_2^\pm$ satisfying $\pm Imz_k^\pm>0$.

2) For any $(x,\xi',\tau)\in\mathcal{Z}_\gamma$, one of the roots of a_γ is real.

3) For any $(x,\xi',\tau)\in\mathcal{E}^-$ the roots of a_γ has in the upper half-plane if $\varphi'_{x_n}<0$ (resp. in the lower half-plane if $\varphi'_{x_n}>0$).

4) For any $(x,\xi',\tau)\in\mathcal{M}$ the roots of $a_{2\mu+\lambda}$ satisfying $\pm Imz_2^\pm>0$ and the roots of a_μ satisfy 3).

3.2 Carleman Estimates in the region $\mathcal{Z}_\mu$

Define the following weighted norms in $H^1(\Omega_0)$, respectively in $H^1(\partial\Omega_0)$;

$$||u||^2_{1,\tau} = \sum_{j=0}^{1} \tau^{2(1-j)}||u||^2_{H^1(\Omega_0)}, \quad |u|^2_{1,\tau} = \sum_{j=0}^{1} \tau^{2(1-j)}|u|^2_{H^1(\partial\Omega_0)} \tag{3.25}$$

and define the norms

$$|u|^2_{1,0,\tau} = |u|^2_{1,\tau} + \left|\frac{\partial u}{\partial n}\right|^2_{L^2} \tag{3.26}$$

The purpose of this section it to get the Carleman estimates in the region $\mathcal{Z}_\mu$. Precisely we take a cutoff function $\chi_0(x,\xi',\tau)$ homogeneous of degree zero in the region $\mathcal{Z}_\mu$ our purpose here is to prove the following Proposition.

Proposition 3.1 *There exist $C>0$ such that for any large enough τ we have*

$$||A(x,D,\tau)u||^2 + \tau|u|^2_{1,0,\tau} + ||u||^2_{1,\tau} \geq C\tau||op(\chi_0)u||^2_{1,\tau} \tag{3.27}$$

whenever $u \in C_0^\infty(K)$.
Furthermore if we assume that $\varphi'_{x_n} > C_0$ ($C_0>0$ large enough) on $\{x_n=0\}\cap supp\chi_0$ then we have

$$\begin{aligned} ||A(x,D,\tau)u||^2 \;&+\; ||u||^2_{1,\tau} + |u|^2_{1,\tau} + \tau|B(x,D,\tau)u|^2_{1-ordB,\tau} \\ &\geq C\Big(\tau||op(\chi_0)u||^2_{1,\tau} + \tau|op(\chi_0)u|^2_{1,\tau}\Big) \end{aligned} \tag{3.28}$$

whenever $u \in C_0^\infty(K)$.

3.2.1 Scalar Estimates

In this section we give a scalar Carleman estimates for the operator with principal symbol $a_\gamma(x,\xi,\tau)$, where $\gamma \in \{\mu, 2\mu+\lambda\}$. This estimations are proved essentially in Lebeau-Robbiano [LR1]and [LR2]. Furthermore we assume that φ satisfies the following assumption

$$H_\varphi : \begin{cases} For \quad any \quad x \in K; \varphi'_{x_n} \neq 0 \\ \{Rea_\gamma, Ima_\gamma\} > 0 \quad whenever \quad a_\gamma(x,\xi,\tau) = 0 \quad (x,\xi,\tau) \in K \times \mathbb{R}^{n+1} \end{cases}$$

Lemma 3.2 *There exist $C>0$ such that for any large enough τ we have*

$$||op(a_\mu)v||^2 + \tau|v|^2_{1,0,\tau} + ||v||^2_{1,\tau} \geq C\tau||op(\chi_0)v||^2_{1,\tau} \tag{3.29}$$

whenever $v \in C_0^\infty(K)$.
If we assume that $\varphi'_{x_n} > 0$ and $D_n v - op(k_1)v = g_1$ on $\{x_n=0\}$ where $k_1(x,\xi',\tau)$ tangential symbol of order 1 then for large enough τ we have

$$||op(a_\mu)v||^2 + ||v||^2_{1,\tau} + \tau|g_1|^2 \geq C\Big(\tau||op(\chi_0)v||^2_{1,\tau} + \tau|op(\chi_0)v|^2_{1,\tau}\Big) \tag{3.30}$$

whenever $v \in C_0^\infty(K)$.

Lemma 3.3 *There exist* $C > 0$ *such that for any large enough* τ *we have*

$$||op(a_{2\mu+\lambda})v||^2 + \tau|v|^2_{1,0,\tau} + ||v||^2_{1,\tau} \geq C\tau^2||op(\chi_0)v||^2_{1,\tau} \tag{3.31}$$

whenever $v \in C_0^\infty(K)$.
If we assume that $D_n v - op(k_2)v = g_2$ *on* $\{x_n = 0\}$ *such that* $k_2(x, \xi', \tau) \neq z_2^+(x, \xi', \tau)$ *for any* $(x, \xi', \tau) \in Supp\chi_0$ *then we have*

$$||op(a_{2\mu+\lambda})v||^2 + ||v||^2_{1,\tau} + \tau|g_2|^2 \geq C\Big(\tau^2||op(\chi_0)v||^2_{1,\tau} + \tau|op(\chi_0)v|^2_{1,\tau}\Big) \tag{3.32}$$

whenever $v \in C_0^\infty(K)$.

Remark 3.1 *We have a similar Lemmas if we assume that a Dirichlet condition in a boundary* $v = g_1$ *on* $x_n = 0$

3.2.2 Estimation for A

By applying Lemma 3.3 and Lemma 3.4 we get the following estimate of $\widetilde{A}$

Lemma 3.4 *There exist* $C > 0$ *such that for any large enough* τ *we have*

$$||op(\widetilde{A})v||^2 + \tau|v|^2_{1,0,\tau} + ||v||^2_{1,\tau} \geq C\tau||op(\chi_0)v||^2_{1,\tau} \tag{3.33}$$

whenever $v \in C_0^\infty(K)$.
Furthermore if we assume that $\varphi'_{x_n} > 0$ *and* $D_n v - op(k)v = g$ *on* $\{x_n = 0\}$ *such that* $k_n(x, \xi', \tau) \neq z_2^+(x, \xi', \tau)$ *for any* $(x, \xi', \tau) \in Supp\chi_0$ *then we have*

$$||op(\widetilde{A})v||^2 + ||v||^2_{1,\tau} + \tau|g|^2 \geq C\Big(\tau||op(\chi_0)v||^2_{1,\tau} + \tau|op(\chi_0)v|^2_{1,\tau}\Big) \tag{3.34}$$

whenever $v \in C_0^\infty(K)$.

In following, we construct on approximate solution of $Pv = u$, with an appropriate boundary condition. Let us now introduce some notations.
Let $op(L)$ a differential operators with symbol $L(x, \xi, \tau)$ and assume that $op(L)$ can be written in local coordinates

$$op(L) = \sum_{j=0}^{m} op(L_j(x, \xi', \tau))D_n^j \tag{3.35}$$

where $L_j(x, \xi', \tau)$ a tangential symbols. Let u^0 denote the function which coincides with u in $x_n \geq 0$ and is zero if $x_n < 0$ and $\gamma_j(u)$ the trace of $D_n^j u$ when $x_n = 0$. Then $op(L)u^0 - (op(L)u)^0$ is supported on $\partial\Omega$ and we set

$$(3.36) op(L^c)u := op(L)u^0 - (op(L)u)^0 = \frac{1}{i}\sum_{l+k+1\leq m} L_{l+k+1}(x, D', \tau)\gamma(u) \otimes D_n^k \delta$$

where δ the Dirac measure and $op(L^c)$ is called Calderon's operator (see Hörmander [H]). If we assume that $op(L)$ is elliptic operator, let $op(Q)$ a parametrix of $op(L)$ then we have by (3.36)

$$u^0 = op(Q)[(op(L)u)^0] + op(Q)(op(L^c)u) + op(R)u^0 \tag{3.37}$$

for regular operator R. Now for $g \in C^\infty(\overline{\Omega})$ we note $op(Q)g = (op(Q)g_0)_{|x_n>0}$ then we have by (3.37)

$$u = op(Q)(op(L)u) + \sum_{l=0}^{m-1} op(T_l)\gamma_l(u) + op(R)u \tag{3.38}$$

where $op(T_l)$ a tangential operator with principal symbol

$$T_l(x,\xi',\tau) = \frac{1}{2i\pi}\int_{\Gamma+} e^{ix_n\xi_n}(L^{-1}(x,\xi,\tau)\sum_{l+k+1\leq m} A_{l+k+1}(x,\xi',\tau)\xi_n^k)d\xi_n \tag{3.39}$$

where Γ^+ contour in the half-plane $Im\xi_n > 0$ surround the roots of $(det(L(x,\xi,\tau))$. Now formally let $Pv = u$ where $P = P_0D_n + P_1$ then we have $Pv^0 = u^0 + P_0\gamma_0(v)\otimes\delta$ applying the parametrix Q of P for the previous equation we get $v = Qu + op(T)\gamma_0(v)$ where T a tangential symbol given by $T(x,\xi',\tau) = \frac{1}{2i\pi}(\int_{\Gamma+} e^{ix_n\xi_n}P^{-1}(x,\xi,\tau)d\xi_n)P_0$ Precisely let $\chi_j, j = 1,...,N$ family of cutoff functions homogeneous of degree zero in the region $\mathcal{Z}_\mu$, and P_j family of operator satisfies (3.20) and $\widetilde{\chi}_j$ equal to 1 in Suppχ_j and supported in the domain of P_j. Finally let $\hat{\chi}_j$ equal to 1 in Supp$\widetilde{\chi}_j$ and supported in the domain of P_j.

Lemma 3.5 *Let $v = (op(Q_j\chi_j)u^0)_{|x_n>0}$ then v satisfies*

$$\begin{cases} op(P_j\widetilde{\chi}_j)v = op(\chi_j)u + op(r)u & in \quad x_n > 0 \\ op(\widetilde{\chi}_j)v_{n-1} = op(\widetilde{\chi}_j)v_n + op(r')u & on \quad x_n = 0 \end{cases} \tag{3.40}$$

where $op(r), op(r')$ are neglectible operators of order $-\infty$

Proof . We have

$$op(P_j\widetilde{\chi}_j)v^0 = op(\chi_j)u^0 + P_{j,0}op(\widetilde{\chi}_j)\gamma_0(v)\otimes\delta + op(r)u^0 \tag{3.41}$$

Applying $op(Q_j\hat{\chi}_j)$ in (3.41) we have

$$op(\widetilde{\chi}_j)v = op(Q_j\chi_j)u + op(T_j)\gamma_0(v) + op(r)u \tag{3.42}$$

where the principal symbol of $op(T_j)$ given by

$$T_j(x,\xi',\tau) = \frac{1}{2i\pi}(\int_{\Gamma+} e^{ix_n\xi_n}P_j^{-1}(x,\xi,\tau)\widetilde{\chi}_j d\xi_n)P_{j,0} \tag{3.43}$$

where Γ^+ contour in the half-plane $Im\xi_n > 0$ that contains in its interior the poles of P_j^{-1}. On one hand the symbol P_j given by

$$P_j(x,\xi,\tau) = H_j(x,\xi',\tau)\hat{P}_j(x,\xi,\tau) \tag{3.44}$$

where

$$\hat{P}_j(x,\xi,\tau) = \begin{pmatrix} Id < \xi',\tau > & 0 & 0 \\ 0 & i\alpha(x,\xi',\tau) & \xi_n + i\tau\varphi'_{x_n} \\ 0 & \xi_n + i\tau\varphi'_{x_n} & i\alpha(x,\xi',\tau) \end{pmatrix}. \tag{3.45}$$

and H_j define by (3.18). This implies that the principal symbol T_j can be given by

$$T_j(x,\xi',\tau) = \frac{1}{2i\pi}\Big(\int_{\Gamma+} e^{ix_n\xi_n}\hat{P}_j^{-1}(x,\xi,\tau)\widetilde{\chi}_j d\xi_n\Big)\hat{P}_{j,0}(x,\xi',\tau) \tag{3.46}$$

Furthermore, the poles of P_j^{-1} are $r_\pm = -i\tau\varphi'_{x_n} \pm i\alpha$ then we have

$$\frac{1}{2i\pi}\Big(\int_{\Gamma+} e^{ix_n\xi_n}\hat{P}_j^{-1}(x,\xi,\tau)\widetilde{\chi}_j d\xi_n\Big) = \frac{1}{2}\widetilde{\chi}_j e^{-x_n r_+}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{3.47}$$

On the other hand by (3.42) we can get

$$op(T_j)\gamma_0(v) = op(r)u \tag{3.48}$$

where $op(r)$ is neglectible operator. Combining (3.48) with (3.47) then we obtain $op(\widetilde{\chi}_j)v_{n-1} = op(\widetilde{\chi}_j)v_n + op(r)u$. Then Lemma 3.5 is proved •

Lemma 3.6 *There exist $C > 0$ such that, with the notation of Lemma 3.5 the following inequality holds*

$$||op(\widetilde{A}\chi\chi_j)v||^2 + \tau^{-1}|u|^2_{1,0,\tau} + ||u||^2 \geq C\tau||op(Q_j\chi_j)u||^2_{1,\tau} \tag{3.49}$$

for each large enough τ

Proof . According to Lemma 3.5 we have

$$\begin{cases} op(P_j\widetilde{\chi}_j)v = op(\chi_j)u + op(r)u & in \quad x_n > 0 \\ op(\widetilde{\chi}_j)v_{n-1} = op(\widetilde{\chi}_j)v_n + op(r')u & on \quad x_n = 0 \end{cases} \tag{3.50}$$

Then, according to the following symbolic calculus

$$\begin{aligned} op(H_j^{-1}\hat{\chi}_j)op(P_j\widetilde{\chi}_j) &= op(H_j^{-1}\hat{\chi}_j)op(H_j\hat{P}_j\widetilde{\chi}_j) \\ &= op(\hat{P}_j\widetilde{\chi}_j) + op(r_0) \end{aligned} \tag{3.51}$$

where $op(r_0)$ is a tangential operator of order zero (because, in $\hat{P}_j$ the coefficient of ξ_n is one), we have

$$(3.52)\quad op(\hat{P}_j\widetilde{\chi}_j)v = [op(H_j^{-1}\chi_j)u + op(r_{-1})u] + [op(r_0)v + op(r)v] \quad in\ x_n > 0$$

where $op(r_{-1})$ a tangential operator of order -1. For the first $(n-2)$ terms of the system (3.52) we apply Lemma 3.2, where $v = op(\widetilde{\chi}_j)v_l$ with $l \in \{1, ..., n-2\}$. Denote by $\widetilde{v} = (v_{n-1}, v_n)$ then we have by (3.50)

$$(3.53)\qquad D_n\widetilde{v} - op(k)\widetilde{v} = \widetilde{\underline{u}} + \widetilde{z}_0$$

where $z_0 = op(r_0)v + op(r)v$, $\underline{u} = op(H_j^{-1}\chi_j)u$, $k(x,\xi',\tau) = diag(r_-(x,\xi',\tau), r_-(x,\xi',\tau))$. According to Lemma 3.2 we obtain

$$(3.54)\qquad ||op(a_\mu)v_{n-1}||^2 + \tau|v_{n-1}|^2_{1,0,\tau} + ||u||^2 \geq C\tau||op(\widetilde{\chi}_j)v_{n-1}||^2_{1,\tau}$$

and by Lemma 3.3 we obtain

$$(3.55)\quad ||op(a_{2\mu+\lambda})v_n||^2 + \tau|u|^2_{0,\tau} + ||u||^2 \geq C\tau^2||op(\widetilde{\chi}_j)v_n||^2_{1,\tau} + \tau|op(\widetilde{\chi}_j)v_n|^2_{1,0,\tau}$$

Using now $op(\widetilde{\chi}_j)v_{n-1} = op(\widetilde{\chi}_j)v_n + op(r)u\ on\ x_n = 0$ then we have by (3.55) the following estimate

$$(3.56)\qquad \tau|v_{n-1}|^2_{1,0,\tau} \leq C(||op(a_{2\mu+\lambda})v_n||^2 + \tau|u|^2_{0,\tau} + ||u||^2$$

Combining (3.56), (3.55) and (3.54) we obtain (3.49) •

Now we prove a simpler estimate which completely neglects the boundary condition.

Lemma 3.7 *there exist $C > 0$, such that for any large enough τ we have*

$$(3.57)\qquad C\tau||op(\chi_0)u||^2_{1,\tau} \leq ||op(A)u||^2 + \tau|u|^2_{1,0,\tau} + ||u||^2_{1,\tau}$$

whenever $u \in C_0^\infty(K)$

Proof . According Lemma 3.6 and (3.40) it is enough to prove that

$$(3.58)\qquad C||op(\widetilde{A}\chi\chi_j)v||^2 \leq ||op(A)u||^2_{-1} + \tau^{-1}|u|^2_{1,0,\tau} + ||u||^2$$

for large enough τ.
Writing

$$(3.59)\qquad op(A)op(P_j\widetilde{\chi}_j) = \sum_{k=0}^{3} op(C_k(x,\xi',\tau))D_n^k$$

where $C_k(x,\xi',\tau)$ a tangential pseudo-differential of order $(3-k)$, then we obtain by(3.35)

$$op(A)op(P_j\widetilde{\chi}_j)v^0 = f^0 + \frac{1}{i}\sum_{l+k+1} op(C_{l+k+1})\gamma_l(v)\otimes D_n^k\delta = f^0 + \frac{1}{i}w_0\otimes\delta - w_1\otimes\delta' - \frac{1}{i}w_2\otimes\delta'' \tag{3.60}$$

where

$$\left\{\begin{array}{l} w_0 = op(C_1)\gamma_0(v) + op(C_2)\gamma_1(v) + op(C_3)\gamma_0(v) \\ w_1 = op(C_2)\gamma_0(v) + op(C_3)\gamma_1(v) \\ w_2 = op(C_3)\gamma_0(v) \end{array}\right. \tag{3.61}$$

Now applying $op(P_j^{-1}\chi_j)$ in (3.60) we obtain

$$op(\widetilde{A}\chi\chi_j)v = op(P_j^{-1}\chi_j)f + \sum_{k=0}^{2} op(T_k)w_k + op(r_0)u \tag{3.62}$$

where $r_0 \in S^0$ and $op(T_k)$ is a tangential pseudo-differential operator with principal symbol

$$T_k(x,\xi',\tau) = \frac{1}{2i\pi}(\int_{\Gamma+} e^{ix_n\xi_n}P_j^{-1}(x,\xi,\tau)\widetilde{\chi}_j\xi_n^k d\xi_n)P_{j,0} \tag{3.63}$$

a simple computation shows that

$$op(\widetilde{A}\chi\chi_j)v = op(P_j^{-1}\chi_j)f + \sum_{k=0}^{2} op(S_k)\gamma_k(v) + op(r_0)u \tag{3.64}$$

where S_k define by

$$\left\{\begin{array}{l} S_0 = T_0C_1 + T_1C_2 + T_2C_3 \\ S_1 = T_0C_2 + T_1C_3 \\ S_2 = T_0C_3 \end{array}\right. \tag{3.65}$$

From (3.53) it is easy to see that

$$op(\widetilde{\chi}_j)D_nv = op(\widetilde{\chi}_jK_1)v + op(r_0)u + op(r_0')v \tag{3.66}$$

where $op(r_0)$, $op(r_0')$ are two tangential operator with symbols of order zero, and the principal symbol of $op(\widetilde{\chi}_jK_1)$ given by

$$K_1(x,\xi',\tau) = \begin{pmatrix} -i\tau\varphi'_{x_n} & -i\alpha \\ -i\alpha & -i\tau\varphi'_{x_n} \end{pmatrix}. \tag{3.67}$$

Combined with (3.52) this gives

$$op(\widetilde{\chi}_j)D_nv = op(\widetilde{\chi}_jr_-)v + op(r_0)u \tag{3.68}$$

and by the first line of (3.50) we get

$$op(\widetilde{\chi}_j)D_n^2 v = op(\widetilde{\chi}_j r_-^2)v + op(r_1')u + op(r_0'')D_n u \qquad on\ x_n = 0 \tag{3.69}$$

with the following estimates

$$||op(r_1)u||^2 \leq \frac{C}{\tau}|u|_{1,\tau}^2, \quad and \quad ||op(r_0'')u||^2 \leq \frac{C}{\tau}|D_n u|_0^2 \tag{3.70}$$

Then we have

$$\sum_{k=0}^{2} op(S_k)\gamma_k(v) = op(Q)\gamma_0(v) + [op(r_1)u + op(r_0')D_n u] \tag{3.71}$$

where

$$\begin{cases} Q = S_0 + r_- S_1 + r_-^2 S_2 \\ ||op(r_1)u||^2 \leq \frac{C}{\tau}|u|_{1,\tau}^2, \quad ||op(r_0')u||^2 \leq \frac{C}{\tau}|D_n u|_0^2 \end{cases} \tag{3.72}$$

But

$$Q = T_0 Q_0 = \hat{T}_0 \hat{Q}_0 \tag{3.73}$$

where $\hat{Q}_0 = \hat{C}_1 + (r_+ + r_-)\hat{C}_2 + (r_+^2 + r_+ r_- + r_-^2)\hat{C}_3$. Using $(r_+ + r_-) = -2i\tau\varphi'_{x_n}$ and $(r_+^2 + r_+ r_- + r_-^2) = -3\tau^2 {\varphi'_{x_n}}^2 + (i\alpha)^2$ we can get by a simple calculus $\hat{Q}_0 = -\tau^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we get from (3.50) $op(Q)v = op(r')u \quad on\ x_n = 0$ where $r' \in S^0$ combining the last equality with (3.71),(3.64) and (3.72) gives (3.58) •

4 End of the Proof of Proposition 3.1

The purpose of this section is to proof Proposition 3.1. The essential ingredient in the proof is to estimate the traces of u by the operators A and B. The subject of this section to proof the following proposition

Proposition 4.1 *There exists $C_0 > 0$ and $C > 0$ such that if $\varphi'_{x_n} > C_0$ for any large enough τ we have*

$$||A(x,D,\tau)u||^2 + \tau|B(x,D,\tau)u|_{1-ordB,\tau}^2 + ||u||_{1,\tau}^2 + |u|_{1,\tau}^2 \geq C\tau|op(\chi_0)u|_{1,0,\tau}^2 \tag{4.1}$$

whenever $u \in C_0^\infty(K)$.

4.1 Preliminaries

Let $u \in C_0^\infty(K)$ denote

$$\widetilde{u} = op(\chi_0)u \, and \qquad \widetilde{f} = op(A)\widetilde{u} \tag{4.2}$$

where $op(A)$ the differential operator with principal symbol

$$A(x,\xi,\tau) = A_0(x)\xi_n^2 + A_1(x,\xi',\tau)\xi_n + A_2(x,\xi',\tau) \tag{4.3}$$

It is easy to see that

$$\left\{ \begin{array}{lll} op(A)\widetilde{u} = \widetilde{f} & in & x_n > 0 \\ \\ op(B)\widetilde{u} = \widetilde{g}_0 & on & x_n = 0 \end{array} \right. \tag{4.4}$$

where $\widetilde{f} = op(\chi_0)f + [op(A); op(\chi_0)]u$
Let us reduce the problem (4.4) to a first order system. Put $v = {}^t(< D', \tau > \tilde{u}, D_n\tilde{u})$ then the system (4.4) is reduced to the following first system

$$\left\{ \begin{array}{lll} D_n v - op(\mathcal{A})v = F & in & x_n > 0 \\ op(\mathcal{B})v = \widetilde{g}_0 & on & x_n = 0 \end{array} \right. \tag{4.5}$$

Where the principal symbol of $op(\mathcal{A})$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & - < \xi', \tau > Id_n \\ - < \xi', \tau >^{-1} A_0^{-1}A_2 & -A_0^{-1}A_1 \end{pmatrix} \tag{4.6}$$

and

$$\mathcal{B} = (< \xi', \tau >^{-1} B_1, B_0) \tag{4.7}$$

with $B(x,\xi,\tau) = B_0(x)\xi_n + B_1(x,\xi',\tau)$ and

$$F = {}^t(0, \tilde{f}) \tag{4.8}$$

and further

$$\begin{aligned} det(\xi_n - \mathcal{A}(x,\xi',\tau)) &= det(A(x,\xi,\tau)) \\ &= \mu^{n-1}(2\mu+\lambda)(a_\mu(x,\xi,\tau))^{n-1}(a_{2\mu+\lambda}(x,\xi,\tau)) \end{aligned} \tag{4.9}$$

Let (x_0, ξ_0', τ_0) fixed in $Supp(\chi_0)$. In this case the eigenvalues of $\mathcal{A}$ are $z_1^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_\mu$ and $z_2^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_{2\mu+\lambda}$ with $\pm Im(z_2^\pm) > 0$ and $z_1^+ \in \mathbb{R}$. Denote $\overline{s}(x_0, \xi_0', \tau_0) = (\overline{s}_1^+, ..., \overline{s}_n^+, \overline{s}_1^-, ..., \overline{s}_n^-)$ where $\{s_j^\pm\}_{j=1,\dots,n}$ from a basis of the generalized eigenspace of $\mathcal{A}(x_0, \xi_0', \tau_0)$ corresponding to eigenvalues with positive or negative imaginary part. Let for $\gamma \in \{\mu, 2\mu+\lambda\}$

$$P_\gamma^\pm(x,\xi',\tau) = \frac{1}{2i\pi}\int_{C_\gamma^\pm} (\xi_n - \mathcal{A}(x,\xi',\tau))^{-1} d\xi_n \tag{4.10}$$

where $C_\gamma^\pm$ is a small circle with the center $-i\tau\varphi'_{x_n} \pm i\alpha_\gamma$. Using this projection operator, we put $s_j^\pm = P_\mu^\pm \overline{s}_j^\pm$, $j = 1, ..., n-1$ and $s_n^\pm = P_{2\mu+\lambda}^\pm \overline{s}_n^\pm$ where $< \xi', \tau >= 1$, and $s(x, \xi', \tau) = (s_1^+, ..., s_n^+, s_1^-, ..., s_n^-)$ as a smooth positively homogeneous function of degree zero and define a pseudo-differential $S(x, D_{x'}, \tau)$ with principal symbol $s(x, \xi', \tau)$. Then by the argument in Taylor [T] (see also Yamamoto [Y]) there exist a pseudo-differential operator $K(x, D_{x'}, \tau)$ of order -1 such that the boundary value problem (4.5) is reduced to the following

$$(4.11) \qquad \begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in\ x_n > 0 \\ \\ op(\widetilde{\mathcal{B}})w = \widetilde{g}_0 & on\ x_n = 0 \end{cases}$$

Where $w = (I+K)^{-1}S^{-1}v$, $\widetilde{F} = (I+K)^{-1}S^{-1}F$, $\widetilde{\mathcal{B}} = \mathcal{B}S(I+K)^{-1}$
and $\mathcal{H} = diag(\mathcal{H}^+, \mathcal{H}^-)$ moreover the eigenvalue of the principal symbol of $\mathcal{H}^-$ have negative imaginary parts. Denote the boundary operator $\mathcal{B}$ of (4.11) by $(\widetilde{\mathcal{B}}^+, \widetilde{\mathcal{B}}^-)$ and E^+ the subspace generate by $(s_1^+, ..., s_n^+)$ then we have

$$(4.12) \qquad E^+ = E^+(x, \xi', \tau) = [Ker(z_1^+ - \mathcal{A})] \oplus [Ker(z_2^+ - \mathcal{A})]$$

4.2 Proof of Proposition 4.1

Lemma 4.1 *Let* $\mathcal{R} = diag(0, -\rho Id_n)$, $\rho > 0$, *then there exist* $C > 0$ *such that*

$$(4.13) \qquad Im(\mathcal{R}\mathcal{H}) = diag(0, e(x, \xi', \tau))$$

and we have

i)$e(x, \xi', \tau) \geq C < \xi', \tau > I_n$ *in* $supp\chi_0$

ii)$-\mathcal{R} + \widetilde{\mathcal{B}}^*\widetilde{\mathcal{B}} \geq C diag(0, Id)$ *on* $\{x_n = 0\} \cap supp\chi_0$

Proof of Proposition 4.1

Denote the function define $G(x_n) = \frac{d}{dx_n}\Big(op(\mathcal{R})w, w\Big)_{L^2(\mathbb{R}^{n-1})}$.
Taking into account (4.10) we have

$$(4.14)\ G(x_n) = -2Im(op(\mathcal{R})op(\mathcal{H})w, w) + 2Im(op(\mathcal{R})w, \widetilde{F}) + (op(\mathcal{R}'_{x_n})w, w)$$

The integration in the normal direction gives

$$\begin{aligned} (4.15) \quad (op(\mathcal{R})w, w)_0 \;&=\; 2\int_0^\infty Im(op(\mathcal{R})op(\mathcal{H})w, w)dx_n \\ &\quad -2Im\int_0^\infty (op(\mathcal{R})w, \widetilde{F}) - \int_0^\infty op(\mathcal{R}'_{x_n})w, w)dx_n \end{aligned}$$

Then according to Lemma 4.2 and Gärding inequality we obtain for $w = (w^+, w^-)$ and large τ

$$Im(op(\mathcal{R})op(\mathcal{H})w, w) \geq C\tau ||w^-||^2 \tag{4.16}$$

and further, for any ε

$$\int_0^\infty |(op(\mathcal{R})w, \widetilde{F})| dx_n \leq \varepsilon C\tau ||w^-||^2 + \frac{C_\varepsilon}{\tau} ||\widetilde{f}||^2. \tag{4.17}$$

Apply now Lemma 4.2 ii) we obtain

$$-(op(\mathcal{R})w, w) + C|\widetilde{\mathcal{B}}w|^2 \geq C|w|^2. \tag{4.18}$$

Combining (4.18) (4.17) (4.16) with (4.15) we get

$$C\tau ||w^-||^2 + C'|w|_0^2 \leq \frac{C}{\tau} ||\widetilde{f}||_0^2 + |\widetilde{\mathcal{B}}w|^2. \tag{4.19}$$

This implies (4.1)

4.3 Proof of Lemma 4.2

First we prove for any $(x, \xi', \tau) \in \ Supp\chi_0$ the restriction $\mathcal{B}^+$ of $\mathcal{B}$ in $E^+(x, \xi', \tau)$ is an isomorphism. The eigenvalues of $\mathcal{A}$ are $z_1^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_\mu$ and $z_2^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_{2\mu+\lambda}$ with multiplicity respectivment $(n-1)$ and 1.
Now let $X = (X_1, X_2) \in \mathbb{C}^n \oplus \mathbb{C}^n$ eigenvector of $\mathcal{A}$ associated to z_0 then X satisfy

$$\begin{cases} <\xi', \tau> X_2 = z_0 X_1 \\ A(z_0).X_1 = 0 \end{cases} \tag{4.20}$$

a) Calculus of eigenvector associated to z_1^+:
Denote by $\Big\{\omega_j^+\Big\}_{j=1,\dots,n-2}$ a basis of $\Big\{\ell_0, \ell_1\Big\}^\perp$ then we have

$$A(z_1^+)\omega_j^+ = 0 \qquad for \ j \in \Big\{1, \dots, n-2\Big\} \tag{4.21}$$

where $A(z_1^+) = (\mu + \lambda)(\ell_1 + i\alpha_\mu \ell_0)^t(\ell_1 + i\alpha_\mu \ell_0)$
Now we set the following vector in $\mathbb{C}^n$

$$\omega_{n-1} =<\xi', \tau>^{-1} \left((i\alpha)^2 \ell_0 + i\alpha_\mu \ell_1\right) \tag{4.22}$$

then we have by a simple calculation

$$A(z_1^+)\omega_{n-1}^+ = 0 \tag{4.23}$$

b) Calculus of eigenvector associated to z_2^+:
We get

$$A(z_2^+) = -\frac{\mu+\lambda}{2\mu+\lambda} Id + (\mu+\lambda)(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0)^t(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0). \tag{4.24}$$

Let ω_n^+ defined by

$$\omega_n^+ = (i\alpha_{2\mu+\lambda})\ell_0 + \ell_1 \tag{4.25}$$

then we have $A(z_2^+)\omega_n^+ = 0$. Using (4.20) we denote

$$\begin{cases} s_j^+ = (\omega_j^+; <\xi', \tau>^{-1} z_1^+ \omega_j^+) \quad j \in \Big\{1, ..., n-1\Big\} \\ s_n^+ = (\omega_n^+; <\xi', \tau>^{-1} z_2^+ \omega_n^+) \end{cases} \tag{4.26}$$

and the principal symbol of $\widetilde{\mathcal{B}}$ by $(b_1^+, ..., b_n^+, b_1^-, ..., b_n^-)$

a) Dirichlet condition: In this case $\mathcal{B} = (<\xi', \tau>^{-1} id_n, 0)$ then $\mathcal{B}S = (\omega_1^+, ..., \omega_n^+, \omega_1^-, ..., \omega_n^-) = (\widetilde{\mathcal{B}}^+, \widetilde{\mathcal{B}}^-)$.
We obtain

$$\widetilde{\mathcal{B}}^+ = (\omega_1^+, ..., \omega_n^+) = (b_1^+, ..., b_n^+) \tag{4.27}$$

Therefore $\widetilde{\mathcal{B}}^+$ is an isomorphism if ω_{n-1}^+ and ω_n^+ are linearly independent. With (4.23) and (4.26), we must see that for any $(x, \xi', \tau) \in Supp\chi_0$

$$(i\alpha)^2 - (i\alpha_\mu)(i\alpha_{2\mu+\lambda}) \neq 0 \tag{4.28}$$

We may show that (4.28) by contradiction. If we assume that $(i\alpha)^2 - (i\alpha_\mu)(i\alpha_{2\mu+\lambda}) = 0$ then we have

$$(i\alpha)^2 = -\frac{\tau^2}{3\mu+\lambda} \tag{4.29}$$

and by (3.23) we get

$$(i\alpha_\mu)^2 = \frac{\tau^2}{\mu} - \frac{\tau^2}{3\mu+\lambda} = \frac{2\mu+\lambda}{\mu(3\mu+\lambda)}\tau^2 > 0 \tag{4.30}$$

Then we get $(i\alpha_\mu) \in \mathbb{R}$, this is implies $Im(z_1^+) = Im(z_1^-) = -\tau\varphi'_{x_n}$, this is impossible because $(x, \xi', \tau) \in \mathcal{Z}_\mu$, then $\widetilde{\mathcal{B}}^+$ is an isomorphism.

b) Neumann Condition: The proof of this part is post pined to section 5 (see Lemma 5.3).
For the second part of Lemma 4.2, we get

$$Im(\mathcal{R}\mathcal{H}) = diag(0, -\rho Im(\mathcal{H})) \tag{4.31}$$

then we obtain i). Let now $w = (w^+, w^-) \in \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ we have $\widetilde{\mathcal{B}}w = \widetilde{\mathcal{B}}^+ w^+ + \widetilde{\mathcal{B}}^- w^-$. Taking into account $\widetilde{\mathcal{B}}^+$ is an isomorphism, then there exist $C' > 0$ such that

$$|w^+|^2 \leq C'(|w^-|^2 + |\widetilde{\mathcal{B}}w|^2) \tag{4.32}$$

This shows that

$$-(\mathcal{R}w, w) = \rho |w^-|^2 \geq C|w|^2 - C''|\widetilde{\mathcal{B}}w|^2 \tag{4.33}$$

for large ρ. This concludes the proof of the lemma 4.2.

5 Carleman Estimates in elliptic region $\mathcal{E}^+$

In this section, our goal is to show the following Carleman estimate in elliptic region

$$\mathcal{E}^+ = \left\{ (x, \xi', \tau) \in K \times S^{n-1};\ q_2 - \frac{\tau^2}{\mu} + \frac{q_1^2}{\tau^2 {\varphi'_{x_n}}^2} > 0 \right\} \tag{5.1}$$

Let χ a cutoff function homogeneous of degree zero in the region $\mathcal{E}^+$

Lemma 5.1 *There exist $C_0 > 0$ and $C_1 >$ such that if $\varphi'_{x_n} > C_0$ for any large enough τ we have*

$$\begin{aligned} ||A(x, D, \tau)u||^2 \quad + \quad & \tau |B(x, D, \tau)u|^2_{0,\tau} + \\ & + ||u||^2_{1,\tau} + |u|^2_{1,\tau} \geq C_1 \tau (||op(\chi)u||^2_{1,\tau} + \tau |op(\chi)u|^2_{1,\tau}) \end{aligned} \tag{5.2}$$

whenever $u \in C_0^\infty(K)$.

Denote

$$\widetilde{u} = op(\chi)u, \quad \widetilde{f} = op(A)\widetilde{u} \tag{5.3}$$

by an argument similar to the one of section 3, we have

$$\begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in\ x_n > 0 \\ \\ op(\widetilde{\mathcal{B}})w = \widetilde{g}_0 & on\ x_n = 0 \end{cases} \tag{5.4}$$

Where $w = (I+K)^{-1}S^{-1}v$, $\widetilde{F} = (I+K)^{-1}S^{-1}F$, $\widetilde{\mathcal{B}} = \mathcal{B}S(I+K)^{-1}$ and $\mathcal{H} = diag(\mathcal{H}^+, \mathcal{H}^-)$ and $\mathcal{H}^\pm$ is an $n \times n$ square matrix, whose components are pseudo-differential operator of order 1 with principal symbol given by $diag(z_1^\mp Id_{n-1}, z_2^\mp)$. The proof of Lemma 5.1 is based on the next Lemma.

Lemma 5.2 *There exist a hermitian matrix $\mathcal{R}(x, \xi', \tau)$, C_1,$C_2 > 0$ such that*

$$Im(\mathcal{R}\mathcal{H}) = e(x, \xi', \tau) \tag{5.5}$$

and we have the following properties

i) $e(x,\xi',\tau) \geq C_1 I$ *in* $\mathrm{supp}(\chi)$

ii)$-\mathcal{R}(x,\xi',\tau) + C_2\widetilde{\mathcal{B}}^*\widetilde{\mathcal{B}} \geq CI$ *on* $\{x_n = 0\} \cap \mathrm{supp}(\chi)$

Proof of Lemma 5.1
Let G the function define by

$$G(x_n) = \frac{d}{dx_n}(op(\mathcal{R})w, w)_{L^2(\mathbb{R}^{n-1})}. \tag{5.6}$$

Using $D_n w - op(\mathcal{H})w = \widetilde{F}$ then we obtain

$$G(x_n) = -2Im(op(\mathcal{R})op(\mathcal{H})w, w) + 2Im(op(\mathcal{R})w, \widetilde{F}) + (op(\mathcal{R}'_{x_n})w, w) \tag{5.7}$$

The integration in the normal direction gives

$$\begin{aligned}(op(\mathcal{R})w, w)_0 &= 2\int_0^\infty Im(op(\mathcal{R})op(\mathcal{H})w, w)dx_n \\ &\quad -2Im\int_0^\infty (op(\mathcal{R})w, \widetilde{F}) - \int_0^\infty (op(\mathcal{R}'_{x_n})w, w)dx_n\end{aligned} \tag{5.8}$$

Taking into account Lemma 5.2 i) we have

$$Im(op(\mathcal{R})op(\mathcal{H})w, w) \geq C\tau||w||^2 \tag{5.9}$$

and by (ii) we obtain

$$(op(\mathcal{R})w, w)_0 \leq C|\mathcal{B}w|^2 - C'|w|^2. \tag{5.10}$$

Combining (5.9) (5.10) with (5.8) we get

$$C\tau||w||^2 + C'|w|_0^2 \leq ||\widetilde{F}||_0^2 + \tau|\mathcal{B}w|^2. \tag{5.11}$$

This completes the proof of Lemma 5.1
Proof of Lemma 5.2
Let $w = (w^+, w^-) \in \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ then we have $\widetilde{\mathcal{B}}w = \widetilde{\mathcal{B}}^+w^+ + \widetilde{\mathcal{B}}^-w^-$. By a similar argument in section 4 we prove that for any (x,ξ',τ) and $\tau \neq 0$ $\widetilde{\mathcal{B}}$ is an isomorphism in the case of Dirichlet boundary condition. In the case of Neumann boundary condition see Lemma 5.3. But in the region $\mathcal{E}^+$ the parameter τ can be degenerate to zero.
Study near $\tau = 0$
For $\tau = 0$ the eigenvalues of $\mathcal{A}$ are $\pm(i\alpha)$, let $\{\omega_j^+\}$, $j = 1, ..., n-2$ a basis of $\{\ell_0, \ell_1\}$ and put

$$\omega_{n-1}^+ = \Lambda^{-1}(\ell_1 + (i\alpha)\ell_0) \; with \quad \Lambda = <\xi', \tau> \tag{5.12}$$

then we have

$$A(i\alpha)\omega_{n-1}^+ = 0 \tag{5.13}$$

and for $j \in \{1, ..., n-1\}$ we set

$$\overline{s}_j^{\pm} = (\omega_j^{\pm}; \pm(i\alpha)\Lambda^{-1}\omega_j^{\pm}) \tag{5.14}$$

and let

$$\overline{s}_n^{+} = ((\lambda + 3\mu)(\ell_1 - (i\alpha)\ell_0); (i\alpha)\Lambda^{-1}(3\lambda + 5\mu)\ell_1 + (\lambda - \mu)(i\alpha)\ell_0) \tag{5.15}$$

Then we have

$$(\mathcal{A} + (i\alpha))\overline{s}_n^{+} = 2(\mu + \lambda)\overline{s}_n^{-} \in Ker(\mathcal{A} - (i\alpha)) \tag{5.16}$$

Then $\{\overline{s}_j^{\pm}\}$ $j = 1, ..., n$ are linearement independent. Let

$$P_{\pm}(x, \xi', \tau) = \frac{1}{2i\pi} \int_{C_{\pm}} (\xi_n - \mathcal{A}(x, \xi', \tau))^{-1} d\xi_n \tag{5.17}$$

where $C^{\pm}$ is a small circle with the center $\pm(i\alpha)$. We define $s_j^{\pm}$ by

$$s_j^{\pm}(x, \xi', \tau) = P_{\pm}(x, \xi', \tau)\overline{s}_j^{\pm} \tag{5.18}$$

then we have $s(x, \xi', \tau) = (s_1^{+}, ..., s_n^{+}, s_1^{-}, ..., s_n^{+})$ a regular function homogeneous of degree zero.
For Dirichlet condition we have $\widetilde{\mathcal{B}}(x_0, \xi_0', 0) = (b_1^{+}, ..., b_n^{+})$ where

$$\begin{cases} b_j^{+} = \omega_j^{+} & j \in \{1, ..., n-1\} \\ b_n^{+} = (3\mu + \lambda)(\ell_1 - (i\alpha)\ell_0) \end{cases} \tag{5.19}$$

then $det(\widetilde{\mathcal{B}}(x_0, \xi_0', 0)) \neq 0$ if and only if $(3\mu + \lambda) \neq 0$.
And for the Neumann condition by easy computation we have by using $\mathcal{B} = (\Lambda^{-1}B_1, B_0)$ where

$$\begin{cases} B_0 = \mu Id + (\mu + \lambda)\ell_0^t\ell_0 \\ \\ B_1 = \mu\ell_1^t\ell_0 + \lambda\ell_0^t\ell_1. \end{cases} \tag{5.20}$$

and $\widetilde{\mathcal{B}} = (b_1^{+}, ..., b_n^{+})$ where

$$\begin{cases} b_j^{+} = \mu z_1^{+}\omega_j^{+} \\ \\ b_{n-1}^{+} = 2\mu(i\alpha)((i\alpha)\ell_0 + \ell_1) \\ \\ b_n^{+} = 2\mu(\mu + \lambda)(i\alpha)(-i\alpha\ell_0 + \ell_1) \end{cases} \tag{5.21}$$

Then we have $det(\widetilde{\mathcal{B}}(x_0, \xi_0', 0)) \neq 0$ if and only if $(\mu + \lambda) \neq 0$.
For the second part of Lemma 5.2 let us search that $\mathcal{R}$ in the form

$$\mathcal{R} = diag(Id_n, -\rho Id_n) \tag{5.22}$$

Then we have

$$Im(\mathcal{R}\mathcal{H}) = diag(Im(\mathcal{H}^+), -\rho Im(\mathcal{H}^-)) = e(x, \xi, \tau) \tag{5.23}$$

where $e(x, \xi, \tau) \geq CId_{2n}$. Now let $w = (w^+, w^-) \in \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ then we have $\widetilde{\mathcal{B}}w = \widetilde{\mathcal{B}}^+ w^+ + \widetilde{\mathcal{B}}^- w^-$ and there exist $C > 0$ such that for any $w \in \mathbb{C}^{2n}$ we get

$$|w^+|^2 \leq C(|w^-|^2 + |\widetilde{\mathcal{B}}w|^2) \tag{5.24}$$

we deduce

$$-(\mathcal{R}w, w) = -|w^+|^2 + \rho|w^-|^2 \geq |w^+|^2 + (\rho - 2C') - 2C'|\widetilde{\mathcal{B}}w|^2 \tag{5.25}$$

Then we have the desireted estimation ii) for large ρ.

Lemma 5.3 *For any* (x, ξ', τ) *in* $\mathcal{Z}_\mu \cup \mathcal{E}^+$ *and* $\tau \neq 0$ *the operator* $\widetilde{\mathcal{B}}^+(x, \xi', \tau)$ *is an isomorphism under the assumption* $\varphi'_{x_n} > C_0$ *for large* $C_0 > 0$

Proof . We will keep some of the notation from section 3. Let $\mathcal{B} = (\Lambda^{-1}B_1, B_0)$ the principal symbol of Neumann operator, where

$$\begin{cases} B_0 = \mu Id + (\mu + \lambda)\ell_0^t \ell_0 \\ \\ B_1 = i\tau\varphi'_{x_n} B_0 + \mu\ell_1^t \ell_0 + \lambda\ell_0^t \ell_1. \end{cases} \tag{5.26}$$

Denote by $\widetilde{\mathcal{B}}^+ = (b_1^+, ..., b_n^+)$ then we have by elementary calculations

$$\begin{cases} b_j^+ = \mu(i\alpha_\mu)\Lambda^{-1}\omega_j^+ & j = 1, ..., n-2 \\ \\ b_{n-1}^+ = \Lambda^{-3}[2\mu(i\alpha_\mu)](i\alpha)^2\ell_0 + \Lambda^{-3}[2\mu(i\alpha)^2 + \tau^2]\ell_1 \\ \\ b_n^+ = \Lambda^{-2}[2\mu(i\alpha)^2 + \tau^2]\ell_0 + \Lambda^{-2}[2\mu(i\alpha_{2\mu+\lambda})]\ell_1 \end{cases} \tag{5.27}$$

Then we get

$$det(\widetilde{\mathcal{B}}^+) = (\mu(i\alpha_\mu))^{n-1}\tau^4\Lambda^{-4}R\Big(\frac{\sqrt{\mu}}{\tau}\alpha(x, \xi', \tau)\Big) \tag{5.28}$$

where R is a function given by

$$R(s) = (1 - 2s^2)^2 - 4s^2\left(\sqrt{s^2 - 1}\sqrt{s^2 - \frac{\mu}{2\mu + \lambda}}\right) \tag{5.29}$$

It is well known that there is only one simple roots $s = s_0$ of $R(s) = 0$, $s > 1$ (see [T1]) and we can prove has no roots in $Rez > 0$. Let Σ the characteristic variety define by

$$\begin{aligned} \Sigma &= \Big\{(x', \xi', \tau) \in T^*\partial\Omega;\ \alpha^2 = \frac{\tau^2}{c_R^2}\Big\} \\ &= \Big\{(x', \xi', \tau) \in T^*\partial\Omega; r(x', \xi') - \tau^2 r(x', \varphi'_{x'}) + 2i\tau q_1 - \frac{\tau^2}{c_R^2} = 0\Big\} \end{aligned} \tag{5.30}$$

where $c_R = \frac{\sqrt{\mu}}{s_0}$ the Rayleigh speed. Therefore $\widetilde{\mathcal{B}}^+$ is elliptic outside Σ but for $(x', \xi', \tau) \in \mathcal{Z}_\mu \cup \mathcal{E}^+$ then we have

$$r(x', \xi') - \tau^2 r(x', \varphi'_{x'}) + \frac{q_1^2}{{\varphi'_{x_n}}^2} \geq \frac{\tau^2}{\mu} + \tau^2 {\varphi'_{x_n}}^2 \tag{5.31}$$

then for $\varphi'_{x_n} > C_0$ we have $\Sigma \cap (\mathcal{Z}_\mu \cup \mathcal{E}^+) = \emptyset$ •

6 Study of the region $\mathcal{M}$ and $\mathcal{Z}_{2\mu+\lambda}$

Our goal in this section is to prove the Carleman estimates in the mixed region $\mathcal{M}$ and $\mathcal{Z}_{2\mu+\lambda}$.

Lemma 6.1 *Assume that $\varphi'_{x_n} > 0$. Then there exists $C > 0$ such that for any large enough τ we have*

$$C\tau |op(\chi)u|^2_{1,0,\tau} \leq ||A(x, D, \tau)u||^2 + \tau |B(x, D, \tau)|^2_{1-ordB,\tau} + ||u||^2_{1,\tau} \tag{6.1}$$

whenever $u \in C_0^\infty(K)$.

Proof . Denote $v =^t (\langle D', \tau\rangle \tilde{u}, D_n \tilde{u})$ where $\widetilde{u} = op(\chi)u$. In the mixed region $\mathcal{M}$ the roots of a_μ denote by z_μ^- and $\tilde{z}_\mu^-$ is in the lower half-plane $Im\xi_n < 0$ and the roots of $a_{2\mu+\lambda}$ satisfying $\pm Im(z_{2\mu+\lambda}) > 0$. By an argument similar to the one of section 3, we have

$$\begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in\ x_n > 0 \\ op(\widetilde{\mathcal{B}})w = g & on\ x_n = 0 \end{cases} \tag{6.2}$$

where $w = (I + K)^{-1} s^{-1} v$, $\widetilde{F} = (I + K)^{-1} S^{-1} F$, $\widetilde{\mathcal{B}} = \mathcal{B} S (I + K)^{-1}$, and $op(\mathcal{H})$ a pseudodifferential operator of order 1 with principal symbol

$$\mathcal{H} = diag(\mathcal{H}^-; z^+_{2\mu+\lambda}) \tag{6.3}$$

where $Im(\mathcal{H}^-) < 0$.
Let $s_j = (\omega_j; \Lambda^{-1} z_\mu^- \omega_j)$, $j = 1, \ldots, n-2$, $s_{n-1} = (\omega_{n-1}; \Lambda^{-1} z_\mu^- \omega_{n-1})$, $s_{2n-2} = (\omega_{n-1}; \Lambda^{-1} \widetilde{z}_\mu^- \omega_{n-1})$, $s_{n-1+j} = (\omega_j; \Lambda^{-1} \widetilde{z}_\mu^- \omega_j)$; $j = 1, \ldots, n-2$, $s_{2n-2} = (\omega_{n-1}; \Lambda^{-1} \widetilde{z}_\mu^- \omega_{n-1})$; $s_{2n-1} = (\omega_n; \Lambda^{-1} z^-_{2\mu+\lambda} \omega_n)$, $s_{2n} = (\omega_n; \Lambda^{-1} z^+_{2\mu+\lambda} \omega_n)$. Since $\mathcal{B} = (\Lambda^{-1} B_1, B_0)$ where

$$\begin{cases} B_0 = \mu Id + (\mu + \lambda) \ell_0 {}^t\ell_0 \\ B_1 = \mu \ell_1 {}^t\ell_0 + \lambda \ell_0 {}^t\ell_1 + i\tau \varphi'_{x_n} B_0 \end{cases} \tag{6.4}$$

from the definition of s_j it follows that $\widetilde{\mathcal{B}} = (b_1, \ldots, b_n)$ where

$$\begin{cases} b_j = \mu \Lambda^{-1} (i\alpha_\mu) \omega_j \ for\ j = 1, \ldots, n-2 \\ b_{n-1} = \mu \Lambda^{-1} [(i\alpha)^2 + (i\alpha_\mu)^2] \ell_1 + 2\mu \Lambda^{-1} (i\alpha_\mu)(i\alpha)^2 \ell_0 \\ b_{n-1+j} = -\mu \Lambda^{-1} (i\alpha_\mu) \omega_j \\ b_{2n-2} = \mu \Lambda^{-1} [(i\alpha)^2 - (i\alpha_\mu)^2] \ell_1 - (2\mu + \lambda)(i\alpha_\mu)(i\alpha_{2\mu+\lambda}) \ell_0 \\ b_{2n-1} = \Lambda^{-1} [-(2\mu + \lambda)(i\alpha_{2\mu+\lambda})^2 - \lambda (i\alpha)^2] \ell_0 \\ b_{2n} = 2\mu \Lambda^{-1} (i\alpha_{2\mu+\lambda}) \ell_1 + \Lambda^{-1} [-\lambda (i\alpha)^2 + (2\mu + \lambda)(i\alpha_{2\mu+\lambda})^2] \ell_0. \end{cases} \tag{6.5}$$

Then we have

$$\widetilde{\mathcal{B}} = (b_1, \ldots, b_{n-2}, b_{n-1}, -b_1, \ldots, -b_{n-2}, b_{2n-2}, b_{2n-1}, b_{2n}). \tag{6.6}$$

Let $C = (b_1, \ldots, b_{n-2}, b_{2n-1}, b_{2n})$, and define an elliptic pseudodifferential operator $C(x, D_{x'}, \tau)$ of order 0 with the principal symbol $C_0(x, \xi', \tau)$. In (6.2) we may assume that boundary operator is $\widetilde{\mathcal{B}}_0 = C^{-1}\widetilde{\mathcal{B}}$ then we have

$$\widetilde{\mathcal{B}}_0 = (e_1, \ldots, e_{n-2}, C_1, -e_1, \ldots, -e_{n-2}, C_2, e_{n-1}, e_n) \tag{6.7}$$

and

$$\begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in\ x_n > 0 \\ op(\widetilde{\mathcal{B}}_0)w = op(C^{-1})g = \widetilde{g} & on\ x_n = 0 \end{cases} \tag{6.8}$$

Now let $K_0 = (0, \ldots, 0, e_n)$ by (6.8) we have w satisfy the following system

$$\begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in\ x_n > 0 \\ w_n = \widetilde{g}_0 & on\ x_n = 0 \end{cases} \tag{6.9}$$

where

$$\begin{aligned} \widetilde{g}_0 &= op(K_0)\widetilde{g} - \sum_{j=1}^{n-1} \beta_j w_j \\ &= op(K_0)\widetilde{g} - op(\beta)\widetilde{w} \end{aligned} \tag{6.10}$$

$w = (\widetilde{w}, w_n)$ and $op(\beta)$ is a pseudodifferential operator of order 0.
Let $\mathcal{R} = (-\lambda I_{2n-1}, 0)$ and define the function $G(x_n)$ by

$$G(x_n) = \frac{d}{dx_n}(op(\mathcal{R})w, w) \tag{6.11}$$

Then we have by integration in the normal direction

$$(op(\mathcal{R})w, w)_0 = 2\int_0^\infty Im(op(\mathcal{R})op(\mathcal{H})w, w)dx_n + \int_0^\infty Im(op(\mathcal{R})w, \widetilde{F}) \tag{6.12}$$

Taking into account (6.9) and (6.11) we get

$$\tau|w|_0^2 \leq C||\widetilde{F}||^2 + \tau|g|^2 \tag{6.13}$$

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Finally the elliptic region $\mathcal{E}^-$ can be handled as $\mathcal{E}^+$, and we prove that we have a better estimates (without boundary terms). Combining proposition 3.2, lemma 5.1 and lemma 6.1 we have theorem 0.3.

Appendix A

In this section we study the outgoing resolvent in the real axis we prove there are no real poles. Let $z \in \mathbb{R}^*$, $u(x,z)$ solve the following problem $(\Delta_e + z^2)u = 0$ in Ω, $Bu = 0$ on $\partial\Omega$ and u is outgoing. Our purpose to prove that u is identically zero. Assume that $z > 0$ and let $a > 0$ such that the obstacle $\mathcal{O} \subset B(0,a)$ and $g = Id$ in ${}^cB(0,a)$ let $R \geq a$ then we have

$$\text{(A.1)} \qquad \int_{\Omega_R} (-E(u,\overline{u}) + z^2|u|^2)dx + \int_{S(0,R)} Bu \cdot \overline{u} d\sigma = 0.$$

Thus we have

$$\text{(A.2)} \qquad lim_{R\to\infty} \int_{S^2} Im(Bu \cdot \overline{u})(R,\sigma) R^2 d\sigma = 0.$$

As a paragraph 1 we set $u = u^{(p)} + u^{(s)}$ then we have

$$\begin{aligned}
\text{(A.3)} \int_{S^2} (Bu.\overline{u})R^2 d\sigma &= \int_{S^2} (Bu^{(p)} + Bu^{(s)})(\overline{u}^{(p)} + \overline{u}^{(s)})R^2 d\sigma \\
&= \int_{S^2} (Bu^{(p)} - i(2\mu+\lambda)k_1 u^{(p)})(\overline{u}^{(p)} + \overline{u}^{(s)})R^2 d\sigma \\
&\quad + \int_{S^2} (Bu^{(s)} - i\mu k_2 u^{(s)})(\overline{u}^{(p)} + \overline{u}^{(s)})R^2 d\sigma \\
&\quad + \int_{S^2} i(2\mu+\lambda)k_1 |u^{(p)}|^2 R^2 d\sigma + \int_{S^2} i\mu k_2 |u^{(s)}|^2 R^2 \\
&\quad + \int_{S^2} (i(2\mu+\lambda)k_1 u^{(p)}\overline{u}^{(s)} + i\mu k_2 u^{(s)}\overline{u}^{(p)})R^2.
\end{aligned}$$

Now using that u is outgoing then u satisfy the radiation condition. According the theorem 2.9 in Kupradze [Ku] (pp. 127) we obtain

$$\text{(A.4)} \qquad lim_{R\to\infty} \int_{S^2} (2\mu+\lambda)k_1 |u^{(p)}|^2 R^2 + \int_{S^2} \mu k_2 |u^{(s)}|^2 R^2 d\sigma = 0.$$

Then we have

$$\text{(A.5)} \qquad lim_{R\to\infty} \int_{S^2} |u^{(p)}|^2 R^2 d\sigma = lim_{R\to\infty} |u^{(s)}|^2 R^2 d\sigma = 0.$$

As is well known if $(\Delta + z^2)u = 0$ and u satisfy the radiation condition then $u \equiv 0$ for $z \in \mathbb{R}$ (see Burq [Bu], Morawetz [Mo]). This argument complete the proof.

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Stabilization of The Elastic Wave Equation

by

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Abstract

In this paper, we study the stabilization problem for the elastic Wave equation in a bounded domain in two different situations. The boundary stabilization is considered with dissipative boundary condition are introduced and the internal stabilization problem. Providing regular initial data we prove, without any assumption on the dynamics, that the energy decay at last the logarithm. In order to prove that result we bound from below the spectrum of the infinitesimal generator of associated semi-group.

Key words. energy decay, Elastic wave equation.
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1 Introduction and Main Results

We are mainly interested in the decaying mode of the energy (stabilization) of the solution of the initial boundary value problem in a connected and compact manifold with compact boundary for the elastic wave equation as time tends to infinity. The goal of this paper is to study the stabilization of the elastic wave equations. We will consider both the boundary stabilization problems with a boundary damping term supported in the boundary or the internal stabilization with a damping term supported in the interior of the domain. It is well known (see Lagnese [L]) that the energy decreases to zero when the dissipation is on a non empty set. Since the works of Bardos-Lebeau-Rauch [BLR], we have a necessary and sufficient condition

which guarantee an exponentially decay rate of the energy for any initial data for general second order scalar boundary value problems. Since singularities propagate along generalized bicharacteristic rays, this can be done whenever a bicharacteristic hits the part which can be controlled. Taylor [T1] gives a rigorous treatment of the singularity for the elastic wave equation with Neumann boundary condition, he prove there are three types of rays that carry singularities. The first types are classical rays reflecting at the boundary according to the lows of geometrical optics. The third type of rays lie on the boundary and singularities propagate along them with a slower propagation speed $C_R > 0$ (the Rayleigh speed). In the present paper we show that even if the geometrical control condition is not fulfilled (i.e. without any assumption on the dynamics) then the energy decays with respect to time at least as fast as the inverse of the logarithm, providing the initial data belong to the domain of A^k (A stays for the infinitesimal generator of the evolution equation). In order to prove that result we bound from below the spectrum of A. This bound is obtained by using a Carleman type estimate for the resolvent of A.
Let (M, g) be a Riemannian, compact manifold with smooth and compact boundary ∂M. We set

$$L(x,D) = \mu\Delta + (\mu+\lambda)\nabla(\mathrm{div}\,.) = \Delta_e \tag{1.1}$$

where μ and λ a real constant satisfied $2\mu + 3\lambda > 0$, and Δ, ∇ and div are the Laplacian, gradian and divergence operator associated to metric g. We consider two classical examples of the elastic wave equation. The first damped by a boundary scalar velocity feedback $a_0 \in C^\infty(\partial M)$ and $a_0 \geq 0$

$$\begin{cases} \partial_t^2 u - L(x,D)u = 0 & in\ M\times\mathbb{R} \\ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1 & in\ M \\ Nu(x,t) + a_0(x)\partial_t u = 0 & on\ \partial M\times\mathbb{R} \end{cases} \tag{1.2}$$

The second damped system by a internal matrix feedback $a \in C^\infty(M)$ satisfies for any $z \in \mathbb{C}$, $a(x)z\bar{z} \geq 0$ and the set $\left\{x \in M;\ a(x)z\bar{z} \geq \delta|z|^2\right\}$ is non empty

$$\begin{cases} \partial_t^2 u - L(x,D)u + a(x)\partial_t u = 0 & in\ M\times\mathbb{R} \\ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1 & in\ M \\ Bu(x,t) = 0 & on\ \partial M\times\mathbb{R} \end{cases} \tag{1.3}$$

Here the boundary operator is of the form

$$B(x,D)u = \begin{cases} u & (Dirichlet\ Condition) \\ Nu = \sigma(u).n & (Neumann\ Condition) \end{cases} \tag{1.4}$$

Where $n(x)$ is the unit outer normal to M at $x \in \partial M$. In the above

$$\sigma(u) = \lambda(\mathrm{div}\,u)Id + 2\mu\varepsilon(u) \tag{1.5}$$

is the stress tensor where $\varepsilon(u) = (\nabla u + {}^t\nabla u)$ the strain tensor.
For $(u_0, u_1) \in H = H^1 \oplus L^2$ we define the energy for the solution of (1.2) or (1.3) as

$$E(u,t) = \frac{1}{2}\int_M |\partial_t u|^2 + \lambda|\operatorname{div} u|^2 + \mu \sum |\varepsilon_{ij}(u)|^2 dx \tag{1.6}$$

and let

$$A_0 = \begin{pmatrix} 0 & -iId \\ -i\Delta_e & 0 \end{pmatrix}. \tag{1.7}$$

Then the system (1.3) is transformed into

$$\frac{dv(t)}{dt} = iA_0 v(t), \quad v(0) = v_0, \quad v = {}^t(u(t,x), \partial_t u(t,x)) \tag{1.8}$$

The domain of A_0 is defined by

$$D(A_0) = \Big\{ u = (u_0, u_1) \in H / A_0 u \in H, \;\; N(u_0) + a_0 u_1 = 0 \; in \; \partial M \Big\}. \tag{1.9}$$

Then, A_0 is a closed operator in H; furthermore, A_0 is skew self-adjoint in H and the immersion $D(A_0) \hookrightarrow H$ is compact. Furthermore $Imz < 0$ is in the set resolvent of A_0 and the spectrum $Sp(A_0)$ of A_0 is discrete.
We can see easily by integration by parts

$$E(u,t) - E(u,0) = -\int_0^t \int_{\partial M} a_0(x) |\partial_s u(x,s)|^2 d\sigma ds \tag{1.10}$$

Now we can able to state our first results

Theorem 1.1 *There exist* $C > 0$ *such that if* $z \in Sp(A_0) \backslash \{0\}$ *then we have* $Im(z) > e^{-C|Rez|}$ *and for* $Imz \leq e^{-C|Rez|}$, $|z| > 1$ *we have*

$$||(A_0 - z)^{-1}||_{\mathcal{L}(H,H)} \leq Ce^{C|Rez|}. \tag{1.11}$$

Theorem 1.2 *For any* $k > 0$ *There exist* $C > 0$ *such that for any initial data* $(u_0, u_1) \in D(A_0^k)$ *the solution* $u(t,x)$, *of* (1.2) *satisfy*

$$E(u,t) \leq \frac{C}{Ln(2+t)^{2k}} \Big|\Big|(u_0, u_1)\Big|\Big|^2_{D(A_0^k)}. \tag{1.12}$$

Let us introduce the operator A defined as

$$A = \begin{pmatrix} 0 & -iId \\ -i\Delta_e & ia \end{pmatrix} \tag{1.13}$$

and let $H^1_B = \Big\{ (u_0, u_1) \in H^1; \; Bu_0 = 0 \; on \, \partial\Omega \Big\}$. The domain of A is defined by $D(A) = (H^1_B \cap H^2) \oplus L^2$. Then we have $(A - z)$ is bijective from $D(A)$ onto H

for z in $\Big\{z \in \mathbb{C}\ ; Imz \notin [0, ||a||_{L^\infty}]\Big\}$. The immersion $D(A) \hookrightarrow H$ is compact, the spectrum of A is constituted a sequence z_j such that $Im(z_j) \in [0, ||a||_{L^\infty}]$.
By integration by parts we have

$$E(u,t) - E(u,0) = -\int_0^t \int_M a(x)\partial_s u(x,s)\overline{\partial_s u(x,s)}dxds \tag{1.14}$$

similarly we have the following Theorems

Theorem 1.3 *There exists* $C > 0$ *such that if* $z \in Sp(A)\backslash\{0\}$ *then we have* $Im(z) > e^{-C|Rez|}$ *and for* $Imz \leq e^{-C|Rez|}$, $|z| > 1$ *we have*

$$||(A - z)^{-1}||_{\mathcal{L}(H,H)} \leq Ce^{C|Rez|}. \tag{1.15}$$

Theorem 1.4 *For any* $k > 0$ *There exists* $C > 0$ *such that for any initial data* $(u_0, u_1) \in D(A^k)$ *the solution* $u(t,x)$, *of* (1.3) *satisfy*

$$E(u,t) \leq \frac{C}{Ln(2+t)^{2k}}\Big|\Big|(u_0, u_1)\Big|\Big|^2_{D(A^k)}. \tag{1.16}$$

Remarks 1.1

i) Theorems 1.1 and 1.3 are the analogous of Lebeau *[L] and* Lebeau-Robbiano *[LR] results in the case of the scalar wave equation. Our method is different and consist to use the Carleman estimate directly for the stationary operator without using the interpolation inequality.*

ii) Theorems 1.2 and 1.4 follows from Theorem 1.1 and 1.3, this is proved in the general case by Burq *(see Theorem 3 of [Bu])*

iii) Theorems 1.1 and 1.3 are optimal if we do not assume condition on the dynamic. We can even show that Theorem 0.1 is optimal under reasonable geometric framework, and the damped term $a(x)$ *is positive on the whole domain* M. *Indeed, in this case the Railegh's rays on the boundary never hits the damped term (see* Kawashita *[K])*

This paper is organized as follow, in section 2 we gives a Carleman inequality adapted to our case, in section 3 we prove our results and the prove of Carleman estimate is in the section 4.

Finally I would like to thank Professor L.Robbiano for useful discussion during the preparation of this work.

2 Carleman Estimates

To prove Theorem 1.1 and Theorem 1.3, we use Carleman estimates to obtain information about the resolvent in bounded domain. The cost is to use phases functions satisfying Hörmander's (see [H]) assumption and thus growing fast.
The point is to show estimations in a bounded domain Ω for the solutions of

$$(2.1) \qquad \begin{cases} P(x,D)u = f & in\ \Omega \\ B(x,D)u = g & on\ \partial\Omega \end{cases}$$

where $P(x,D)$ the differential operator with principal symbol

$$(2.2) \qquad P(x,\xi) = \mu^t\xi.\xi + (\mu+\lambda)\xi^t\xi - \tau^2 Id$$

and the boundary operator

$$(2.3) \qquad B(x,D)u = \begin{cases} u & (Dirichlet\ Condition) \\ \sigma(u).n + ia_0\tau u & (Neumann\ Condition) \end{cases}$$

Let $\varphi(x)$ a real function in $C^\infty(\mathbb{R}^n)$ we define the operator

$$(2.4) \qquad P(x,D,\tau) = e^{\tau\varphi}P(x,D)e^{-\tau\varphi}$$

with principal symbol given by

$$(2.5) \qquad P(x,\xi,\tau) = P(x,\xi + i\tau\varphi').$$

We introduce the scalar partial differential operator

$$(2.6) \qquad a_\gamma(x,D) = \Delta + \frac{\tau^2}{\gamma}, \qquad \gamma \in \{\mu, 2\mu+\lambda\},$$

and we define $a_\gamma(x,D,\tau)$ by $e^{\tau\varphi}a_\gamma(x,D)e^{-\tau\varphi}$ with principal symbol

$$(2.7) \qquad a_\gamma(x,\xi,\tau) = a_\gamma(x,\xi+i\tau\varphi').$$

We assume that φ satisfies Hörmander's assumption for the operator a_γ

$$(2.8)\ \exists\ C\ \forall x \in M,\ \forall\, \xi \in T^*M\backslash\{0\}; \quad a_\gamma(x,\xi;\tau) = 0 \Longrightarrow \Big\{Rea_\gamma; Ima_\gamma\Big\} > C,$$

where $\{,\}$ denote the Poisson brackets, Furthermore we assume that

$$(2.9) \qquad \nabla\varphi \neq 0, \quad in\ \overline{M} \quad and \quad \partial_n\varphi \neq 0 \quad on\ \partial M.$$

Finally we set $B(x,D,\tau) = e^{\tau\varphi}B(x,D)e^{-\tau\varphi}$. Then we have the following Carleman type estimates

Proposition 2.1 *Let φ satisfy* (2.8) *and* (2.9) *there exists $C>0$, and τ_0 such that for any $u\in C^\infty(\overline{M})$ we have*

$$(2.10)\ \int_M |P(x,D,\tau)u|^2+\tau\int_{\partial M}(\tau^2|u|^2+|\nabla u|^2)d\sigma\geq C\tau\int_M(\tau^2|u|^2+|\nabla u|^2)$$

for large $\tau>\tau_0$.

Proposition 2.2 *Furthermore if we assume that $\partial_n\varphi<-C_0$ (where $C_0>0$ large constant fixed in section 4) in $\Sigma\subset\partial M$ and $Bu=g$ on Σ, then there exist $C>0$ and τ_0 such that for any $u\in C^\infty(\overline{M})$ we have*

$$(2.11)\ \int_M |P(x,D,\tau)u|^2\ +\tau\int_{\partial M\backslash\Sigma}\tau^2|u|^2+\tau\int_\Sigma|g|^2d\sigma+ \\ \tau\int_{\partial M\backslash\Sigma}|B(x,D,\tau)u|^2d\sigma\geq C\tau\int_M(\tau^2|u|^2+|\nabla u|^2)$$

for large enough $\tau>\tau_0$.

2.1 Construction and properties of the phases functions.

The purpose of this section is to construct two phases φ_1, φ_2 which satisfy the Hörmander's assumption, excepting in a finite number of balls, such that on a ball where one of them do not satisfies these conditions the second does and is strictly greater.

Proposition 2.3 *(see* Burq *[Bu] and [Be]) Let Σ_1 and Σ_2 two non empty parts of the boundary ∂M such that $\Sigma_1\cup\Sigma_2=\partial M$. There exist two functions $\psi_{1,2}\in C^\infty(\bar{M})$ satisfying $\partial_n\psi_{i|\Sigma_1}<0$ and $\partial_n\psi_{i|\Sigma_2}>0$, having only no degenerate critical points, such that when $\nabla\psi_i=0$ then $\nabla\psi_{i+1}\neq0$ and $\psi_{i+1}>\psi_i$ (where $\psi_3=\psi_1$).*

Then there exist a finite number of points $x_{ij}\in\Omega$; $i=1,2$ $j=1,...,N_i$ and $\varepsilon>0$ such that $B(x_{ij},2\varepsilon)\subset\Omega$, $B(x_{1j},2\varepsilon)\cap B(x_{2j},2\varepsilon)=\emptyset$ and $\psi_{i+1}>\psi_i$ (where $\psi_3=\psi_1$). Denote $\Omega_i=\Omega\cap(\cup_jB(x_{ij},\varepsilon))^c$, and we take $\varphi_i=e^{\beta\psi_i}$ then for large β the phases φ_i satisfy Hörmander assumption.

3 Proof of Mains Results.

3.1 Proof of Theorem 1.3

We prove here the existence and estimation of the norm of the resolvent $(A-z)^{-1}$ for $z\in\left\{z\in\mathbb{C};0<Im(z)<\frac{1}{C}e^{-C|Rez|};\ |z|>1\right\}$.

Let B_{4r} a ball of radius $4r > 0$, in M such that $a(x) > 0$ in B_{4r} we note $\Omega_0 = M\backslash\overline{B_r}$. Let $f = (f_0, f_1) \in H$ and $u = (u_0, u_1) \in D(A)$ such that $(A - z)u = f$ then we have

$$\left\{ \begin{array}{l} -zu_0 - iu_1 = f_0 \\ -i\Delta_e u_0 + ia(x)u_1 - zu_1 = f_1 \end{array} \right. \tag{3.1}$$

then we get

$$\left\{ \begin{array}{l} (\Delta_e + z^2 - iaz)u_0 = g_0 \\ u_1 = if_0 + izu_0. \end{array} \right. \tag{3.2}$$

where $g_0 = if_1 + iaf_0 - zf_0$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that χ supported in B_{2r}^c, equal to 1 in B_{3r}^c, denote $v_0 = \chi u_0$, then we get

$$(\Delta_e + z^2 - iaz)v_0 = \chi g_0 + [\Delta_e, \chi]u_0 := \widetilde{g}_0 \tag{3.3}$$

and $v_0 \equiv 0$ on ∂B_r. Now let $\Sigma_1 = \partial M$ and $\Sigma_2 = \partial B_r$ thus $\partial\Omega_0 = \Sigma_1 \cup \Sigma_2$. Let χ_i, $i = 1, 2$, two cutoff functions equal to 1 in $\left(\bigcup_j B(x_{ij}, 2\varepsilon)\right)^c$ and supported in $\left(\bigcup_j B(x_{ij}, \varepsilon)\right)^c$. We applique Proposition 2.2 to the function $e^{\tau\varphi_i}\chi_i u_0$ in a domain $\Omega_i = \Omega_0 \cap \left(\bigcup_j B(x_{ij}, \varepsilon)\right)^c$ we get, for $\tau^2 = Re(z^2)$, and

$$g = [\Delta_e, \chi]e^{\tau\varphi}\chi v_0 + \chi\widetilde{g}_0 - iIm(z^2)v_0 \tag{3.4}$$

the following estimate

$$\int_{\Omega_i} e^{2\tau\varphi_i}|g|^2 \geq C\tau \int_{\Omega_i} (\tau^2|\chi_i v_0|^2 + |\nabla(\chi_i v_0)|^2)e^{2\tau\varphi_i}. \tag{3.5}$$

Using the properties of the phases $\varphi_i > \varphi_{i+1}$ in $\left(\bigcup_j B(x_{i,j+1}, 2\varepsilon)\right)$ then we have for large τ

$$\begin{aligned} \int_{\Omega_0} & (e^{2\tau\varphi_1} + e^{2\tau\varphi_2})(|\widetilde{g}|^2 + \\ & +2|Imz||Rez||v_0|^2) \geq C\tau \int_{\Omega_0} (\tau^2|v_0|^2 + |\nabla v_0|^2)(e^{2\tau\varphi_1} + e^{2\tau\varphi_2}), \end{aligned} \tag{3.6}$$

and for $|Imz| \leq C$ we can have

$$\begin{aligned} \int_M |u_0|^2 + |\nabla u_0|^2 &\leq Ce^{C\tau}\Big(\int_M (|g|^2 + \int_{B_{4r}} |u_0|^2 + \int_{\Omega_0} |[\Delta_e, \chi]u_0|^2\Big) \\ &\leq Ce^{C\tau}\Big(|g|^2 + \int_M a(x)u_0.\overline{u}_0 + \int_{\Omega_0} |[\Delta_e, \chi]u_0|^2\Big) \end{aligned} \tag{3.7}$$

We set $\widetilde{\chi}$ a cutoff function equal to 1 in a neighborhood of B_{3r} and supported in B_{4r} then we have

$$(-1 + \Delta_e)\chi u_0 = \widetilde{\chi}(z^2 - 1 + iazu_0) + \widetilde{\chi}g_0 + [\Delta_e, \widetilde{\chi}]u_0 \tag{3.8}$$

and hence we get

$$\begin{aligned} ||u_0||^2_{H^1(B_{3r})} &\le C||(-1+\Delta_e)\widetilde{\chi}u_0||^2_{H^{-1}} \\ &\le C\Big(||g_0||^2_{L^2} + (1+|z|^2)^2||u_0||^2_{L^2(B_{4r})}\Big). \end{aligned} \tag{3.9}$$

On the other hand integrating by parts gives

$$\begin{aligned} \int_M g_0\overline{u}_0 &= \int_M (\Delta_e + z^2 - iaz)u_0\overline{u}_0 \\ &= \int_M z^2|u_0|^2 + \int_M E(u_0,\overline{u}_0) - 2iz\int_M a(x)u_0\overline{u}_0 \end{aligned} \tag{3.10}$$

This implies

$$\tau\int_M a(x)u_0\overline{u}_0 \le 2|Imz||Rez|||u_0||^2_{L^2} + ||g_0||_{L^2}||u_0||_{L^2} \tag{3.11}$$

Combining (3.11), (3.9), and (3.7), we obtain

$$||u_0||^2_{H^1_0} \le Ce^{C\tau}\Big(||g_0||^2_{L^2} + ||g_0||_{L^2}||u_0||_{L^2} + |Imz||Rez|||u_0||^2_{L^2}\Big) \tag{3.12}$$

these imply that for $|Imz| \le \frac{1}{2C}e^{-C\tau}$

$$||u_0||^2_{H^1_0} \le Ce^{C\tau}\Big(||f_0||^2_{H^1_0} + ||f_1||^2_{L^2}\Big). \tag{3.13}$$

Using (3.2) we can get also

$$||u_1||^2_{L^2} \le Ce^{C\tau}\Big(||f_0||^2_{H^1_0} + ||f_1||^2_{L^2}\Big). \tag{3.14}$$

Which completes the proof.

3.2 Proof of Theorem 1.1

Let $f = (f_0, f_1) \in H$, $u \in D(A_0)$ then $u = (u_0, u_1)$ such that $N(u_0) + a_0u_1 = 0$ on ∂M. The equation $(A_0 - z)u = f$ is equivalent to

$$\begin{cases} -zu_0 - iu_1 = f_0 \\ -i\Delta_e u_0 - zu_1 = f_1 & in\ \ M \\ N(u_0) + a_0u_1 = 0 & on\ \ \partial M \end{cases} \tag{3.15}$$

then we have

$$\begin{cases} (\Delta_e + z^2)u_0 = \widetilde{g}_1 & in\ \ M \\ N(u_0) + a_0u_0 = \widetilde{g}_0 & on\ \ \partial M \\ u_1 = if_0 + izu_0. \end{cases} \tag{3.16}$$

where $\widetilde{g}_0 = -ia_0 f_0$. Let $\Sigma_2 \subset \Big\{x \in \partial M; a_0(x) > \delta\Big\}$ and $\Sigma_1 = \partial M \backslash \Sigma_2$, and χ_i, $i = 1, 2$, two cutoff functions equal to 1 in $\Big(\bigcup_j B(x_{ij}, 2\varepsilon)\Big)^c$ and supported in $\Big(\bigcup_j B(x_{ij}, \varepsilon)\Big)^c$. We applique Proposition 2.2 to the function $e^{\tau\varphi_i}\chi_i u_0$ in a domain $\Omega_i = M \cap \Big(\bigcup_j B(x_{ij}, \varepsilon)\Big)$ we get for $\tau^2 = Re(z^2)$ and

$$\left\{\begin{array}{l} g_i = [\Delta_e, \chi_i] e^{\tau\varphi_i}\chi_i u_0 + \chi_i \widetilde{g}_1 - iIm(z^2) u_0 \\ \\ N(u_0) + ia_0\tau u_0 = \widetilde{g}_0 + Im(z) a_0 u_0 := h_0 \quad on \ \partial M \end{array}\right. \tag{3.17}$$

the following inequality

$$\begin{aligned} &\int_{\Omega_i} (e^{\tau\varphi_1} + e^{\tau\varphi_2})(|\widetilde{g}_1|^2 + |Im(z^2)u_0|^2) + \tau \int_{\Sigma_1} (e^{\tau\varphi_1} + e^{\tau\varphi_2})|h_0|^2 + \\ &\qquad + \tau \int_{\Sigma_2} \tau^2 |u_0|^2 (e^{\tau\varphi_1} + e^{\tau\varphi_2}) \geq C\tau \int_{\Omega_i} (\tau^2|u_0|^2 + |\nabla u_0|^2)(e^{\tau\varphi_1} + e^{\tau\varphi_2}). \end{aligned} \tag{3.18}$$

Using the properties of the phases $\varphi_i > \varphi_{i+1}$ in $\Big(\bigcup_j B(x_{i,j+1}, 2\varepsilon)\Big)$ and $Bu_0 + ia_0\tau u_0 = \widetilde{g}_0 + Im(z) a_0 u_0$ we have for large τ

$$\begin{aligned} \int_M |u_0|^2 + |\nabla u_0|^2 &\leq Ce^{C\tau}\Big(||\widetilde{g}_1||^2 + ||f_0||^2 + \int_{\Sigma_2} |u_0|^2\Big) \\ &\leq Ce^{C\tau}\Big(||\widetilde{g}_1||^2_{L^2} + ||f_0||^2_{H^1} + \int_{\partial M} a_0(x)|u_0|^2\Big) \end{aligned} \tag{3.19}$$

On the other hand, by integration by part we get

$$\begin{aligned} \int_M \widetilde{g}_1 \overline{u}_0 &= \int_M (\Delta_e + z^2) u_0 \overline{u}_0 \\ &= z^2 \int_M |u_0|^2 + \int_M E(u_0, \overline{u}_0) + \int_{\partial M} Nu_0\overline{u}_0 + z^2||u_0||^2_{L^2} + \\ &\quad + \int_M E(u_0, \overline{u}_0) - iz \int_{\partial M} a_0(x)|u_0|^2 + \int_{\partial M} \widetilde{g}_0.\overline{u}_0. \end{aligned} \tag{3.20}$$

Then we have

$$\tau \int_{\partial M} a_0(x)|u_0|^2 d\sigma \leq \Big(||\widetilde{g}_1||_{L^2}||u_0||_{L^2} + 2|Imz||Rez|||u_0||^2_{L^2} + \\ + ||g_0||_{H^1}||u_0||_{H^1}\Big). \tag{3.21}$$

Combining (3.21) with(3.19) we obtain

$$||u_0||^2_{H^1} \leq Ce^{C\tau}\Big(||\widetilde{g}_1||^2_{L^2} + ||f_0||^2_{H^1} + ||\widetilde{g}_1||_{L^2}||u_0||_{L^2} + ||g_0||_{H^1}||u_0||_{H^1} + 2|Imz||Rez|||u_0||^2_{L^2}\Big). \tag{3.22}$$

Now assume that $|Imz| \leq \frac{1}{C}e^{-C\tau}$ then we have by the previous estimate

$$||u_0||^2_{H^1} \leq Ce^{C\tau}\Big(||f_0||^2_{H^1} + ||f_1||^2_{L^2}\Big). \tag{3.23}$$

Using (3.16) we can get

$$\begin{aligned} ||u_1||^2_{L^2} &\leq ||f_1||^2_{L^2} + |z|^2||u_0||^2_{L^2} \\ &\leq Ce^{C\tau}||f||^2_H \end{aligned} \tag{3.24}$$

these imply that

$$||u||^2 \leq Ce^{C|Rez|}||(A_0 - z)||^2 \tag{3.25}$$

then $(A_0 - z)$ is injection then bijective in $D(A_0)$ and we get

$$||(A_0 - z)^{-1}||_{\mathcal{L}(H,H)} \leq Ce^{-C|Rez|} \tag{3.26}$$

for any $z \in \Big\{z \in \mathbb{C}, \quad |Imz| < \frac{1}{C}e^{-C|Rez|}; \quad |Rez| > 1\Big\}$. The proof is complete.

4 Proof of Carleman Estimates

This section is devoted to prove estimates of Carleman's type near the boundary for solutions to boundary value problem of the form

$$\begin{cases} P(x, D)u = f & in\ \Omega_0 \\ B(x, D)u = g & on\ \partial\Omega_0 \end{cases} \tag{4.1}$$

Where $P(x, D)$ a partial differential operator with principal symbol given by

$$P(x, \xi) = \mu|\xi|^2 + (\mu + \lambda)\xi^t\xi - \tau^2 Id \tag{4.2}$$

and $B(x, D)u = Nu + ia_0\tau u$. For Dirichlet condition or $Bu = Nu$ the proposition 2.2 is proved in [Be] (see Theorem 0.3)

4.1 Reduction of the Problems.

4.1.1 Reduction of the Laplacian.

Let Ω_0 be a bounded smooth domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^∞. In a neighborhood of a given $x_0 \in \partial\Omega_0$ we denote by $x = (x', x_n)$ the system of normal geodesic coordinates where $x' \in \partial\Omega_0$ and $x_n \in \mathbb{R}$ are characterized by

$$|x_n| = dist(x, \partial\Omega_0); \quad \Omega_0 = \{x_n > 0\}; \quad dist(x', x) = dist(x, \partial\Omega_0). \tag{4.3}$$

In this system of coordinates there exist $\ell(x,\xi)$ such that

$$\ell(x,\xi)=\ell_0(x)\xi_n+\ell_1(x,\xi') \tag{4.4}$$

and the principal symbol of Laplace operator takes the form

$$\widetilde{a}={}^t\ell(x,\xi)\ell(x,\xi)=\xi_n^2+r(x,\xi') \tag{4.5}$$

where $r(x,\xi')$ is a quadratic form, such that there exist $C>0$

$$r(x,\xi')\geq C|\xi'|^2, \qquad for\ any\ x\in K,\ \ \xi'\in T^*(\partial\Omega_0) \tag{4.6}$$

where K is a fixed compact in $\overline{\Omega}_0$ finally we have

$$ {}^t\ell_1\ell_1=r(x,\xi'),\ \ {}^t\ell_0\ell_1=0 \tag{4.7}$$

Let $\varphi(x)$ be a $C^\infty(\mathbb{R}^n)$ function with values in $\mathbb{R}$, defined in a neighborhood of K. We define the operator

$$a(x,D,\tau)=e^{\tau\varphi}\widetilde{a}(x,D)e^{-\tau\varphi}:=op(a) \tag{4.8}$$

Denote by

$$a(x,\xi,\tau)=a(x,\xi+i\tau\varphi'). \tag{4.9}$$

the principal symbol of the operator, and we set

$$op(\widetilde{q}_2)=\frac{1}{2}(op(a)+op(a)^*);\qquad op(\widetilde{q}_1)=\frac{1}{2i}(op(a)-op(a)^*), \tag{4.10}$$

its real and imaginary part. Then we have

$$\left\{\begin{array}{l} op(a)=op(\widetilde{q}_2)+iop(\widetilde{q}_1)\\ \widetilde{q}_2=\xi_n^2+q_2(x,\xi',\tau);\qquad \widetilde{q}_1=2i\tau\xi_n\varphi'_{x_n}+2q_1(x,\xi',\tau).\end{array}\right. \tag{4.11}$$

Where q_1 and q_2 a tangential symbols of order respectively 1 and 2 given by

$$\left\{\begin{array}{l} q_2(x,\xi,\tau)=r(x,\xi')-\tau^2{\varphi'_{x_n}}^2-\tau^2r(x,\varphi'_{x'})\\ q_1(x,\xi,\tau)=\widetilde{r}(x,\xi',\tau\varphi'_{x'})\end{array}\right. \tag{4.12}$$

and $\widetilde{r}(x,\xi',\eta')$ the bilinear form attached to the quadratic form $r(x,\xi')$.

4.1.2 Reduction of the elasticity system

In the system of normal geodesic coordinates the principal symbol of elasticity operator can be written as

$$L(x,\xi)=\mu\,{}^t\ell(x,\xi)\ell(x,\xi)Id+(\lambda+\mu)\ell(x,\xi)\,{}^t\ell(x,\xi) \tag{4.13}$$

where $\ell(x,\xi)$ defined by (4.5) and $\ell(x,\xi)^t\ell(x,\xi)$ the orthogonal projection onto the space spanned by $\ell(x,\xi)$. The principal symbol of $P(x,D,\tau) = e^{\tau\varphi}L(x,D)e^{-\tau\varphi} - \tau^2 Id$ is given by

$$P(x,\xi,\tau) = \sum_{j=0}^{2} P_{2-j}(x,\xi',\tau)\xi_n^j \tag{4.14}$$

where $P_j(x,\xi',\tau)$ a tangential symbols in $S^j(\mathbb{R}^n \times \mathbb{R}^{n-1})$ and defined by

$$\begin{cases} P_0 = \mu Id + (\mu+\lambda)\ell_0^t\ell_0 \\ \\ P_1 = 2i\tau\varphi'_{x_n}P_0 + (\mu+\lambda)(\ell_0^t\ell_1 + \ell_1^t\ell_0) \\ \\ P_2 = \mu\ell_1^t\ell_1 + i\tau\varphi'_{x_n}P_1(x,\xi) - \tau^2\varphi'^2_{x_n}P_0 + (\mu+\lambda)\ell_1^t\ell_1 - \tau^2 Id. \end{cases} \tag{4.15}$$

For fixed $(x,\xi') \in T^*(\partial\Omega_0)$ let $\alpha(x,\xi',\tau) \in \mathbb{C}$ such that

$$a(x,\xi,\tau) = (\xi_n + i\tau\varphi'_{x_n} + i\alpha)(\xi_n + i\tau\varphi'_{x_n} - i\alpha) \tag{4.16}$$

then we have also by

$$r(x,\xi' + i\tau\varphi'_{x'}) = -(i\alpha)^2 = {}^t\ell(x,\xi' + i\tau\varphi'_{x'})\ell(x,\xi' + i\tau\varphi'_{x'}). \tag{4.17}$$

the determinant of $P(x,\xi,\tau)$ is given by

$$det(P(x,\xi,\tau)) = \mu^{n-1}(2\mu+\lambda)a_\mu(x,\xi,\tau)a_{2\mu+\lambda}(x,\xi,\tau). \tag{4.18}$$

where $a_\gamma(x,\xi,\tau) = a(x,\xi,\tau) - \frac{\tau^2}{\gamma}$.

4.2 Study of the eigenvalues

The proof of Carleman's estimate rely on a cutting argument based on the nature of the roots with respect ξ_n of $a_\gamma(x,\xi',\xi_n,\tau)$.
Let use now introduce the following micro-local regions:

$$E^+ = \left\{(x,\xi',\tau) \in K \times S^{n-1};\ q_2 - \frac{\tau^2}{\mu} + \frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2} > 0\right\}, \tag{i}$$

$$Z_\gamma = \left\{(x,\xi',\tau) \in K \times S^{n-1}; q_2 - \frac{\tau^2}{\gamma} + \frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2} = 0\right\}, \tag{ii}$$

$$E^- = \left\{(x,\xi',\tau) \in K \times S^{n-1};\ q_2 - \frac{\tau^2}{2\mu+\lambda} + \frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2} < 0\right\}, \tag{iii}$$

(iv) $M = \left\{ (x, \xi', \tau) \in K \times S^{n-1};\ q_2 - \dfrac{\tau^2}{\mu} + \dfrac{q_1^2}{\tau^2 {\varphi'_{x_n}}^2} < 0 < q_2 - \dfrac{\tau^2}{2\mu + \lambda} + \dfrac{q_1^2}{\tau^2 {\varphi'_{x_n}}^2} \right\}.$

And for fixed (x, ξ', τ) let $\alpha_\gamma(x, \xi', \tau) \in \mathbb{C}$ such that

$$\begin{aligned} (4.19) \qquad a_\gamma(x, \xi, \tau) &= a(x, \xi, \tau) - \frac{\tau^2}{\gamma} \\ &= (\xi_n + i\tau\varphi'_{x_n} + i\alpha_\gamma)(\xi_n + i\tau\varphi'_{x_n} - i\alpha_\gamma). \end{aligned}$$

Taking into account (4.16) (4.23) we have

$$(4.20) \qquad (i\alpha_\gamma)^2 = (i\alpha)^2 + \frac{\tau^2}{\gamma}.$$

then we get by (4.12) and (4.17)

$$(4.21) \qquad \alpha_\gamma^2 = \tau^2 {\varphi'_{x_n}}^2 + q_2 - \frac{\tau^2}{\gamma} + 2i\tau q_1$$

For a fixed (x, ξ', τ) decompose $a_\gamma(x, \xi, \tau)$ as polynomial in ξ_n then we have the following Lemma

Lemma 4.1

1) For any $(x, \xi', \tau) \in E^+$ the roots of a_μ and $a_{2\mu+\lambda}$ denote by $z_1^\pm, z_2^\pm$ satisfying $\pm Im z_k^\pm > 0$.

2) For any $(x, \xi', \tau) \in Z_\gamma$, one of the roots of a_γ is real.

3) For any $(x, \xi', \tau) \in E^-$ the roots of a_γ for $\gamma \in \{\mu, 2\mu + \lambda\}$ has in the upper half-plane if $\varphi'_{x_n} < 0$ and in the lower half-plane if $\varphi'_{x_n} > 0$.

4) For any $(x, \xi', \tau) \in M$ the roots of $a_{2\mu+\lambda}$ satisfying $\pm Im z_2^\pm > 0$ and the roots of a_μ satisfy 3).

Proof . The imaginary parts of the roots of a_γ are

$$(4.22) \qquad -\tau\varphi'_{x_n} \pm Re(\alpha_\gamma)$$

but the lines $Re(z) = \pm\tau\varphi'_{x_n}$ can be transformed by $z \to z' = z^2$ in the parabola defined by $Re z' = \tau^2 {\varphi'_{x_n}}^2 - \frac{|Im z'|^2}{4{\varphi'_{x_n}}^2}$. Then the Lemma is proved if we change z' by α_γ^2 and using (4.21) •

4.3 Proof of the Carleman estimate in the region E^+

The purpose of this section is to proof the Carleman estimate in the region E^+. First let us introduce the following norms in the Sobolev spaces $H^k(\Omega)$ and $H^k(\partial\Omega)$

$$||u||_{k,\tau}^2 = \sum_{j=0}^{k} \tau^{2(k-j)} ||u||_{H^j(\Omega)}^2 \qquad |u|_{k,\tau}^2 = \sum_{j=0}^{k} \tau^{2(k-j)} |u|_{H^j(\partial\Omega)}^2 \tag{4.23}$$

and we set

$$|u|_{1,0,\tau}^2 = |u|_{1,\tau}^2 + \left|\frac{\partial u}{\partial n}\right|_{L^2}^2. \tag{4.24}$$

Let $\chi(x,\xi',\tau)$ a homogeneous cutoff function of degree 0 in the region E^+.

Proposition 4.1 *There exist* $C_0 > 0$ *and* $C_1 >$ *such that if* $\varphi'_{x_n} > C_0$ *for any large enough* τ *we have*

$$\begin{aligned} ||P(x,D,\tau)u||^2 + \tau|B(x,D,\tau)u|_{0,\tau}^2 &+ \\ + ||u||_{1,\tau}^2 + |u|_{1,\tau}^2 &\geq C_1\tau(||op(\chi)u||_{1,\tau}^2 + \tau|op(\chi)u|_{1,\tau}^2) \end{aligned} \tag{4.25}$$

whenever $u \in C_0^\infty(K)$.

4.3.1 Preliminaries

We consider the boundary operator $B(x,D)u = \sigma(u).n + a_0 i\tau$ and the conjugate operator $e^{\tau\varphi}B(x,D)e^{-\tau\varphi}$ with principal symbol given by

$$B(x,\xi,\tau) = B_0(x)\xi_n + B_1(x,\xi',\tau) \tag{4.26}$$

where B_0 and B_1 two tangential symbols define by

$$\begin{cases} B_0 = \mu Id + (\mu+\lambda)\ell_0^t\ell_0 \\ \\ B_1 = i\tau\varphi'_{x_n}B_0 + \mu\ell_1^t\ell_0 + \lambda\ell_0^t\ell_1 - a_0\tau Id. \end{cases} \tag{4.27}$$

Let $u \in C_0^\infty(K)$ denote

$$\widetilde{u} = op(\chi)u \qquad \widetilde{f} = op(P)\widetilde{u} \tag{4.28}$$

where $op(P)$ the differential operator with principal symbol

$$P(x,\xi,\tau) = P_0(x)\xi_n^2 + P_1(x,\xi',\tau)\xi_n + P_2(x,\xi',\tau) \tag{4.29}$$

It is easy to see that

$$\begin{cases} op(P)\widetilde{u} = \widetilde{f} \quad in \quad x_n > 0 \\ \\ op(B)\widetilde{u} = \widetilde{g} \quad on \quad x_n = 0 \end{cases} \tag{4.30}$$

where $\widetilde{f} = op(\chi)f + [op(P); op(\chi)]u$ and $\widetilde{g} = op(\chi)g + [op(B); op(\chi)]u$.
Let us reduce the problem (4.30) to a first order system. Put $v = {}^t(< D', \tau > \tilde{u}, D_n\tilde{u})$ then the system (4.30) is reduced to the following first system

$$(4.31) \qquad \begin{cases} D_n v - op(\mathcal{A})v = F & in \quad x_n > 0 \\ op(\mathcal{B})v = \widetilde{g} & on \quad x_n = 0 \end{cases}$$

Where the principal symbol of $op(\mathcal{A})$ is given by

$$(4.32) \qquad \mathcal{A} = \begin{pmatrix} 0 & - < \xi', \tau > \\ - < \xi', \tau >^{-1} P_0^{-1}P_2 & -P_0^{-1}P_1 \end{pmatrix}$$

and $\mathcal{B}$ the tangential symbol

$$(4.33) \qquad \mathcal{B} = (< \xi', \tau >^{-1} B_1, B_0)$$

with B_0 and B_1 are defined by (2.3) and $F = {}^t(0, \tilde{f})$.
For (x_0, ξ'_0, τ_0) fixed in $\mathrm{supp}(\chi)$ then the eigenvalues of the matrix $\mathcal{A}$ are $z_1^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_\mu$ and $z_2^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_{2\mu+\lambda}$ with $\pm Im(z_k^\pm) > 0$.
Denote by $S = (s_1^+, ..., s_n^+, s_1^-, ..., s_n^-)$ a basis of the generalized eigenspace of the matrix $\mathcal{A}(x_0, \xi'_0, \tau_0)$ corresponding to eigenvalues with positive or non positive imaginary parts. Then we can extended S as a smooth positively homogeneous function of degree zero in a small conic neighborhood of (x_0, ξ'_0, τ_0) . Let $S(x, D_{x'}, \tau)$ the pseudo-differential operator with principal symbol $S(x, \xi', \tau)$. Then by the argument in TAYLOR [T2] (see also YAMAMOTO [Y]) there exist a pseudo-differential operator $K(x, D_{x'}, \tau)$ of order -1 such that the boundary value problem (4.35) is reduced to the following

$$(4.34) \qquad \begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in \ x_n > 0 \\ \\ op(\widetilde{\mathcal{B}})w = \widetilde{g}_0 & on \ x_n = 0 \end{cases}$$

Where $w = (I + K)^{-1}S^{-1}v$, $\widetilde{F} = (I + K)^{-1}S^{-1}F$, $\widetilde{\mathcal{B}} = \mathcal{B}S(I + K)^{-1}$
and $op(\mathcal{H})$ a tangential operator of order 1 with principal symbol $\mathcal{H} = diag(\mathcal{H}^+, \mathcal{H}^-)$ such that

$$(4.35) \qquad \pm Im(\mathcal{H}^\pm) \geq C < \xi', \tau > .$$

We can now state the Lemma which will be the key ingredient in the proof of the Proposition 4.4.

Lemma 4.2 *There exist a hermitian matrix* $\mathcal{R}(x, \xi', \tau)$, $C_1, C_2 > 0$ *such that*

$$(4.36) \qquad Im(\mathcal{R}\mathcal{H}) = e(x, \xi', \tau)$$

and we have the following properties

i) $e(x, \xi', \tau) \geq C_1 I$ *in* $\mathrm{supp}(\chi)$

$ii) -\mathcal{R}(x,\xi',\tau) + C_2\widetilde{\mathcal{B}}^*\widetilde{\mathcal{B}} \geq CI \qquad on \ \ \{x_n = 0\} \cap \mathrm{supp}(\chi)$

Proof . Let $\widetilde{\mathcal{B}}$ the principal symbol of $op(\widetilde{\mathcal{B}})$ where $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{B}}^+, \widetilde{\mathcal{B}}^-)$, and $\widetilde{\mathcal{B}}^+ = \widetilde{\mathcal{B}}S^+$ the projection of $\widetilde{\mathcal{B}}$ in eigenspace generated by the vectors of S^+. First we prove $\widetilde{\mathcal{B}}^+$ is an isomorphism. The eigenvalues of $\mathcal{A}$ are $z_1^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_\mu$ and $z_2^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_{2\mu+\lambda}$ such that $\pm Im(z_k^\pm) > 0$ and with multiplicity respectively $(n-1)$ and 1.
Now let $X = (X_1, X_2) \in \mathbb{C}^n \oplus \mathbb{C}^n$ eigenvector of $\mathcal{A}$ associated to z_0 then X satisfy

$$(4.37) \qquad \begin{cases} <\xi',\tau> X_2 = z_0 X_1 \\ P(z_0).X_1 = 0 \end{cases}$$

a) Calculus of eigenvector associated to z_1^+:
Denote by $\Big\{\omega_j\Big\}_{j=1,\dots,n-2}$ a basis of $\Big\{\ell_0,\ell_1\Big\}^\perp$ then we have

$$(4.38) \qquad P(z_1^+)\omega_j^+ = 0 \qquad for \ j \in \Big\{1,\dots,n-2\Big\}$$

where

$$(4.39) \qquad P(z_1^+) = (\mu+\lambda)(\ell_1 + i\alpha_\mu\ell_0)^t(\ell_1 + i\alpha_\mu\ell_0)$$

Now we set the following vector in $\mathbb{C}^n$

$$(4.40) \qquad \omega_{n-1} = <\xi',\tau>^{-1} \left((i\alpha)^2\ell_0 + i\alpha_\mu\ell_1\right)$$

then we have by a simple calculation

$$(4.41) \qquad P(z_1^+)\omega_{n-1}^+ = 0$$

b) Calculus of eigenvector associated to z_2^+:
We get

$$(4.42) \qquad P(z_2^+) = -\frac{\mu+\lambda}{2\mu+\lambda}Id + (\mu+\lambda)(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0)^t(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0).$$

Let now

$$(4.43) \qquad \omega_n^+ = (i\alpha_{2\mu+\lambda})\ell_0 + \ell_1$$

then we have $P(z_2^+)\omega_n^+ = 0$, using $(3,14)$ we denote

$$(4.44) \qquad \begin{cases} s_j^+ = (\omega_j^+; <\xi',\tau>^{-1} z_1^+\omega_j^+) \quad j \in \Big\{1,\dots,n-1\Big\} \\ \\ s_n^+ = (\omega_n^+; <\xi',\tau>^{-1} z_2^+\omega_n^+) \end{cases}$$

and by $(b_1^+,...,b_n^+,b_1^-,...,b_n^-)$ the principal symbol of $\widetilde{\mathcal{B}}$. By elementary calculus we get $\widetilde{\mathcal{B}}^+ = \widetilde{\mathcal{B}}S^+ = (b_1^+,...,b_n^+)$ where

$$(4.45)\qquad \begin{cases} b_j^+ = (\mu(i\alpha_\mu) - a_0\tau)\omega_j^+ \\ \\ b_{n-1}^+ = [2\mu(i\alpha_\mu) - a_0\tau)](i\alpha)^2\ell_0 + [2\mu(i\alpha)^2 - a_0\tau(i\alpha_\mu) + \tau^2]\ell_1 \\ \\ b_n^+ = [2\mu(i\alpha)^2 - a_0\tau(i\alpha_{2\mu+\lambda}) + \tau^2]\ell_0 + [2\mu(i\alpha_{2\mu+\lambda}) - a_0\tau)]\ell_1 \end{cases}$$

Then we get

$$(4.46)\qquad det(\widetilde{\mathcal{B}}^+) = (\mu(i\alpha_\mu) - a_0\tau)^{n-1}R_0\Big(\frac{\sqrt{\mu}}{\tau}\alpha(x,\xi',\tau)\Big)$$

where R_0 is a function given by

$$(4.47)\qquad \begin{aligned} R_0(s) \;=\; & (1-2s^2)^2 + (\frac{a_0^2}{\mu} - 4s^2)\Big(\sqrt{s^2-1}\sqrt{s^2 - \frac{\mu}{2\mu+\lambda}}\Big) \\ & -\frac{a_0^2}{\mu}s^2 + \frac{ia_0}{\sqrt{\mu}}(1-2s^2)\Big(\sqrt{s^2-1} + \sqrt{s^2 - \frac{\mu}{2\mu+\lambda}}\Big). \end{aligned}$$

We define the characteristic manifold by

$$(4.48)\qquad \Sigma = \Big\{(x',\xi',\tau) \in T^*\partial\Omega, \quad R_0\Big(\frac{\sqrt{\mu}}{\tau}\alpha(x,\xi',\tau)\Big) = 0\Big\}.$$

Therefore $\widetilde{\mathcal{B}}^+$ is elliptic outside Σ.
Let z_0 a root of R_0 such that $\frac{\sqrt{\mu}}{\tau}\alpha(x,\xi',\tau) = z_0$ then we have

$$(4.49)\qquad \begin{aligned} \alpha^2(x,\xi',\tau) &= \tau^2{\varphi'_{x_n}}^2 + q_2 + 2i\tau q_1 \\ &= \frac{\tau^2}{\mu}z_0^2 \end{aligned}$$

then we have in the region E^+.

$$(4.50)\qquad \frac{\tau^2}{\mu}Rez_0^2 - \tau^2{\varphi'_{x_n}}^2 - \frac{\tau^2}{\mu} + \frac{(Imz_0^2)^2}{4{\varphi'_{x_n}}^2}\tau^2 > 0.$$

this is impossible if we put $\varphi'_{x_n} > C_0$ for large $C_0 > 0$. Then $E^+ \cap \Sigma = \emptyset$, and $\mathcal{B}^+$ is an isomorphism in the elliptic region E^+.
Now let us search that the symmetrieser $\mathcal{R}$ in the form $diag(Id, \rho Id)$, then we have

$$(4.51)\qquad \begin{aligned} Im(\mathcal{R}\mathcal{H}) &= diag\Big(Im(\mathcal{H}^+), -\rho Im(\mathcal{H}^-)\Big) \\ &:= e(x,\xi',\tau) \end{aligned}$$

where $e(x,\xi',\tau) \geq CId_{2n}$.
Let $w = (w^+,w^-) \in \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ then we have $\widetilde{\mathcal{B}}w = \widetilde{\mathcal{B}}^+w^+ + \widetilde{\mathcal{B}}^-w^-$ and there exist $C > 0$ such that for any $w \in \mathbb{C}^{2n}$ we get

$$(4.52)\qquad |w^+|^2 \leq C(|w^-|^2 + |\mathcal{B}w|^2)$$

we deduce

$$
\begin{aligned}
(4.53) \qquad -(\mathcal{R}w, w) &= -|w^+|^2 + \rho|w^-|^2 \\
&\geq |w^+|^2 + (\rho - 2C')|w^-|^2 - 2C'|\mathcal{B}w|^2.
\end{aligned}
$$

Then we have the desirable estimation, for large ρ •

4.3.2 Proof of Proposition4.4

Let G the function define by

$$
(4.54) \qquad G(x_n) = \frac{d}{dx_n}(op(\mathcal{R})w, w)_{L^2(\mathbb{R}^{n-1})}.
$$

Using $D_n w - op(\mathcal{H})w = \widetilde{F}$ then we obtain

$$
(4.55)\; G(x_n) = -2Im(op(\mathcal{R})op(\mathcal{H})w, w) + 2Im(op(\mathcal{R})w, \widetilde{F}) + (op(\mathcal{R}'_{x_n})w, w)
$$

The integration in the normal direction gives

$$
\begin{aligned}
(4.56) \quad (op(\mathcal{R})w, w)_0 &= 2\int_0^\infty Im(op(\mathcal{R})op(\mathcal{H})w, w)dx_n \\
&\quad -2Im\int_0^\infty (op(\mathcal{R})w, \widetilde{F}) - \int_0^\infty op(\mathcal{R}'_{x_n})w, w)dx_n
\end{aligned}
$$

Taking into account Lemma 4.2 i) we have

$$
(4.57) \qquad Im(op(\mathcal{R})op(\mathcal{H})w, w) \geq C\tau||w||^2
$$

and by (ii) we obtain

$$
(4.58) \qquad (op(\mathcal{R})w, w)_0 \leq C|\mathcal{B}w|^2 - C'|w|^2.
$$

Combining (4.62) (4.61) with (4.60) we get

$$
(4.59) \qquad C\tau||w||^2 + C'|w|_0^2 \leq ||\widetilde{F}||_0^2 + \tau|\mathcal{B}w|^2.
$$

this complete the proof of Proposition4.4.

Remarks 4.1 *In the elliptic region E^+, τ can be degenerate to zero. But for $\tau = 0$ the problem $L(x, D)u = \mu\Delta u + (\mu + \lambda)\nabla(\operatorname{div} u)$ with Neumann boundary condition $N(u) = \sigma(u).n$ is a coercive elliptic boundary problem, and we can prove the Lemma 4.2 in this case (see Lemma 5.2 of [Be]).*

4.4 Carleman estimate in a non elliptic regions

The purpose of this section is to get the Carleman estimates in the non elliptic region. Precisely we take a cutoff function $\chi_0(x,\xi',\tau)$ homogeneous of degree zero in a neighborhood of the regions $Z_\gamma \cup M$. Our purpose here is to prove the following Lemma.

Lemma 4.3 *There exist* $C>0$ *such that for any large enough* τ *we have*

$$||P(x,D,\tau)u||^2+\tau|u|^2_{1,0,\tau}+||u||^2_{1,\tau}\geq C\tau||op(\chi_0)u||^2_{1,\tau} \tag{4.60}$$

whenever $u\in C_0^\infty(K)$.
Furthermore if we assume that $\varphi'_{x_n}>C_0$ *on* $\{x_n=0\}\cap supp\chi_0$ *then we have*

$$\begin{aligned}||P(x,D,\tau)u||^2+||u||^2_{1,\tau}+|u|^2_{1,\tau}+\tau|B(x,D,\tau)u|^2&\\ \geq C\Big(\tau||op(\chi_0)u||^2_{1,\tau}+\tau|op(\chi_0)u|^2_{1,\tau}\Big)&\end{aligned} \tag{4.61}$$

whenever $u\in C_0^\infty(K)$.

The first part of the previous Lemma is proved in [Be] (see proposition 3.1). Then we study only the case when the cutoff function supported in a neighborhood of the region Z_γ. For the proof of the second part we need the next Lemma.

Lemma 4.4 *Assume that* $\varphi'_{x_n}>C_0$ *on* $\{x_n=0\}\cap supp\chi_0$ *then we have*

$$||P(x,D,\tau)u||^2+\tau|B(x,D,\tau)u|^2+||u||^2_{1,\tau}+|u|^2_{1,\tau}\geq C\tau|op(\chi_0)u|^2_{1,0,\tau} \tag{4.62}$$

whenever $u\in C_0^\infty(K)$.

Let

$$\widetilde{u}=op(\chi_0)u,\quad \widetilde{f}=op(P)\widetilde{u} \tag{4.63}$$

by an argument similar to the one of section 4.3, we have

$$\left\{\begin{array}{ll} D_n w-op(\mathcal{H})w=\widetilde{F} & in\ x_n>0\\ \\ op(\widetilde{\mathcal{B}})w=\widetilde{g}_0 & on\ x_n=0\end{array}\right. \tag{4.64}$$

Where $w=(I+K)^{-1}S^{-1}v$, $\widetilde{F}=(I+K)^{-1}S^{-1}F$, $\widetilde{\mathcal{B}}=\mathcal{B}S(I+K)^{-1}$
and $op(\mathcal{H})$ is an $n\times n$ square matrix, whose components are tangential pseudo-differential operators of order 1 with principal symbol $\mathcal{H}=diag(\mathcal{H}^+,\mathcal{H}^-)$, such that $\mathcal{H}^-$ satisfy $-Im(\mathcal{H}^-)\geq C<\xi',\tau>$. The proof of Lemma 4.4 is based on the next Lemma

Lemma 4.5 *Let* $\mathcal{R}=diag(0,-\rho Id_n)$ *then there exist* $C>0$ *such that*

i)$Im(\mathcal{R}\mathcal{H})=diag(0,e(x,\xi',\tau))$

ii)$e(x,\xi',\tau)\geq C<\xi',\tau>$ *in* $supp\chi_0$

iii)$-\mathcal{R}+\widetilde{\mathcal{B}}^*\widetilde{\mathcal{B}}\geq C diag(0,Id)$ *on* $\{x_n=0\}\cap supp\chi_0$

4.4.1 Proof of Lemma 4.4

Denote the function define $G(x_n) = \frac{d}{dx_n}\Big(op(\mathcal{R})w, w\Big)_{L^2(\mathbb{R}^{n-1})}$.
Taking into account $D_n w - op(\mathcal{H})w = \widetilde{F}$ then we obtain

(4.65) $G(x_n) = -2Im(op(\mathcal{R})op(\mathcal{H})w, w) + 2Im(op(\mathcal{R})w, \widetilde{F}) + (op(\mathcal{R}'_{x_n})w, w)$

The integration in the normal direction gives

$$\begin{aligned}(4.66)\quad (op(\mathcal{R})w, w)_0 \;=\; & 2\int_0^\infty Im(op(\mathcal{R})op(\mathcal{H})w, w)dx_n \\ & -2Im\int_0^\infty (op(\mathcal{R})w, \widetilde{F}) - \int_0^\infty op(\mathcal{R}'_{x_n})w, w)dx_n\end{aligned}$$

Taking into account Lemma 4.5 and Gärding inequality we have for large τ

$$(4.67)\qquad Im(op(\mathcal{R})op(\mathcal{H})w, w) \geq C\tau||w^-||^2$$

and further, for any ε

$$(4.68)\qquad \int_0^\infty |(op(\mathcal{R})w, \widetilde{F})|dx_n \leq \varepsilon C\tau||w^-||^2 + \frac{C_\varepsilon}{\tau}||\widetilde{f}||^2.$$

Apply now Lemma 4.5 we obtain

$$(4.69)\qquad -(op(\mathcal{R})w, w) + C|\widetilde{\mathcal{B}}w|^2 \geq C|w|^2.$$

Combining (4.73) (4.72) with (4.71) we get

$$(4.70)\qquad C\tau||w^-||^2 + C'|w|_0^2 \leq \frac{C}{\tau}||\widetilde{f}||_0^2 + |\widetilde{\mathcal{B}}w|^2.$$

This complete the proof of Lemma 4.4.

4.5 Carleman estimate in the region E^-

The purpose of this section is to proof the Carleman estimate in the region E^-. Let the cutoff function $\chi_1(x, \xi', \tau)$ homogeneous of degree zero in the region E^-, then we have the following Lemma.

Lemma 4.6 *Assume that* $\varphi'_{x_n} > 0$ *on* $supp\chi_1 \cap \{x_n = 0\}$, *then there exist* $C > 0$ *and* τ_0 *such that for any* $\tau \geq \tau_0$ *we have*

$$(4.71)\quad ||P(x, D, \tau)u||^2 + ||u||^2_{1,\tau} + |u|^2_{1,\tau} \geq C\tau\Big(||op(\chi_1)u||^2_{1,\tau} + |op(\chi_1)u|^2_{1,\tau}\Big),$$

whenever $u \in C_0^\infty$.

Proof . Let

$$\widetilde{u} = op(\chi_1)u, \qquad \widetilde{f} = op(P)\widetilde{u}, \tag{4.72}$$

by an argument similar to the one of section 4.3, we have

$$\begin{cases} D_n w - op(\mathcal{H}^-)w = \widetilde{F} & in\ x_n > 0 \\ \\ op(\widetilde{\mathcal{B}})w = \widetilde{g}_0 & on\ x_n = 0 \end{cases} \tag{4.73}$$

Where $w = (I+K)^{-1}S^{-1}v$, $\tilde{F} = (I+K)^{-1}S^{-1}F$, $\tilde{\mathcal{B}} = \mathcal{B}S(I+K)^{-1}$
and $op(\mathcal{H}^-)$ is an $n \times n$ square matrix, whose components are tangential pseudo-differential operators of order 1 with principal symbol $\mathcal{H}^-$, such that $Im(\mathcal{H}^-) \geq C < \xi', \tau >$, let $G(x_n) = -\frac{d}{dx_n}(w,w)$, using 4.73 then we have

$$G(x_n) = -2Im(op(\mathcal{H}^-)w, w) + 2Im(w, \widetilde{F}). \tag{4.74}$$

The integration in x_n gives

$$-|w|_0^2 = 2\int_0^\infty -Im(op(\mathcal{H}^-)w, w)dx_n - 2Im\int_0^\infty (w, \tilde{F}). \tag{4.75}$$

By Gärding inequality we have

$$-Im(op(\mathcal{H}^-)w, w) \geq C\tau||w||^2 \tag{4.76}$$

Furthermore, for any ε

$$\int_0^\infty |(w, \widetilde{F})|dx_n \leq \varepsilon C\tau||w||^2 + \frac{C_\varepsilon}{\tau}||\widetilde{f}||^2. \tag{4.77}$$

Combining (4.77) (4.76) with (4.75) we get

$$C\tau||w||^2 + C'|w|_0^2 \leq \frac{C}{\tau}||\widetilde{f}||_0^2 + |\widetilde{\mathcal{B}}w|^2. \tag{4.78}$$

This complete the proof.
Combining Lemma 4.6, Lemma 4.4 and Proposition 4.4 we get Proposition 2.2 for large τ •

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